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ABSOLUTE ORDER AND INVOLUTIONS

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Abstract

We study the restriction of the absolute order on a Coxeter group W to an interval $[1, w]_T$, where $w \in W$ is an involution. We characterise and classify those involutions w for which $[1, w]_T$ is a lattice, using the notion of *involutive parabolic subgroups*.

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1. Introduction

Let (W, S) be a Coxeter system with set $T = \bigcup_{w \in W} wSw^{-1}$ of reflections. Let $\ell_T : W \longrightarrow \mathbb{Z}_{\geq 0}$ denote the length function on W with respect to the set T of generators, called the *reflection length*. The *absolute order* on W, denoted $\leq_T$, is defined by

$$u \leq_T v \;\Leftrightarrow\; \ell_T(u) + \ell_T(u^{-1}v) = \ell_T(v).$$

The reflection length was first considered by Carter [9] to classify conjugacy classes in the Weyl groups. Both the reflection length and absolute order play a crucial role in the construction of dual Artin groups attached to finite Coxeter groups. In this context, denoting $[u, v]_T := \{w \in W \mid u \leq_T w \leq_T v\}$, such groups are built as so-called *interval groups* from the intervals $[1, c]_T$, where c is a Coxeter element in W, that is, a product of the elements of S in some order. An important feature is that, when W is finite, the poset $[1, c]_T$ is a lattice, isomorphic if W is the symmetric group to the lattice of noncrossing partitions [4]. The absolute order therefore also allows one to generalise the lattice of noncrossing partitions to every finite Coxeter group and is thus also actively studied in connection with combinatorial structures related to noncrossing partitions (see, for instance, [1, 29]). In the interval group setting, the lattice property ensures that the attached interval group is a so-called *Garside group*. Such a group is torsion-free, admits a solution to the word and conjugacy problems, has a finite $K(\pi, 1)$, a nontrivial computable centre, etc. In the case of a finite Coxeter group W, the interval group attached to $[1, c]_T$ turns out to be isomorphic to the Artin group of

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the same type. This motivates the study of the poset $[1, c]_T$, also in nonspherical type, as a tool to understand Artin groups of nonspherical type. For more on the topic, we refer to [2, 24]. For which W and which choice of Coxeter element c the poset $[1, c]_T$ is a lattice remains a difficult, open question in the field. A solution to this question is known only for finite, affine, universal and rank three Coxeter systems.

In fact, every element $w \in W$ such that $[1, w]_T$ is a lattice gives rise to an interval Garside group. It is thus natural to seek elements w satisfying this property. We summarise the known results on this question:

- if W is finite [2, 7, 25], universal [3] or of rank three [10, 19], and c is a (parabolic) Coxeter element, then $[1, c]_T$ is a lattice;
- if W is of type $\widetilde{A}_n$ [13], then for suitable choices of Coxeter elements c, the interval $[1, c]_T$ is a lattice and for other choices, it is not;
- if W is of type $\widetilde{C}_n$ [14], then for every Coxeter element c, the interval $[1, c]_T$ is a lattice;
- if W is of affine type and not covered by the three previous points, then for any choice of Coxeter element c, the interval $[1, c]_T$ is not a lattice [23];
- in type A_n, the interval $[1, w]_T$ is a lattice for every $w \in W$ (as every element is a parabolic Coxeter element in this case);
- in types B_n and D_n, there is a full classification of elements w such that $[1, w]_T$ is a lattice [21] (unlike type A, there are elements for which the lattice property fails);
- if W is arbitrary and $w \in W$ is such that $\ell_T(w) \leq 3$, then $[1, w]_T$ is a lattice [19].

The aim of this paper is to classify the involutions w such that $[1, w]_T$ is a lattice. We develop uniform tools to answer this question. More precisely, given an involution w in a Coxeter group W, denote by $P(w)$ its parabolic closure, which turns out to be finite. Since $P(w)$ is a reflection subgroup of W, it is again a Coxeter group, with set of generators T_w. The poset $[1, w]_T$ fully lies inside $P(w)$ and coincides with $[1, w]_{T_w}$. We can thus restrict our study to finite Coxeter groups which are parabolic closures of involutions. We then show in Proposition 3.10 that $[1, w]_T$ coincides with the set of involutions in $P(w)$, and in Theorem 3.11 that $[1, w]_T$ is a lattice if and only if the set of parabolic closures of involutions in $P(w)$ is stable by intersection. Finally, using these results, we classify in Section 4 the involutions w such that $[1, w]_T$ is a lattice.

In a forthcoming joint paper with E. Chavli, we will identify and study the attached interval Garside groups.

