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Wave solutions for the deterministic host-vector epidemic

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1. *Introduction*

Recent papers by Atkinson and Reuter (1), Brown and Carr (5), and Barbour (2) have proved several important results concerning wave solutions of the usual deterministic model for the spatial spread of an epidemic, such as measles or influenza. In particular Atkinson and Reuter showed that non-trivial wave solutions, in a population along a line, only exist provided the contact distribution function has an exponentially bounded tail, and that the speed of propagation c must be at least some critical value c_0. They constructed a solution for $c > c_0$, which Barbour later showed to be the unique solution, modulo translation, at that speed. Brown and Carr showed that a solution was also possible at speed c_0, though it has not been possible to show uniqueness at this speed.

Results analogous to those of Atkinson and Reuter and Barbour have been obtained, using different methods, by Diekmann (6, 7) and Diekmann and Kaper (10) for more general situations, which include the spread of an advantageous gene and an epidemic in which the infectivity of an individual is a function of the time since infection.

The related problems concerning the asymptotic speed of propagation, limiting behaviour and pandemic theorem have been investigated by Diekmann (8, 9) and Thieme (13, 14, 15).

Similar results can be conjectured for more complex models involving more than one type of individual, such as host-vector and carrier-borne epidemics and rabies involving several species of animals. The pandemic theorem for the host-vector epidemic has been considered by Radcliffe (12).

As a first step in the analysis of such models we confine our attention to possible wave solutions for the host-vector epidemic. Sections 2, 3, 4 and 5 set up the equations, derive conditions for the existence of non-trivial solutions, prove results on the existence of certain transforms, and motivate the use of a 'characteristic equation'. Section 6 shows the existence of a unique solution, modulo translation, for each speed $c > c_0$. The non-existence of solutions for $c < c_0$ is shown in section 7. Finally the case $c = c_0$ is discussed in section 8.

2. *Models for the deterministic host-vector epidemic*

This paper considers a host-vector epidemic arising from a viral or asexual parasitic disease, which is transferred from infected vector to uninfected host and from infected host to uninfected vector. A stochastic model for this situation has been suggested by Radcliffe (11). For application to a specific disease, in order to obtain a realistic

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though simplified model, the characteristics of the disease should be reflected in the choice of the parameters in the model. Two possible models are considered and in each case it is shown that we are led to study a pair of integral equations.

Model 1

Consider two populations of uniform densities on the real line R, each of which consists of susceptible, infected and removed individuals. Denote the proportions of susceptible, infected and removed in the ith population at position s and time t by $x_i(s,t)$, $y_i(s,t)$ and $z_i(s,t)$, $(i = 0, 1)$; so that $x_i(s,t)+y_i(s,t)+z_i(s,t) = 1$. The density of individuals in the ith population is σ_i. The rate of infection from infected individuals in that population is λ_i, and the contact distribution representing the distance r over which infection occurs has density $p_i(r)$. The removal rate for infected individuals in the ith population is μ_i. We consider a model described by the equations

$$\left.\begin{aligned} \frac{\partial}{\partial t}x_i(s,t) &= -\sigma_{1-i}\lambda_{1-i}x_i(s,t)\,\bar{y}_{1-i}(s,t), \\ \frac{\partial}{\partial t}y_i(s,t) &= \sigma_{1-i}\lambda_{1-i}x_i(s,t)\,\bar{y}_{1-i}(s,t)-\mu_i y_i(s,t), \\ \frac{\partial}{\partial t}z_i(s,t) &= \mu_i y_i(s,t) \quad (i = 0, 1) \end{aligned}\right\} \tag{1}$$

where

$$\bar{y}_i(s,t) = \int_{-\infty}^{\infty} y_i(s-r,t)\,p_i(r)\,dr.$$

We look for wave solutions travelling with velocity $c > 0$. Let $x_i(s,t) = x_i(u)$, where $u = s-ct$, and $x_i'(u) = dx_i(u)/du$, together with equivalent relations for y_i and z_i. The equations become

$$\left.\begin{aligned} cx_i'(u) &= \sigma_{1-i}\lambda_{1-i}x_i(u)\,\bar{y}_{1-i}(u), \\ cy_i'(u) &= -\sigma_{1-i}\lambda_{1-i}x_i(u)\,\bar{y}_{1-i}(u)+\mu_i y_i(u), \\ cz_i'(u) &= -\mu_i y_i(u) \quad (i = 0, 1) \end{aligned}\right\} \tag{2}$$

with $\bar{y}_i(u) = \int_{-\infty}^{\infty} y_i(u-r)\,p_i(r)\,dr$. We specify that all individuals are susceptible before the wave, so that $x_i(\infty) = 1$ and $y_i(\infty) = z_i(\infty) = 0$ for $i = 0, 1$.

A differential equation in $x_i(u)$ is now obtained. If (2) has a solution then

$$\begin{aligned} y_i'(u)-\mu_i c^{-1}y_i(u) &= -x_i'(u), \\ (y_i(u)\,e^{-\mu_i u/c})' &= -x_i'(u)\,e^{-\mu_i u/c}, \\ y_i(u) &= \int_0^{\infty} x_i'(u+s)\,e^{-\mu_i s/c}\,ds. \end{aligned} \tag{3}$$

Then from (2) we obtain

$$\begin{aligned} x_{1-i}'(u) &= c^{-1}\sigma_i\lambda_i x_{1-i}(u)\int_{-\infty}^{\infty} p_i(r)\int_0^{\infty} x_1'(u-r+s)\,e^{-\mu_i s/c}\,ds\,dr \\ &= c^{-1}\sigma_i\lambda_i x_{1-i}(u)\int_{-\infty}^{\infty} x_1'(u+t)\int_0^{\infty} p_i(s-t)\,e^{-\mu_i s/c}\,ds\,dt. \end{aligned}$$

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This may be written as

$$x'_{1-i}(u) = x_{1-i}(u)\int_{-\infty}^{\infty} x'_i(u+t)\,v_i(t)\,dt, \tag{4}$$

where $$v_i(t) = c^{-1}\sigma_i\int_0^{\infty} p_i(s-t)\,\lambda_i e^{-\mu_i s/c}\,ds, \quad (i = 0, 1).$$

Note that from (2) $x_i(u)$ is monotone increasing. It is also non-negative with $x_i(\infty) = 1$, for $i = 0, 1$.

Defining $y_i(u)$ by (3) and $z_i(u) = 1 - x_i(u) - y_i(u)$, we may reverse these steps and show that any non-negative monotone increasing solution $x_i(u)$ to (4) with $x_i(\infty) = 1$, $(i = 0, 1)$, leads to a solution $x_i(u)$, $y_i(u)$ and $z_i(u)$, $(i = 0, 1)$ satisfying (2) with the boundary conditions $x_i(\infty) = 1$ and $y_i(\infty) = z_i(\infty) = 0$, $(i = 0, 1)$.

Model 2

Again we consider two populations of uniform densities σ_0 and σ_1, each now only consisting of susceptible and infected individuals. Instead of infected individuals being removed at a steady rate, the infectivity is allowed to vary with the time since infection. Let the rate of infection of an individual in population i from an individual in population $(1-i)$ who was infected time τ ago be $\lambda_{1-i}(\tau)$.

The proportions of susceptibles in population i at position s and time t is $x_i(s,t)$ and the contact distribution has density $p_i(r)$. Let $I_i(s,t,\tau)\,d\tau$ be the proportion of individuals at position s and time t who were infected in the time interval $(t-\tau-d\tau, t-\tau)$. Note that $I_i(s,t,\tau) = I_i(s,t-\tau,0)$. The equations for this model are

$$\left.\begin{aligned} \frac{\partial}{\partial t}x_{1-i}(s,t) &= -\sigma_i x_{1-i}(s,t)\int_{-\infty}^{\infty}\int_0^{\infty} I_i(s-r,t,\tau)\,p_i(r)\,\lambda_i(\tau)\,d\tau\,dr,\\ I_i(s,t,0) &= -\frac{\partial}{\partial t}x_i(s,t) \quad (i = 0, 1). \end{aligned}\right\} \tag{5}$$

Again we look for wave solutions travelling with velocity $c > 0$. Let $x_i(s,t) = x_i(u)$ and $I_i(s,t,\tau) = I_i(u,t)$ where $u = s - ct$. The equations become

$$\left.\begin{aligned} x'_{1-i}(u) &= c^{-1}\sigma_i x_{1-i}(u)\int_{-\infty}^{\infty}\int_0^{\infty} I_i(u-r+c\tau,0)\,p_i(r)\,\lambda_i(\tau)\,d\tau\,dr,\\ I_i(u,0) &= cx'_i(u). \end{aligned}\right\} \tag{6}$$

We specify that all individuals are susceptible before the wave, i.e. $x_i(\infty) = 1$, for $i = 0, 1$.

From equations (6) we obtain a differential equation in $x_i(u)$,

$$x'_{1-i}(u) = x_{1-i}(u)\int_{-\infty}^{\infty} x'_i(u+t)\,v_i(t)\,dt, \tag{7}$$

where $$v_i(t) = c^{-1}\sigma_i\int_0^{\infty} p_i(s-t)\,\lambda_i(s/c)\,ds, \quad (i = 0, 1). \tag{8}$$

The condition on $I_i(u,0)$ in (6) implies that $x_i(u)$ is monotone increasing. Any non-negative, monotone increasing solution to (7) with $x_i(\infty) = 1$ $(i = 0, 1)$ is a solution to (6) with $I_i(u,0) \geqslant 0$.

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Observe that equations (4) are a special case of equations (7) with $\lambda_i(\tau) = \lambda_i e^{-\mu_i \tau}$ with identical conditions on $x_i(u)$. Both may therefore be treated simultaneously.

We now relate the solutions of (7) to the solutions of each of two pairs of equations; firstly

$$x_{1-i}'(u) = x_{1-i}(u) \int_{-\infty}^{\infty} (1 - x_i(u+t))\, v_i'(t)\, dt, \quad (i = 0, 1), \tag{9}$$

with $x_i(u)$ non-negative, monotone increasing and $x_i(\infty) = 1$, $(i = 0, 1)$; secondly

$$w_{1-i}(u) = \int_{-\infty}^{\infty} (1 - e^{-w_i(u+t)})\, v_i(t)\, dt, \quad (i = 0, 1), \tag{10}$$

with $w_i(u)$ monotone decreasing and $w_i(\infty) = 0$, $(i = 0, 1)$.

