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Impact of Linear Operator on the Convergence of HAM Solution: a Modified Operator Approach

S. T. Hussain[1,2,*], S. Nadeem[2] and M. Qasim[3]

[1] *Department of Mechanical and Materials Engineering, Spencer Engineering Building, N6A 5B9, University of Western Ontario, London, Ontario, Canada*
[2] *Department of Mathematics, Quaid-I-Azam University 45320, Islamabad 44000, Pakistan*
[3] *Department of Mathematics, COMSATS Institute of Information Technology, Chak Shahzad, Islamabad, Pakistan*

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Abstract. The linear operator plays an important role in the computational process of Homotopy Analysis Method (HAM). In HAM frame any kind of linear operator can be chosen to develop a solution. Hence, it is easy to introduce the modified/physical parameter dependent linear operators. The effective use of these operators has been judged through solving fluid flow problems. Modification in linear operators affects the solution and improves the computational efficiency of HAM for larger values of parameters. The convergence rate of the solution is rapid and several times higher resulting in lesser computational time.

AMS subject classifications: 76-XX, 76A05, 65B99, 76M25

Key words: Optimal homotopy analysis method (OHAM), modified operator, rapid convergence, large physical parameters, stretching surface.

1 Introduction

Homotopy Analysis Method (HAM) is being used to obtain the solutions of highly nonlinear differential equations. Firstly this technique was introduced by Liao in his Ph.D thesis and developed it further by introducing a convergence control parameter in the deformation equation. His book [1] thoroughly describes the application of HAM. This technique has several advantages over the perturbation methods and other analytic techniques. For example; no small or large parameter restriction, freedom to choose the linear

*Corresponding author.
Email: sthqau@gmail.com (S. T. Hussain)

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operator, existence of efficient scheme to examine the convergence of obtained solutions, uniform applicability to the ordinary and partial differential equations and successful application for solving the highly non-linear system of differential equations [2–11]. The applicability and use of this method over the years resulted in a great improvement therein. Marinca and Herisanu [12, 13] first introduced a number of convergence control parameters and thereafter used the concept of optimization for obtaining the best values. Although this concept is rigorous theoretically yet it is less efficient computationally. Liao by taking into account computational efficiency, introduced three parameter based Optimal Homotopy Analysis Method (OHAM). In his recent paper [14] he, however suggested that the basic OHAM (i.e., which contains one convergence control parameter) is computationally more efficient than three parameters OHAM, finite parameters OHAM and infinite parameters OHAM. Recently Zhao et al. [15], introduced the modified HAM, orthonormal functions have been used to control the rapid growth of terms in higher order approximations. This approach reduced the standard computational time as compared to the standard HAM. Shehzad et al. [16] discussed the magnetohydrodynamic flow of casson fluid over a stretching sheet. They plotted the h-curve at $M = 0.5$ and showed the convergence of solution for small value of M. The variation of velocity profile for $M = 0$ to 2 was discussed and tables were drawn for maximum M equals to 1.6. Hayat et al. [17] discussed the flow and heat transfer phenomenon in magneto hydrodynamic fluid over a permeable stretching sheet in the presence of radiations. They examined the convergence of solution by assuming M and Pr equal to 1. Due to use of simple linear operators for momentum and heat transfer equation the convergent solutions for large values of M and Pr were difficult to obtain. The maximum values of M and Pr chosen in the article were 3 and 10 respectively. Nadeem and Saleem [18] discussed the flow of Nano fluid over a rotating cone in the presence of magnetic field. Their study reports the maximum value of M equals to 7 and Pr equals to 6. Ellahi and Riaz [19] modelled the third grade fluid flow with variable velocity and solved the governing equations by using the HAM. In their article the maximum value of M was reported to be 4. Xu and Liao [20] obtained the series solution for the unsteady MHD flow of non-Newtonian fluid due to impulsively stretching sheet and discussed the velocity variations for M equals to 0, 1 and 2. Liao [21] obtained the solutions for the Blasius flow problem in the larger domain of Prandtl number between 0 and 50. First he introduced an arbitrary parameter in the linear operator and later on chose its suitable value. The result obtained through this technique could only match up to three decimal points with numerical results. Although introduction of parameter in the linear operator was a good thought yet no precise way was suggested to choose the best value. One possible way could be the optimization of this parameter against the total squared residual error.

Although HAM is a strong analytic technique to solve highly nonlinear problems yet for large physical parameters it was not providing the best correlation with numerical solutions. Liao in his recent book [22] obtained the solutions by choosing different kind of basis set and linear operators.

In HAM frame any kind of linear operator and basis set can be chosen to develop a

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solution. This helps in introducing physical parameter dependent basis set and linear operator. The aim of the present article is to explore the option of getting convergent solution for large physical parameters and to strengthen the Liao's claim that the HAM solutions are uniformly valid for small as well as large values of physical parameters. The physical parameter chosen for the present study are magnetic parameter and Prandtl number. For momentum and heat equation mathematical modelling is being presented separately so as the effect of modified operator approach can be examined for different type of equations and boundary conditions. A detailed analysis is also being done to check the effectiveness of modified operator approach with the help of several numerical tables.

2 Mathematical formulation of equations

In this section two problems are being formulated for non-Newtonian Williamson fluid flow over a linearly stretching sheet. Problem one involves magnetic parameter while problem two involves the Prandtl number.

