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General dynamical systems and conditional stability

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1. The notion of a general dynamical system was introduced by Barbashin (1). In this paper we consider a general dynamical system in a metric space following Zubov (6) where Lyapunov's method has been extended to investigate the stability properties using a single Lyapunov functional.

Defining, in a natural way, the concepts of conditional stability and boundedness of a compact set with respect to a general dynamical system in a metric space, we obtain sufficient conditions in terms of several Lyapunov functionals such that these concepts hold. These notions, as is to be expected, include as special cases the usual notions of stability and boundedness of a compact set. We follow the comparison technique used in studying the stability properties of differential equations (2, 3, 4). Thus, the study of the stability properties of a general dynamical system in a metric space is reduced to the study of similar properties of solutions of a system of differential equations defined in Euclidean space. Examples are worked out to illustrate the fruitfulness of using several Lyapunov functions or functionals.

2. Let E denote a locally compact, separable metric space with a metric distance $d(p, q), p, q \in E$, and let I denote the half-line $0 \leqslant t < \infty$. We consider a general dynamical system in E defined as follows:

By a general dynamical system or simply a system in E, we mean a two-parameter family of transformations $f(t, t_0, .)$ of E into the set of all non-empty subsets of E, satisfying the following properties:

(i) For each $p_0 \in E$ and $t_0 \in I$, there is defined a set $f(t, t_0, p_0) \subset E$ for $t \geqslant t_0$ and the set $f(t, t_0, p_0)$ is non-empty;

(ii) $f(t_0, t_0, p_0) = \{p_0\}$;

(iii) for any element $p_1 \in f(t_1, t_0, p_0)$, there is defined a set $f(t, t_1, p_1)$ such that

$$\bigcup f(t, t_1, p_1) = f(t, t_0, p_0) \quad \text{for} \quad t \geqslant t_1 \geqslant t_0,$$
$$p_1 \in f(t_1, t_0, p_0).$$

For a fixed $p_0 \in E$, $f(t, t_0, p_0)$ is called a *motion*, while the set defined in (i) above is called the *trajectory* of this motion. The set $A \subset E$ is called *invariant* with respect to a general dynamical system in E, if all the trajectories of this system initiated in A are contained in A.

Let the set $A \subset E$ be compact in E. Denote by $d(p, A)$, the distance from the point $p \in E$ to the set A, defined by $\inf_{p \in A} d(p, q)$. Let $S(\alpha)$ and $\bar{S}(\alpha)$ denote the set of points

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$\{p: d(p, A) < \alpha\}$ and $\{p: d(p, A) \leqslant \alpha\}$ respectively. Let $f(t, t_0, p_0)$ be a motion of general dynamical system and $\rho(t, t_0, p_0)$ denote $\sup_{p \in f(t, t_0, p_0)} d(p, A)$. It is obvious that

$$\rho(t_0, t_0, p_0) = d(p_0, A).$$

Let M be a set such that $A \subset M \subset E$,

In order to unify our results on conditional stability and boundedness of the compact set A with respect to the system in E, we list below the following conditions. For any $\epsilon > 0$ and $t_0 \in I$, there exists a positive function $\delta = \delta(t_0, \epsilon) < \epsilon$, that is continuous in t_0 for each ϵ, such that

$$\rho(t, t_0, p_0) < \epsilon \quad (t \geqslant t_0), \tag{2·1}$$

whenever $$p_0 \in \bar{S}(\delta) \cap M.$$

The δ in (2·1) is independent of t_0. (2·2)

For any $\epsilon > 0$, $\alpha \geqslant 0$ and $t_0 \in I$, there exists a positive number $T = T(t_0, \epsilon, \alpha)$ such that

$$\rho(t, t_0, p_0) < \epsilon \quad (t \geqslant t_0 + T), \tag{2·3}$$

whenever $$p_0 \in \bar{S}(\alpha) \cap M.$$

The T in (2·3) is independent of t_0. (2·4)

The conditions (2·1) and (2·3) hold simultaneously. (2·5)

The conditions (2·2) and (2·4) hold simultaneously. (2·6)