In order to do this we need to impose some conditions on $v_i(t)$. We consider it more appropriate to place the conditions on $p_i(r)$ and $\lambda_i(\tau)$, rather than on $v_i(t)$ directly. $p_i(r)$ is assumed to be continuous for all r. A set of conditions for $\lambda_i(\tau)$, which are valid for Model 1 and will suffice for the remainder of this paper, are that $\lambda_i(\tau)$ is bounded everywhere and is convex for τ sufficiently large. In addition $\lambda_i'(\tau)$ exists, is continuous and bounded, and changes sign at most a finite number of times.

LEMMA 1. *There is a one to one correspondence between the non-negative monotone increasing solutions* $x_i(u)$ *(with* $x_i(\infty) = 1$, $(i = 0, 1)$*) of each of equations* (7) *and* (9) *and the monotone decreasing solutions* $w_i(u)$ *(with* $w_i(\infty) = 0$, $(i = 0, 1)$*) of equations* (10)*, given by* $w_i(u) = -\log x_i(u)$.

Proof. We first show the equivalence of solutions $x_i(u)$ to equations (7) and (9). Observe that $v_i(t)$ is differentiable, since $\lambda_i'(s)$ exists and is bounded and $p_i(t)$ is continuous, and

$$v_i'(s) = \sigma_i c^{-1} p_i(-s)\, \lambda_i(0) + \sigma_i c^{-2} \int_{-s}^{\infty} p_i(t)\, \lambda_i'((s+t)/c)\, dt. \tag{11}$$

Integrating (7) by parts, and noting that $v_i(-\infty) = 0$ and $v_i(+\infty)$ is finite, we obtain equations (9). This integration can be reversed. Hence solutions $x_i(u)$, with the stated conditions, to (7) and (9) are identical.

It is now sufficient to show the equivalence of the solutions to equations (9) and (10). Let $x_i(u)$, $(i = 0, 1)$, be non-negative monotone solutions to (9) with $x_i(\infty) = 1$. Then $w_i(u) = -\log x_i(u)$, $(i = 0, 1)$, are monotone decreasing to zero as $u \to \infty$. Equations (9) may be written in terms of $w_i(u)$,

$$w_{1-i}'(u) = -\int_{-\infty}^{\infty} (1 - e^{-w_i(u+s)})\, v_i'(s)\, ds, \quad (i = 0, 1). \tag{12}$$

Integrating from u to ∞, and noting that $w_i(\infty) = 0$ and $v_i(-\infty) = 0$, we obtain equations (10).

Now consider any monotone decreasing solution $w_i(u)$ to (10) with $w_i(\infty) = 0$, $(i = 0, 1)$. Let $x_i(u) = e^{-w_i(u)}$. Then $x_i(u)$ is non-negative, monotone increasing with $x_i(\infty) = 1$, $(i = 0, 1)$. Writing equations (10) in terms of $x_i(u)$ we obtain

$$-\log x_{1-i}(u) = \int_{-\infty}^{\infty} (1 - x_i(t))\, v_i(t-u)\, dt, \quad (i = 0, 1). \tag{13}$$

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Had we imposed the conditions on the $v_i(t)$ directly we could use this equation to establish the differentiability of $x_i(u)$. Since the conditions are mainly on $\lambda_i(\tau)$ it is convenient to rewrite the equations in the form

$$-\log x_{1-i}(u) = c^{-1}\sigma_i \int_u^\infty D_i(t)\,\lambda_i[(t-u)/c]\,dt, \tag{14}$$

where

$$D_i(t) = \int_{-\infty}^{\infty} (1 - x_i(s)\,p_i(t-s))\,ds.$$

Since $p_i(t)$ is continuous and integrable and $x_i(s)$ is bounded, it is easily shown that $D_i(t)$ is continuous. For any given u

$$\frac{d}{du}\int_u^\infty D_i(t)\,\lambda_i[(t-u)/c]\,dt = -D_i(u)\,\lambda_i(0) + \lim_{\delta u \to 0} \int_u^\infty D_i(t)\left\{\frac{\lambda_i[(t-u-\delta u)/c] - \lambda_i[(t-u)/c]}{\delta u}\right\} dt.$$

We may split up the range of integration on the right into (u, a) and (a, ∞), where a is sufficiently large such that $\lambda_i(\tau)$ is convex for $\tau > (a-u-1)/c$. Applying dominated convergence and monotone convergence respectively to the integrals over these ranges we obtain the result

$$\frac{d}{du}\int_u^\infty D_i(t)\,\lambda_i[(t-u)/c]\,dt = -D_i(u)\,\lambda_i(0) - c^{-1}\int_u^\infty D_i(t)\,\lambda_i'[(t-u)/c]\,dt.$$

Hence $x_{1-i}(u)$ is differentiable and

$$\begin{aligned} x_{1-i}'(u)/x_{1-i}(u) &= \sigma_i c^{-1}\{D_i(u)\,\lambda_i(0) + c^{-1}\int_u^\infty D_i(t)\,\lambda_i'[(t-u)/c]\,dt\} \\ &= \int_{-\infty}^{\infty} (1 - x_i(s))\,v_i'(s-u)\,ds. \end{aligned}$$

Thus we can differentiate equations (10) to obtain equations (9), which concludes the proof of the lemma.

In view of this lemma, our analysis of possible wave solutions for both models 1 and 2 is based on the fundamental integral equations (10) together with the conditions that $w_i(u)$ decreases monotonically to zero as $u \to \infty$ for $i = 0, 1$.

Equations (9), with the equivalent conditions on $x_i(u)$, will be used in the last section when we use a limiting argument to prove the existence of a non-trivial wave solution at the critical speed c_0.

3. *A condition for the existence of a non-trivial solution*

In this section we obtain a condition on the rates of infection which must be satisfied for a solution, other than the trivial solution, $x_i(u) \equiv 1$, $(i = 0, 1)$ to be possible.

Define $\gamma_i = \int_{-\infty}^{\infty} v_i(s)\,ds = \sigma_i \int_0^\infty \lambda_i(\tau)\,d\tau$, $(i = 0, 1)$. Clearly γ_0 and γ_1 do not depend on the speed of propagation c. It is possible for either γ_0 or γ_1 to be infinite, and in fact both will be infinite for the model of the simple epidemic described by equations (1) with $\mu_0 = \mu_1 = 0$.

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THEOREM 1. *A necessary condition for the existence of a non-trivial solution to equations* (10) *is that* $\gamma_0\gamma_1 > 1$.

Proof. Since $x_i(u) = e^{-w_i(u)}$, $(i = 0, 1)$ are non-negative and monotone increasing, each will tend to a limit as $u \to -\infty$. Let $l_i = \lim_{u\to-\infty} x_i(u)$, $(i = 0, 1)$, where $0 \leqslant l_i \leqslant 1$. The case $l_0 = l_1 = 1$ corresponds to the trivial solution $x_i(u) \equiv 1$, $(i = 0, 1)$. Consider equation (10). The left hand side converges to $-\log l_{1-i}$ as $u \to -\infty$. Since the integrand on the right solution is non-negative, $l_{1-1} = 1$ implies that $l_i = 1$. Thus for a non-trivial solution it is necessary that $l_i < 1$, $(i = 0, 1)$. In this case the right hand side of the equation will tend, by monotone convergence, to $(1-l_i)\gamma_i$ for γ_i finite, and to infinity if γ_i is infinite. Hence for a non-trivial solution of equations (10) to be possible, there must exist $0 \leqslant l_i < 1$, $(i = 0, 1)$, satisfying the equations

$$\gamma_i(l_i - 1) = \log l_{1-i} \quad (i = 0, 1). \tag{15}$$

When γ_i is infinite, $l_{1-i} = 0$ and hence $l_i = e^{-\gamma_{1-i}}$. Thus a unique solution of (15) in the range $0 \leqslant l_i < 1$, $(i = 0, 1)$ occurs when at least one of γ_0 and γ_1 is infinite.

When γ_0 and γ_1 are finite, consider

$$f_i(l_i) = e^{\gamma_i(l_i-1)} - (1 + \gamma_{1-i}^{-1}\log l_i) = 0, \quad (i = 0, 1).$$

This function $f_i(l_i)$ is convex, $f_i(0+) = \infty$, and $f_i(1) = 0$. Hence if $f_i'(1) > 0$, that is $\gamma_0\gamma_1 > 1$, there is a unique non-trivial solution in the range $0 \leqslant l_i < 1$, $(i = 0, 1)$. If $f_i'(1) \leqslant 0$, that is $\gamma_0\gamma_1 \leqslant 1$, no solution in this range is possible. Hence for all γ_0 and γ_1, a non-trivial solution of equations (9) is only possible if $\gamma_0\gamma_1 > 1$, which proves the theorem.

This condition, expressed in terms of the parameters of the simpler model described by equations (1) is $\sigma_0\sigma_1\lambda_0\lambda_1 > \mu_0\mu_1$. This is the condition, noted by Bartlett (4), for a major epidemic to occur starting from a small initial infection.

4. *On the regions of convergence of certain transforms*

Let $$W_i(\lambda) = \int_{-\infty}^{\infty} e^{\lambda u}\, w_i(u)\, du \quad \text{and} \quad V_i(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda u}\, v_i(u)\, du, \quad (i = 0, 1).$$

We prove theorems concerning the regions of convergence of these transforms in the complex plane. These results are crucial to the proofs of existence, uniqueness and non-existence of wave solutions at different speeds in Sections 6 and 7. The convergence of $W_i(\lambda)$ also implies that, for non-trivial solutions to be possible $p_i(r)$, $(i = 0, 1)$ must have exponentially bounded tails at $+\infty$.

The following lemma, which is stated without proof, is essentially the same as Diekmann and Kaper ((10), lemma 4·4). Let N_+ denote the set of positive integers, and $R_+ = \{x \in R : x \geqslant 0\}$.

LEMMA 2. *Let* $f : R \to R$ *be non negative and* $f \in L_1(R)$. *Define* $f^{(0)}(x) = f(x)$ *and recursively* $f^{(k)}(x) = \int_x^{\infty} f^{(k-1)}(y)\, dy$ *for those values* $k \in N_+$ *for which* $f^{(k-1)} \in L_1(R_+)$. *Then for any* $r \in R$

$$\int_r^{\infty} (x-r)^k f(x)\, dx = k! \int_r^{\infty} f^{(k)}(x)\, dx,$$

in the sense that if one side converges so does the other.

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This lemma is used in the following theorem to prove the existence of $W_i(\lambda)$ in some region. The second part of the theorem is not required until a later section, but is included here since it has a similar proof.

THEOREM 2. *Let $w_i(u)$ be a non-trivial solution of equations* (10) *with $0 \leqslant w_i(u) \leqslant 1$ and $w_i(u)$ decreasing monotonically to zero as $u \to \infty$.*

(i) There exists a positive real δ such that $W_i(\lambda)$ converges for $\lambda \in S_\delta$ where $S_\delta = \{\lambda \in C : 0 < \operatorname{Re} \lambda < \delta\}$.