2.1 MHD non-Newtonian Williamson fluid flow

The boundary layer equations for the two dimensional flow of Williamson fluid are developed by Nadeem et al. [23]. In the presence of transverse magnetic field (body force component) the momentum equation would be

$$div\mathbf{V}=0, \tag{2.1a}$$

$$\rho\frac{d\mathbf{V}}{dt}=div\mathbf{S}+\rho\mathbf{b}, \tag{2.1b}$$

where $\mathbf{V}(u(x,y),v(x,y),0)$ is the velocity vector, ρ is fluid density, $\mathbf{S}$ is Cauchy stress tensor, $\mathbf{b}$ is body force vector. For Williamson fluid model Cauchy stress tensor S is defined in [24]. The interaction of magnetic field and velocity will give rise to the body force $\mathbf{J}\times\mathbf{B}$ defined in Eq. (2.2), in which $\mathbf{J}$ is the current density and σ is the electrical conductivity.

$$\mathbf{J}=\sigma(\mathbf{E}+\mathbf{V}\times\mathbf{B}) \quad \text{or} \quad \sigma(\mathbf{V}\times\mathbf{B}), \quad \text{when} \ \ \mathbf{E}=0. \tag{2.2}$$

After using above equations and applying boundary layer approximations we get

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.3a}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+\sqrt{2}\nu\Gamma\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B^2 u}{\rho}. \tag{2.3b}$$

The corresponding boundary conditions to the flow problem are

$$u=U_w=ax, \quad v=0 \quad \text{at} \ \ y=0, \quad u\to 0 \ \ \text{as} \ \ y\to\infty. \tag{2.4}$$

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After introducing the following transformations in above equations, we get

$$u=axf'(\eta),\quad v=-\sqrt{av}f(\eta),\quad \eta=\sqrt{\frac{a}{\nu}}y. \tag{2.5}$$

Eq. (2.3a) is identically satisfied and Eqs. (2.3b)-(2.4) take the following form

$$f'''-f'^2+ff''+\lambda f''f'''-Mf'=0. \tag{2.6}$$

The corresponding boundary conditions in non dimensional form are

$$f=0,\quad f'=1\quad \text{at}\ \ \eta=0,\quad f'=0\quad \text{as}\quad \eta\to\infty. \tag{2.7}$$

In above equations λ is non Newtonian parameter and M is magnetic parameter

$$\lambda=\Gamma x\sqrt{\frac{2a^3}{\nu}},\quad M=\frac{\sigma B_\circ^2}{\rho a}. \tag{2.8}$$

2.2 Heat transfer in non-Newtonian Williamson fluid flow

In this subsection heat transfer equation is modelled for non-Newtonian Williamson fluid over a linearly stretching surface. Heat equation for the Williamson fluid is

$$\rho\frac{de}{dt}=\mathbf{S}\cdot\mathbf{L}-div\mathbf{q}. \tag{2.9}$$

Where e is internal energy, $\mathbf{q}$ is heat flux and $\mathbf{L}=grad\mathbf{V}$. In the absence of viscous dissipation effects

$$\mathbf{S}\cdot\mathbf{L}=0,\quad e=c_pT,\quad \mathbf{q}=-k\nabla T, \tag{2.10}$$

above equation take the form

$$\rho c\left(\frac{\partial T}{\partial t}+V\cdot\nabla T\right)=\nabla\cdot k\nabla T. \tag{2.11}$$

The steady two dimensional boundary layer equation with corresponding boundary conditions is:

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}, \tag{2.12a}$$

$$T=T_w\quad \text{at}\ \ \eta=0,\quad T=T_\infty\ \text{as}\ \ \eta\to\infty, \tag{2.12b}$$

where $\alpha=k/\rho c$ is thermal diffusivity and T_w is the constant wall temperature. By introducing the velocity transformations and the following transformation for temperature in Eq. (2.12)

$$\theta=\frac{T-T_\infty}{T_w-T_\infty}. \tag{2.13}$$

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The equation would be

$$\theta''+\Pr f\theta'=0, \tag{2.14a}$$

$$\theta=1 \text{ at } \eta=0, \quad \theta=0 \text{ as } \eta\to\infty. \tag{2.14b}$$

In order to differentiate the velocity equation for the first and second problem, we will rewrite the Eq. (2.6) in the absence of magnetic field in terms of new non dimensional function $g(\eta)$ similar to the function $f(\eta)$. The non-dimensional system of equations describing the heat and fluid flow phenomenon of Williamson fluid in the absence of magnetic field is

$$\theta''+\Pr g\theta'=0, \tag{2.15a}$$

$$\theta=1 \text{ at } \eta=0, \quad \theta=0 \text{ as } \eta\to\infty, \tag{2.15b}$$

and

$$g'''-g'^2+gg''+\lambda g''g'''=0, \tag{2.16a}$$

$$g=0, \quad g'=1 \text{ at } \eta=0, \quad g'=0 \text{ as } \eta\to\infty. \tag{2.16b}$$

3 Solution technique

3.1 Homotopy Analysis Method (HAM)

For solution of first and second problem the physical parameter dependent basis set and linear operator are chosen, which are:

$$\{\eta^k\exp(-nM\eta);\, k\geq 0,\, n\geq 0\}, \quad \{\eta^i\exp(-r\eta),\, \eta^i\exp(-r\Pr\eta);\, i\geq 0,\, r\geq 0\}. \tag{3.1}$$

Keeping in view the above expressions and the boundary conditions (2.7), the modified linear operator (physical parameter dependent operator) and the initial guess for first problem is

$$\mathbf{L}_f(f)=\frac{d^3f}{d\eta^3}-M\frac{df}{d\eta}, \tag{3.2a}$$

$$f_0(\eta)=\frac{1-\exp(-\sqrt{M}\eta)}{\sqrt{M}}. \tag{3.2b}$$

In second problem, to keep the simplicity and to examine the validity of new basis set for Prandtl number, set M is equal to 1. The simple operator for velocity and Prandtl dependent operator for temperature are:

$$\mathbf{L}_g(g)=\frac{d^3g}{d\eta^3}-\frac{dg}{d\eta}, \qquad \mathbf{L}_\theta(\theta)=\frac{d^2\theta}{d\eta^2}+\Pr\frac{d\theta}{d\eta}, \tag{3.3a}$$

$$g_0(\eta)=1-\exp(-\eta), \qquad \theta_0(\eta)=\exp(-\Pr\eta), \tag{3.3b}$$

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and

$$L_f[C_1+C_2\exp(\sqrt{M}\eta)+C_3\exp(-\sqrt{M}\eta)]=0, \tag{3.4a}$$
$$L_g[C_4+C_5\exp(\eta)+C_6\exp(-\eta)]=0, \tag{3.4b}$$
$$L_\theta[C_7+C_8\exp(-\mathrm{Pr}\eta)]=0. \tag{3.4c}$$

C_i $(i=1-8)$ are the arbitrary constants.

