SOME PROBABILITY LIMIT THEOREMS WITH STATISTICAL APPLICATIONS

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1. *Introduction.* In fundamental papers Bernstein (3) and Loève (8) have proved central limit theorems for wide classes of dependent variables. Their theorems are stated in terms of conditional distributions. In the case of d_n-dependent variables (see § 3) they assume the existence, as the 'conditioning' variates vary, of finite upper bounds for certain conditional absolute moments higher than the second. More recently, Hoeffding and Robbins (7) have proved central limit theorems for m-dependent variables with finite third absolute moments, and Moran (10) has given a direct generalization of the Lindeberg-Lévy theorem for stationary discrete linear scalar processes.

In the present paper extensions of the Lindeberg-Lévy theorem are given in § 3 for stationary m-dependent scalar variables and for certain other types of stationary dependent scalar variables, some of which are of importance in the statistical study of stationary stochastic processes. In proving the extensions mentioned above a generalization, given in § 2, of a convergence theorem given by Cramér ((4), p. 254) is used. Extensions of these results to vector variables are considered in § 4. Some applications of these extensions and generalization are given in § 5.

2. *A convergence theorem.* We use the following extension of the usual definition of convergence in probability ((4), p. 252):

Definition 1. *Let $X_{h,k,r}$ $(h, k = 1, 2, 3, \ldots)$ be a triple sequence of random variables, r assuming a finite or an infinite number of values. Suppose that a random variable (or a constant) X exists such that, for any $\epsilon > 0$, $P(|X_{h,k,r} - X| \leqslant \epsilon) \to 1$ uniformly in r as $h, k \to \infty$. Then we say that $X_{h,k,r}$ converges in probability to X uniformly in r as $h, k \to \infty$.*

Theorem 1. *Let $\{W_n\}$ $(n = 1, 2, 3, \ldots)$ be a sequence of random variables. Suppose that, for each k such that $n = hk + r$ $(h, k$ integers; $0 \leqslant r < k)$, there exist random variables $U_{n,k}$ and $V_{n,k}$, distribution functions (d.f.'s) $F_k(x)$ and $F(x)$, and a constant V such that*

(a) $W_n = U_{n,k} + V_{n,k}$,

(b) the d.f. of $U_{n,k} \to F_k(x)$ as $h \to \infty$ (for all r and for $k \geqslant K$) if x is a continuity point of $F_k(x)$,

(c) $F_k(x) \to F(x)$ as $k \to \infty$ if x is a continuity point of $F(x)$, and

(d) $V_{n,k}$ converges in probability to V uniformly in r as $h, k \to \infty$.

Then the d.f. of $W_n \to F(x - V)$ as $n \to \infty$ if $x - V$ is a continuity point of $F(x)$.

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Proof in the case of scalar variables. Since, by (*a*), $W_n - V = U_{n,k} + (V_{n,k} - V)$ we assume, without loss of generality, that $V = 0$. Then we have to show that the d.f. of $W_n \to F(x)$ as $n \to \infty$ if x is a continuity point of $F(x)$. Note that h, k, r are integers; $k > 0$, $h \geqslant 0$, $0 \leqslant r < k$. Let $\delta, \epsilon > 0$. We have

(i) by Boole's inequality (11),

$$P(W_n \leqslant x) - P(W_n \leqslant x, |V_{n,k}| \leqslant \epsilon)$$
$$\leqslant P(W_n \leqslant x) - \{P(W_n \leqslant x) + P(|V_{n,k}| \leqslant \epsilon) - 1\} = 1 - P(|V_{n,k}| \leqslant \epsilon) < \delta$$

for all r, $h \geqslant H_1(\delta, \epsilon)$, say, and $k \geqslant K_1(\delta, \epsilon)$, say, by (*d*) and Definition 1;

(ii) $P(W_n \leqslant x, |V_{n,k}| \leqslant \epsilon)$, by (*a*), lies between

$$P(U_{n,k} \leqslant x - \epsilon, \quad |V_{n,k}| \leqslant \epsilon) \quad \text{and} \quad P(U_{n,k} \leqslant x + \epsilon, \quad |V_{n,k}| \leqslant \epsilon),$$

which, by an argument similar to that used in (i), differ from $P(U_{n,k} \leqslant x - \epsilon)$ and $P(U_{n,k} \leqslant x + \epsilon)$ respectively by less than δ for all $r, h \geqslant H_1(\delta, \epsilon)$ and $k \geqslant K_1(\delta, \epsilon)$;