(ii) If for a real positive β, $V_0(\beta) V_1(\beta) > 1$ and $W_i(\beta)$, $(i = 0, 1)$ converge, then the abscissa of convergence of $W_i(\lambda)$ is to the right of β.

Proof. From equations (11) and (12) we obtain the inequality

$$0 \leqslant [-w'_{1-i}(u)] \leqslant \int_{-\infty}^{\infty} |v'_i(s)|\, ds \leqslant \sigma_i c^{-1}\left(\lambda_i(0) + \int_0^{\infty} |\lambda'_i(\tau)|\, d\tau\right).$$

Let $L_i = \max_\tau \lambda_i(\tau)$. From the conditions on $\lambda'_i(\tau)$ there exist a finite number k of intervals, $(0, a_1), (a_1, a_2), \ldots, (a_{k-1}, \infty)$, such that $\lambda'_i(\tau)$ does not change sign within each interval, for each i. Hence we obtain the inequality

$$0 \leqslant [-w'_{1-i}(u)] \leqslant \sigma_i c^{-1}(\lambda_i(0) + k L_i).$$

Thus $w_i(u) = O(|u|)$ as $u \to -\infty$. If γ_{1-i} is finite, in fact $w_i(u)$ tends to a finite limit. Hence for any non-negative β, $\int_{-\infty}^{r} e^{(\lambda+\beta)u} w_i(u)\, du$, $(i = 0, 1)$ converges for $\operatorname{Re} \lambda > 0$ and any finite r. In order to prove the theorem, it suffices to prove the convergence of $\int_r^{\infty} e^{(\lambda+\beta)u} w_i(u)\, du$, $(i = 0, 1)$ for some $r < \infty$ and $0 < \operatorname{Re} \lambda < \delta$. The cases $\beta = 0$ and $\beta > 0$ prove parts (i) and (ii) respectively.

If $\beta = 0$, since $w_i(u)$ is a non-trivial solution $\gamma_0 \gamma_1 > 1$, and if $\beta > 0, V_0(\beta) V_1(\beta) > 1$. Thus there exists a finite real α such that $\int_{-\alpha}^{\alpha} e^{-\beta u} v_i(u)\, du = \gamma_i^*$, $(i = 0, 1)$ and $\gamma_0^* \gamma_1^* > 1$. Define

$$v_i^*(u) = \begin{cases} e^{-\beta u} v_i(u) & |u| \leqslant \alpha \\ 0 & |u| > \alpha. \end{cases}$$

Putting $f_i(u) = e^{\beta u} w_i(u)$, equations (10) may be expressed in the form

$$f_{1-i}(u) = \theta \int_{-\infty}^{\infty} f_i(u+s) v_i^*(s)\, ds + \psi_i(u), \tag{16}$$

where
$$\psi_i(u) = \int_{-\infty}^{\infty} e^{\beta(u+s)}(1 - e^{-w_i(u+s)})\,(e^{-\beta s} v_i(s) - v_i^*(s))\, ds$$
$$+ \int_{-\infty}^{\infty} e^{\beta(u+s)}(1 - e^{-w_i(u+s)} - \theta w_i(u+s))\, v_i^*(s)\, ds, \quad (i = 0, 1),$$

and θ is chosen so that $0 < \theta < 1$ and $\theta^2 \gamma_0^* \gamma_1^* > 1$. The first term in $\psi_i(u)$ is positive. Since $\lim_{u \to \infty} (1 - e^{-w_i(u)})/w_i(u) = 1$, there exists an $L > 0$ such that $(1 - e^{-w_i(u)} - \theta w_i(u)) > 0$, $(i = 0, 1)$, for all $u > L$. Hence $\psi_i(u) > 0$, $(i = 0, 1)$, for $u > L + \alpha$.

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Define

$$n_i(x) = \begin{cases} \int_x^\infty v_i^*(y)\,dy & x \geqslant 0 \\ -\int_{-\infty}^x v_i^*(y)\,dy & x < 0, \quad (i = 0, 1). \end{cases}$$

Next we integrate equation (16) from r to x for some $r > L+\alpha$. Thus

$$\begin{aligned}
\int_r^x f_{1-i}(u)\,du - \int_r^x \psi_i(u)\,du &= \theta \int_r^x \int_{-\infty}^\infty f_i(u+s)\,v_i^*(s)\,ds\,du. \\
&= \theta \int_r^x \int_{-\alpha}^\alpha f_i(u+s)\,v_i^*(s)\,ds\,du \\
&= \theta \int_r^x \left\{ \left[f_i(u+s) \int_{-\alpha}^s v_i^*(t)\,dt \right]_{-\alpha}^\alpha - \int_{-\alpha}^\alpha f_i'(u+s) \left[\int_{-\alpha}^s v_i^*(t)\,dt \right] ds \right\} du \\
&= \theta \int_r^x \left\{ \gamma_i^* f_i(u+\alpha) - \left[\int_{-\alpha}^0 (-n_i(s)) f_i'(u+s)\,ds + \int_0^\alpha [\gamma_i^* - n_i(s)] f_i'(u+s)\,ds \right] \right\} du \\
&= \theta \int_r^x \left\{ \gamma_i^* f_i(u+\alpha) - \gamma_i^* [f_i(u+\alpha) - f_i(u)] + \int_{-\alpha}^\alpha n_i(s) f_i'(u+s)\,ds \right\} du \\
&= \theta \gamma_i^* \int_r^x f_i(u)\,du + \theta \int_{-\alpha}^\alpha n_i(s) \int_{r+s}^{x+s} f_i'(t)\,dt\,ds \\
&= \theta \gamma_i^* \int_r^x f_i(u)\,du - \theta \int_{-\infty}^\infty [f_i(u+r) - f_i(u+x)]\,n_i(u)\,du.
\end{aligned}$$

From this one obtains the relation, for $i = 0, 1$,

$$\begin{aligned}
(\theta^2 \gamma_0^* \gamma_1^* - 1) \int_r^x f_i(u)\,du + \int_r^x \psi_{1-i}(u)\,du + \theta \gamma_{1-i}^* \int_r^x \psi_i(u)\,du \\
= \theta \int_{-\infty}^\infty [f_{1-i}(u+r) - f_{1-i}(u+x)]\,n_{1-i}(u)\,du + \theta^2 \gamma_{1-i}^* \\
\times \int_{-\infty}^\infty [f_i(u+r) - f_i(u+x)] n_i(u)\,du \qquad (17)
\end{aligned}$$

We now consider the two cases, (i) $\beta = 0$ and (ii) $\beta > 0$, separately.

Case (i), $\beta = 0$. The terms on the left hand sides of equations (17) are all positive for $r > L+\alpha$, and as $x \to \infty$ the right hand sides tend by monotone convergence to the finite limits

$$\theta \int_{-\infty}^\infty f_{1-i}(u+r)\,n_{1-i}(u)\,du + \theta^2 \gamma_{1-i}^* \int_{-\infty}^\infty f_i(u+r)\,n_i(u)\,du, \quad (i = 0, 1).$$

We conclude that

$$\lim_{x\to\infty} \int_r^x f_i(u)\,du \quad \text{and} \quad \lim_{x\to\infty} \int_r^x \psi_i(u)\,du, \quad (i = 0, 1)$$

exist and obtain the bounds

$$\begin{aligned}
\int_r^\infty f_i(u)\,du &\leqslant (\theta^2 \gamma_0^* \gamma_1^* - 1)^{-1} \theta \left[\int_{-\infty}^\infty f_{1-i}(u+r)\,n_{1-i}(u)\,du + \theta \gamma_{1-i}^* \int_{-\infty}^\infty f_i(u+r)\,n_i(u)\,du \right] \\
&\leqslant (\theta^2 \gamma_0^* \gamma_1^* - 1)^{-1} \theta \left[f_{1-i}(r-\alpha) \int_{-\alpha}^\alpha |n_{1-i}(u)|\,du + \theta \gamma_{1-i}^* f_i(r-\alpha) \int_{-\alpha}^\alpha |n_i(u)|\,du \right] \\
&= m_{i,i} f_i(r-\alpha) + m_{i,1-i} f_{1-i}(r-\alpha),
\end{aligned}$$

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where $$m_{i,i} = \theta^2\gamma^*_{1-i}(\theta^2\gamma^*_0\gamma^*_1 - 1)^{-1}\int_{-\alpha}^{\alpha} |n_i(u)|\,du$$

and $$m_{1-i,i} = m_{i,i}/(\theta\gamma^*_{1-i}).$$

Case (ii), $\beta > 0$. Let $F_i = \int_{-\infty}^{\infty} f_i(u)\,du$. This integral exists and hence $\int_r^{\infty} f_i(u)\,du$ exists and is bounded above by F_i, $(i = 0, 1)$. Integrating equation (16) from r to x we obtain the inequality

$$\begin{aligned}\int_r^x \psi_i(u)\,du &= \int_r^x f_{1-i}(u)\,du - \theta\int_{-\infty}^{\infty} v_i^*(s)\int_{r+s}^{x+s} f_i(t)\,dt\\ &\leqslant F_{1-i} + \theta F_i\gamma_i^*,\end{aligned}$$

for all x. Since $\int_r^x \psi_i(u)\,du$ is monotone increasing in x for $r > L+\alpha$, it follows that $\int_r^{\infty} \psi_i(u)\,du$ exists for $r > L+\alpha$.

The remainder of the proof treats cases (i) and (ii) concurrently. Suppose $\beta \geqslant 0$, and define sequences $f_i^{(k)}(u) = \int_u^{\infty} f_i^{(k-1)}(s)\,ds$ with $f_i^{(0)}(u) = f_i(u)$, together with similar sequences for $\psi_i^{(k)}(u)$, $(i = 0, 1)$. We have established the existence of $f_i^{(1)}(u)$ and $\psi_i^{(1)}(u)$ for $u > L+\alpha$. Observe that these are decreasing functions of u.

