Math. Proc. Camb. Phil. Soc. (1987), **102**, 443
Printed in Great Britain

On $T(6)$-groups

BY MOHAMED S. EL-MOSALAMY
Department of Mathematics, Aswan University, Egypt
AND STEPHEN J. PRIDE
Department of Mathematics, University of Glasgow, Glasgow, G12 8QW

(*Received* 20 *November* 1986; *revised* 18 *March* 1987)

1. *Introduction*

This paper is concerned with group presentations satisfying the small cancellation condition $T(6)$. The definition of this condition is given in §1·2, together with some examples. Before giving the definition, however, we describe (in §1·1) some material which, to a certain extent, motivated our paper. In §1·3 we state our main theorem, which provides new solutions to the word and conjugacy problems for $T(6)$-groups.

We make several standing assumptions regarding presentations. (*These assumptions will not be repeated.*) Let $\langle \mathbf{x}; \mathbf{r} \rangle$ be a group presentation. We will assume that $\mathbf{x}$ and $\mathbf{r}$ are finite, and that each element of $\mathbf{r}$ is cyclically reduced and *of length at least* 3. If R, R' are distinct elements of $\mathbf{r}$ then we will assume that no cyclic permutation of $R^{\pm 1}$ is equal to R'. The set of all cyclic permutations of elements of $\mathbf{r} \cup \mathbf{r}^{-1}$ will be denoted by $\mathbf{r}^*$.

1·1. *Background to the paper*

Let $G = \langle \mathbf{x}; \mathbf{r} \rangle$. Given a sequence of words $(W_0, W_1, \ldots, W_k)$ in the generators $\mathbf{x}$ of G, we say that W_0 is *dependent* on $(W_1, \ldots, W_k)$ if there is a subset $\{i_1, i_2, \ldots, i_l\}$ of $\{1, \ldots, k\}$ and words $U_1, \ldots, U_l$ such that

$$W_0(U_1 W_{i_1} U_1^{-1}) \ldots (U_l W_{i_l} U_l^{-1}) = 1 \quad \text{in } G.$$

If m is a positive integer or ∞ then the *dependence problem* $DP(m)$ for G asks for an algorithm to decide for any sequence $(W_0, W_1, \ldots, W_k)$ $(0 \leqslant k < m)$ whether or not W_0 is dependent on $(W_1, \ldots, W_k)$. The problems $DP(1)$, $DP(2)$ are usually called the *word problem* and *conjugacy problem* for G, respectively.

In [5] the second author showed that for groups given by certain presentations called 'hyperbolic presentations', DP(∞) is solvable ([5], theorem 2). Now presentations satisfying the $T(7)$ condition are hyperbolic ([5], theorem 4, corollary). On the other hand, there are $T(6)$-presentations which are not hyperbolic (for example, $\langle a, b, c;\ a^3, b^3, c^3, abc \rangle$), so the question naturally arises as to whether $DP(\infty)$ is solvable for groups given by $T(6)$-presentations. This paper arose from trying to settle this question.

Now it turns out that there is an intimate connection between the dependence problems and tesselations of spheres, which we now explain. A *diagram* over $\langle \mathbf{x}; \mathbf{r} \rangle$ is a triple (S, Θ, ϕ), where S is a finitely tesselated sphere, Θ is a subset of the set of regions of S, ϕ is a labelling of the oriented edges of S by elements of $\mathbf{x} \cup \mathbf{x}^{-1}$ (where if an oriented edge is labelled by $x \in \mathbf{x} \cup \mathbf{x}^{-1}$ then the oppositely oriented edge is labelled by x^{-1}). For each region D we assume that a vertex v has been chosen on ∂D.

https://doi.org/10.1017/S0305004100067499 Published online by Cambridge University Press

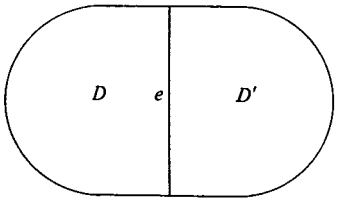


Fig. 1

Then the *label on* D, denoted $\phi(D)$, is the label on the *clockwise* boundary cycle of D starting at v. We require that if D is not in Θ then $\phi(D) \in \mathbf{r}^*$. The diagram is said to be *reduced* if for any two regions D, D' not in Θ, which have an edge e in common, the label on the anticlockwise boundary cycle of D starting with e is not the inverse of the label on the clockwise boundary of D' starting with e.