(iii) $P(U_{n,k} \leqslant x - \epsilon)$ and $P(U_{n,k} \leqslant x + \epsilon)$, by (*b*), differ from $F_k(x - \epsilon)$ and $F_k(x + \epsilon)$ respectively by less than δ for each $r, k \geqslant K$ and $h \geqslant H_2(k, \delta, \epsilon, x, r)$, say, and hence, for all r, $k \geqslant K$ and

$$h \geqslant H_3(k, \delta, \epsilon, x) = \max_{0 \leqslant r < k} H_2(k, \delta, \epsilon, x, r),$$

if $x - \epsilon$ and $x + \epsilon$ are continuity points of $F_k(x)$;

(iv) $F_k(x - \epsilon)$ and $F_k(x + \epsilon)$, by (*c*), differ from $F(x - \epsilon)$ and $F(x + \epsilon)$ respectively by less than δ for $k \geqslant K_2(\delta, \epsilon, x)$, say, if $x - \epsilon$ and $x + \epsilon$ are continuity points of $F(x)$; and

(v) $F(x - \epsilon)$ and $F(x + \epsilon)$ differ from $F(x)$ by less than δ for $\epsilon < E(\delta, x)$, say, if x is a continuity point of $F(x)$.

The numbers of discontinuity points of $F_k(x), F(x)$ are at most enumerable and, hence, $\epsilon = \epsilon(\delta, x) > 0$ can be chosen to satisfy the conditions imposed on ϵ in (iii)–(v). Fix $\epsilon = \epsilon(\delta, x)$. $K_1(\delta, \epsilon)$, K, $K_2(\delta, \epsilon, x)$ are now functions of δ, x only and, hence, a positive integer $k = k(\delta, x) \geqslant \max(K_1, K, K_2)$ can be chosen to satisfy the conditions imposed on k in (i)–(iv). Fix $k = k(\delta, x)$. $H_1(\delta, \epsilon), H_3(k, \delta, \epsilon, x)$ are now functions of δ, x only and hence, an integer $H(\delta, x) \geqslant \max(H_1, H_3)$ can be chosen such that all $h \geqslant H(\delta, x)$ satisfy the conditions imposed on h in (i)–(iii). Combining (i)–(v) we now have that $P(W_n \leqslant x)$ differs from $F(x)$ by less than 5δ for all r in $0 \leqslant r < k$, a fixed $k = k(\delta, x) > 0$ and all $h \geqslant H(\delta, x)$ and, thus, for all $n \geqslant N(\delta, x) = kH$. Hence the d.f. of $W_n \to F(x)$ as $n \to \infty$ if x is a continuity point of $F(x)$.

In applications of Theorem 1 we sometimes split $V_{n,k}$ into a sum of a finite number of variables $V_{n,k;1}, V_{n,k;2}, \ldots, V_{n,k;q}$, which converge in probability to constants $V_1, V_2, \ldots, V_q$ respectively uniformly in r as $h, k \to \infty$. By a simple extension of Slutsky's theorem ((4), p. 255) we then have that $V_{n,k}$ converges in probability to $V = V_1 + V_2 + \ldots + V_q$ uniformly in r as $h, k \to \infty$. Alternatively, we can obtain this result directly by using Definition 1, Boole's inequality, and the fact that

$$|V_{n.k} - V| \leqslant |V_{n,k;1} - V_1| + |V_{n,k;2} - V_2| + \ldots + |V_{n,k;q} - V_q|.$$

3. *The central limit theorem for some stationary scalar processes.* The Lindeberg-Lévy theorem ((4), p. 215) is extended to the case of stationary m-dependent scalar variables in Theorem 2 and to the case of certain stationary linear scalar processes in Theorem 3.