For each $\alpha \geqslant 0$, $t_0 \in I$, there exists a positive function $\beta = \beta(t_0, \alpha)$ that is continuous in t_0 for each α, such that

$$\rho(t, t_0, p_0) < \beta \quad (t \geqslant t_0), \tag{2·7}$$

whenever $$p_0 \in \bar{S}(\alpha) \cap M.$$

The β in (2·7) is independent of t_0. (2·8)

For each $\alpha \geqslant 0$ and $t_0 \in I$, there exist positive numbers B and $T = T(t_0, \alpha)$ such that

$$\rho(t, t_0, p_0) < B \quad (t \geqslant t_0 + T), \tag{2·9}$$

whenever $$p_0 \in \bar{S}(\alpha) \cap M.$$

The T in (2·9) is independent of t_0. (2·10)

The conditions (2·7) and (2·9) hold simultaneously. (2·11)

The conditions (2·8) and (2·10) hold simultaneously. (2·12)

Remark. If $M = E$ these definitions reduce to the usual definitions of stability and boundedness of the compact set A with respect to the general dynamical system in E. We are not assuming that the set A is invariant with respect to this system. In fact, in certain cases, the invariance of A is implied by the conditions. For example, if (2·1) holds, it is clear, since $\delta \to 0$ as $\epsilon \to 0$, that A is an invariant (positively) set for the system in E. Whereas this need not, in general, be the case with respect to the conditions (2·3) and ((2·7)–(2·12)). Example 2, given at the end, illustrates this point.

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3. Let R^N denote the real N-dimensional Euclidean space. Let W be an N-vector and the function $W = W(t, r)$ be defined and continuous on the product space $I \times R^N$. Let J_N denote the set of positive integers from 1 to N. If $K \in J_{N-1}$, then the set of positive integers from $(K+1)$ to N will be denoted by $\tilde{J}_K$. Let $r = (r_1, \ldots, r_N) \in R^N$. If for each $i \in J_N$, $W_i(t, r)$ is non-decreasing in $r_1, r_2, \ldots, r_{i-1}, r_{i+1}, \ldots, r_N$, for each $t \in I$, then it is known (5) that the differential system

$$r' = W(t, r), \quad (' = d/dt), \quad r(t_0) = r_0, \tag{3·1}$$

has the maximal solution (in the sense of componentwise majorization) existing to the right of t_0.

Let V be an N-vector and the vector-valued function $V = (V_1, \ldots, V_N)$ be defined and continuous on the product space $I \times E$, such that for each $i \in J_N$,

$$V_i(t, p) \geqslant 0 \quad \text{for each} \quad (t, p) \in I \times E. \tag{3·2}$$

For each $i \in J_N$ define $$V_i(t, t_0, p_0) = \sup_{p \in f(t, t_0, p_0)} V_i(t, p),$$

and $$V_i^*(t, t_0, p_0) = \limsup_{h \to 0^+} \frac{1}{h} [V_i(t+h, t_0, p_0) - V_i(t, t_0, p_0)]. \tag{3·3}$$

Denote by $V(t, t_0, p_0)$ the N-dimensional vector whose ith component, $i \in J_N$, is equal to $V_i(t, t_0, p_0)$.

With respect to the function defined above we state the following Lemma.

LEMMA. *Let there exist a vector functional $V(t, p)$ defined and continuous on $I \times E$ such that* (3·2) *holds and the function $V_i^*(t, t_0, p_0)$ of* (3·3) *satisfies, for each $i \in J_N$, the inequality*

$$V_i^*(t, t_0, p_0) \leqslant W_i(t, V(t, t_0, p_0)) \quad (t \geqslant t_0), \tag{3·4}$$

where $W(t, r)$ is the same function as defined above satisfying the stated monotonic conditions. Assume that $r(t, t_0, r_0)$ is the maximal solution of the differential system (3·1) *existing to the right of t_0. Then for each $i \in J_N$,*

$$V_i(t, t_0, p_0) \leqslant r_i(t, t_0, r_0) \quad (t \geqslant t_0), \tag{3·5}$$

whenever $$V_i(t_0, t_0, p_0) = V_i(t_0, p_0) \leqslant r_i(t_0, t_0, r_0) = r_{i0}.$$

Sketch of the proof. Let $m = m(t) = m_i(t)$ be an N-dimensional vector such that for each $i \in J_N$, $m_i(t) = V_i(t, t_0, p_0)$. From the definitions of $V_i^*(t, t_0, p_0)$ and (3·4) we deduce that

$$\limsup_{h \to 0^+} \frac{m_i(t+h) - m_i(t)}{h} \leqslant W_i(t, m(t)).$$

Using the notion of maximal solution for the systems and the monotonic property of W, the Lemma can be easily established following the standard arguments used in (5).