Using arguments similar to those of § V.1 of [**2**], one can prove the following.

(*) *W_0 is dependent on $(W_1, \ldots, W_k)$ in G if and only if there is a reduced diagram $(S, \{D_0, D_1, \ldots, D_l\}, \phi)$ over $\langle \mathbf{x}; \mathbf{r} \rangle$ with $\phi(D_0) = W_0$ and $(\phi(D_1), \ldots, \phi(D_l))$ a subsequence of $(W_1, \ldots, W_k)$.*

In Section 2 below, we examine the geometry of pairs (S, Θ) where S is a tesselated sphere and Θ is a subset of the set of regions of S. For certain types of tesselations we obtain a formula involving a term

$$2 - |\Theta|.$$

Now this is non-negative if and only if $|\Theta| \leqslant 2$. As a consequence we will be able to use our formula to solve the word and conjugacy problems for $T(6)$-groups, but we will not be able to use it to solve the problems $DP(m)$ for $m > 2$. It should be mentioned that solutions to the word and conjugacy problems for $T(6)$-groups are already known ([**2**], §§ V.6, V.7). However, the solutions given here are new, and appear to be simpler.

1·2. *Definition of $T(6)$-presentations, and some examples*

For further information concerning the material in the first part of this section, the reader should consult [**1**] or [**5**], §5.

A *1-complex* consists of two disjoint sets $\mathbf{v}$ (vertices), $\mathbf{e}$ (edges) together with functions $\iota: \mathbf{e} \to \mathbf{v}, \tau: \mathbf{e} \to \mathbf{v}, {}^{-}: \mathbf{e} \to \mathbf{e}$ satisfying $\iota(e) = \tau(\bar{e})$, $\bar{\bar{e}} = e$, $\bar{e} \neq e$ for all $e \in \mathbf{e}$. A *closed path* in this 1-complex is a finite sequence of edges $e_1 e_2 \ldots e_m$ with $\tau(e_i) = \iota(e_{i+1})$ for $i = 1, \ldots, m$ (where subscripts are computed mod m). The closed path is said to be *cyclically reduced* if $e_i \neq \bar{e}_{i+1}$ for $i = 1, 2, \ldots, m$. We call m the *length* of the path.

Consider a presentation $\mathscr{P} = \langle \mathbf{x}; \mathbf{r} \rangle$. We associate with $\mathscr{P}$ a 1-complex $\mathscr{P}^{\mathrm{st}}$ (called the *star-complex* of $\mathscr{P}$) as follows. The vertex set is $\mathbf{x} \cup \mathbf{x}^{-1}$, and the edge set is $\mathbf{r}^*$. For a given edge R of $\mathscr{P}^{\mathrm{st}}$ we define $\iota(R)$ to be the first symbol of R and $\tau(R)$ to be the inverse of the last term of R. We define $\bar{R}$ to be R^{-1}. Note that since each element of $\mathbf{r}$ is cyclically reduced $\mathscr{P}^{\mathrm{st}}$ has no closed paths of length 1.

For a given positive integer q we say that $\mathscr{P}$ satisfies the condition $T(q)$ if there is no cyclically reduced closed path in $\mathscr{P}^{\mathrm{st}}$ of length m, with $3 \leqslant m < q$.

Now if $q > 4$ and if $\mathscr{P}$ satisfies $T(q)$ then there can be no cyclically reduced closed

https://doi.org/10.1017/S0305004100067499 Published online by Cambridge University Press

path in $\mathscr{P}^{st}$ of length 2 (for otherwise its square would be a cyclically reduced closed path of length 4, contradicting $T(5)$). Thus we see that *$\mathscr{P}$ satisfies $T(q)$ $(q > 4)$ if and only if there are no cyclically reduced closed paths in $\mathscr{P}^{st}$ of length less than q.* The important consequence of this as regards the present paper is the following: *if $\mathscr{P}$ satisfies $T(q)$ $(q > 4)$ and if (S, Θ, ϕ) is a reduced diagram over $\mathscr{P}$ then the valence of each vertex not on the boundary of any region in Θ is at least q.* (See [1] in this connection.)