2. Preliminaries

Let (W, S) be a Coxeter system. Let V be its geometric representation and $T := \bigcup_{w \in W} wSw^{-1}$ its set of reflections. (For basics on Coxeter groups, we refer to [5, 20].) We denote by $\Phi = \Phi^+ \sqcup \Phi^-$ the corresponding root system, and by $t \leftrightarrow \alpha_t$ the 1-to-1 correspondence between reflections and positive roots. Recall that there is a function $N : W \longrightarrow \mathcal{P}(T)$ such that $N(s) = \{s\}$ for all $s \in S$ and $N(xy) = N(x) \,\Delta\, xN(y)x^{-1}$, where Δ denotes symmetric difference. One has $|N(w)| = \ell_S(w)$

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for all $w \in W$, where ℓ_S stands for the length function on W with respect to the generating set S. The elements of $N(w)$ are called the *(left) inversions* of w and may be characterised geometrically as those $t \in T$ such that $w^{-1}(\alpha_t) \in \Phi^-$, equivalently, such that $\ell_S(tw) < \ell_S(w)$. If W is finite, we say that (W, S) is a finite Coxeter system, and denote by w_0 its longest element, which is involutive and satisfies $N(w_0) = T$.

The *reflection length* $\ell_T(w)$ of an element $w \in W$ is the smallest number k for which there exist reflections $t_1, t_2, \dots, t_k \in T$ such that $w = t_1 t_2 \cdots t_k$. If $t_1, t_2, \dots, t_k \in T$ and $w = t_1 t_2 \cdots t_k$ with $k = \ell_T(w)$, we say that the word $t_1 t_2 \cdots t_k$ is a *T-reduced expression* (versus an *S-reduced expression* when $t_i \in S$ and $k = \ell_S(w)$) of $w \in W$. The *absolute order* $\leq_T$ on W is then defined as $u \leq_T v$ if and only if $\ell_T(u) + \ell_T(u^{-1}v) = \ell_T(v)$, that is, there is a T-reduced expression of v having a T-reduced expression of u as a prefix (equivalently, as a suffix, since T is stable by conjugation). It is natural to seek geometric interpretations of $\ell_T(w)$ and algorithms to determine it. When W is finite, we have the following result.

THEOREM 2.1 (Carter's lemma, [9, Lemmas 1–3]). *Let (W, S) be a finite Coxeter group. For $w \in W$, let $V^w = \{v \in V \mid w(v) = v\}$, which we write as H_t if $w = t \in T$. Then:*

(1) *$\ell_T(w) = \dim(V) - \dim(V^w)$ for all $w \in W$;*
(2) *for all $t \in T$, $w \in W$, we have $t \leq_T w \Leftrightarrow V^w \subseteq H_t$;*
(3) *given $w \in W$ and $t_1, t_2, \dots, t_k \in T$ such that $w = t_1 t_2 \cdots t_k$, we have $\ell_T(w) = k$ if and only if the roots $\{\alpha_{t_i}\}_{i=1}^k$ are linearly independent.*

Let $\mathrm{Mov}(w) := \mathrm{Im}(w - \mathrm{Id}_V)$. It is called the *moved space* of w and is stable by w. We have $\mathrm{Mov}(w)^\perp = V^w$ (see [6, Proposition 1]). In particular, $\ell_T(w) = \dim(\mathrm{Mov}(w))$.

Theorem 2.1 does not generalise to arbitrary W, where, typically, the reflection length can be unbounded in general [15]. In affine types, one can derive a formula analogous to that in point (1) (see [22]).

For arbitrary Coxeter systems, we have the following result.

THEOREM 2.2 (Dyer, [17, Theorem 1.1]). *Let (W, S) be an arbitrary Coxeter system. Let $w \in W$. Let $s_1 s_2 \cdots s_k$ be any S-reduced expression of w. Then, $\ell_T(w)$ is equal to the minimal number of letters to delete in the word $s_1 s_2 \cdots s_k$ to get a word representing the identity.*

Let $I \subseteq S$. Recall that the subgroup $(W_I := \langle I \rangle, I)$ is itself a Coxeter group, called a *standard parabolic subgroup*. A *parabolic subgroup* is a subgroup of the form $wW_I w^{-1}$ for some $w \in W$. Parabolic subgroups are examples of reflection subgroups, that is, subgroups generated by a subset of T, which themselves carry a canonical structure of a Coxeter group [12, 16]. Recall that if $W' \subseteq W$ is a reflection subgroup of W, then its set T' of reflections with respect to its canonical Coxeter group structure is given by $W' \cap T$. Parabolic subgroups can also be described as centralisers of subspaces of V.

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Parabolic subgroups are stable by intersection; hence, given $w \in W$, one can consider its *parabolic closure* $P(w)$, defined as the smallest parabolic subgroup containing w.

The following result is folklore (see, for instance, [18, Section 2.4]) and will be useful later on.

PROPOSITION 2.3. *Assume that W is finite. Let $w, x \in W$, $t \in T$.*

(1) $P(w) = C_W(V^w) := \{w \in W \mid w(v) = v \text{ for all } v \in V^w\}$ *and* $\operatorname{rank}(P(w)) = \ell_T(w)$.
(2) *If $w \leq_T x$, then $V^x \subseteq V^w$; hence, $P(w) \subseteq P(x)$.*
(3) *We have $t \leq_T w \Leftrightarrow t \in P(w)$.*

Since $P(w)$ is a reflection subgroup of W and hence has set of reflections $P(w) \cap T$, the set of elements of T such that $t \leq_T w$ is precisely the set of reflections of $P(w)$. In particular, it is a generating set.

3. Involutive parabolic subgroups and lattices of involutions

Let (W, S) be a Coxeter system.