4. Let r_0 be such that

$$r_{i0} = \begin{cases} r_{i0} & \text{for each} \quad i \in J_K, \quad r_{i0} > 0, \\ 0 & \text{for each} \quad i \in \tilde{J}_K. \end{cases} \tag{4·1}$$

Define $\|r\| = \sum_{i \in J_N} |r_i|$, $r \in R^N$, so that $\|r_0\| = \sum_{i \in J_K} r_{i0}$ for r_0 satisfying (4·1). Similarly $\|V(t, t_0, p_0)\| = \sum_{i \in J_N} V_i(t, t_0, p_0)$.

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Let $r(t, t_0, r_0)$ be the maximal solution of the differential system (3·1). Corresponding to the conditions (2·1) to (2·12) stated above, if we say that the differential system (3·1) has the property (2·1s), we mean the following condition holds, where r_0 satisfies (4·1).

Given $\epsilon > 0$ and $t_0 \geqslant 0$, there exists a positive (2·1s) function $\delta = \delta(t_0, \epsilon)$ that is continuous in t_0 for each δ, such that

$$\|r(t, t_0, r_0)\| < \epsilon \quad (t \geqslant t_0),$$

whenever

$$\|r_0\| \leqslant \delta.$$

Conditions (2·2s) to (2·12s) can be formulated similarly.

We state the following assumptions to be used later whenever necessary.

M is the set of points $p \in E$, defined by $V_i(t, p) \equiv 0$ for each $i \in J_K$. (4·2)

The function $b(u)$ is continuous and non-decreasing in u and $b(u) > 0$ for $u > 0$, and

$$b(\rho(t, t_0, p_0)) \leqslant \|V(t, t_0, p_0)\| \quad (t \geqslant t_0). \tag{4·3}$$

For $p \in E$, $\|V(t, p)\| \to 0$ as $d(p, A) \to 0$ for each $t \in I$. (4·4)

For $p \in E$, $\|V(t, p)\| \to 0$ as $d(p, A) \to 0$ uniformly in $t \in I$. (4·4*)

$$b(u) \to \infty \quad \text{as} \quad u \to \infty. \tag{4·5}$$

Then we have the following theorems on conditional stability and boundedness of the compact set A with respect to the dynamical system in E.

THEOREM 1. *Let the assumptions of the Lemma hold together with* (4·2), (4·3) *and* (4·4). *Then the condition* (2·1s) *implies condition* (2·1).

Proof. For any $\epsilon > 0$, if $\rho(t, t_0, p_0) = \epsilon$, we get from (4·3) that

$$b(\epsilon) \leqslant \|V(t, t_0, p_0)\|. \tag{4·6}$$

If (2·1s) holds, given $b(\epsilon) > 0$ and $t_0 \geqslant 0$, there exists a positive function $d = d(t_0, \epsilon)$ that is continuous in t_0 for each ϵ, such that

$$\|r(t, t_0, r_0)\| < b(\epsilon) \quad (t \geqslant t_0), \tag{4·7}$$

provided that $\|r_0\| \leqslant d$. Suppose that $f(t, t_0, p_0)$ is a motion of the general dynamical system in E and $V(t, t_0, p_0)$ denotes the function defined in section (3). Then it follows from the Lemma that

$$\|V(t, t_0, p_0)\| \leqslant \|r(t, t_0, r_0)\| \quad (t \geqslant t_0), \tag{4·8}$$

whenever, for each $i \in J_N$,

$$V_i(t_0, p_0) \leqslant r_{i0}. \tag{4·9}$$

Now choose r_{i0} to satisfy the following conditions:

$$r_{i0} = 0 \quad \text{for each} \quad i \in J_K \tag{4·10}$$

and

$$\sum_{i \in J_K} r_{i0} \leqslant d. \tag{4·11}$$

In view of (4·9) and the fact that $V_i(t, p) \geqslant 0$, $p \in f(t, t_0, p_0)$, (4·10) implies that $p_0 \in M$ because of (4·2). Further from the monotonic property of $b(u)$, (4·3), (4·9) and (4·11)