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We define d_n-dependent variables as follows*(7):

DEFINITION 2. *Let d_n be a function of n (or a constant). Suppose $\{X_i\}$ $(i = 1, 2, 3, \ldots)$ is a sequence of random variables such that the two sets of variables $(X_1, X_2, \ldots, X_r)$ and $(X_s, X_{s+1}, \ldots, X_n)$ are independent whenever $s - r > d_n$. Then we say that $\{X_i\}$ $(i = 1, 2, 3, \ldots)$ is a sequence of d_n-dependent variables or is a d_n-dependent process.*

THEOREM 2. *Let $\{X_i\}$ $(i = 1, 2, 3, \ldots)$ be a sequence of stationary m-dependent scalar variables with means zero and $\mathscr{E}(X_i X_j) = C_{i-j}$. Then the d.f. of the random variable $(X_1 + X_2 + \ldots + X_n)/\sqrt{n} \to$ the normal d.f. with mean zero and variance $\sum_{-m}^{m} C_p$ as $n \to \infty$.*

Proof. We note that $C_p = 0$ for $|p| > m$. Let h, k, r be as in Theorem 1 and let $k \geqslant 2m$. Define

$$\left.\begin{aligned} Y_i &= X_{(i-1)k+1} + X_{(i-1)k+2} + \ldots + X_{ik-m} \\ Z_i &= X_{ik-m+1} + X_{ik-m+2} + \ldots + X_{ik} \end{aligned}\right\} \quad (i = 1, 2, \ldots, h),$$

$$U_{n,k} = (Y_1 + Y_2 + \ldots + Y_h)/\sqrt{n}, \quad V_{n,k;1} = (Z_1 + Z_2 + \ldots + Z_h)/\sqrt{n},$$

$$V_{n,k;2} = (X_{hk+1} + X_{hk+2} + \ldots + X_{hk+r})/\sqrt{n}, \quad V_{n,k} = V_{n,k;1} + V_{n,k;2}.$$

The Y_i are stationary and mutually independent. Hence the d.f. of $U_{n,k} \to F_k(x)$ as $h \to \infty$, where $F_k(x)$ is the normal d.f. with mean zero and variance

$$\sigma_k^2 = \lim_{h \to \infty} (h/n)\, \mathscr{E}(Y_i^2) = \sum_{p=-m}^{m} (1/k)\,(k - m - |p|)\, C_p.$$

Now, as $k \to \infty$, $\sigma_k^2 \to \sum_{-m}^{m} C_p$, and, hence, $F_k(x) \to$ the normal d.f. $F(x)$ with mean zero and variance $\sum_{-m}^{m} C_p$. The Z_i also are mutually independent and stationary. Hence

$$\mathscr{E}(V_{n,k;1}^2) = (h/n)\, \mathscr{E}(Z_i^2) \leqslant (1/k)\, \mathscr{E}(Z_i^2) \to 0$$

uniformly in r as $h, k \to \infty$, since the variance of Z_i is independent of h, k. Also, with $s = \min(m, r)$,

$$\begin{aligned} \mathscr{E}(V_{n,k;2}^2) &= \sum_{p=-s}^{s} (1/n)\,(r - |p|)\, C_p \\ &\leqslant (2m + 1)\,(k/kh)\, \mathscr{E}(X_i^2) \to 0 \end{aligned}$$

uniformly in r as $h, k \to \infty$. Thus $V_{n,k;1}, V_{n,k;2}$ converge in probability to zero and, hence, $V_{n,k}$ to $V = 0$ uniformly in r as $h, k \to \infty$.

Applying Theorem 1 to $W_n = U_{n,k} + V_{n,k} = (X_1 + X_2 + \ldots + X_n)/\sqrt{n}$, we now obtain Theorem 2.

DEFINITION 3. *Let $\{Y_j\}$ $(j = \ldots, -1, 0, 1, 2, \ldots)$ be an m-dependent scalar process and A_t $(t = 0, 1, 2, \ldots)$ be scalar constants such that for every integer i, $\sum_{t=0}^{n} A_t Y_{i-t}$ converges in probability to a variable X_i as $n \to \infty$. Then we say that $\{X_i\}$ $(i = \ldots, -1, 0, 1, 2, \ldots)$, where $X_i = \sum_{t=0}^{\infty} A_t Y_{i-t}$ is a discrete linear scalar process with m-dependent scalar residuals.*

* When d_n is a constant m and i can take all integral values (positive, negative, and zero); for m-dependence, $(X_a, X_{a+1}, \ldots, X_r)$ and $(X_s, X_{s+1}, \ldots, X_n)$ should be independent whenever $s - r > m$, for all integral a, r, s, n $(a \leqslant r < s \leqslant n)$.