Our concern here will be with the $T(6)$ condition. The reader who is familiar with small cancellation theory will be used to thinking of this condition in conjunction with another condition $C(3)$. However, a $T(6)$-presentation in which each defining relator has length at least 3 automatically satisfies $C(3)$ (see [5], §5). This is the reason why we imposed the standing assumption at the start of our paper that all defining relators are to have length at least 3. Without this assumption it would have been necessary to consider 'degenerate' situations. These can, in fact, be quite easily dealt with, but we have chosen to avoid becoming involved with annoying technicalities.

We now give some examples of $T(6)$-presentations.

Example 1. Probably the best-known examples are the standard presentations of surface groups:

$$\langle x_1, \ldots, x_n ; x_1^2 x_2^2 \ldots x_n^2 \rangle \quad (n \geqslant 3),$$

$$\langle x_1, y_1, \ldots, x_n, y_n ; x_1 y_1 x_1^{-1} y_1^{-1} \ldots x_n y_n x_n^{-1} y_n^{-1} \rangle \quad (n \geqslant 2).$$

Example 2. Recall that a presentation $\langle \mathbf{x} ; \mathbf{r} \rangle$ is said to be *positive* if each relator $R \in \mathbf{r}$ is a positive word on $\mathbf{x}$ (that is, only positive powers of elements of $\mathbf{x}$ occur in R). We give a way of combining positive $T(6)$-presentations together to get new ones: Let $\langle x_1, \ldots, x_n ; \mathbf{p} \rangle$, $\langle y_1, \ldots, y_n ; \mathbf{q} \rangle$, $\langle z_1, \ldots, z_n ; \mathbf{r} \rangle$ be three positive $T(6)$-presentations. Then

$$\langle x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n ; \mathbf{p}, \mathbf{q}, \mathbf{r}, x_i y_i z_i \quad (1 \leqslant i \leqslant n) \rangle$$

is a positive $T(6)$-presentation.

As a specific illustration, if we start with the positive $T(6)$-presentations $\langle x ; x^k \rangle$, $\langle y ; y^l \rangle$, $\langle z ; z^m \rangle$ $(k, l, m \geqslant 3)$, then we obtain the positive $T(6)$-presentation

$$\langle x, y, z ; x^k, y^l, z^m, xyz \rangle,$$

which is a presentation of a triangle group.

Note that the presentations $\langle x_1, \ldots, x_n ; x_1^2 \ldots x_n^2 \rangle$ $(n \geqslant 3)$ in Example 1 are positive $T(6)$-presentations.

Example 3. Consider a presentation

$$\langle s_1, \mathbf{x} ; s_1 U s_1^{-1} V^{-1} \rangle$$

where U and V are non-empty cyclically reduced words on $\mathbf{x}$ of the same length. Write $U = u_1 u_2 \ldots u_n$, $V = v_1 v_2 \ldots v_n$ where the u's and v's belong to $\mathbf{x} \cup \mathbf{x}^{-1}$. Then the presentation is Tietze equivalent to the $T(6)$-presentation

$$\langle s_1, t_1, \ldots, s_n, t_n, \mathbf{x} ; s_i u_i t_i^{-1}, t_i^{-1} v_i s_{i+1} \quad (1 \leqslant i \leqslant n, \text{ subscripts computed mod } n) \rangle.$$

https://doi.org/10.1017/S0305004100067499 Published online by Cambridge University Press