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we deduce that $\rho(t_0, t_0, p_0) = d(p_0, A) \leqslant b^{-1}(d) = \delta_1$. Also due to (4·4), there exists a $\delta_2 = \delta_2(t_0, d) = \delta_2(t_0, \epsilon)$ such that

$$\sup_{d(p_0, A) \leqslant \delta_2} \|V(t_0, p_0)\| \leqslant d.$$

Let $\delta = \min(\delta_1, \delta_2)$. It then follows from the choice of r_{i0} and δ, that $p_0 \in \bar{S}(\delta) \cap M$ implies that $V(t, t_0, p_0)$ satisfies (4·8). If possible, given $p_0 \in \bar{S}(\delta) \cap M$, suppose that for some value of $t = t_1 \geqslant t_0$ $\rho(t_1, t_0, p_0) = \epsilon$. Then using the relations (4·6), (4·8) and (4·7) we are led to the contradiction

$$b(\epsilon) \leqslant \|V(t_1, t_0, p_0)\| \leqslant \|r(t_1, t_0, r_0)\| < b(\epsilon),$$

which proves the theorem.

Let the condition (4·4) be strengthened to (4·4*), other assumptions in the Theorem 1 remaining the same, we deduce from the fact that d and δ are independent of t_0 in this case, the following corollary:

COROLLARY 1. *Let the assumptions of the Lemma hold together with* (4·2), (4·3) *and* (4·4*). *Then the condition* (2·2*s*) *implies condition* (2·2).

THEOREM 2. *Let the assumptions of the Lemma hold together with* (4·2), (4·3) *and* (4·4). *Then the condition* (2·3*s*) *implies condition* (2·3).

Proof. Let $\epsilon > 0$, $\alpha \geqslant 0$ and $t_0 \geqslant 0$ be given. Let $\rho(t_0, t_0, p_0) = d(p_0, A) \leqslant \alpha$. Then because of (4·4) we can choose a number $\alpha_1 = \alpha_1(t_0, \alpha)$ such that

$$\sup_{d(p_0, A) = \rho(t_0, t_0, p_0) \leqslant \alpha} \|V(t_0, t_0, p_0)\| \leqslant \alpha_1. \tag{4·12}$$

Since (4·9) implies (4·8), we choose r_{i0}, $(i \in J_N)$, such that (4·10) and $\sum_{i \in J_K} r_{i0} \leqslant \alpha$ hold. As in Theorem 1, one concludes from (4·2), (4·10) and $\rho(t_0, t_0, p_0) \leqslant \alpha$, that whenever $p_0 \in \bar{S}(\alpha) \cap M$, (4·8) holds.

Since (2·3*s*) holds, given $b(\epsilon) > 0$, $\alpha_1 \geqslant 0$ and $t_0 \geqslant 0$, there exists a positive number $T = T(t_0, \alpha, \epsilon)$ such that

$$\|r(t, t_0, r_0)\| < b(\epsilon) \quad (t \geqslant t_0 + T), \tag{4·13}$$

if $\|r_0\| \leqslant \alpha_1$, where α_1 is given by (4·12). Let $\{t_n\}$ be a sequence such that $t_n \to \infty$ as $n \to \infty$ and $t_n \geqslant t_0 + T$. Then the assumption that $\rho(t_n, t_0, p_0) \geqslant \epsilon$, $(t_n \geqslant t_0 + T)$, $p_0 \in \bar{S}(\alpha) \cap M$, leads to the contradiction

$$b(\epsilon) \leqslant \|V(t_n, t_0, p_0)\| \leqslant \|r(t_n, t_0, r_0)\| < b(\epsilon),$$

because of (4·3), (4·8) and (4·12). This contradiction proves Theorem 2.

As in Theorem 1, if the assumption (4·4) used in Theorem 2 is strengthened to (4·4*) since α and T are now independent of t_0, we deduce the following corollary.