DEFINITION 3.1. Let $P \subseteq W$ be a parabolic subgroup. We say that P is *involutive* if there is an element $x \in W$ such that $x^2 = 1$ and $P(x) = P$.

LEMMA 3.2. *If P is involutive, then P is finite.*

PROOF. Let $x \in W$ such that $x^2 = 1$ and $P = P(x)$. In particular, x has finite order. By [8, Proposition 1.3], there are $I \subseteq S$ and $w \in W$ such that W_I is finite and $x \in wW_Iw^{-1}$. Since wW_Iw^{-1} is a parabolic subgroup containing x, we have $P \subseteq wW_Iw^{-1}$. The latter being finite, we deduce that P is finite. □

The above lemma, together with Proposition 2.3, will allow us to restrict to finite Coxeter groups when studying $[1, w]_T$ with w an involution.

PROPOSITION 3.3. *Let (W, S) be a finite Coxeter system and let $u \in W$ be an involution. Let $t \in T$. Then,*

$$t \leq_T u \;\Leftrightarrow\; tu = ut \quad \text{and} \quad t \in N(u).$$

PROOF. Assume that $t \leq_T u$. Then, $V^u \subseteq H_t$. Thus, $\mathbb{R}\alpha_t = (H_t)^\perp \subseteq (V^u)^\perp$. However, since u is involutive, $(V^u)^\perp$ is precisely the (-1)-eigenspace of u. Thus, $u(\alpha_t) = -\alpha_t$ and hence, $t \in N(u)$. Moreover, since $\mathbb{R}\alpha_t$ is stable by u, we see that $H_t = (\mathbb{R}\alpha_t)^\perp$ is also stable by u. Since $\mathbb{R}\alpha_t$ and H_t are the eigenspaces of t, it follows that $tu = ut$.

Conversely, assume that $tu = ut$ and that $t \in N(u)$. Since t and u commute with each other, they are simultaneously diagonalisable. It follows that u preserves $\mathbb{R}\alpha_t$ and H_t, which are the eigenspaces of t, so that $u(\alpha_t) \in \{\pm\alpha_t\}$. Since $t \in N(u)$, this forces $u(\alpha_t) = -\alpha_t$. Writing $H_t = E_1 \oplus E_{-1}$, where u acts by Id_V on E_1 and by -1 on E_{-1},

$$V = E_1 \oplus E_{-1} \oplus \mathbb{R}\alpha_t,$$

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where ut acts by Id_V on $E_1 \oplus \mathbb{R}\alpha_t$ and by $-\mathrm{Id}_V$ on E_{-1}. Now, $V^u = E_1$ while $V^{ut} = E_1 \oplus \mathbb{R}\alpha_t$. Hence, $\ell_T(u) = \ell_T(ut) + 1$, which concludes the proof. □

COROLLARY 3.4. *Let (W, S) be a finite Coxeter system and $t \in T$. Then,*

$$t \leq_T w_0 \iff w_0 t = t w_0.$$

PROOF. The corollary follows immediately from the previous proposition together with the fact that $N(w_0) = T$. □

COROLLARY 3.5. *Let (W, S) be a finite Coxeter system and $u \in W$. Then, $u^2 = 1$ if and only if u is the longest element in its parabolic closure $P(u)$. In particular, if P is an involutive parabolic subgroup, then there is a unique involution $u \in P$ such that $P = P(u)$.*

PROOF. Assume that u is an involution. Let $t \in P(u)$. By Proposition 2.3(3), $t \leq_T u$. By Proposition 3.3, $t \in N(u)$. It follows that every reflection in $P(u)$ is an inversion of u, which happens if and only if u is the longest element of $P(u)$. The converse is immediate as longest elements of finite Coxeter groups are always involutive. □

PROPOSITION 3.6. *Let (W, S) be a finite Coxeter system and $u \in W$ be an involution. Let $v \in W$ such that $v \leq_T u$. Then, v is an involution and $vu = uv$.*

PROOF. We begin by showing the second statement. Since $v \leq_T u$, we have $V^u \subseteq V^v$. Hence, v preserves V^u and, thus, it also preserves $(V^u)^\perp = \mathrm{Mov}(u)$. Since V^u and $\mathrm{Mov}(u)$ are the eigenspaces of u, it follows that $uv = vu$.

For the first statement, we begin by observing that

$$V = \mathrm{Mov}(u) \oplus V^u = \mathrm{Mov}(v) \oplus \mathrm{Mov}(v^{-1}u) \oplus V^u, \tag{3.1}$$

where the second equality holds by [6, Corollary 1]. Since u is involutive, it acts by $-\mathrm{Id}_V$ on $(V^u)^\perp = \mathrm{Mov}(u) = \mathrm{Mov}(v) \oplus \mathrm{Mov}(v^{-1}u)$. It suffices to show that v acts by $-\mathrm{Id}_V$ on $\mathrm{Mov}(v)$ to conclude that v is involutive. Hence, let $x \in \mathrm{Mov}(v)$. We have

$$v^{-1}(v(x) + x) = v^{-1}(v(x) - u(x)) = -(v^{-1}u(x) - x),$$

but $v^{-1}(v(x)+x) \in \mathrm{Mov}(v)$ since $v(\mathrm{Mov}(v)) = \mathrm{Mov}(v)$, while $-(v^{-1}u(x)-x) \in \mathrm{Mov}(v^{-1}u)$. The decomposition (3.1) forces $v^{-1}(v(x) + x) = 0$ and hence, $v(x) = -x$. □

In [26, Propositions 2.4 and 2.5], it is proven that every involution in a finite Coxeter group admits a T-reduced decomposition made of reflections which are pairwise commuting (such decompositions also appeared in the work of Deodhar [11] and Springer [27], without the notion of reflection length). We deduce from the above proposition that this result holds for every T-reduced decomposition.