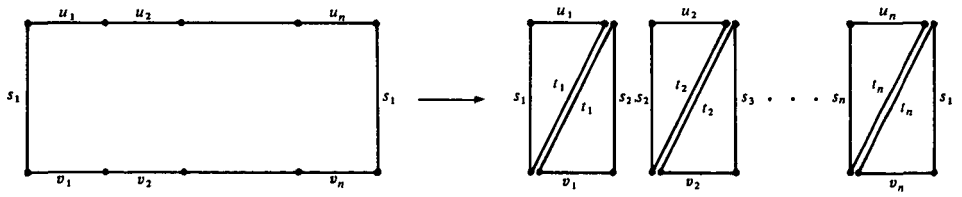


Fig. 2

This example is of interest in connection with the problem of whether finitely presented small cancellation groups are residually finite ([**6**]; [**7**], problem F.5). Many one-relator groups with a presentation of the form

$$\langle t, \mathbf{x}; tWt^{-1} = Z \rangle$$

(W, Z words on $\mathbf{x}$) are *not* residually finite [**3**], but *can this happen if W and Z are cyclically reduced and of the same length*?

Further interesting examples of $T(6)$-groups (derived from finite projective planes) have been found by M. Edjvet and J. Howie [**8**].

1·3. *The word and conjugacy problems for $T(6)$-groups*

A *triangulated presentation* is a presentation in which each defining relator has length 3.

Given a presentation $\langle \mathbf{x}; \mathbf{r} \rangle$ we can triangulate it as follows. Let $R = x_1 x_2 \dots x_n$ be an element of $\mathbf{r}$ of length greater than 3, where each x_i belongs to $\mathbf{x} \cup \mathbf{x}^{-1}$. Adjoin new generators $t_1, t_2, \dots, t_{n-3}$ to the presentation, and replace R by the $n-2$ relators $x_1 x_2 t_1^{-1}$, $t_1 x_3 t_2^{-1}$, $\dots$, $t_{n-3} x_{n-1} x_n$.

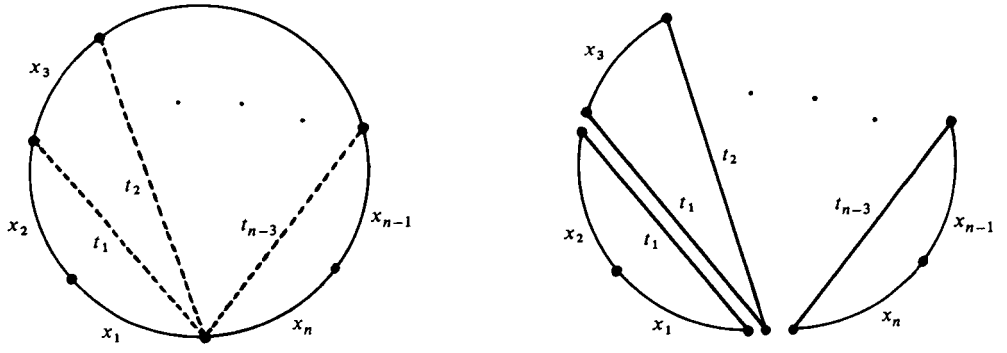


Fig. 3

Carrying out the above procedure for each defining relator of length greater than 3, we arrive at a presentation $\langle \mathbf{x}, \mathbf{t}; \mathbf{r}_\Delta \rangle$ called the *triangulation* of $\langle \mathbf{x}; \mathbf{r} \rangle$. Note for future reference that if G, G_Δ are the groups defined by $\langle \mathbf{x}; \mathbf{r} \rangle$ and $\langle \mathbf{x}, \mathbf{t}; \mathbf{r}_\Delta \rangle$ respectively, then the mapping $x \to x$ ($x \in \mathbf{x}$) induces an isomorphism from G to G_Δ. Note also for future reference that if $\mathbf{D} = (S, \Theta, \phi)$ is a reduced diagram over $\langle \mathbf{x}; \mathbf{r} \rangle$ then we can triangulate the regions not in Θ in an obvious way to obtain a reduced diagram $\mathbf{D}_\Delta$ over $\langle \mathbf{x}, \mathbf{t}; \mathbf{r}_\Delta \rangle$.

https://doi.org/10.1017/S0305004100067499 Published online by Cambridge University Press

Let $\langle \mathbf{y}; \mathbf{s} \rangle$ be a triangulated presentation. We define certain operations on words W on $\mathbf{y}$ as follows. (In (II)–(V) below a, b, c, d, e (sometimes subscripted) denote elements of $\mathbf{y} \cup \mathbf{y}^{-1}$.)