If we let $x \to \infty$ in equations (17) and the integrate over r, an equation identical to equation (17) is obtained with $f_i^{(1)}$ and $\psi_i^{(1)}$ replacing $f_i^{(0)}$ and $\psi_i^{(0)}$. It follows, in a similar manner to case (1) above, that $f_i^{(2)}(r)$ and $\psi_i^{(2)}(r)$ exist for $r > L+\alpha$ and

$$\int_r^{\infty} f_i^{(1)}(u)\,du \leqslant m_{i,i}f_i^{(1)}(r-\alpha) + m_{i,1-i}f_{1-i}^{(1)}(r-\alpha) \quad (i = 0, 1).$$

Therefore by induction we can prove the relation, for $k \in N_+$,

$$\int_r^{\infty} f_i^{(k)}(u)\,du \leqslant m_{i,i}f_i^{(k)}(r-\alpha) + m_{i,1-i}f_{1-i}^{(k)}(r-\alpha), \quad (i = 0, 1).$$

Since $f_i^{(k)}(u)$ is monotone decreasing the following inequality holds

$$\begin{aligned}f_i^{(k)}(r-\alpha) &\leqslant \int_r^{\infty} f_i^{(k-1)}(u)\,du + \alpha f_i^{(k-1)}(r-\alpha)\\ &\leqslant (m_{i,i}+\alpha)f_i^{(k-1)}(r-\alpha) + m_{i,1-i}f_{1-i}^{(k-1)}(r-\alpha).\end{aligned}$$

Applying Lemma 2 we obtain, for $k \in N_+$, the inequality

$$\frac{1}{k!}\int_r^{\infty} (u-r)^k f_i(u)\,du = \int_r^{\infty} f_i^{(k)}(u)\,du \leqslant \mathbf{m}_i'(\mathbf{M}+\alpha\mathbf{I})^{(k-1)}\mathbf{f}^{(1)}(r-\alpha),$$

where

$$\mathbf{m}_i = \begin{pmatrix} m_{i,0} \\ m_{i,1}\end{pmatrix}, \quad \mathbf{f}^{(1)}(u) = \begin{pmatrix} f_0^{(1)}(u) \\ f_1^{(1)}(u)\end{pmatrix} \quad \text{and} \quad \mathbf{M} = \begin{pmatrix} m_{0,0} & m_{0,1} \\ m_{1,0} & m_{1,1}\end{pmatrix}, \quad (i = 0, 1).$$

Note that

$$f_i^{(1)}(r-\alpha) \leqslant \begin{cases} (m_{i,i}+\alpha)f_i^{(0)}(r-\alpha) + m_{i,1-i}f_{1-i}^{(0)}(r-\alpha), & \beta = 0.\\ F_i, & \beta > 0.\end{cases}$$

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Let δ be the minimum of the reciprocals of the absolute values of the eigenvalues of $(\mathbf{M}+\alpha\mathbf{I})$.

Then

$$\left|\int_r^\infty e^{\lambda(u-r)} f_i(u)\,du\right| \leqslant \int_r^\infty e^{\mathrm{Re}\lambda(u-r)} f_i(u)\,du = \sum_{k=0}^{\infty} \frac{(\mathrm{Re}\,\lambda)^k}{k!}\int_r^\infty (u-r)^k f_i(u)\,du,$$

which converges when $\mathrm{Re}\,\lambda < \delta$. Thus the integrals $\int_r^\infty e^{\lambda(u-r)+\beta u}\, w_i(u)\,du$, $(i = 0, 1)$, converge for $\mathrm{Re}\,\lambda < \delta$, from which the theorem follows.

The proof of Theorem 2 is valid for all values of μ_0 and μ_1 in equations (1). For the simple epidemic, that is $\mu_0 = \mu_1 = 0$, it can be shown, by a method similar to that of Atkinson and Reuter ((1), p. 317), that $w_i(u) = O(e^{-\delta u})$, for some positive real δ, as $u \to \infty$. This gives a simpler proof of part (1) of Theorem 2 for this special case.

Define $\Delta_{w_i} = \sup\{\lambda \in R : W_i(\lambda) < \infty\}$ and $\Delta_{v_i} = \sup\{\lambda \in R : V_i(\lambda) < \infty\}$. Hence $W_i(\lambda)$ is analytic in the strip $\{\lambda \in C : 0 < \mathrm{Re}\,\lambda < \Delta_{w_i}\}$ and $V_i(\lambda)$ is analytic in the strip $\{\lambda \in C : 0 < \mathrm{Re}\,\lambda < \Delta_{v_i}\}$, since they are both transforms of non-negative functions. Let

$$\Delta_w = \min\{\Delta_{w_0}, \Delta_{w_1}\} \quad \text{and} \quad \Delta_v = \min\{\Delta_{v_0}, \Delta_{v_1}\}.$$

THEOREM 3. *If $w_i(u)$, $(i = 0, 1)$ is a non-trivial solution of equations* (10), *then* $\Delta_{w_0} = \Delta_{w_1} = \Delta_w$ *and* $\Delta_w \leqslant \Delta_v$.

Proof. Taking the transforms of equations (10), we obtain

$$W_{1-i}(\lambda) = V_i(\lambda)\int_{-\infty}^{\infty} e^{\lambda r}(1 - e^{-w_i(r)})\,dr, \quad (i = 0, 1) \tag{18}$$

where equality holds in the sense that if one side converges so does the other. Since the region of convergence of $\int_{-\infty}^{\infty} (1 - e^{-w_i(r)})\, e^{\lambda r}\,dr$ is the same as that of $W_i(\lambda)$, it follows that $\Delta_{w_{1-i}} \leqslant \Delta_{w_i}$ and $\Delta_{w_{1-i}} \leqslant \Delta_{v_i}$, $(i = 0, 1)$. The theorem follows immediately.

COROLLARY. *Necessary conditions for the existence of a non-trivial solution to equations* (10) *with $w_i(u) > 0$ for some u, $(i = 0, 1)$, and $w_i(u) \to 0$ as $u \to \infty$ are that $p_i(r) = O(e^{-\delta r})$ as $r \to \infty$, $(i = 0, 1)$, for some positive real δ.*

Proof. The result follows by considering the transforms of equations (8), that is

$$V_i(\lambda) = \sigma_i P_i(\lambda)\,\Lambda_i(c\lambda), \quad (i = 0, 1),$$

where $$P_i(\lambda) = \int_{-\infty}^{\infty} e^{\lambda r} p_i(r)\,dr \quad \text{and} \quad \Lambda_i(\lambda) = \int_0^\infty e^{-\lambda r}\lambda_i(r)\,dr.$$

5. *The characteristic equation*

Let $K_c(\lambda) = V_0(\lambda)\,V_1(\lambda)$.

It is shown that if there exists a non-trivial solution to equation (10) such that $\Delta_w < \Delta_v$, then the characteristic equation $K_c(\lambda) = 1$ has its smallest root at $\lambda = \Delta_w$. Theorem 2 implies that $w_i(u) = O(e^{-\delta u})$ as $u \to \infty$. When the root of the characteristic equation is simple, this result will be sharpened in Section 6 to $w_i(u) \sim A_i e^{-\Delta_w u}$,

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$(i = 0, 1)$ for some finite positive reals A_0 and A_1. The properties of $K_c(\lambda)$ are examined and are used to motivate the theorems on the existence and non-existence of non-trivial wave solutions which are formulated and proved in Sections 6 and 7.

THEOREM 4. *Let $w_i(u)$, $(i = 0, 1)$, be a non-trivial wave solution of equations (10) with $w_i(u) \to 0$ as $u \to \infty$. If $\Delta_w < \Delta_v$, then Δ_w is the smallest positive root of $K_c(\lambda) = 1$.*

Proof. From Theorem 2, $\Delta_w > 0$. Equation (18) can be rewritten in the form

$$W_{1-i}(\lambda) = W_i(\lambda) V_i(\lambda) - R_i(\lambda), \quad (i = 0, 1),$$

where
$$R_i(\lambda) = V_i(\lambda) \int_{-\infty}^{\infty} [w_i(u) - (1 - e^{-w_i(u)})] e^{\lambda u} du.$$

Hence, for $0 < \lambda < \Delta_w$,

$$(1 - K_c(\lambda)) W_{1-i}(\lambda) = -[V_i(\lambda) R_{1-i}(\lambda) + R_i(\lambda)], \quad (i = 0, 1). \tag{19}$$

Since $R_i(\lambda) > 0$, $V_i(\lambda) > 0$ and $W_i(\lambda)$ exist, $(i = 0, 1)$ in the range $0 < \lambda < \Delta_w$, the equation $K_c(\lambda) = 1$ cannot have a solution in this range.

It is now shown that the abscissa of convergence of $R_i(\lambda)$ is to the right of Δ_w, $(i = 0, 1)$. From equations (10), we obtain the inequality

$$\begin{aligned} w_{1-i}(u) e^{\lambda u} &= \int_{-\infty}^{\infty} (1 - e^{-w_i(u+r)}) e^{\lambda(u+r)} v_i(r) e^{-\lambda r} dr \\ &\leqslant \int_{-\infty}^{\infty} w_i(u+r) e^{\lambda(u+r)} v_i(r) e^{-\lambda r} dr \\ &\leqslant \sup\{e^{-\lambda r} v_i(r) : r \in R\} \quad W_i(\lambda). \end{aligned}$$

Observe that

$$\begin{aligned} e^{-\lambda r} v_i(r) &= c^{-1} \sigma_i \int_0^{\infty} [p_i(s-r) e^{\lambda(s-r)}] [\lambda_i(s/c) e^{-\lambda s}] ds \\ &\leqslant c^{-1} \sigma_i L_i P_i(\lambda) \quad \text{for} \quad \lambda > 0. \end{aligned}$$

Hence $w_{1-i}(u) e^{\lambda u}$ is bounded for any λ such that $0 < \lambda < \Delta_w$. Consider any real ν and λ such that $0 \leqslant \nu < \lambda < \Delta_w$. In addition let $\nu < \Delta_v - \Delta_w$ if $K_c(\Delta_v)$ is infinite and $\nu \leqslant \Delta_v - \Delta_w$ if $K_c(\Delta_v)$ is finite, we have

$$\begin{aligned} 0 \leqslant R_i(\Delta_w + \nu) &= V_i(\Delta_w + \nu) \int_{-\infty}^{\infty} [w_i(u) - (1 - e^{-w_i(u)})] e^{(\Delta_w + \nu) u} du \\ &\leqslant \tfrac{1}{2} V_i(\Delta_w + \nu) \int_{-\infty}^{\infty} w_i^2(u) e^{(\Delta_w + \nu) u} du \\ &= \tfrac{1}{2} V_i(\Delta_w + \nu) \int_{-\infty}^{\infty} w_i(s) e^{\lambda s} w_i(s) e^{(\Delta_w + \nu - \lambda) s} ds \\ &\leqslant \tfrac{1}{2} V_i(\Delta_w + \nu) \sup\{w_i(s) e^{\lambda s} : s \in R\} W_i(\Delta_w + \nu - \lambda). \end{aligned} \tag{20}$$

Thus $R_i(\lambda)$ converges for $0 < \lambda < 2\Delta_w$ provided a further restriction on λ also holds. This restriction on λ is that $\lambda < \Delta_v$ if $K_c(\Delta_v)$ is infinite and $\lambda \leqslant \Delta_v$ if $K_c(\Delta_v)$ is finite. (Observe that if Δ_v and $K_c(\Delta_v)$ are finite, the inequality is still valid when $\Delta_w = \Delta_v$ and $\nu = 0$. This will be used subsequently in the paper).