COROLLARY 3.7. *Let (W, S) be a finite Coxeter system and $u \in W$ an involution. Let $t_1 t_2 \cdots t_k$ be a T-reduced expression of u. Then, the reflections $t_1, t_2, \ldots, t_k$ are pairwise commuting.*

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PROOF. Let $1 \le i < j \le k$. Since $u = t_i t_j^{t_j t_i}(t_1 \cdots t_{i-1})^{t_j}(t_{i+1} \cdots t_{j-1})(t_{j+1} \cdots t_k)$ and the conjugate of a reflection is again a reflection, we have $t_i t_j \le_T u$. We deduce from the previous proposition that $t_i t_j$ is an involution and hence, $t_i t_j = t_j t_i$. □

REMARK 3.8. A classical tool in the study of posets $[1, c]_T$, where c is a (parabolic) Coxeter element, is the so-called *Hurwitz action* on T-reduced decompositions of c [2, Section 1.6]. It is an action of the braid group on $\ell_T(c)$ strands on such decompositions. A property which has many important consequences in the study of dual Artin groups is the transitivity of this action. It can be defined on T-reduced decompositions of an arbitrary $w \in W$, but in the case of an involution, this tool is quite irrelevant. Except in particular cases, it has no chance to be transitive since when you have a T-reduced decomposition made of pairwise commuting reflections, the action just modifies the order of the factors. In particular, there are, in general, several Hurwitz orbits, resulting, for example, in relations of the form $t_1 t_2 = t_3 t_4$ with $|\{t_1, t_2, t_3, t_4\}| = 4$ occurring in the case of involutions. For instance, in type B_2 with $w = w_0$ and $S = \{s, t\}$, we have $w_0 = t(sts) = s(tst)$. Such relations are not captured by the Hurwitz action.

PROPOSITION 3.9. *Let (W, S) be a finite Coxeter system and let $u \in W$ be an involution. Let $P(u)$ denote the parabolic closure of u. Let $v \in W$. The following are equivalent:*

(1) $v \le_T u$;
(2) $v^2 = 1$ *and* $v \in P(u)$;
(3) $v^2 = 1$ *and* $V^u \subseteq V^v$.

If any of these conditions holds, then $uv = vu$.

PROOF. Suppose that condition (1) holds. We deduce condition (2) and also the last statement using Propositions 3.6 and 2.3(2).

Assume condition (2). Since $v \in P(u) = C_W(V^u)$, we have $V^u \subseteq V^v$, which gives condition (3).

Assume condition (3). We have $V^u \subseteq V^v$ and hence, $V^v = V^u \oplus (V^v \cap \mathrm{Mov}(u))$ yielding

$$V = V^v \oplus \mathrm{Mov}(v) = (V^u \oplus (V^v \cap \mathrm{Mov}(u))) \oplus \mathrm{Mov}(v).$$

Since u and v are involutive and $\mathrm{Mov}(v) = (V^v)^\perp \subseteq (V^u)^\perp = \mathrm{Mov}(u)$, we see that u (respectively v) acts by 1 on V^u (respectively by 1), by -1 on $V^v \cap \mathrm{Mov}(u)$ (respectively by 1) and by -1 on $\mathrm{Mov}(v)$ (respectively by -1). Thus, $v^{-1}u$ is involutive with $V^{v^{-1}u} = V^u \oplus \mathrm{Mov}(v)$ and $\mathrm{Mov}(v^{-1}u) = V^v \cap \mathrm{Mov}(u)$. We deduce that

$$\begin{aligned}\ell_T(v) + \ell_T(v^{-1}u) &= \dim(\mathrm{Mov}(v)) + \dim(V^v \cap \mathrm{Mov}(u)) = \dim(V) - \dim(V^u) \\ &= \dim(\mathrm{Mov}(u)) = \ell_T(u),\end{aligned}$$

which shows that $v \le_T u$, yielding condition (1). □

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PROPOSITION 3.10. *Let (W,S) be a finite Coxeter system. Let $u \in W$ be an involution. Then, $[1,u]_T$ is the set of involutions of $P(u)$. Moreover, denoting by $\mathcal{IP}(u)$ the set of involutive parabolic subgroups of $P(u)$, we have an isomorphism of posets*

$$\varphi : [1,u]_T \longrightarrow (\mathcal{IP}(u), \subseteq), \quad v \mapsto P(v).$$

PROOF. The first statement is an immediate consequence of the equivalence between points (1) and (2) of Proposition 3.9.