COROLLARY 2. *Let the assumptions of the Lemma hold together with* (4·2), (4·3) *and* (4·4*). *Then the condition* (2·4*s*) *implies condition* (2·4).

THEOREM 3. *Let the assumptions of the Lemma hold together with* (4·2), (4·3), (4·4) *and* (4·5). *Then the condition* (2·7*s*) *implies condition* (2·7).

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Proof. Let $\alpha \geqslant 0$ and $t_0 \geqslant 0$ be given. Then as in Theorem 2, we conclude that whenever $p_0 \in \bar{S}(\alpha) \cap M$, (4·8) holds. Since (2·7s) holds, given $\alpha_1 \geqslant 0$ and $t_0 \geqslant 0$, there exists a $\beta = \beta(t_0, \alpha_1)$ such that

$$\|r(t, t_0, r_0)\| \leqslant \beta \quad (t \geqslant t_0), \tag{4·14}$$

whenever

$$\|r_0\| \leqslant \alpha_1.$$

Choose r_0 satisfying (4·10) and $\sum_{i \in J_K} r_{i0} \leqslant \alpha_1$, where α_1 is defined by (4·12). Since $b(u) \to \infty$ as $u \to \infty$ there exists a $L = L(t_0, \alpha_1)$ such that

$$\beta \leqslant b(L). \tag{4·15}$$

Now assume that there is a $t_1 \geqslant t_0$, such that $p_0 \in \bar{S}(\alpha) \cap M$, $\rho(t_1, t_0, p_0) = L$. This is due to relations (4·3), (4·8), (4·14) and (4·15) and leads to the contradiction

$$b(L) \leqslant \|V(t_1, t_0, p_0)\| \leqslant \|r(_1, t_0, r_0)\| < \beta \leqslant b(L),$$

which proves the Theorem 3.

If the assumption (4·4) in Theorem 3 is replaced by (4·4*), since α_1 and L are now independent of t_0, we deduce the following corollary:

COROLLARY 3. *Let the assumptions of the Lemma hold together with* (4·2), (4·3), (4·4*) *and* (4·5). *Then the condition* (2·8s) *implies condition* (2·8).

THEOREM 4. *Let the assumptions of the Lemma hold together with* (4·2), (4·3), (4·4) *and* (4·5). *Then the condition* (2·9s) *implies condition* (2·9).

Proof. Let $\alpha \geqslant 0$ and $t \geqslant 0$ be given. Then as in Theorem 2, we conclude that whenever $p_0 \in \bar{S}(\alpha) \cap M$, (4·8) holds. As (2·9$s$) holds, given $\alpha_1 \geqslant 0$ and $t_0 \geqslant 0$, there exist positive numbers B and $T = T(t_0, \alpha_1)$ such that

$$\|r(t, t_0, r_0)\| < B \quad (t \geqslant t_0 + T), \tag{4·16}$$

whenever

$$\|r_0\| \leqslant \alpha_1.$$

Choose r_0 satisfying (4·10) such that

$$\sum_{i \in J_K} r_{i0} \leqslant \alpha_1, \quad \text{where } \alpha_1 \text{ is defined by (4·12).}$$

Since $b(u) \to \infty$ as $u \to \infty$, there is an $L = L(t_0, \alpha_1)$ such that

$$B \leqslant b(L). \tag{4·17}$$

Now assume that there is a sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and $t_n \geqslant t_0 + T$. Let us assume $\rho(t_n, t_0, p_0) \geqslant L$. Then the relations (4·3), (4·8), (4·16) and (4·17) lead to the contradiction

$$b(L) \leqslant \|V(t_n, t_0, p_0)\| \leqslant \|r(t_n, t_0, r_0)\| < B \leqslant b(L).$$

This contradiction proves Theorem 4.

Replacing the assumption (4·4) in Theorem 4, by (4·4*), as α_1 and L are independent of t_0 in this case, we deduce the following corollary:

COROLLARY 4. *Let the assumptions of the Lemma hold together with* (4·2), (4·3), (4·4*) *and* (4·5). *Then the condition* (2·10s) *implies condition* (2·10).