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Observe that if our two-sided transforms, with real parameter λ, of continuous non-negative functions exists in an interval $\lambda_0 - \delta < \lambda < \lambda_0$, then the limit as $\lambda \uparrow \lambda_0$ of the transforms are equal to the transforms with parameter λ_0. This can easily be seen by splitting each range of integration into $(-\infty, 0)$ and $[0, \infty)$ and applying monotone convergence.

Thus, since $\Delta_w < \Delta_v$, the right hand side of equations (19) tend to finite negative limits as $\lambda \uparrow \Delta_w$. Hence $\lim_{\lambda \uparrow \Delta_w} (1 - K_c(\lambda)) W_{1-i}(\lambda)$ is finite and negative for $i = 0, 1$. If $\lim_{\lambda \uparrow \Delta_w} W_i(\lambda) = W_i(\Delta_w)$ is finite for $i = 0, 1$, then $K_c(\Delta_w) > 1$, but Theorem 2(ii) then states that the abscissae of convergence of $W_i(\lambda)$ for $i = 0, 1$, are to the right of $\lambda = \Delta_w$, which is a contradiction. Hence $\lim_{\lambda \uparrow \Delta_w} W_i(\lambda)$ is infinite for some i. It then follows that $K_c(\Delta_w) = 1$, (and necessarily $\lim_{\lambda \uparrow \Delta_w} W_i(\lambda)$ is infinite for both $i = 0, 1$), and the theorem is proved.

We now consider the properties of $K_c(\lambda)$ and discuss the relevance of these properties to the existence, uniqueness and non-existence of waves at different speeds.

Properties of $K_c(\lambda)$

(i) $K_c(0) > 1$.

(ii) $K_c(\lambda)$ is a convex function of λ for $0 < \lambda < \Delta_v$.

(iii) When Δ_v is infinite, $K_c(\lambda)$, either tends to zero or infinity as $\lambda \to \infty$.

(iv) The abscissa of convergence of $K_c(\lambda)$, in the right half of the complex plane is the same for all c.

(v) $K_c(\lambda)$ is a decreasing function of c, $(c \geqslant 0)$.

(vi) $\inf_{\lambda, c} K_c(\lambda) = 0$.

Proof. (i) This result follows immediately since $K_c(0) = V_0(0) V_1(0) = \gamma_0 \gamma_1 > 1$.

(ii)

$$K_c(\lambda) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\lambda(s+t)} v_0(s) v_1(t) \, ds \, dt$$

$$= \int_{-\infty}^{\infty} e^{-\lambda x} v(x) \, dx,$$

where $$v(x) = \int_{-\infty}^{\infty} v_0(s) v_1(x - s) \, ds \geqslant 0 \quad \text{for all } x.$$

Thus $$K_c''(\lambda) = \int_{-\infty}^{\infty} x^2 e^{-\lambda x} v(x) \, dx > 0, \quad \text{for} \quad 0 < \lambda < \Delta_v.$$

Thus $K_c(\lambda)$ is a convex function of λ in this range.

(iii) When $v(x) = 0$ for $x < 0$, then $K_c(\lambda) \to 0$ as $\lambda \to \infty$; otherwise $K_c(\lambda) \to \infty$ as $\lambda \to \infty$.

(iv) Now $K_c(\lambda) = \sigma_0 \sigma_1 P_0(\lambda) P_1(\lambda) \Lambda_0(c\lambda) \Lambda_1(c\lambda)$. Since $\Lambda_0(c\lambda) \Lambda_1(c\lambda)$ is finite and positive for all $0 < \lambda < \infty$, Δ_r is the abscissa of convergence of $P_0(\lambda) P_1(\lambda)$, which is the same for all c.

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(v) Since

$$\frac{\partial K_c(\lambda)}{\partial c} = -\lambda\sigma_0\sigma_1 P_0(\lambda)\, P_1(\lambda) \int_0^\infty \int_0^\infty (x+y)\, e^{-c\lambda(x+y)} \lambda_0(x)\, \lambda_1(y)\, dx\, dy < 0,$$

$K_c(\lambda)$ is a strictly decreasing function of c for all $c > 0$ and $0 < \lambda < \Delta_v$.

(vi) For all λ, $\Lambda_0(c\lambda)\,\Lambda_1(c\lambda) \to 0$ as $c \to \infty$ and $P_0(\lambda)\, P_1(\lambda)$ is finite in the range $0 < \lambda < \Delta_v$ hence $\inf_{\lambda, c} K_c(\lambda) = 0$.

The properties of $K_c(\lambda)$ are now used to obtain conditions for the existence, uniqueness and non-existence of wave solutions at different speeds. They are further used to classify the speeds at which non-trivial solutions can be obtained.

Motivation for the formulation of the theorems

For fixed c, when the smallest positive root of $K_c(\lambda) = 1$ does not lie on the abscissa of convergence and the root is simple, then properties (i) and (ii) of $K_c(\lambda)$ imply that $dK_c(\lambda)/d\lambda < 0$ at this root. This condition is used to construct a solution at speed c in Theorem 5 and to show that the solution is unique modulo translation in Theorem 7.

When $K_c(\lambda) = 1$ does not have a solution at speed c for $0 < \lambda \leqslant \Delta_v$ then either $1 < K_c(\Delta_v) < \infty$ and hence from property (iii) $\Delta_v < \infty$, or $K_c(\Delta_v) = \infty$ and properties (i) and (ii) imply that there exists a λ in the range $0 < \lambda < \Delta_v$ such that $dK_c(\lambda)/d\lambda > 0$. These conditions are used to prove the non-existence of a non-trivial wave solution at speed c in Theorem 8.

The only other possibilities are either that $K_c(\lambda) = 1$ has a double root within the range $0 < \lambda < \Delta_v$, or that $K_c(\lambda) = 1$ has no solution for $0 < \lambda < \Delta_v$ but $K_c(\Delta_v) = 1$ with, from property (iii) $\Delta_v < \infty$.

Classification of possible speeds

Properties (iv) and (v) imply that if $K_c(\lambda) = 1$ has a simple root in the range $0 < \lambda < \Delta_v$ at speed c then the same is true for any higher speed. Additionally if $K_c(\lambda) = 1$ has no root in the range $0 < \lambda \leqslant \Delta_v$ at speed c then the same holds for all smaller speeds. From condition (vi), $K_c(\lambda) = 1$ has a simple root for some values of c. Let $c_0 = \inf\{c: K_c(\lambda) = 1$ has a simple root$\}$. Hence we show the existence of non-trivial solutions which are unique modulo translation for all speeds $c > c_0$. Since $\lim_{c \to 0} K_c(\lambda) = \sigma_0\sigma_1\gamma_0\gamma_1 P_0(\lambda)\, P_1(\lambda)$ it is easily seen that the case $c_0 = 0$ corresponds to

$$\inf_\lambda [\sigma_0\sigma_1\gamma_0\gamma_1 P_0(\lambda)\, P_1(\lambda)] \leqslant 1.$$

In this case $K_c(\lambda)$ has either one or two simple roots in the range $0 < \lambda < \Delta_v$ for each $c > 0$. The case with one simple root for all $c > 0$ will certainly occur when $p_i(x) = 0$ for $x < 0$, for $i = 0, 1$. Then $\Delta_v = \infty$ and $K_c(\lambda) \to 0$ as $\lambda \to \infty$ for all c.

The case $c_0 > 0$ corresponds to one of two situations. Either $K_{c_0}(\lambda) = 1$ has a double root in the range $0 < \lambda < \Delta_v$, or $K_{c_0}(\lambda) = 1$ has no solution for $0 < \lambda < \Delta_v$ but $K_{c_0}(\Delta_v) = 1$. From property (v) no solution to $K_c(\lambda) = 1$ can thus occur for $c < c_0$ and we show the non-existence of non-trivial solutions for speeds $c < c_0$. The existence of wave solutions at the critical speed c_0, where $c_0 > 0$ is discussed in the last section.

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6. *Existence of wave solutions*

In the previous section we saw that for all $c > c_0$, where $c_0 \geqslant 0$, $K_c(\lambda) = 1$ has its smallest positive root $\lambda = \alpha$ simple and $K_c(\lambda)$ is strictly decreasing at the root and exists in a region beyond the root. Hence there exists a $\delta > 0$ such that $K_c(\lambda) < 1$ in the interval $\alpha < \lambda < \alpha + \delta$. We now show using a method similar to that of Atkinson and Reuter (1) that a non-trivial solution can be constructed under this condition. This solution is shown to be unique modulo translation.

THEOREM 5. *If the characteristic equation $K_c(\lambda) = 1$ has a positive root at $\lambda = \alpha$, and if $K_c(\lambda)$ exists and is < 1 in some interval $\alpha < \lambda < \alpha + \delta$, then equations* (10) *have a non-trivial monotone decreasing solution $w_i(u)$ with $w_i(\infty) = 0$, for $i = 0, 1$.*

Proof. Define $w_i^{(0)}(u) = A_i e^{-\alpha u}$, $(i = 0, 1)$, where $A_i \geqslant 2$ and $A_1 = V_0(\alpha) A_0$ and hence, since $K_c(\alpha) = V_0(\alpha) V_1(\alpha) = 1$, $A_0 = V_1(\alpha) A_1$. We now define $w_i^{(n)}(u)$ recursively for all positive integers n by

$$w_{1-i}^{(n)}(u) = \int_{-\infty}^{\infty} (1 - \exp\{-w_i^{(n-1)}(u+t)\}) v_i(t)\, dt, \quad (i = 0, 1). \tag{21}$$

The sequence $\{w_i^{(n)}(u)\}$ is shown to tend to a limit $w_i(u)$ as $n \to \infty$, for each $i = 0, 1$, which is a monotone decreasing solution to (10) with $w_i(\infty) = 0$. From (21) with $n = 1$ we obtain

$$\begin{aligned} w_{1-i}^{(1)}(u) &\leqslant \int_{-\infty}^{\infty} w_i^{(0)}(u+t) v_i(t)\, dt \\ &= A_i \int_{-\infty}^{\infty} e^{-\alpha(u+t)} v_i(t)\, dt \\ &= A_i e^{-\alpha u} V_i(\alpha) \\ &= A_{1-i} e^{-\alpha u} \\ &= w_{1-i}^{(0)}(u). \end{aligned}$$