We now prove the second statement. Since $P(u)$ is parabolic, the parabolic subgroups of $P(u)$ are exactly the parabolic subgroups of W included in $P(u)$ (this is easily seen, for instance, using the description of parabolic subgroups as centralisers of subspaces in the geometric representation). Let $v_1, v_2 \in [1,u]_T$, which by the already proven statement are involutions. If $\varphi(v_1) = \varphi(v_2)$, then $v_1 \in P(v_2)$ and $v_2 \in P(v_1)$. Using the equivalence between Proposition 3.9(1) and (2), we obtain $v_1 \leq_T v_2$ and $v_2 \leq_T v_1$, and hence, $v_1 = v_2$. The map is thus injective. It is surjective since, if P is an involutive parabolic subgroup of $P(u)$, then there is an involution $v \in P(u)$ such that $P = P(v)$. By Proposition 3.9, again we deduce that $v \leq_T u$ and hence, $v \in [1,v]_T$. Finally, the map φ is an isomorphism of posets since, using again Proposition 3.9, we have

$$v \leq_T u \Leftrightarrow v \in P(u) \Leftrightarrow P(v) \subseteq P(u). \qquad \square$$

THEOREM 3.11. *Let (W,S) be a finite Coxeter system. Let $u \in W$ such that $u^2 = 1$. Then, $[1,u]_T$ is a lattice if and only if the set $\mathcal{IP}(u)$ of involutive parabolic subgroups of $P(u)$ is stable by intersection. In this case, in the notation of Proposition 3.10, for $v, w \in [1,u]_T$, we have $\varphi(v \wedge w) = P(v) \cap P(w)$.*

PROOF. Assume that the involutive parabolic subgroups of $P(u)$ are stable by intersection. It suffices to show the existence of meets since the poset is finite and has a maximal element $P(u)$ (see [28, Proposition 3.3.1]). Hence, let $v, w \in [1,u]_T$. Then, by Proposition 3.9, $v, w \in P(u)$ and $v^2 = w^2 = 1$. In particular, the parabolic subgroups $P(v)$ and $P(w)$ are involutive, and hence by assumption, the same holds for $P = P(v) \cap P(w)$. Thus, there is $z \in P$ such that $P = P(z)$ and $z^2 = 1$. By Proposition 3.9, $z \leq_T v, w$. Now, let $z' \in W$ be such that $z' \leq_T v$, $z' \leq_T w$. In particular, $z' \in P(w) \cap P(y) = P(z)$. Since $z \in [1,u]_T$, by Proposition 3.6, it is an involution. Using Proposition 3.9, we deduce that $z' \leq_T z$.

Conversely, assume that the involutive parabolic subgroups of $P(u)$ are not stable by intersection. Then, there are two involutions $v, w \in P(u)$ such that $P := P(v) \cap P(w)$ is not involutive. Consider the longest element $z_1 = w_{0,P}$ of P. Since it is an involution, by Proposition 3.9, we have $z_1 \leq_T v$ and $z_1 \leq_T w$. Since P is not involutive, we have $P(z_1) \subsetneq P$. Since both $P(z_1)$ and P are generated by reflections, there is $t \in P \cap T$ such that $t \notin P(z_1)$. Hence, $t \not\leq_T z_1$, and by Corollary 3.4 applied to the Coxeter group P, it follows that $tz_1 \neq z_1 t$. Setting $z_2 := tz_1 t$, we see that z_2 is an involution lying in P and distinct from z_1, with $\ell_T(z_1) = \ell_T(z_2)$. By Proposition 3.9, $z_2 \leq_T v, w$. We can now easily deduce that v and w cannot have a meet for $\leq_T$. Assume for a contradiction

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that $Z \in [1,u]_T$ is a meet of v and w. Then, Z is an involution, and $Z \leq_T v$ and $Z \leq_T w$; hence, $Z \in P(v) \cap P(w) = P$ and $z_1, z_2 \leq_T Z$. Since $\ell_T(z_1) = \ell_T(z_2)$ and $z_1 \neq z_2$, we have $z_1 <_T Z$. Hence, there is $t \in P \cap T$ such that $z_1 \leq_T z_1 t \leq_T Z$. Since Z is an involution, by Proposition 3.6, $z_1 t$ is also an involution. We thus have $z_1 t = t z_1$ and hence, $t \leq_T z_1$ by Corollary 3.4, which contradicts $z_1 \leq_T z_1 t$.

The last statement follows immediately from the isomorphism of posets in Proposition 3.10, since the meet in the lattice of involutive parabolic subgroups is clearly given by the intersection. □

Given an involution u of a finite Coxeter group W, we thus obtain that $[1,u]_T$ is a lattice if and only if the involutive parabolic subgroups of $P(u)$ are stable by intersection. If $P(u)$ is not irreducible, then as we shall see in Lemma 4.1 below, $P(u)$ will decompose as a direct product $P(u_1) \times P(u_2) \times \cdots \times P(u_k)$, where each u_i is an involution, and $u = u_1 u_2 \cdots u_k$. The lattice will then be given by the direct product of the lattices $[1,u_i]_T$. Classifying involutions u such that $[1,u]_T$ is a lattice thus reduces to classifying finite irreducible and involutive Coxeter groups in which involutive parabolic subgroups are stable by intersection. The first step is to determine the finite irreducible and involutive Coxeter groups, which will be easy with the help of the next proposition.