Detailed HAM [25–27] procedure is given for both problems in the subsequent sub-sections. Usual simple operators and guess are obtained assuming $M=Pr=1$ in E-qs. (3.2a)-(3.4). However, it doesnt necessarily imply in the governing Eqs. (2.6) and (2.15).

3.1.1 Zeroth order deformation equation

If $q\in[0,1]$ is the embedding parameter and the non-zero auxiliary convergence control parameters are h_f, h_g, h_θ then the problems at the zeroth order deformation are:

$$(1-q)L_f[f(\eta;q)-f_0(\eta)]=qh_fN_f[f(\eta;q)], \tag{3.5a}$$
$$(1-q)L_g[g(\eta;q)-g_0(\eta)]=qh_gN_g[g(\eta;q)], \tag{3.5b}$$
$$(1-q)L_\theta[\theta(\eta;q)-\theta_0(\eta)]=qh_\theta N_\theta[g(\eta;q),\theta(\eta;q)], \tag{3.5c}$$

and

$$(f(\eta;q))_{\eta=0}=0, \qquad \left(\frac{\partial f(\eta;q)}{\partial\eta}\right)_{\eta=0}=1, \qquad \left(\frac{\partial f(\eta;q)}{\partial\eta}\right)_{\eta\to\infty}=0, \tag{3.6a}$$
$$(g(\eta;q))_{\eta=0}=0, \qquad \left(\frac{\partial g(\eta;q)}{\partial\eta}\right)_{\eta=0}=1, \qquad \left(\frac{\partial g(\eta;q)}{\partial\eta}\right)_{\eta\to\infty}=0, \tag{3.6b}$$
$$(\theta(\eta;q))_{\eta=0}=1, \qquad (\theta(\eta;q))_{\eta\to\infty}=0, \tag{3.6c}$$

where the nonlinear operators are defined as:

$$N_f[f(\eta;q)]=\frac{\partial^3 f(\eta;q)}{\partial\eta^3}-\left(\frac{\partial f(\eta;q)}{\partial\eta}\right)^2+f(\eta;q)\frac{\partial^2 f(\eta;q)}{\partial\eta^2}+\lambda\frac{\partial^2 f(\eta;q)}{\partial\eta^2}\cdot\frac{\partial^3 f(\eta;q)}{\partial\eta^3} -M\frac{\partial f(\eta;q)}{\partial\eta}, \tag{3.7a}$$
$$N_g[g(\eta;q)]=\frac{\partial^3 g(\eta;q)}{\partial\eta^3}-\left(\frac{\partial g(\eta;q)}{\partial\eta}\right)^2+g(\eta;q)\frac{\partial^2 g(\eta;q)}{\partial\eta^2}+\lambda\frac{\partial^2 g(\eta;q)}{\partial\eta^2}\cdot\frac{\partial^3 g(\eta;q)}{\partial\eta^3}, \tag{3.7b}$$
$$N_\theta[\theta(\eta;q)]=\frac{\partial^2\theta(\eta;q)}{\partial\eta^2}+\mathrm{Pr}f(\eta;q)\frac{\partial\theta(\eta;q)}{\partial\eta}, \tag{3.7c}$$

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when $q=0$ and $q=1$, then

$$f(\eta;0)=f_0(\eta), \qquad f(\eta;1)=f(\eta), \tag{3.8a}$$
$$g(\eta;0)=g_0(\eta), \qquad (\eta;1)=g(\eta), \tag{3.8b}$$
$$\theta(\eta;0)=\theta_0(\eta), \qquad \theta(\eta;1)=\theta(\eta). \tag{3.8c}$$

With the help of Taylor series expansion

$$f(\eta;q)=f_0(\eta)+\sum_{m=1}^{\infty} f_m(\eta)q^m, \tag{3.9a}$$
$$g(\eta;q)=g_0(\eta)+\sum_{m=1}^{\infty} g_m(\eta)q^m, \tag{3.9b}$$
$$\theta(\eta;q)=\theta_0(\eta)+\sum_{m=1}^{\infty} \theta_m(\eta)q^m, \tag{3.9c}$$

and

$$f_m(\eta)=\frac{1}{m!}\frac{\partial^m f(\eta;q)}{\partial q^m}\Big|_{q=0}, \tag{3.10a}$$
$$g_m(\eta)=\frac{1}{m!}\frac{\partial^m g(\eta;q)}{\partial q^m}\Big|_{q=0}, \tag{3.10b}$$
$$\theta_m(\eta)=\frac{1}{m!}\frac{\partial^m \theta(\eta;q)}{\partial q^m}\Big|_{q=0}, \tag{3.10c}$$

auxiliary parameters are chosen in such a way that Eqs. (3.9) converge at $q=1$, Thus

$$f(\eta)=f_0(\eta)+\sum_{m=1}^{\infty} f_m(\eta), \quad g(\eta)=g_0(\eta)+\sum_{m=1}^{\infty} g_m(\eta), \quad \theta(\eta)=\theta_0(\eta)+\sum_{m=1}^{\infty} \theta_m(\eta). \tag{3.11}$$