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The process defined above is a generalization of the discrete linear process (1) or process of linear regression (12) (which is the case $m = 0$). A particular case is the discrete process set up by 'equally spaced' variables from a stationary 'continuous autoregressive process' 'defined' by a linear differential equation of order $m+1$ ((1), § 5). The residuals associated with this process are m-dependent.

THEOREM 3. *Let* (*a*) $\{Y_i\}$ $(i = \ldots, -1, 0, 1, 2, \ldots)$ *be a stationary m-dependent scalar process with zero mean and finite variance and* (*b*) A_t $(t = 0, 1, 2, \ldots)$ *be scalar constants such that* $\sum_0^\infty A_t$ *is absolutely convergent. Then* (i) $\{X_i\}$ $(i = \ldots, -1, 0, 1, 2, \ldots)$, *where* $X_i = \sum_{t=0}^\infty A_t Y_{i-t}$, *is a stationary discrete linear scalar process with m-dependent scalar residuals, zero mean and finite variance, and* (ii) *the d.f. of the random variable* $(X_1 + X_2 + \ldots + X_n)/\sqrt{n} \to$ *the normal d.f. with zero mean and variance* $\sum_{-\infty}^\infty C_p$ *as* $n \to \infty$, *where* $C_{i-j} = \mathscr{E}(X_i X_j)$.

In proving this theorem we use the following lemma (6).

LEMMA 1. *Let* $\{S_i\}$ $(i = 1, 2, 3, \ldots)$ *be a sequence of random variables with zero means such that* (*a*) $\mathscr{E}(S_n^2) = \sigma_n^2$ *is finite and* (*b*) $\mathscr{E}(S_{n+p} - S_n)^2 \to 0$ *uniformly in* p (> 0) *as* $n \to \infty$. *Then, as* $n \to \infty$, (i) $\sigma_n^2 \to \sigma^2$ (*finite*) *and* (ii) S_n *converges in probability to a variable* S *with zero mean and variance* σ^2.

Proof of the theorem. From (*b*) of the theorem it follows that $\sum_0^\infty A_t^2$ is convergent. Let $S_n = \sum_{t=0}^n A_t Y_{i-t}$. Then, by (*a*) of the theorem,

$$\mathscr{E}(S_n) = 0, \quad \mathscr{E}(S_n^2) \leqslant (2m+1)\,\mathscr{E}(Y_i^2) \sum_0^\infty A_t^2 < \infty,$$

and

$$\mathscr{E}(S_{n+p} - S_n)^2 \leqslant (2m+1)\,\mathscr{E}(Y_i^2) \sum_n^\infty A_t^2 \to 0$$

uniformly in p as $n \to \infty$. Thus the conditions of the lemma are satisfied and S_n converges in probability to a variable $S = X_i$ (by Definition 3) with zero mean and finite variance as $n \to \infty$. This gives (i) of the theorem, $\{X_i\}$ being a stationary process since, by (*a*) of the theorem, the distribution of

$$(X_{i+j}, X_{i+k}, \ldots, X_{i+q}) = \sum_{t=0}^\infty A_t (Y_{i+j-t}, Y_{i+k-t}, \ldots, Y_{i+q-t})$$

$(i, j, k, \ldots, q$ integers) is independent of i. We now proceed to the proof of (ii)—a direct generalization of Theorem 2.