PROPOSITION 3.12. *Let (W,S) be a finite, irreducible Coxeter system and let $u \in W$ be an involution. The following are equivalent:*

(1) $W = P(u)$;
(2) $t \leq_T u$ *for all* $t \in T$;
(3) $u \in Z(W)$;
(4) $\ell_T(u) = \operatorname{rank}(W)$.

In this case, $u = w_0$ and $[1,u]_T$ is the set of involutions of W.

PROOF. (1) $\Leftrightarrow$ (2). Since T generates W, this equivalence is obtained by using Proposition 2.3(3).

(2) $\Leftrightarrow$ (3). Assume statement (2). Then, $t \leq_T u$ for all $t \in T$. By Proposition 3.3, we deduce that $tu = ut$ for all $t \in T$, and hence that $u \in Z(W)$ since T generates W. Conversely, assume that $u \in Z(W)$. Then, ut is an involution for all $t \in T$. Let $T = T_1 \cup T_2$, where $T_1 = \{t \in T \mid t \leq_T u\}$ and $T_2 = \{t \in T \mid t \not\leq_T u\}$. Since $u \neq 1$, we have $T_1 \neq \emptyset$. Assume that $T_2 \neq \emptyset$ and let $t_i \in T_i$, $i = 1,2$. Then, $ut_1 \leq_T u \leq_T ut_2$ and, hence, ut_2 is an involution such that $t_1 t_2 \leq_T ut_2$. By Proposition 3.6, $t_1 t_2$ is an involution and hence, $t_1 t_2 = t_2 t_1$. The reflections in T_1 therefore commute with the reflections in T_2. We thus obtain a decomposition as a direct product $W = \langle T_1 \rangle \times \langle T_2 \rangle$, which is impossible since W is irreducible. Hence, $T_2 = \emptyset$ and $T_1 = T$, yielding statement (2).

The equivalence between statements (1) and (4) follows from Proposition 2.3(1). □

It is well known that if the centre of a finite, irreducible Coxeter group W is nontrivial, then $Z(W) = \{1, w_0\}$ (see, for instance, [5, Ch. V, Section 4, Exercise 3]).

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Thus, from the previous proposition, the finite, irreducible and involutive Coxeter groups coincide with those with a nontrivial centre, or those for which the longest element has maximal reflection length, given by

$$A_1,\ I_2(2k)\ (k \geq 2),\ B_n\ (n \geq 3),\ D_{2k}\ (k \geq 2),\ E_7,\ E_8,\ F_4,\ H_3,\ H_4. \tag{3.2}$$

REMARK 3.13. In [26], several results on involutions in finite Coxeter groups are collected and proven. In particular, Serre defines the *degree* of an involution u as the codimension of V^u; hence, the degree is exactly the reflection length. He calls an involution *maximal* if it has maximal degree. Using different methods, Serre shows that maximal involutions are always conjugate [26, Corollaire 3.4] and that for every involution $v \in W$, there is a maximal involution $u \in W$ such that $v \leq_T u$ [26, Théorème 3.7]. The setting of Proposition 3.12 corresponds to the case where there is a unique maximal involution, in which case, it acts by $-\mathrm{Id}_V$ on V. Indeed, let u and W be as in Proposition 3.12 and assume that any of the four equivalent conditions of the proposition holds. Since $\ell_T(u) = \mathrm{rank}(W) = \dim(V)$, we have $V = \mathrm{Mov}(u)$ and since u is involutive, it follows that u acts on V as $-\mathrm{Id}_V$. In this case, Serre's result also gives $v \leq_T u$ for every involution $v \in W$.

4. Classification of involutions yielding a lattice

The aim of this section is to classify the involutions u in a Coxeter group W such that $[1, u]_T$ is a lattice. We begin by the following preliminary lemma, which allows us to reduce to the case where W is finite, irreducible and involutive.

LEMMA 4.1. *Let (W, S) be a Coxeter system and $u \in W$ an involution. Then, $P(u)$ is finite and there are involutions $u_1, u_2, \ldots, u_k \in W$, commuting with each other, unique up to reordering, such that $u = u_1 u_2 \cdots u_k$, $P(u_i)$ is irreducible for each i and*

$$P(u) = P(u_1) \times P(u_2) \times \cdots \times P(u_k).$$

The poset $[1, u]_T$ is then the direct product of the posets $[1, u_i]_T$.

PROOF. The fact that $P(u)$ is finite is given by Lemma 3.2. Since $P(u)$ is a parabolic subgroup, it is a Coxeter group and we can decompose it into irreducible components $P_1, P_2, \ldots, P_k$, which are themselves parabolic, and this decomposition is unique up to the order of the factors. We thus have $P(u) = P_1 \times P_2 \times \cdots P_k$, and u decomposes uniquely according to this decomposition as $u = u_1 u_2 \cdots u_k$, where $u_i \in P_i$ and u_i is involutive for all i. It remains to show that $P_i = P(u_i)$. If not, there is i such that $P(u_i) \subsetneq P_i$. However, then, u lies in $P_1 \times \cdots \times P_{i-1} \times P(u_i) \times P_{i+1} \times \cdots \times P_k$, which is strictly included in $P(u)$ and parabolic, which is a contradiction. We thus get the existence of the decomposition, and the uniqueness follows from the uniqueness of the decomposition of $P(u)$ into irreducible components and Corollary 3.5.