3.1.2 m^{th} order deformation equations

m^{th} order approximations are defined as

$$L_f[f_m(\eta)-\chi_m f_{m-1}(\eta)]=h_f R_m^f(\eta), \tag{3.12a}$$
$$L_g[g_m(\eta)-\chi_m g_{m-1}(\eta)]=h_g R_m^g(\eta), \tag{3.12b}$$
$$L_\theta[\theta_m(\eta)-\chi_m \theta_{m-1}(\eta)]=h_\theta R_m^\theta(\eta), \tag{3.12c}$$

and

$$f_m(0)=f_m'(0)=f_m'(\infty)=0, \tag{3.13a}$$
$$g_m(0)=g_m'(0)=g_m(\infty)=0, \tag{3.13b}$$
$$\theta_m(0)=\theta_m(\infty)=0, \tag{3.13c}$$

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where

$$R_m^f(\eta)=f'''_{m-1}+\sum_{n=0}^{m-1}[f_{m-1-n}f''_n-f'_{m-1-n}f'_n]+\lambda\sum_{n=0}^{m-1}f''_{m-1-n}f'''_n-Mf'_{m-1}, \tag{3.14a}$$

$$R_m^g(\eta)=g'''_{m-1}+\sum_{r=0}^{m-1}[g_{m-1-r}g''_r-g'_{m-1-r}g'_r]+\lambda\sum_{r=0}^{m-1}g''_{m-1-r}g'''_r, \tag{3.14b}$$

$$R_m^\theta(\eta)=\theta''_{m-1}+\Pr\sum_{r=0}^{m-1}[g_{m-1-r}\theta'_r], \tag{3.14c}$$

$$\chi_m=\begin{cases}0, & m\leq 1,\\ 1, & m>1.\end{cases} \tag{3.14d}$$

The general solution for Eqs. (3.12) are defined as:

$$f_m(\eta)=f_m^*(\eta)+C_1+C_2\exp(\sqrt{M}\eta)+C_3\exp(-\sqrt{M}\eta), \tag{3.15a}$$

$$g_m(\eta)=g_m^*(\eta)+C_4+C_5\exp(-\eta)+C_6\exp(\eta), \tag{3.15b}$$

$$\theta_m(\eta)=\theta_m^*(\eta)+C_7+C_8\exp(-\Pr\eta), \tag{3.15c}$$

where $f_m^*(\eta)$, $g_m^*(\eta)$, $\theta_m^*(\eta)$ are special functions.

Constants can be determined using conditions (3.13) and solutions are obtained with the help of OHAM. Any order approximation for $m=1,2,3,\cdots$, can be obtained with the help of computational software Mathematica.

3.1.3 Optimal convergence control parameters

It is worth mentioning that our solutions $f(\eta)$, $g(\eta)$ and $\theta(\eta)$ contain unknown convergence control (auxiliary) parameters (h_f,h_g,h_θ). To find out the optimal values of convergence control parameters the exact squared residual errors at the mth order approximation are as follows

$$\varepsilon_m^f=\int_0^\infty\left\{N_f\left[\sum_{i=0}^m f_i(\eta)\right]\right\}^2 d\eta, \tag{3.16a}$$

$$\varepsilon_m^g=\int_0^\infty\left\{N_g\left[\sum_{i=0}^m g_i(\eta)\right]\right\}^2 d\eta, \tag{3.16b}$$

$$\varepsilon_m^\theta=\int_0^\infty\left\{N_\theta\left[\sum_{i=0}^m g_i(\eta),\sum_{i=0}^m \theta_i(\eta)\right]\right\}^2 d\eta, \tag{3.16c}$$

and

$$\varepsilon_m^t=\varepsilon_m^g+\varepsilon_m^\theta, \tag{3.17}$$

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where ε_m^f and ε_m^t are the total squared residual errors for first and second problem. ε_m^f, ε_m^g, ε_m^θ, are the corresponding individual errors at m^{th} iteration. It is obvious that more quickly total squared residual error approaches to zero, faster the corresponding solution converges. Thus at m^{th} iteration the optimal values of convergence control parameters are given by the minimum of ε_m^f and ε_m^t corresponding to the set of these equations

$$\frac{\partial \varepsilon_m^f}{\partial h_f}=0, \quad \frac{\partial \varepsilon_m^t}{\partial h_g}=\frac{\partial \varepsilon_m^t}{\partial h_\theta}=0. \tag{3.18}$$

The total squared residual error defined by Eqs. (3.16a) and (3.17) takes too much CPU time to calculate the error even at low order approximations. In order to increase the computational efficiency Liao [14] introduced the discrete squared residual error approach. The discrete squared residual errors at the m^{th} iteration are defined as

$$E_m^f \approx \frac{1}{N+1}\sum_{j=0}^{N}\left\{N_f\left[\sum_{i=0}^{m} f_i(\eta_j)\right]\right\}^2, \tag{3.19a}$$

$$E_m^g \approx \frac{1}{N+1}\sum_{j=0}^{N}\left\{N_g\left[\sum_{i=0}^{m} g_i(\eta_j)\right]\right\}^2, \tag{3.19b}$$

$$E_m^\theta \approx \frac{1}{N+1}\sum_{j=0}^{N}\left\{N_\theta\left[\sum_{i=0}^{m} g_i(\eta_j),\sum_{i=0}^{m}\theta_i(\eta_j)\right]\right\}^2, \tag{3.19c}$$

where $\eta_j=j\Delta\eta$, $\Delta\eta=0.5$ and $N=20$ in above equations. The total discrete squared residual errors are defined by E_m^f and E_m^t

$$E_m^t = E_m^\theta + E_m^g. \tag{3.20}$$

Where the minimum is obtained corresponding to the following constraints:

$$\frac{\partial E_m^f}{\partial h_f}=0, \quad \frac{\partial E_m^t}{\partial h_g}=\frac{\partial E_m^t}{\partial h_\theta}=0. \tag{3.21}$$

Discrete squared residual errors approach is used to obtain the optimal values of convergence control parameters.