Suppose that $w_i^{(n)}(u) \leqslant w_i^{(n-1)}(u)$, for all u, $(i = 0, 1)$, and all integers n such that $1 \leqslant n \leqslant N$. Since $(1 - e^{-w})$ decreases as w decreases we obtain

$$\begin{aligned} w_{1-i}^{(N+1)}(u) &= \int_{-\infty}^{\infty} (1 - \exp\{-w_i^{(N)}(u+t)\}) v_i(t)\, dt \\ &\leqslant \int_{-\infty}^{\infty} (1 - \exp\{-w_i^{(N-1)}(u+t)\}) v_i(t)\, dt \\ &= w_{1-i}^{(N)}(u). \end{aligned}$$

Hence by induction $\{w_i^{(n)}(u)\}$ is a monotone decreasing sequence. Observe that $w_i^{(0)}(u)$ is non-negative and monotone decreasing. Equations (21) imply, using induction on n, that the same is true for all $w_i^{(n)}(u)$. Hence $\{w_i^{(n)}(u)\}$ is a monotone decreasing sequence of monotone decreasing functions of u, which is bounded below by the zero function. The sequence therefore tends to a limit $w_i(u)$, as $n \to \infty$, which is monotone decreasing and non-negative, for each $i = 0, 1$. Since $0 \leqslant w_i^{(n)}(u) \leqslant w_i^{(0)}(u)$ for all n, and $w_i^{(0)}(\infty) = 0$, it immediately follows that $w_i(\infty) = 0$, $(i = 0, 1)$.

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It now remains to prove that $w_i(u)$, $(i = 0, 1)$ is not the trivial solution. There exists a δ such that $0 < K_c(\alpha+\delta) < 1$ and $0 < \delta < \alpha$. For this δ we show that for all non-negative integers n

$$w_i^{(n)}(u) \geqslant A_i e^{-\alpha u} - B_i e^{-(\alpha+\delta)u}, \quad (i = 0, 1), \tag{22}$$

where $$B_i = \tfrac{1}{2}(A_{1-i}^2 V_{1-i}(\alpha+\delta) + A_i^2 K_c(\alpha+\delta))/(1 - K_c(\alpha+\delta)).$$

The inequalities obviously hold for $n = 0$. We assume they hold for $n \leqslant N$. From (21) with $n = (N+1)$ we have

$$\begin{aligned} w_{1-i}^{(N+1)}(u) &= \int_{-\infty}^{\infty} (1 - \exp\{-w_i^{(N)}(t)\})\, v_i(t-u)\, dt \\ &\geqslant \int_{0}^{\infty} (1 - \exp\{-w_i^{(N)}(t)\})\, v_i(t-u)\, dt \\ &\geqslant \int_{0}^{\infty} [w_i^{(N)}(t) - \tfrac{1}{2}(w_i^{(N)}(t))^2]\, v_i(t-u)\, dt, \end{aligned}$$

since $(1 - e^{-w}) \geqslant w - \frac{1}{2}w^2$ for $w > 0$. Using (22) for $n = N$ and noting that

$$(w_i^{(N)}(t))^2 \leqslant (w_i^{(0)}(t))^2 = A_i^2 e^{-2\alpha t} \leqslant A_i^2 e^{-(\alpha+\delta)t},$$

we obtain

$$\begin{aligned} w_{1-i}^{(N+1)}(u) &\geqslant \int_{0}^{\infty} [A_i e^{-\alpha t} - (B_i + \tfrac{1}{2}A_i^2)\, e^{-(\alpha+\delta)t}]\, v_i(t-u)\, dt \\ &= \int_{-\infty}^{\infty} [A_i e^{-\alpha t} - (B_i + \tfrac{1}{2}A_i^2)\, e^{-(\alpha+\delta)t}]\, v_i(t-u)\, dt \\ &\quad + \int_{-\infty}^{0} e^{-\alpha t}[(B_i + \tfrac{1}{2}A_i^2)\, e^{-\delta t} - A_i]\, v_i(t-u)\, dt. \end{aligned}$$

The second term on the right hand side will be positive if $(B_i + \frac{1}{2}A_i^2) \geqslant A_i$, i.e. $B_i \geqslant A_i(1 - \frac{1}{2}A_i)$. Since $A_i \geqslant 2$ and B_i is positive this condition clearly holds. Hence

$$\begin{aligned} w_{1-i}^{(N+1)}(u) &\geqslant \int_{-\infty}^{\infty} [A_i e^{-\alpha t} - (B_i + \tfrac{1}{2}A_i^2)\, e^{-(\alpha+\delta)t}]\, v_i(t-u)\, dt \\ &= A_i e^{-\alpha u} V_i(\alpha) - (B_i + \tfrac{1}{2}A_i^2)\, V_i(\alpha+\delta)\, e^{-(\alpha+\delta)u} \\ &= A_{1-i} e^{-\alpha u} - B_{1-i} e^{-(\alpha+\delta)u}. \end{aligned}$$

Hence the inequalities (22) hold for $n = N+1$ and thus by induction for all non-negative integers n. Thus $w_i(u)$, $(i = 0, 1)$, satisfy the inequalities

$$w_i(u) \geqslant A_i e^{-\alpha u} - B_i e^{-(\alpha+\delta)u} \quad \text{for all } u, \quad (i = 0, 1).$$

For u sufficiently large $(A_i - B_i e^{-\delta u})$ is positive and hence $w_i(u) > 0$, proving that $w_i(u)$, $(i = 0, 1)$, is not the trivial solution. Note that in our construction theorem

$$A_i e^{-\alpha u} \geqslant w_i(u) \geqslant A_i e^{-\alpha u} - B_i e^{-(\alpha+\delta)u} \quad \text{all } u \quad (i = 0, 1).$$

Hence $w_i(u) \sim A_i e^{-\alpha u}$ as $u \to \infty$, where $\alpha = \Delta_w$.

We now state without proof a lemma which is almost identical to lemma 6·1 of Diekmann and Kaper (10). This is used in a similar manner to their theorem 6·2 to

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show that any monotone decreasing solution $w_i(u)$ to (10) with $w_i(\infty) = 0$, $(i = 0, 1)$, if suitably translated has $w_i(u) \sim A_i e^{-\Delta_w u}$ as $u \to \infty$. From this result we can show the uniqueness of our constructed solution modulo translation.

LEMMA 3. *If the real-valued function w is non-negative monotone decreasing on R_+, and there exists Δ_w such that $W(\lambda) = \int_0^\infty e^{\lambda x} w(x)\,dx$ converges for $\operatorname{Re}\lambda < \Delta_w$ and if for some constant A and some function $h: R \to R$,*

$$\lim_{x \uparrow \Delta_w} [W(x+iy) + A(-\Delta_w + x + iy)^{-1}] = h(y),$$

uniformly on compact subsets of R, then

$$\lim_{x \to \infty} w(x)\, e^{\Delta_w x} = A.$$

THEOREM 6. *If $K_c(\lambda) = 1$ has its smallest positive root at $\lambda = \Delta_w < \Delta_v$ with $K_c'(\Delta_w) < 0$ and if $w_i(u)$, $(i = 0, 1)$, is a monotone decreasing solution of equations (10) with $w_i(\infty) = 0$, then $w_i(u)\, e^{\Delta_w u} \to C_i$ as $u \to \infty$, $(i = 0, 1)$ for some positive constants C_0 and C_1 with $C_1 = V_0(\Delta_w)\, C_0$.*

Proof. In Theorem 4 we showed that for a monotone decreasing solution $w_i(u)$ to (10)

$$(1 - K_c(\lambda))\, W_i(\lambda) = -[V_{1-i}(\lambda)\, R_i(\lambda) + R_{1-i}(\lambda)] \quad (i = 0, 1),$$

for $0 < \lambda < \Delta_w$. Since $\Delta_w < \Delta_v$, the right hand sides exist and are positive at $\lambda = \Delta_w$ and are analytic in a neighbourhood of the line $\operatorname{Re}\lambda = \Delta_w$. Also $K_c(\lambda)$ is analytic at $\lambda = \Delta_w$. $K_c(\lambda)$ is the transform of a continuous non-negative function (namely the convolution of $v_0(t)$ and $v_1(t)$). Hence $\operatorname{Re}(K_c(\lambda)) < K_c(\operatorname{Re}(\lambda))$ when λ is not real. Since $K_c(\Delta_w) = 1$, the function $(K_c(\lambda) - 1)$ cannot have a zero elsewhere on the line $\operatorname{Re}\lambda = \Delta_w$ other than the zero at $\lambda = \Delta_w$. The zeros of the function $(K_c(\lambda) - 1)$ which lie in the vertical strip $\Delta_w - \epsilon < \operatorname{Re}\lambda < \Delta_w + \epsilon$, where $\epsilon \leqslant \min(\Delta_w, \Delta_v - \Delta_w)$, in fact lie in a rectangle by the Riemann–Lebesgue lemma. Also, since $K_c(\lambda)$ is analytic, there are only finitely many in this rectangle. By taking ϵ sufficiently small we can ensure that there are no zeros in the strip except at $\lambda = \Delta_w$. Since $(K_c(\lambda) - 1)$ has a simple root at $\lambda = \Delta_w$, the function $[V_{1-i}(\lambda)\, R_i(\lambda) + R_{1-i}(\lambda)]/(1 - K_c(\lambda))$ is analytic within the strip except at the point $\lambda = \Delta_w$, where it has a simple pole with residue

$$C_i = \frac{[V_{1-i}(\Delta_w)\, R_i(\Delta_w) + R_{1-i}(\Delta_w)]}{(-K_c'(\Delta_w))}.$$

Thus

$$f(\lambda) = \left[-\frac{[V_{1-i}(\lambda)\, R_i(\lambda) + R_{1-i}(\lambda)]}{(1 - K_c(\lambda))} + \frac{C_i}{(\lambda - \Delta_w)} \right]$$

is analytic within the strip. Hence

$$\lim_{x \uparrow \Delta_w} [W_i(x+iy) + C_i(x + iy - \Delta_w)^{-1}] = f(\Delta_w + iy)$$

uniformly on compact subsets of R. We may now apply Lemma 3 and obtain

$$\lim_{u \to \infty} w_i(u)\, e^{\Delta_w u} = C_i, \quad (i = 0, 1),$$

where $C_i > 0$ and $C_{1-i} = V_i(\Delta_w)\, C_i$.