The last statement follows from the decomposition $P(u) = P(u_1) \times \cdots \times P(u_k)$ and the fact that $\ell_T(u) = \sum_{i=1}^k \ell_T(u_i)$; this last claim holds since by Proposition 2.3(1),

$$\ell_T(u) = \mathrm{rank}(P(u)) = \sum_{i=1}^k \mathrm{rank}(P(u_i)) = \sum_{i=1}^k \ell_T(u_i). \qquad \square$$

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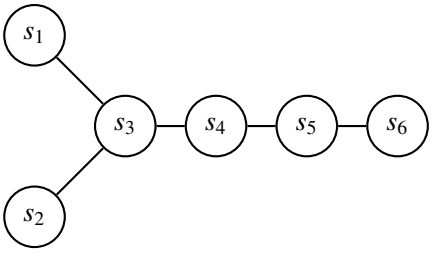


FIGURE 1. Labelling of the Dynkin diagram of type D_6.

We can now prove the main result, checking which finite irreducible Coxeter groups among those listed in (3.2) satisfy the condition of Theorem 3.11.

THEOREM 4.2. *The finite, irreducible, involutive Coxeter groups W such that $[1, w_0]_T$ is a lattice are those of type $A_1, I_2(2k)$ $(k \geq 2), B_n$ $(n \geq 3), D_4, H_3$.*

PROOF. We go through the list given in (3.2) and check for each family whether involutive parabolic subgroups are stable by intersection or not; applying Theorem 3.11 then completes the proof.

We first note that the cases B_n $(n \geq 3)$ and D_{2k} $(k \geq 2)$ have already been treated by Kallipoliti in [21, Theorems 7 and 8], who characterised all the elements w in types B_n and D_n such that $[1, w]_T$ is a lattice. We can also reobtain it using the framework developed in this paper by checking that involutive parabolic subgroups are stable by intersection in these cases: in type B_n, using the list in (3.2), we see that involutive parabolic subgroups are of type $B_k \times (A_1)^\ell$ and the intersection of subgroups of such type is still of this type; in D_4, proper involutive parabolic subgroups are of the form $(A_1)^\ell$ for $\ell \leq 3$ and hence, also stable by intersection. In type D_{2k} with $k \geq 3$, one can consider a standard parabolic subgroup P_1 of type D_4 and a simple reflection $s \in D_{2k} \backslash P_1$ commuting with every simple reflection in P_1 except one. Then, $P_2 := sP_1s$ is again an involutive parabolic subgroup and $P_1 \cap P_2$ has type A_3; hence, is not involutive. For instance, in D_6, labelling the Dynkin diagram as in Figure 1, we set $P_1 = \langle s_1, s_2, s_3, s_4 \rangle$, $P_2 = s_5P_1s_5 = \langle s_1, s_2, s_3, s_5s_4s_5 \rangle$ and $P_1 \cap P_2 = \langle s_1, s_2, s_3 \rangle$: indeed $P_1 \cap P_2$ is parabolic, of rank at most three since P_1 and P_2 have rank four and are distinct, and $\langle s_1, s_2, s_3 \rangle$ is a rank-three parabolic subgroup included in $P_1 \cap P_2$. Note that the argument also works for types E_7 and E_8.

The case A_1 is trivial, and in type $I_2(2k)$ with $k \geq 2$, proper involutive parabolic subgroups are of type A_1; hence, stability by intersection is clear (but in this case, every element w of the group has reflection length at most 2 and hence, $[1, w]_T$ is necessarily a lattice).

The case H_3 is already covered by our previously obtained result [19, Theorem 2.2] since $\ell_T(w_0) = 3$ in this case. Nevertheless, we can reobtain it easily since proper involutive parabolic subgroups in type H_3 are of type $(A_1)^\ell$ for $\ell \leq 2$; hence, stability by intersection is clear.

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For types F_4 and H_4, we can use an argument similar to that used for D_{2k} ($k \geq 3$). Consider a standard parabolic subgroup P_1 of type B_3 (respectively H_3) in F_4 (respectively H_4). Let s be the unique simple reflection which remains, that is, not lying in P_1. Then, $P_2 := sP_1s$ has the same type as P_1 and hence, is involutive, but $P_1 \cap P_2$ has type A_2 (respectively $I_2(5)$); hence, involutive parabolic subgroups are not stable by intersection. □

Combining Theorem 4.2 with Lemma 4.1 gives the following corollary.