3.2 Numerical solution

Numerical solution of the governing nonlinear ordinary differential equations (2.15) and (2.16) are obtained by shooting technique [28–31], by converting the Boundary Value Problem (BVP) into Initial Value Problem (IVP). The shooting parameters are selected in such a way that the far field boundary conditions, i.e., $\eta\to\infty$ is satisfied at a finite value

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say $\eta_{\max}$. With the introduction of new set of dependent variables w_1, w_2, w_3, p_1 and p_2, the first order system is:

$$\begin{cases} w_1 = g, \\ w_2 = w_1' = g', \\ w_3 = w_2' = g'', \\ w_3' = \dfrac{w_2^2 - w_1 w_3}{1 + \lambda w_3}, \end{cases} \tag{3.22}$$

and

$$\begin{cases} p_1 = \theta, \\ p_2 = p_1' = \theta', \\ p_2' = -\Pr[w_1 p_2], \end{cases} \tag{3.23}$$

with boundary conditions

$$w_1(0) = 0, \quad w_2(0) = 1, \quad p_1(0) = 1. \tag{3.24}$$

In order to solve (3.22)-(3.24) as an IVP, values of $w_3(0)$ i.e., $g''(0)$ and $p_2(0)$ i.e., $\theta'(0)$ are required but no such values are available. Thus, the initial guess values of $g''(0)$ and $\theta'(0)$ have been chosen and fourth order Runge-Kutta method is applied to obtain the solution. The calculated values of $f'(\eta)$ and $\theta(\eta)$ at the far field boundary condition $\eta_\infty = \eta_{\max}(=15)$ have been compared with the given boundary condition $g'(\eta_\infty) = 0$ and $\theta(\eta_\infty) = 0$. The values of $g''(0)$ and $\theta'(0)$ have been adjusted using Secant method for better approximation. The step-size has been taken as $\Delta\eta = 0.01$ and accuracy to the fifth decimal place as the criterion of convergence. All the numerical computations are conducted in MATHEMATICA software.

4 Results and discussions

In OHAM, the values of convergence control parameters are obtained by minimizing the discrete squared residual errors against the convergence control parameters. Basic optimal HAM has been used to obtain the values of h_f, h_g and h_θ with the help of Mathematica package Bvph2.0 available on Liao's website.

4.1 Optimal convergence control parameters

Several tables have been drawn to show the optimal values of h_f, h_g and h_θ against different values of physical parameters.

Tables 1 and 2 present the optimal values of h_f for MHD viscous and MHD non Newtonian cases respectively. Here p denotes the number of iteration where total squared residual error is minimized. Higher the value of p is, better will be the solutions.

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Table 1: Local optimal convergence control parameters against M (Viscous case: $\lambda=0$).

	M dependent linear operator			Simple linear operator		
M	p	h_f	E_m^f	p	h_f	E_m^f
0.8	8	-0.6569	7.5369×10^{-10}	8	-0.7505	3.0288×10^{-12}
2	8	-0.8257	1.4288×10^{-14}	8	-0.5354	3.2676×10^{-7}
10	8	-0.9601	8.7540×10^{-25}	30	-0.1726	1.2877×10^{-5}
100	8	-0.9960	8.9232×10^{-41}	50	-0.0199	13.4632
1000	8	-0.9996	4.7267×10^{-57}	50	-0.0020	8220.71

Table 2: Local Optimal convergence control parameters against M (Non-Newtonian: $\lambda=0.1$).

	M dependent linear operator			Simple linear operator		
M	p	h_f	E_m^f	p	h_f	E_m^f
0.8	8	-0.7185	8.3142×10^{-8}	8	-0.8374	2.3720×10^{-9}
2	8	-0.9658	1.0371×10^{-9}	8	-0.5812	1.5833×10^{-5}
10	8	-1.3743	6.3990×10^{-8}	30	-0.1799	6.3268×10^{-3}
20	8	-1.7079	2.6781×10^{-4}	30	-0.0957	0.4992
25	12	-1.8443	1.2939×10^{-3}	30	-0.0773	1.3446
30	20	-1.9787	3.8109×10^{-2}	30	-0.0648	2.7277

In Table 1, for M Dependent Linear Operator (MDLO), set $p=8$ and for Simple Linear Operator (SLO) higher values of p have been chosen so as to obtain better convergent solution. If value of M is less than 1 i.e., 0.8 total error is of order 10^{-10} in MDLO and 10^{-12} in SLO. Gradually M has been increased from 0.8 to 1000 the resulting total error in MDLO of order 10^{-57} while in SLO it is 8220.71. This showed that the total error decreased enormously for large values of M in case of MDLO while it has increased in case of SLO. In viscous case initial guess for MDLO came very close to the exact solution obtained by Pavlov [32]. This point prompted the idea of choosing physical parameter (M) dependent basis set for HAM solution.

In Table 2, at value of $M=0.8$ total error in MDLO is of order 10^{-8} and in SLO it is 10^{-9}. The value of M has been raised gradually in MDLO from 0.8 to 20 while the value of p remained fixed at 8. The total error is of the order of 10^{-4}. The value of M in MDLO has been further increased to 25 and 30 with increase in the value of p from 8 to 12 and 20 respectively. The total error in MDLO is 10^{-3} and 10^{-2} respectively. While with SLO approach, the value of p when kept static at 8 and 30 the total error increased from 10^{-9} to 2.727. This showed that although effectiveness of MDLO has been reduced from $M=1000$ to 30 yet again total error obtained with SLO is high at each value, despite the fact that the value of p has been raised to 30 in SLO as compared to 20 in MDLO.