We consider first the *case* $m = 0$, when $\{X_i\}$ is a stationary discrete linear process. Let h, k, r be as in Theorem 1. Define

$$P_{i,k} = \sum_{t=0}^{k-1} A_t Y_{i-t}, \quad Q_{i,k} = \sum_{t=k}^\infty A_t Y_{i-t},$$

$$U_{n,k} = (1/\sqrt{n}) \sum_{i=1}^n P_{i,k}, \quad V_{n,k} = (1/\sqrt{n}) \sum_{i=1}^n Q_{i,k}.$$

Then

$$X_i = P_{i,k} + Q_{i,k}$$

and

$$W_n = (X_1 + X_2 + \ldots + X_n)/\sqrt{n} = U_{n,k} + V_{n,k}.$$

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The $P_{i,k}$ are stationary k-dependent variables with zero means and, by Theorem 2, the d.f. of $U_{n,k} \to$ the normal d.f. $F_k(x)$ with zero mean and variance

$$(A_0 + A_1 + \ldots + A_{k-1})^2 \mathscr{E}(Y_i^2)$$

as $h \to \infty$. Also, by (*b*) of Theorem 3, $F_k(x) \to$ the normal d.f. $F(x)$ with zero mean and variance $(A_0 + A_1 + A_2 + \ldots)^2 \mathscr{E}(Y_i^2)$ as $k \to \infty$. Further, $V_{n,k}$ converges in probability to $V = 0$ uniformly in r as $h, k \to \infty$, since

$$\begin{aligned}\mathscr{E}(V_{n,k}^2) &= (1/n)\,\mathscr{E}(Y_i^2)\{A_k^2 + (A_k + A_{k+1})^2 + \ldots + (A_k + A_{k+1} + \ldots + A_{k+n-2})^2 \\ &\quad + \sum_{i=0}^{\infty} (A_{k+i} + A_{k+i+1} + \ldots + A_{k+i+n-1})^2\} \\ &= (1/n)\,\mathscr{E}(Y_i^2)\{n(A_k^2 + A_{k+1}^2 + A_{k+2}^2 + \ldots) + 2(n-1)(A_k A_{k+1} + A_{k+1} A_{k+2} \\ &\quad + A_{k+2} A_{k+3} + \ldots) + 2(n-2)(A_k A_{k+2} + A_{k+1} A_{k+3} + A_{k+2} A_{k+4} + \ldots) + \ldots\} \\ &\leqslant \mathscr{E}(Y_i^2)\{|A_k| + |A_{k+1}| + |A_{k+2}| + \ldots\}^2 \to 0\end{aligned}$$

uniformly in r as $h, k \to \infty$. Applying Theorem 1 to W_n we now obtain (ii) of Theorem 3, since

$$\begin{aligned}\sum_{-\infty}^{\infty} C_p &= \mathscr{E}(Y_i^2)\{(A_0^2 + A_1^2 + A_2^2 + \ldots) + 2(A_0 A_1 + A_1 A_2 + A_2 A_3 + \ldots) \\ &\quad + 2(A_0 A_2 + A_1 A_3 + A_2 A_4 + \ldots) + \ldots\} \\ &= \mathscr{E}(Y_i^2)(A_0 + A_1 + A_2 + \ldots)^2,\end{aligned}$$

by (*b*).

We next consider the *case* $m > 0$. The proof is a modification of that in the previous case, the only changes being that (1) the P_{ik} are now $(k+m)$-dependent, and (2) $\mathscr{E}(Y_i^2)$ is replaced by either $\sum_{p=-m}^{m} \mathscr{E}(Y_i Y_{i+p})$ or $(2m+1)\theta\mathscr{E}(Y_i^2)$ or $(2m+1)\mathscr{E}(Y_i^2)$ as is relevant, where $|\theta| \leqslant 1$.

4. *Extensions to vector variables.* The results proved in §§ 2, 3 can all be modified to give us analogous results for the case of vector variables. The statement of Theorem 1 does not need any modification in the case of vector variables. The modifications of Theorems 2 and 3 give us Theorems 2A and 3A, extensions of the Lindeberg-Lévy theorem ((4), p. 316) for stationary m-dependent vector variables and for certain stationary linear vector processes (see (9) for a particular case) defined as follows.