COROLLARY 4.3. *Let (W, S) be a Coxeter system and $u \in W$ an involution. Then, $[1, u]_T$ is a lattice if and only if every irreducible factor of $P(u)$ is of one of the following types: $A_1, I_2(2k)$ $(k \geq 2), B_n$ $(n \geq 3), D_4, H_3$.*

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References

[1] D. Armstrong, *Generalized Noncrossing Partitions and Combinatorics of Coxeter Groups*, Memoirs of the American Mathematical Society, 202, no. 949 (American Mathematical Society, Providence, RI, 2009).
[2] D. Bessis, 'The dual braid monoid', *Ann. Sci. Éc. Norm. Supér. (4)* **36** (2003), 647–683.
[3] D. Bessis, 'A dual braid monoid for the free group', *J. Algebra* **302** (2006), 275–309.
[4] P. Biane, 'Some properties of crossings and partitions', *Discrete Math.* **175**(1–3) (1997), 41–53.
[5] N. Bourbaki, *Éléments de Mathématique, Fasc. XXXIV, Groupes et Algèbres de Lie, Chapitre IV: Groupes de Coxeter et systèmes de Tits, Chapitre V: Groupes engendrés par des réflexions, Chapitre VI: Systèmes de racines*, Actualités Scientifiques et Industrielles, 1337 (Hermann, Paris, 1968).
[6] T. Brady and C. Watt, 'A partial order on the orthogonal group', *Comm. Algebra* **30**(8) (2002), 3749–3754.
[7] T. Brady and C. Watt, 'Non-crossing partition lattices in finite real reflection groups', *Trans. Amer. Math. Soc.* **360**(4) (2008), 1983–2005.
[8] B. Brink and R. Howlett, 'A finiteness property and an automatic structure for Coxeter groups', *Math. Ann.* **296**(1) (1993), 179–190.
[9] R. W. Carter, 'Conjugacy classes in the Weyl group', *Compositio Math.* **25** (1972), 1–59.
[10] E. Delucchi, G. Paolini and M. Salvetti, 'Dual structures on Coxeter and Artin groups of rank three', *Geom. Topol.* **28**(9) (2024), 4295–4336.
[11] V. V. Deodhar, 'On the root system of a Coxeter group', *Comm. Algebra* **10**(6) (1982), 611–630.
[12] V. V. Deodhar, 'A note on subgroups generated by reflections in Coxeter groups', *Arch. Math. (Basel)* **53**(6) (1989), 543–546.
[13] F. Digne, 'Présentations duales des groupes de tresses de type affine $\tilde{A}$', *Comment. Math. Helv.* **81**(1) (2006), 23–47.
[14] F. Digne, 'A Garside presentation for Artin groups of type $\tilde{C}_n$', *Ann. Inst. Fourier (Grenoble)* **62**(2) (2012), 641–666.
[15] K. Duszenko, 'Reflection length in non-affine Coxeter groups', *Bull. Lond. Math. Soc.* **44**(3) (2012), 571–577.
[16] M. J. Dyer, 'Reflection subgroups of Coxeter systems', *J. Algebra* **135**(1) (1990), 57–73.
[17] M. J. Dyer, 'On minimal lengths of expressions of Coxeter group elements as products of reflections', *Proc. Amer. Math. Soc.* **129**(9) (2001), 2591–2595.

https://doi.org/10.1017/S0004972725100531 Published online by Cambridge University Press

[18] T. Gobet, 'Dual Garside structures and Coxeter sortable elements', *J. Comb. Algebra* **4**(2) (2020), 167–213.
[19] T. Gobet, 'On maximal dihedral reflection subgroups and generalized noncrossing partitions', *Proc. Amer. Math. Soc.* **152**(10) (2024), 4095–4101.
[20] J. E. Humphreys, *Reflection Groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, 29 (Cambridge University Press, Cambridge, 1990).
[21] M. Kallipoliti, 'The absolute order on the hyperoctahedral group', *J. Algebraic Combin.* **34**(2) (2011), 183–211.
[22] J. B. Lewis, J. McCammond, T. K. Petersen and P. Schwer, 'Computing reflection length in an affine Coxeter group', *Trans. Amer. Math. Soc.* **371**(6) (2019), 4097–4127.
[23] J. McCammond, 'Dual Euclidean Artin groups and the failure of the lattice property', *J. Algebra* **437** (2015), 308–343.
[24] G. Paolini, 'The dual approach to the $K(\pi, 1)$ conjecture', *Collection Soc. Math. France Séminaires et Congrès* **34** (2025), 177–201.
[25] N. Reading, 'Noncrossing partitions and the shard intersection order', *J. Algebraic Combin.* **33**(4) (2011), 483–530.
[26] J.-P. Serre, 'Groupes de Coxeter finis: involutions et cubes', *Enseign. Math.* **68**(102) (2022), 99–133.
[27] T. A. Springer, 'Some remarks on involutions in Coxeter groups', *Comm. Algebra* **10**(6) (1982), 631–636.
[28] R. Stanley, *Enumerative Combinatorics. Volume 1*, Cambridge Studies in Advanced Mathematics, 49 (Cambridge University Press, Cambridge, 2012).
[29] C. Stump, H. Thomas and N. Williams, *Cataland: Why the Fuß?*, Memoirs of the American Mathematical Society, 305, no. 1535 (American Mathematical Society, Providence, RI, 2025).

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