Tables 3 and 4 compare Pr Dependent Linear Operator (PrDLO) and SLO effects on total squared residual error for second problem. Table 3 shows the total error for viscous case. For examining the convergence of solution for viscous case Pr values have been chosen up to 150 because Prandtl number is not very large for viscous fluids. At value of

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Table 3: Local Optimal convergence control parameters against Pr (Viscous case: $\lambda=0$).

	Pr dependent linear operator				simple linear operator			
Pr	p	h_g	h_θ	E_m^t	p	h_g	h_θ	E_m^t
0.8	8	-0.3898	-0.9695	1.5598×10^{-8}	8	-0.8632	-1.0283	3.6413×10^{-13}
2	8	-0.7680	-1.0142	1.1182×10^{-10}	8	-0.4915	-0.7023	1.0350×10^{-10}
10	8	-0.9252	-1.0990	4.1684×10^{-4}	20	-0.2319	-0.2014	1.8833×10^{-4}
50	18	-0.2308	-1.2652	1.3622×10^{-3}	30	0.1334	-0.0389	3.0722
100	20	-0.3008	-1.3219	2.9661×10^{-3}	30	-0.1259	-0.0177	40.9117
150	30	-0.8371	-1.5934	1.0642×10^{-3}	50	-0.3875	-0.0125	73.2734

Table 4: Local Optimal convergence control parameters against Pr (non-Newtonian: $\lambda=0.2$).

	Pr dependent linear operator				simple linear operator			
Pr	p	h_g	h_θ	E_m^t	p	h_g	h_θ	E_m^t
0.8	8	-1.1471	-0.9697	2.2699×10^{-14}	8	-1.0970	-1.0927	4.0474×10^{-13}
2	8	-0.9938	-1.0279	2.8163×10^{-11}	8	-0.9001	-0.6781	3.8394×10^{-10}
10	8	-0.5155	-1.0976	3.7104×10^{-4}	15	-0.8161	-0.2017	1.1714×10^{-3}
50	25	-1.1881	-1.4869	6.8011×10^{-7}	30	-0.5622	-0.0427	2.4114
100	25	-0.4325	-1.4432	2.7220×10^{-4}	30	-0.4163	-0.0192	35.8488
150	30	-0.8671	-1.5932	1.1867×10^{-3}	30	-0.3361	-0.0118	120.1960
200	30	-0.2075	-1.2458	1.3496×10^{-2}	30	-0.2785	-0.0082	256.894

$Pr=0.8$ total error in PrDLO is of order 10^{-8} and in SLO it is 10^{-13}. The value of Pr has been increase gradually from 2 to 150 as against the value of p has been increased from 8 to 30. The total error at the maximum level of p value i.e., 30 is of order 10^{-3}. While in SLO at maximum p value i.e., 50 the total error is 73.2734. For Pr equal to 0.8 better total error has been obtained with the help of SLO while for Pr values greater than 1 PrDLO has given better results.

Table 4 shows the optimal values of convergence control parameters for non-Newtonian case. Prandtl number is larger for non-Newtonian fluids hence Pr values up to 200 have been chosen. At value of Pr equal to 0.8 total error in PrDLO is of order 10^{-14} and in SLO it is 10^{-13}. The value of Pr has been increase gradually from 2 to 200 as against the value of p has been increased from 8 to 30. The total error at the maximum level of p value i.e., 30 is of order 10^{-2}. While in SLO at the same value of p i.e., 30 the total error is 256.894. The last two rows of Table 4 gave a very clear comparison of PrDLO and SLO i.e., total error obtained by SLO is several hundred times greater than the error obtained by PrDLO at same number of iteration.

4.2 Convergence of solution

Optimal values of convergence control parameters have been used (from Tables 1-4) corresponding to the values of M (Tables 5-8) and Pr (Tables 9-12) (with precision up to 5 decimal places) to produce the results in this section.

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Table 5: Convergence table for different values of M (Viscous case: $\lambda=0$).

	M dependent linear operator ($f''(0)$)			simple linear operator ($f''(0)$)				Numerical
M [33]	$m=10$	$m=20$	$m=30$	$m=10$	$m=20$	$m=25$	$m=30$	$f''(0)$
0.8	−1.34164	−1.34164	−1.34164	−1.34164	−1.34164	−1.34164	−1.34164	−1.34164
2	−1.73205	−1.73205	−1.73205	−1.73193	−1.73205	−1.73205	−1.73205	−1.73205
10	−3.31662	−3.31662	−3.31662	−3.16303	−3.29888	−3.31044	−3.31436	−3.31662
100	−10.04987	−10.04987	−10.04987	−4.86163	−6.39039	−6.92211	−7.35394	−10.04987
1000	−31.63858	−31.63858	−31.63858	−5.13695	−7.13347	−7.93188	−8.64475	−31.63858

Table 6: Pade convergence table for Simple Linear operator (Viscous case: $\lambda=0$).

M	Pade [16/16]	Pade [18/18]	Pade [20/20]	Numerical
10	−3.31662	−3.31662	−3.31662	−3.31662
100	−9.98862	−9.91423	−9.75087	−10.04987
1000	−22.91997	−20.83897	−18.38197	−31.63858

Table 7: Convergence table for different values of M (Non Newtonian case: $\lambda=0.1$).