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Following corollaries can be immediately deduced from the thorems and corollaries proved above.

COROLLARY 5. *If the conditions of the Theorems* 1 *and* 2 *hold simultaneously, then the condition* (2·5*s*) *implies condition* (2·5).

COROLLARY 6. *If the conditions of the Corollaries* 1 *and* 2 *hold simultaneously, then the condition* (2·6*s*) *implies condition* (2·6).

COROLLARY 7. *If the conditions of the Theorems* 3 *and* 4 *hold simultaneously, then the condition* (2·11*s*) *implies condition* (2·11).

COROLLARY 8. *If the conditions of the Corollaries* 3 *and* 4 *hold simultaneously, then the condition* (2·12*s*) *implies condition* (2·12).

5. We give below two simple examples to illustrate the results.

Example 1. Consider the following differential system in R^3 ($t \in I$).

$$\left.\begin{aligned} x_1' &= (1+\sin t)\,x_1 + (1-\sin t)\,(x_2 - x_3), \\ x_2' &= (1-\cos t)\,(x_1 - x_3) + (1+\cos t)\,x_2, \\ x_3' &= (\sin t - \cos t)\,(x_1 - x_2) + (\sin t + \cos t)\,x_3. \end{aligned}\right\} \tag{5·1}$$

Corresponding to the solutions of (5·1) we define a dynamical system in R^3, whose motion is the single point set $\{x(t, t_0, x_0)\}$ where

$$\begin{aligned} x_1(t, t_0, x_0) &= \tfrac{1}{2}[\exp(2t - 2t_0) + \exp\{2(\cos t_0 - \cos t)\}]\,x_{10} \\ &\quad + \tfrac{1}{2}[\exp\{2(t-t_0)\} - \exp\{2(\cos t_0 - \cos t)\}]\,(x_{20} - x_{30}), \\ x_2(t, t_0, x_0) &= \tfrac{1}{2}[\exp\{2(t-t_0)\} + \exp\{2(\sin t - \sin t_0)\}]\,x_{20} \\ &\quad + \tfrac{1}{2}[\exp\{2(t-t_0)\} - \exp\{2(\sin t - \sin t_0)\}]\,(x_{10} - x_{30}), \\ x_3(t, t_0, x_0) &= \tfrac{1}{2}[\exp\{2(\sin t - \sin t_0)\} + \exp\{2(\cos t_0 - \cos t)\}]\,x_{30} \\ &\quad + \tfrac{1}{2}[\exp\{2(\cos t_0 - \cos t)\} - \exp\{2(\sin t - \sin t_0)\}]\,(x_{10} - x_{20}), \end{aligned}$$

where $x_0 = (x_{10}, x_{20}, x_{30})$. Let $A = \{0\}$. The vector function $V(t, x)$ be defined by $V_1(t, x) = (-x_1 + x_2 + x_3)^2$, $V_2(t, x) = (x_1 - x_2 + x_3)^2$ and $V_3(t, x) = (x_1 + x_2 - x_3)^2$. As the motion is a single point set the function $V(t, t_0, x_0)$ will be the same as $V(t, x)$ with $x = x(t, t_0, x_0)$. Hence the function $V^*(t, t_0, x_0)$ of (3·3) is given by

$$\begin{aligned} V_1^*(t, t_0, x_0) &= 4V_1 \cos t, \\ V_2^*(t, t_0, x_0) &= 4V_2 \sin t, \\ V_3^*(t, t_0, x_0) &= 4V_3 \end{aligned}$$

so that the differential system (3·1) reduces to

$$\left.\begin{aligned} r_1' &= 4r_1 \cos t, \\ r_2' &= 4r_2 \sin t, \\ r_3' &= 4r_3; \end{aligned}\right\} \tag{5·2}$$

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and the monotonic property of W is satisfied. Let K in (4·1) be equal to 2 and $r_0 = (r_{10}, r_{20}, 0)$. The solution of (5·2) through r_0 is given by

$$\left.\begin{aligned} r_1(t, t_0, r_0) &= r_{10} \exp[4(\sin t - \sin t_0)], \\ r_2(t, t_0, r_0) &= r_{20} \exp[4(\cos t_0 - \cos t)], \\ r_3(t, t_0, r_0) &\equiv 0. \end{aligned}\right\} \tag{5·3}$$