DEFINITION 3A. *Let* $\{Y_j\}$ $(j = \ldots, -1, 0, 1, 2, \ldots)$ *be an* m*-dependent* l*-dimensional vector process and* A_t $(t = 0, 1, 2, \ldots)$ *be* $g \times l$ *matrix constants such that for every integer* i, $\sum_{t=0}^{n} A_t Y_{i-t}$ *converges in probability to a variable* X_i *as* $n \to \infty$. *Then we say that* $\{X_i\}$ $(i = \ldots, -1, 0, 1, 2, \ldots)$, *where* $X_i = \sum_{t=0}^{\infty} A_t Y_{i-t}$, *is a discrete linear* g*-dimensional vector process with* m*-dependent* l*-dimensional vector residuals.*

The proofs of Theorems 2 and 3 can be modified in the usual manner to give us the analogous results in the vector case. In proving Theorem 3A we use in place of Lemma 1 its analogue for the vector case, which we can obtain by modifying the usual proof of Lemma 1. Theorems 2A and 3A, which we use in § 5, are as follows:

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THEOREM 2A. *Let* $\{X_i\}$ $(i = 1, 2, 3, \ldots)$ *be a sequence of stationary m-dependent vector variables with mean vectors zero and* $\mathscr{E}(X_i X_j') = C_{i-j}$. *Then the d.f. of the random variable* $(X_1 + X_2 + \ldots + X_n)/\sqrt{n} \to$ *the normal d.f. with mean vector zero and covariance matrix* $\sum_{-m}^{m} C_p$ *as* $n \to \infty$.

THEOREM 3A. *Let* (*a*) $\{Y_i\}$ $(i = \ldots, -1, 0, 1, 2, \ldots)$ *be a stationary m-dependent l-dimensional vector process with zero mean vector and finite covariance matrix and* (*b*) A_t $(t = 0, 1, 2, \ldots)$ *be* $g \times l$ *matrix constants such that* $\sum_{0}^{\infty} A_t$ *is absolutely convergent. Then* (i) $\{X_i\}$ $(i = \ldots, -1, 0, 1, 2, \ldots)$, *where* $X_i = \sum_{t=0}^{\infty} A_t Y_{i-t}$, *is a stationary discrete linear g-dimensional vector process with m-dependent l-dimensional vector residuals, zero mean vector and finite covariance matrix and* (ii) *the d.f. of the random variable* $(X_1 + X_2 + \ldots + X_n)/\sqrt{n} \to$ *the normal d.f. with zero mean vector and covariance matrix* $\sum_{-\infty}^{\infty} C_p$ *as* $n \to \infty$, *where* $C_{i-j} = \mathscr{E}(X_i X_j')$.

5. *Some applications in statistics.* The asymptotic normality of sample automoments of stationary m-dependent variables is discussed in § 5·1, and of serial correlation coefficients of stationary independent (0-dependent) processes in § 5·2. Conditions for the validity of certain extensions of Quenouille's test are discussed in § 5·3.

Theorem 1 can be used to set up sufficient conditions, to be discussed elsewhere, for the asymptotic normality of the maximum-likelihood estimator of an unknown parameter when the observations used in estimation are realized values of a stationary m-dependent process.

5·1. *Method of automoments.* This is a general method, analogous to the method of moments, of estimation which can be used when the sample of observations consists of realized values of dependent variables. It consists in equating a number, equal to that of the unknown parameters, of realized values of sample automoments to their expectations, which are the corresponding population automoments (assumed finite), the sample automoments being so chosen that we can obtain estimates of all the parameters to be estimated.

When the variables are stationary m-dependent, the weak law of large numbers for stationary dependent variables shows that the sample automoments are consistent estimators of the corresponding population automoments, assumed finite. Theorem 2A shows that, provided that the variances of the terms that enter into the automoments exist finitely, the vector variable consisting of the sample automoments is asymptotically normally distributed about the corresponding vector for the population. The covariance matrix associated with the asymptotic distribution can be worked out in any particular case.

To take an example, suppose $\{Z_i\}$ $(i = 1, 2, 3, \ldots)$ is a stationary 1-dependent process, and the automoments $\overline{Z_i}$, $\overline{Z_i^2}$, $\overline{Z_i Z_{i+1}}$, where $\overline{Z_i^k} = (1/n) \sum_{1}^{n} Z_i^k$, $\overline{Z_i Z_{i+k}} = (1/n) \sum_{1}^{n} Z_i Z_{i+k}$,

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are used as estimators of their expectations. Then the vector variables $\{Z_i, Z_i^2, Z_iZ_{i+1}\}$ $(i = 1, 2, 3, \ldots)$ are 2-dependent, and we obtain that the vector variable

$$\sqrt{n}\{\overline{Z_i} - \mathscr{E}(Z_i), \quad \overline{Z_i^2} - \mathscr{E}(Z_i^2), \quad \overline{Z_iZ_{i+1}} - \mathscr{E}(Z_iZ_{i+1})\}$$

is asymptotically normally distributed provided that $\mathscr{E}(Z_i^4)$ is finite.