	M dependent linear operator ($f''(0)$)				Pade convergence table ($f''(0)$)			Numerical
M	$m=10$	$m=20$	$m=25$	$m=30$	Pade [16/16]	Pade [18/18]	Pade [20/20]	$f''(0)$
0.8	−1.40753	−1.40759	−1.40759	−1.40759	−1.40759	−1.40759	−1.40759	−1.40759
2	−1.84792	−1.84792	−1.84792	−1.84792	−1.84792	−1.84792	−1.84792	−1.84792
10	−3.84350	−3.84350	−3.84350	−3.84350	−3.84350	−3.84350	−3.84350	−3.84350
20	−5.86814	−5.87119	−5.87132	−5.87129	−5.87130	−5.87130	−5.87130	−5.87130
25	−6.94077	−6.95969	−6.96324	−6.96169	−6.96226	−6.96226	−6.96226	−6.96226
30	−8.19928	−8.30872	−8.38721	−8.33151	−8.37086	−8.37087	−8.37088	−8.37088

Table 5 shows the convergence of solution as M varies from 0.8 to 1000 for viscous case. For all values of M in MDLO case the solution remained convergent from 10th iteration onward while for SLO convergence was only achieved at 10th and 20th iteration for M equal to 0.8 and 2 respectively up to five decimal points. Convergence could not be achieved for other values of M in SLO case. In Table 6, Pade approximation has been used in SLO case to check the convergence for M equals 10,100 and 1000. Convergence could only be achieved in one case i.e., M is equal to 10. In other two cases convergence could not be achieved again. Furthermore, the results obtained through MDLO approach were matching up to five decimal places with the numerical results in each case.

The convergence of solution for non-Newtonian case with MDLO approach is given in Table 7. Convergence has been obtained for M equals to 0.8, 2 and 10 without using the Pade approximation. However, for values of M equals to 20, 25 and 30 Pade approximation has been applied to achieve convergence up to five decimal places. Moreover, the results against all values of M are matched with numerical results up to five decimal places.

The convergence of solution for non-Newtonian case with SLO approach is given in Table 8. Convergence has only been obtained for M equals to 0.8 without using the Pade approximation. However, after use of Pade approximation convergence has been achieved for value of M equals to 2 up to five decimal places. No convergence could be achieved for all the remaining values of M despite use of Pade approximation. Moreover,

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Table 8: Convergence table for different values of M with SLO (Non Newtonian: $\lambda=0.1$).

	simple linear operator ($f''(0)$)				Pade convergence table ($f''(0)$)			Numerical
M	$m=10$	$m=20$	$m=25$	$m=30$	Pade [16/16]	Pade [18/18]	Pade [20/20]	$f''(0)$
0.8	−1.40759	−1.40759	−1.40759	−1.40759	−1.40759	−1.40759	−1.40759	−1.40759
2	−1.84596	−1.84790	−1.84791	−1.84792	−1.84792	−1.84792	−1.84792	−1.84792
10	−3.41252	−3.71012	−3.76444	−3.79404	−3.84334	−3.84287	−3.84102	−3.84350
20	−4.10457	−4.86666	−5.07773	−5.22827	−5.84239	−5.81479	−5.76041	−5.87130
25	−4.28115	−5.20825	−5.48637	−5.69550	−6.82823	−6.74991	−6.62298	−6.96226
30	−4.40744	−5.46485	−5.79915	−6.05921	−7.82987	−7.66197	−7.42814	−8.37088

Table 9: Convergence table for different values of Pr (Viscous case: $\lambda=0$).

	Pr dependent linear operator ($\theta'(0)$)				Pade convergence table ($\theta'(0)$)			Numerical
Pr	$m=10$	$m=20$	$m=25$	$m=30$	Pade [16/16]	Pade [18/18]	Pade [20/20]	$\theta'(0)$
0.8	−0.49896	−0.49896	−0.49896	−0.49896	−0.49896	−0.49896	−0.49896	−0.49896
2	−0.91135	−0.91135	−0.91135	−0.91135	−0.91135	−0.91135	−0.91135	−0.91135
10	−2.30868	−2.30800	−2.30800	−2.30800	−2.30800	−2.30800	−2.30800	−2.30800
50	−5.46161	−5.42609	−5.42859	−5.42835	−5.42830	−5.42830	−5.42830	−5.42830
100	−7.86139	−7.74877	−7.75640	−7.76825	−7.76565	−7.76565	−7.76565	−7.76565
150	−9.27802	−9.61847	−9.52870	−9.54936	−9.55904	−9.55903	−9.55903	−9.55903

Table 10: Convergence table for different values of Pr with SLO (Viscous case: $\lambda=0$).

	simple linear operator ($\theta'(0)$)				Pade convergence table ($\theta'(0)$)			Numerical
Pr	$m=10$	$m=20$	$m=25$	$m=30$	Pade [16/16]	Pade [18/18]	Pade [20/20]	$\theta'(0)$
0.8	−0.49896	−0.49896	−0.49896	−0.49896	−0.49896	−0.49896	−0.49896	−0.49896
2	−0.91135	−0.91135	−0.91135	−0.91135	−0.91135	−0.91135	−0.91135	−0.91135
10	−2.28946	− −2.30774	−2.30805	−2.30805	−2.30800	−2.30800	−2.30800	−2.30800
50	−3.79174	−4.55967	−4.77750	−4.93537	−5.42830	−5.42828	−5.42792	−5.42830
100	−3.96251	−5.06460	−5.44566	−5.75794	−7.76546	−7.76402	−7.75381	−7.76565
150	−4.15945	−5.41646	−5.87222	−6.25702	−9.55710	−9.54810	−9.50401	−9.55903

the results against two values of M i.e., 0.8 and 2 could only be matched with numerical results up to five decimal places.

The convergence rate of MDLO approach was faster than the SLO approach e.g., at value of M equals to 2, convergence was achieved at 10th iteration with MDLO and it was achieved in SLO at 20th and 30th iteration (Tables 5, 7 and 8).

Comparison of $\theta'(0)$ convergence with PrDLO and SLO is given in Tables 9 and 10, respectively. Convergence of $f''(0)$ has not been shown because its value is -1 for all values of Prandtl number in Tables 9 and 10. In Table 9, PrDLO gave convergent solution up to Pr value equals to 10 without using Pade approximation. Convergence for Pr value equals to 0.8 and 2 was achieved at 10th iteration while it was achieved at 20th iteration for Pr value equals to 10. Convergence has been achieved for remaining values from 50 to 150 with use of Pade approximation. Moreover, the convergence results against all values of Pr are matched with numerical results up to five decimal places.