Let the distance in R^3 be the usual Euclidean one so that

$$d(x, A) = \sqrt{(x_1^2 + x_2^2 + x_3^2)}.$$

Let $b(r) = r^2$ so that (4·3) is satisfied as

$$\sum_{i=1}^{3} V_i(t, x) = x_1^2 + x_2^2 + x_3^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2.$$

Also (4·4) is satisfied as $\|V(t, x)\| \to 0$ as $d(x, A) \to 0$. The set M of (4·2) is the set of points in the plane $x_1 + x_2 - x_3 = 0$. From (5·3) it is obvious that condition (2·1s) holds. Hence by Theorem 1 condition (2·1) is satisfied.

Example 2. Consider the differential system.

$$\left.\begin{aligned} x_1' &= \frac{t^2 + t - 1}{t + 1} x_1 + \frac{t^2 + t + 1}{t + 1}(x_2 - x_3), \\ x_2' &= (t + 1)(x_1 - x_3) + (t - 1) x_2 + e^{-2t} |x_2 + x_3 - x_1|^{\frac{1}{2}}, \\ x_3' &= \frac{t}{t + 1}(x_1 - x_2) + \frac{t + 2}{t + 1} x_3 + e^{-2t} |x_2 + x_3 - x_1|. \end{aligned}\right\} \tag{5·4}$$

The general solution of the system is given by

$$x_1(t, t_0, x_0) = \tfrac{1}{2}(x_{10} + x_{20} - x_{30}) \exp(t^2 - t_0^2) + \tfrac{1}{2}(x_{10} - x_{20} + x_{30}) \left(\frac{t_0 + 1}{t + 1}\right)^2,$$

$$x_2(t, t_0, x_0) = \tfrac{1}{2}(x_{10} + x_{20} - x_{30}) \exp(t^2 - t_0^2) \pm \tfrac{1}{2} e^{-2t} [|x_{20} + x_{30} - x_{10}|^{\frac{1}{2}} e^{t_0} = t - t_0]^2,$$

$$x_3(t, t_0, x_0) = \tfrac{1}{2}(x_{10} - x_{20} + x_{30}) \left(\frac{t_0 + 1}{t + 1}\right)^2 \pm \tfrac{1}{2} e^{-2t} [|x_{20} + x_{30} - x_{10}|^{\frac{1}{2}} e^{t_0} + t - t_0]^2,$$

where $x_0 = (x_{10}, x_{20}, x_{30})$. The signs in $x_2(t, t_0, x_0)$ and $x_3(t, t_0, x_0)$ are both positive for the points x_0 such that $x_{20} + x_{30} > x_{10}$. These are both negative if $x_{20} + x_{30} < x_{10}$. However, the case when x_0 is on the plane $x_2 + x_3 = x_1$ is interesting as the solution is non-unique. The dynamical system defined by the solutions is such that $\{x(t, t_0, x_0)\}$ is no longer a single point set if x_0 is on the above plane.

Taking the same Lyapunov vector-function $V(t, x)$ as in Example 1, we deduce that the function $W(t, r)$ of (3·1) takes the form

$$W_1 = -4r_1 + e^{-2t} r_1^{\frac{1}{2}}, \quad W_2 = \frac{-4r_2}{t + 1} \quad \text{and} \quad W_3 = 4tr_3.$$

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Clearly the monotonic restriction on W is satisfied and the maximal solution of (3·1) satisfying (4·1) with $K = 2$, $r_0 = (r_{10}, r_{20}, 0)$ is given by

$$r(t, t_0, r_0) = \begin{cases} r_1(t, t_0, r_0) = e^{-4t}[r_{10}^{\frac{1}{4}} e^{t_0} + e^{-t_0} - e^t]^4, \\ r_2(t, t_0, r_0) = r_{20}\left(\dfrac{t_0+1}{t+1}\right)^4, \\ r_3(t, t_0, r_0) \equiv 0. \end{cases}$$

Obviously (5·6) has the property (2·3s). Hence by Theorem 2, we conclude that the dynamical system has the property (2·3), provided M is the same set as defined in Example 1.

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