5·2. *Serial correlation coefficients.* Hoeffding and Robbins (7) show that, if $\{Z_i\}$ $(i = 1, 2, 3, \ldots)$ is a stationary independent process with zero mean and unit variance, then $\sqrt{n}\,\overline{Z_iZ_{i+1}}/\overline{Z_i^2}$ is asymptotically normally distributed with zero mean and unit variance, provided that the *sixth* moment of Z_i is finite. They show further that the same asymptotic property holds for r_1, where

$$r_k = \sqrt{n}\,(\overline{Z_iZ_{i+k}} - \overline{Z}_i^2)/(\overline{Z_i^2} - \overline{Z}_i^2)$$

is a sample serial correlation coefficient with lag k.

We prove below that the above property holds, without assuming that any moment higher than the *second* exists finitely. Consider

$$r_k = (\sqrt{n}\,\overline{Z_iZ_{i+k}} - n^{\frac{1}{2}}\overline{Z_i}.n^{\frac{1}{2}}\overline{Z_i})/(\overline{Z_i^2} - \overline{Z}_i^2).$$

Since $\sqrt{n}\,\overline{Z_i}$ is asymptotically normally distributed with zero mean and unit variance ((4), p. 215), $n^{\frac{1}{2}}\overline{Z_i}$ is asymptotically zero, and, hence, $n^{\frac{1}{2}}\overline{Z_i}.n^{\frac{1}{2}}\overline{Z_i}$ is asymptotically zero ((4), p. 254), and since the variables Z_iZ_{i+k} $(i = 1, 2, 3, \ldots)$ are stationary, k-dependent, standardized and uncorrelated, by Theorem 2, $\sqrt{n}\,\overline{Z_iZ_{i+k}}$ is asymptotically normally distributed with zero mean and unit variance. Hence, the numerator in the expression for r_k is asymptotically normally distributed with zero mean and unit variance; and, since $\overline{Z_i^2}$ is asymptotically unity ((4), p. 253) and $\overline{Z_i}$ asymptotically zero, the denominator is asymptotically unity. Hence r_k (and r_1, in particular) is asymptotically normally distributed with zero mean and unit variance.

A similar proof shows that, under the same conditions, $\sqrt{n}\,\overline{Z_iZ_{i+k}}/\overline{Z_i^2}$ is asymptotically normally distributed with zero mean and unit variance.

5·3. *Quenouille's test.* Extensions of Quenouille's test for autoregressive schemes were discussed in a previous paper (2). In the case of the alternative test, based on $H_{-t}H_tr_t$, Theorem 3A shows that the existence of (1) finite second moments of the residuals Y_u when $m = 0$ and (2) finite fourth moments when $m > 0$ is sufficient for the joint distribution of the $H_{-t}H_tC_t$ and hence of the $H_{-t}H_tr_t$ to be asymptotically normal and thus for the validity of this test when no estimation is involved. (In this connexion it should be noted that the first component Y_u^2 in the vector Z_u ((2), p. 111) can be omitted without the argument given being invalidated. If Y_u^2 is not omitted we have to assume the existence of finite fourth moments of the Y_u even when $m = 0$.) In the case of moving averages ($p = 0$, $h_t = H_t = 1$) the adaptations of Quenouille's test and the alternative test are identical, so that in this case (if $m > 0$) the existence of finite fourth moments of the Y_u is sufficient for the joint distribution of the $h_t^2r_t$ $(= r_t)$ to be asymptotically normal and thus for the validity of these tests.

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the present proofs are simpler, Theorem 1 being a simplification of a theorem used there. I am also indebted to the referee for the reference (6) and for some helpful criticisms and suggestions.

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