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THEOREM 7. *If $K_c(\lambda)$ has its smallest positive root at $\lambda = \Delta_w < \Delta_v$ and if $K_c'(\Delta_w) < 0$, then the solution $w_i(u)$ to equations* (10) *which decreases monotonically to zero* $(i = 0, 1)$ *is unique modulo translation.*

Proof. If $w_i(u)$ is a solution to (10) its translate $w_i(u+t)$ is also a solution. By Theorem 6 $w_i(u)\, e^{\Delta_w u} \to C_i$ as $u \to \infty$ for some $C_i > 0$ and hence $w_i(u+t)\, e^{\Delta_w u} \to C_i e^{-\Delta_w t}$. Hence we can choose a suitable translate of any solution so that C_0 is any positive value. C_1 is determined by C_0.

Let $w_i(u)$ and $w_i^*(u)$, $(i = 0, 1)$, be two solutions. By choosing a suitable translate of the second solution we have $w_i(u)\, e^{\Delta_w u} \to C_i$ and $w_i^*(u)\, e^{\Delta_w u} \to C_i$ as $u \to \infty$, $(i = 0, 1)$ where $C_{1-i} = V_i(\Delta_w) C_i$. We now prove that $w_i(u) = w_i^*(u)$ for all u, $(i = 0, 1)$, so that there is a unique solution to (10) modulo translation.

Consider $D_i(u) = (w_i(u) - w_i^*(u))\, e^{\Delta_w u}$, which is continuous with limit zero at $\pm\infty$.

Let

$$\sup\{|D_i(u)| : u \in R\} = |D_i| < \infty, \quad (i = 0, 1).$$

The inequality

$$|D_{1-i}| \leqslant \int_{-\infty}^{\infty} |e^{w_i(t)} - e^{w_i^*(t)}|\, e^{\Delta_w u}\, v_i(t-u)\, dt$$

holds for some u. If $|D_i| \neq 0$ then for some u

$$\begin{aligned} |D_{i-1}| &< \int_{-\infty}^{\infty} |w_i(t) - w_i^*(t)|\, e^{\Delta_w t}\, e^{-\Delta_w(t-u)}\, v_i(t-u)\, dt \\ &\leqslant |D_i|\, V_i(\Delta_w). \end{aligned}$$

Thus for $|D_i| \neq 0$, $(i = 0, 1)$,

$$|D_{1-i}| < |D_i|\, V_i(\Delta_w) < |D_{1-i}|\, V_{1-i}(\Delta_w)\, V_i(\Delta_w) = |D_{1-i}|,$$

which is a contradiction. Hence at least one of $|D_0|$ and $|D_1|$ is zero. If $|D_i| = 0$, $w_i(t) = w_i^*(t)$ for all t, and hence from (10), $|D_{1-i}| = 0$. Hence $w_i(u) \equiv w_i^*(u)$ for all u, $(i = 0, 1)$, so that the solution constructed in Theorem 5 is unique modulo translation.

The asymptotic behaviour of $w_i(u)$ has been determined in Theorem 6 by the use of complex analysis. However, it is interesting to note that a proof of Theorem 6 can be constructed using Barbour's probabilistic argument (2). Let $w_i^*(u;\lambda) = w_i(u)\, e^{\lambda u}$, then

$$w_{1-i}^*(u;\lambda) = V_i(\lambda) \int_{-\infty}^{\infty} w_i^*(u+t;\lambda)\, dF_i(t;\lambda) + f_i(u;\lambda),$$

where

$$dF_i(t;\lambda) = (V_i(\lambda))^{-1}\, v_i(t)\, e^{-\lambda t}\, dt$$

and

$$f_i(u;\lambda) = \int_{-\infty}^{\infty} e^{\lambda u}(1 - e^{-w_i(u+t)} - w_i(u+t))\, v_i(t)\, dt.$$

Let $S_n = S_0 + \Sigma_{r=1}^{n} (A_r + B_r)$ for some S_0, where $\{A_r, r = 1, 2, \ldots, n\}$ and $\{B_r, r = 1, 2, \ldots, n\}$ are $2n$ independent random variables, and A_r are identically distributed with distribution function $F_0(u;\lambda)$ and B_r are identically distributed with distribution function $F_1(u;\lambda)$. The σ-field $\sigma(A_1, A_2, \ldots, A_n; B_1, B_2, \ldots, B_n)$ is denoted by F_n. A

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suitable pair of super-martingales on which to base the proof are $(Y_i^{(n)}, F_n, n \geqslant 0)$, $(i = 0, 1)$, where

$$Y_i^{(n)} = w_i^*(S_n; \lambda) + \sum_{j=0}^{n-1} \left\{ V_{1-i}(\lambda) \int_{-\infty}^{\infty} f_i(S_j + u; \lambda)\, dF_{1-i}(u; \lambda) + f_{1-i}(S_j; \lambda) \right\} \quad (i = 0, 1).$$

7. *Non-existence of waves*

When $K_c(\lambda) = 1$ does not have a root in the range $0 < \lambda \leqslant \Delta_v$, we show that a non-trivial wave solution cannot exist. This shows the non-existence of waves for speeds $c < c_0$, where $c_0 > 0$. Case (ii) of Theorem 8 uses a technique similar to that of Atkinson and Reuter (1).

THEOREM 8. *If $K_c(\lambda) > 1$ for all $0 < \lambda \leqslant \Delta_v$ then there exists no non-trivial monotone decreasing solution $w_i(u)$ to equations* (10) *with $w_i(\infty) = 0$, $(i = 0, 1)$.*

Proof. Since $K_c(\lambda) = 1$ does not have a solution, Theorem 4 implies $\Delta_w = \Delta_v$. From the properties of $K_c(\lambda)$ described in Section 5, it follows that there are two cases, (i) Δ_w is finite, and $K_c(\Delta_w) > 1$ and is finite and (ii) $K_c(\Delta_w)$ is infinite. In each case we assume that a non-trivial solution exists and obtain a contradiction.

Case (i)

From Theorem 4, taking the limits of equations (19) we obtain

$$\lim_{\lambda \uparrow \Delta_w} (1 - K_c(\lambda))\, W_i(\lambda) = - \lim_{\lambda \uparrow \Delta_w} [V_{1-i}(\lambda)\, R_i(\lambda) + R_{1-i}(\lambda)], \quad (i = 0, 1).$$

In addition inequalities (20) with $\Delta_w = \Delta_v$ and $\nu = 0$ state that for any $0 < \lambda < \Delta_w$

$$R_i(\Delta_w) \leqslant \tfrac{1}{2} V_i(\Delta_w)\, W_i(\Delta_w - \lambda) \sup (w_i(u)\, e^{\lambda u} : u \in R).$$

Hence $[V_{1-i}(\Delta_w)\, R_i(\Delta_w) + R_{1-i}(\Delta_w)]$ is finite, and consequently $\lim_{\lambda \uparrow \Delta_w} (1 - K_c(\lambda))\, W_i(\lambda)$ is finite, $(i = 0, 1)$, using the result on the limits of our transforms given at the end of Theorem 4.

Since $(1 - K_c(\Delta_w))$ is non-zero, this implies $W_i(\Delta_w)$ is finite for $i = 0, 1$. We can then apply Theorem 2(ii) with $\beta = \Delta_w$ and obtain the result that the abscissa of convergence of $W_i(\lambda)$ is to the right of Δ_w, which contradicts the definition of Δ_w.

Case (ii)

When $K_c(\Delta_w)$ is infinite there exists a positive $\theta < \Delta_w$ such that $K_c'(\theta) > 0$. Taking transforms of (10) we obtain

$$W_i(\lambda) = V_{1-i}(\lambda) \int_{-\infty}^{\infty} (1 - e^{-w_{1-i}(u)})\, e^{\lambda u}\, du, \quad 0 < \lambda < \Delta_w, \quad (i = 0, 1).$$

Multiplying by $e^{-\lambda E_i}$, differentiating with respect to λ, multiplying by $e^{\lambda E_i}$ and setting $\lambda = \theta$ we obtain

$$\int_{-\infty}^{\infty} (u - E_i)\, w_i(u)\, e^{\theta u}\, du = V_{1-i}(\theta) \int_{-\infty}^{\infty} \left(u - E_i + \frac{V_{1-i}'(\theta)}{V_{1-i}(\theta)} \right) (1 - e^{-w_{1-i}(u)})\, e^{\theta u}\, du, \quad (i = 0, 1).$$

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Let $E_0 = E_1 - V_0'(\theta)/V_0(\theta)$, then $E_0 - V_1'(\theta)/V_1(\theta) = E_1 - K_c'(\theta)/K_c(\theta) < E_1$ and hence

$$\int_{-\infty}^{\infty} (u - E_0)\, w_0(u)\, e^{\theta u} du > V_1(\theta) \int_{-\infty}^{\infty} (u - E_1)\,(1 - e^{-w_1(u)})\, e^{\theta u} du, \tag{23}$$

and

$$\int_{-\infty}^{\infty} (u - E_1)\, w_1(u)\, e^{\theta u} du = V_0(\theta) \int_{-\infty}^{\infty} (u - E_0)\,(1 - e^{-w_0(u)})\, e^{\theta u} du.$$

Now $[(1-e^{-w})/w]' = [1 + w - e^{w}]/w^2 e^{w} < 0$ for $w > 0$, and $w_i(u)$, $(i = 0, 1)$ are decreasing functions of u; hence $r_i(u) = (1 - e^{-w_i(u)})/w_i(u)$ increases from $(1 - l_i)/\log l_i$ to one as u increases from $-\infty$ to ∞.