In Table 10, SLO gave convergent solution for Pr values equals to 0.8 and 2 without using Pade approximation. Convergence for Pr value equals to 0.8 and 2 was achieved at 10th iteration while convergence could not be achieved in first 30 iterations for Pr value equals to 10. Convergence has been achieved for Pr value equals to 10 with the help

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Table 11: Convergence table for different values of Pr (Non Newtonian case: $\lambda=0.2$).

	Pr dependent linear operator ($\theta'(0)$)				Pade convergence table ($\theta'(0)$)			Numerical
Pr	$m=10$	$m=20$	$m=25$	$m=30$	Pade [16/16]	Pade [18/18]	Pade [20/20]	$\theta'(0)$
0.8	−0.49142	−0.49142	−0.49142	−0.49142	−0.49142	−0.49142	−0.49142	−0.49142
2	−0.90119	−0.90119	−0.90119	−0.90119	−0.90119	−0.90119	−0.90119	−0.90119
10	−2.29290	−2.29476	−2.29475	−2.29475	−2.29475	−2.29475	−2.29475	−2.29475
50	−5.49879	−5.41688	−5.41320	−5.41358	−5.41349	−5.41349	−5.41349	−5.41349
100	−8.01479	−7.72334	−7.75366	−7.75157	−7.75045	−7.75045	−7.75045	−7.75045
150	−10.55570	−9.50474	−9.53543	−9.54945	−9.54366	−9.54366	−9.54366	−9.54365
200	−9.63887	−11.3212	−11.1242	−11.0279	−11.05547	−11.05541	−11.05540	−11.05540

Table 12: Convergence table for different values of Pr with SLO (Non Newtonian case: $\lambda=0.2$).

	simple linear operator ($\theta'(0)$)				Pade convergence table ($\theta'(0)$)			Numerical
Pr	$m=10$	$m=20$	$m=25$	$m=30$	Pade [16/16]	Pade [18/18]	Pade [20/20]	$\theta'(0)$
0.8	−0.49142	−0.49142	−0.49142	−0.49142	−0.49142	−0.49142	−0.49142	−0.49142
2	−0.90119	−0.90119	−0.90119	−0.90119	−0.90119	−0.90119	−0.90119	−0.90119
10	−2.27524	−2.29449	−2.29481	−2.29480	−2.29475	−2.29475	−2.29475	−2.29475
50	−3.88728	−4.64232	−4.84883	−4.99533	−5.41356	−5.41334	−5.41230	−5.41349
100	−4.07300	−5.19146	−5.57332	−5.88374	−7.74954	−7.74607	−7.72927	−7.75045
150	−4.05111	−5.28542	−5.73591	−6.11774	−9.53836	−9.52266	−9.46341	−9.54365
200	−3.97658	−5.24733	−5.72628	−6.14054	−11.03933	−11.00005	−10.87373	−11.05540

of Pade approximation. However convergence could not be achieved even after use of Pade approximation for Pr values from 50 to 150. Moreover, the results against Pr values equals to 0.8, 2 and 10 could only be matched with numerical results up to five decimal places.

Both the tables showed that convergence was faster, accurate up to five decimal and achievable for larger values of Pr with PrDLO approach as compared to SLO approach.

Solution convergence for non-Newtonian case is shown in Tables 11 and 12. Convergence of $f''(0)$ has not been shown because its value is -1.07621 for all values of Prandtl number in Tables 11 and 12.

In Table 11, PrDLO gave convergent solution up to Pr value equals to 10 without using Pade approximation. Convergence for Pr value equals to 0.8 and 2 was achieved at 10th iteration while it was achieved at 25th iteration for Pr value equals to 10. Convergence has been achieved for Pr values equals to 50, 100 and 150 with the use of Pade approximation up to five decimal points and for Pr value equals to 200, convergence could only be achieved up to four decimal points. Moreover, the results against values of Pr equals to 0.8, 2, 10, 50 and 100 has been matched with numerical results up to five decimal points but it could only be matched up to four decimal points for Pr values equals to 150 and 200.

In Table 12, SLO gave convergent solution for Pr values equals to 0.8 and 2 without using Pade approximation. Convergence for Pr value equals to 0.8 and 2 was achieved at 10th iteration while convergence could not be achieved in first 30 iterations for Pr value equals to 10. Convergence has been achieved for Pr value equals to 10 with the help of Pade approximation. However, convergence could not be achieved even after use of Pade approximation for Pr values from 50 to 200. Again for non-Newtonian case, both the

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tables showed that convergence was faster, accurate up to five/four decimal points and achievable for larger values of Pr with PrDLO approach as compared to SLO approach. Moreover, the results against most of the values of Pr with PrDLO approach have been matched with numerical results up to five decimal places.

5 Conclusions

In this article impact of linear operator on the convergence of HAM solution has been discussed with an idea about use of physical parameters in the linear operator. To prove that physical parameters in the linear operator improve the convergence of solution several times more as compared to simple linear operator, detailed computational analysis have been carried out and their results have been shown in Tables 1-12. On the basis of achieved results, it is concluded that:

- Better total squared residual error can be obtained by employing modified linear operator (i.e., physical parameter dependent linear operator) in both Newtonian and non-Newtonian case.
- Simple and modified linear operator results are similar for small values of physical parameters in both Newtonian and non-Newtonian cases.
- Simple linear operator approach failed to give the convergent solution for larger values of physical parameters.
- Modified operator approach can provide solutions evenly for small as well as larger values of physical parameters.
- Computational efficiency of HAM greatly depends upon the choice of linear operator.
- In modified linear operator approach convergence was faster, accurate up to five decimal and achievable for larger values physical parameters as compared to simple linear operator approach.

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