If $K_c(\theta) < \gamma_0\gamma_1$ we can choose E_1 such that $r_i(E_i) = G_i$, $(i = 0, 1)$ and $G_0 G_1 = 1/K_c(\theta)$, since $\lim_{E_1 \to -\infty} [r_0(E_0)\, r_1(E_1)] = 1/\gamma_0\gamma_1$. Then the following inequalities hold

$$\int_{-\infty}^{\infty} (u - E_i)\, w_i(u)\, e^{\theta u}(1 - G_i^{-1} r_i(u))\, du < 0, \quad (i = 0, 1). \tag{24}$$

If $K_c(\theta) \geqslant \gamma_0\gamma_1$, then $r_0(-\infty)\, r_1(-\infty) \geqslant 1/K_c(\theta)$ and hence we can choose positive reals G_0 and G_1 such that $r_i(-\infty) \geqslant G_i$, $(i = 0, 1)$, and $G_0 G_1 = 1/K_c(\theta)$. Since $(1 - G_i^{-1} r_i(u)) < 0$ for all u and is a decreasing function of u, by taking E_1 sufficiently large and negative (24) will still hold. A sufficient condition is

$$(-E_1) > \max\{(W_1(\theta))^{-1}\,|W_1'(\theta)|, [(W_0(\theta))^{-1}\,|W_0'(\theta)| - (V_0(\theta))^{-1}\, V_0'(\theta)]\}.$$

From (24) we obtain the inequality

$$\int_{-\infty}^{\infty} (u - E_i)\, w_i(u)\, e^{\theta u} du < G_i^{-1} \int_{-\infty}^{\infty} (u - E_i)\,(1 - e^{-w_i(u)})\, e^{\theta u} du, \quad (i = 0, 1).$$

Using these inequalities together with (23) we have

$$\begin{aligned}
\int_{-\infty}^{\infty} (u - E_0)\, w_0(u)\, e^{\theta u} du &> V_1(\theta) \int_{-\infty}^{\infty} (u - E_1)\,(1 - e^{-w_1(u)})\, e^{\theta u} du \\
&> G_1 V_1(\theta) \int_{-\infty}^{\infty} (u - E_1)\, w_1(u)\, e^{\theta u} du \\
&= G_1 K_c(\theta) \int_{-\infty}^{\infty} (u - E_0)\,(1 - e^{-w_0(u)})\, e^{\theta u} du \\
&> \int_{-\infty}^{\infty} (u - E_0)\, w_0(u)\, e^{\theta u} du.
\end{aligned}$$

Since this integral is finite we obtain a contradiction.

Note. In case (ii) it is necessary to consider the two possibilities $K_c(\theta) \geqslant \gamma_0\gamma_1$ and $K_c(\theta) < \gamma_0\gamma_1$. The former occurs if $K_c'(0) \geqslant 0$ and the latter occurs if $\gamma_0\gamma_1$ is infinite.

8. *The existence of a wave solution at the critical speed*

Our treatment of the case $c = c_0$ is incomplete. The existence of a non-trivial wave solution when $c = c_0$, where $c_0 > 0$, is established by a method similar to that of Brown and Carr (5). When $K_{c_0}(\lambda) = 1$ has a double root at $\lambda = \alpha < \Delta_v$ it can be shown that

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$w_i(u) \sim A_i u e^{-\alpha u}$ as $u \to \infty$, but ascertaining the asymptotic behaviour when $\lambda = \Delta_v$ is the only positive root of the characteristic equation is more intractable. Even in the first of these two cases we were unable to prove uniqueness at speed $c = c_0 > 0$.

THEOREM 9. *If $c_0 > 0$, then equations (9) with $c = c_0$ have a non-trivial monotone increasing solution $x_i(u)$ with $x_i(\infty) = 1$ and $x_i(-\infty) = l_i$, $(i = 0, 1)$.*

Proof. Equations (9) can be written in the form

$$cx'_{1-i}(u) = \sigma_i x_{1-i}(u)\,[\lambda_i(\infty) - \lambda_i(0)\,\bar{x}_i(u) - s_i(u)] \quad (i = 0, 1), \tag{25}$$

where $\bar{x}_i(u) = \int_{-\infty}^{\infty} x_i(u+s)\,p_i(-s)\,ds$ and $s_i(u) = \int_0^{\infty} \bar{x}_i(u+cr)\,\lambda_i'(r)\,dr$.

Consider a monotone sequence $\{c_n\}$ such that $c_n > c_0$ for all n and $c_n \downarrow c_0$ as $n \to \infty$. Since $c_n > c_0$ there exists a non-trivial solution $x_i^{(n)}(u)$, $(i = 0, 1)$, to (9) with $c = c_n$, which is unique modulo translation. As $x_0^{(n)}(-\infty) = l_0$ and $x_0^{(n)}(\infty) = 1$ we can choose the solution $x_i^{(n)}(u)$, $(i = 0, 1)$ so that $x_0^{(n)}(0) = \frac{1}{2}(1 + l_0)$. This will anchor our limiting solution so that it is non-trivial.

Letting (n) indicate the corresponding functions when $c = c_n$, we show that the sequences $\{x_i^{(n)}(u)\}$, $\{\bar{x}_i^{(n)}(u)\}$, $\{s_i^{(n)}(u)\}$ and $\{x_i^{(n)\prime}(u)\}$, for $i = 0, 1$, are uniformly bounded and equicontinuous.

$$|x_i^{(n)}(u)| \leqslant 1,$$

$$|\bar{x}_i^{(n)}(u)| \leqslant \int_{-\infty}^{\infty} p_i(-s)\,ds = 1,$$

$$|s_i^{(n)}(u)| \leqslant \int_0^{\infty} |\lambda_i'(r)|\,dr \leqslant kL_i,$$

$$\begin{aligned} |x_i^{(n)\prime}(u)| &\leqslant \frac{\sigma_{1-i}}{c_n} |x_i^{(n)}(u)|\,\{\lambda_{1-i}(\infty) + \lambda_{1-i}(0)\,|x_{1-i}^{(n)}(u)| + |s_{1-i}^{(n)}(u)|\} \\ &\leqslant \sigma_{1-i} c_0^{-1}(k+2)\,L_{1-i}; \end{aligned}$$

hence they are uniformly bounded. To show that they are equicontinuous we show the derivatives are all uniformly bounded. $x_i^{(n)\prime}(u)$ has already been shown to be uniformly bounded.

$$|\bar{x}_i^{(n)\prime}(u)| \leqslant \int_{-\infty}^{\infty} |x_i^{(n)\prime}(u+s)\,|p_i(-s)\,ds \leqslant \sigma_{1-i} c_0^{-1}(k+2)\,L_{1-i},$$

$$|s_i^{(n)\prime}(u)| \leqslant \int_0^{\infty} |\bar{x}_i^{(n)\prime}(u + c_n r)|\,|\lambda_i'(r)|\,dr \leqslant \sigma_{1-i} c_0^{-1} k(k+2)\,L_0 L_1,$$

$$\begin{aligned} |x_i^{(n)\prime}(u)| &\leqslant \sigma_{1-i} c_n^{-1}\{|x_i^{(n)\prime}(u)|\,[\lambda_{1-i}(\infty) + \lambda_{1-i}(0)\,|(\bar{x}_{1-i}^{(n)} u)| + s_{1-i}^{(n)}(u)] \\ &\quad + |x_i^{(n)}(u)|\,[\lambda_{1-i}(0)\,|\bar{x}_{1-i}^{(n)\prime}(u)| + |s_{1-i}^{(n)\prime}(u)|]\}, \\ &\leqslant \sigma_{1-i} c_0^{-1}\{\sigma_{1-i} c_0^{-1}[(k+2)\,L_{1-i}]^2 + L_0 L_1 \sigma_i c_0^{-1}(k+2)\,(k+1)\}; \end{aligned}$$

hence the sequences of functions are equicontinuous.

Using Ascoli's theorem and a nested subsequence argument there exists a subsequence of $\{c_n\}$ which we again denote by $\{c_n\}$ such that $c_n \downarrow c_0$ as $n \to \infty$ and the corres-

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ponding sequences $\{x_i^{(n)}(u)\}, \{\bar{x}_i^{(n)}(u)\}, \{s_i^{(n)}(u)\}$ and $\{x_i^{(n)\prime}(u)\}$ converge uniformly on every bounded interval and hence pointwise on R to $x_i(u)$, $b_i(u)$, $s_i(u)$ and $d_i(u)$, $(i = 0, 1)$. Note that $x_0(0) = \frac{1}{2}(1 + l_0)$. It therefore follows that $x_i(u)$ is differentiable and $d_i(u) = x_i'(u)$ (see Bartle (3), theorem 19·12). Using the dominated convergence theorem,

$$b_i(u) = \lim_{n\to\infty} \int_{-\infty}^{\infty} x_i^{(n)}(u+s)\, p_i(-s)\, ds = \int_{-\infty}^{\infty} x_i(u+s)\, p_i(-s)\, ds = \bar{x}_i(u);$$

and
$$s_i(u) = \lim_{n\to\infty} \int_0^{\infty} \lambda_i'(r)\, \bar{x}_i^{(n)}(u + c_n r)\, dr = \int_0^{\infty} \lambda_i'(r) [\lim_{n\to\infty} \bar{x}_i^{(n)}(u + c_n r)]\, dr,$$

where
$$\lim_{n\to\infty} \bar{x}_i^{(n)}(u + c_n r) = \int_{-\infty}^{\infty} p_i(-s) [\lim_{n\to\infty} x_i^{(n)}(u + c_n r + s)]\, ds.$$

It can easily be shown that $\lim_{n\to\infty} x_i^{(n)}(u + c_n r) = x_i(u + c_0 r)$ and hence

$$s_i(u) = \int_0^{\infty} \lambda_i'(r)\, \bar{x}_i(u + c_0 r)\, dr.$$

Hence $x_i(u)$, $(i = 0, 1)$, satisfy the equations

$$c_0 x_{1-i}'(u) = \sigma_i x_{1-i}(u) [\lambda_i(\infty) - \lambda_i(0)\, \bar{x}_i(u) - s_i(u)], \quad (i = 0, 1),$$

and hence equations (9) with $c = c_0$.

Since $x_i^{(n)}(u)$ is monotone increasing and $l_i \leqslant x_i^{(n)}(u) \leqslant 1$ it immediately follows that $l_i \leqslant x_i(u) \leqslant 1$ and $x_i(u)$ is monotone increasing, $(i = 0, 1)$. Also $x_0(0) = \frac{1}{2}(1 + l_0)$. Integrating equations (9) from u to ∞ for $c = c_0$ we obtain

$$\log x_{1-i}(u) - \log x_{1-i}(\infty) = \int_{-\infty}^{\infty} (1 - x_i(u+s))\, v_i(s)\, ds, \quad (i = 0, 1).$$

Letting $u \to \infty$, since the left hand side tends to zero and the integrand on the right is monotone decreasing in u and non-negative, we obtain $x_i(\infty) = 1$, $(i = 0, 1)$. Now letting $u \to -\infty$, equations (15) are obtained with l_i replaced by $x_i(-\infty)$. These equations are only satisfied by $x_i(-\infty) = 1$, $(i = 0, 1)$ or by $x_i(-\infty) = l_i$, $(i = 0, 1)$. Since $x_0(u)$ is monotone increasing and $x_0(0) = \frac{1}{2}(1 + l_0) < 1$, $x_0(-\infty) \neq 1$ and hence $x_i(-\infty) = l_i$, $(i = 0, 1)$.

Hence we have obtained a monotone increasing solution $x_i(u)$ to (9) for $c = c_0$ with $x_i(-\infty) = l_i$ and $x_i(\infty) = 1$, $(i = 0, 1)$.

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