(I) Cyclically permute W.

(II) Delete an adjacent pair aa^{-1} (free reduction).

(III) Replace a subword ab by c if $abc^{-1} \in \mathbf{s}^*$.

(IV) Replace a subword ab by cd if there are elements aec^{-1}, $bd^{-1}e \in \mathbf{s}^*$.

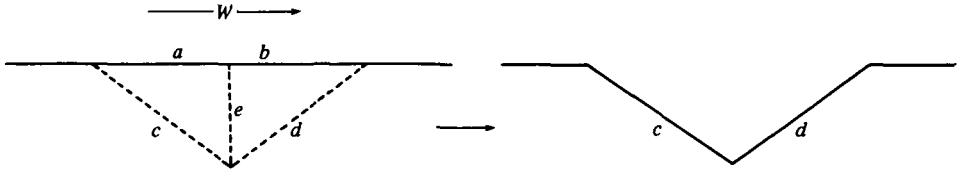


Fig. 4

(V) If $W = a_1 a_2 \dots a_n$, then change W to $b_1 b_2 \dots b_n$ if there are elements $d_n a_1 c_1^{-1}$, $c_1 d_1^{-1} b_1^{-1}$, $d_1 a_2 c_2^{-1}$, $c_2 d_2^{-1} b_2^{-1}, \dots, d_{n-1} a_n c_n^{-1}$, $c_n d_n^{-1} b_n^{-1}$ in $\mathbf{s}^*$.

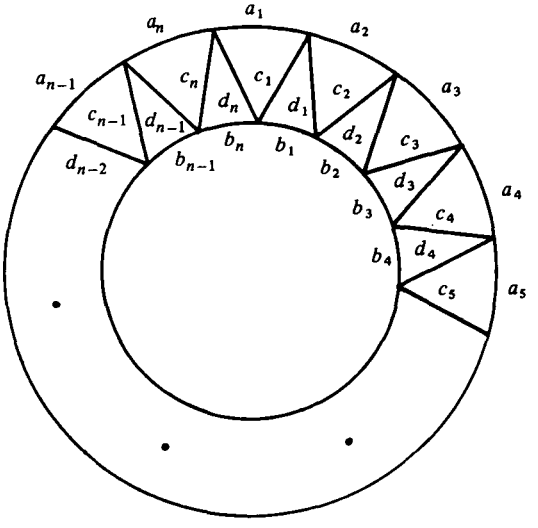


Fig. 5

THEOREM. *Let $G = \langle \mathbf{x}; \mathbf{r} \rangle$, where the presentation satisfies the $T(6)$ condition.*

(i) *A word on $\mathbf{x}$ is equal to 1 in G if and only if it can be converted to the empty word by a finite sequence of operations* (I)–(IV) *in the triangulation of $\langle \mathbf{x}; \mathbf{r} \rangle$.*

(ii) *Two words on $\mathbf{x}$ are conjugate in G if and only if they can be converted to the same word by a finite sequence of operations* (I)–(V) *in the triangulation of $\langle \mathbf{x}; \mathbf{r} \rangle$.*

Since the operations (I)–(V) do not increase the length of a word, and since there are only finitely many words of any given length, the above theorem can easily be used to provide solutions to the word problem and conjugacy problem for $T(6)$-groups. Details are left to the reader.

https://doi.org/10.1017/S0305004100067499 Published online by Cambridge University Press

2. *On tesselations of spheres*

Throughout this section we will consider pairs (S, Θ), where S is a finitely tesselated sphere and Θ is a subset of the set of regions of S. The regions in Θ will be called *distinguished regions*. A vertex on the boundary of a distinguished region will be called a *distinguished vertex*. We will denote the number of vertices, non-distinguished vertices, edges, regions of S by V, V^0, E, F, respectively. The valence of a vertex v will be denoted by $d(v)$.

Consider a vertex v with $d(v) = n$. Reading around a neighbourhood of v we encounter regions $D_1, D_2, \ldots, D_n$ (which may not all be distinct). We write $d^*(v) = k$ if exactly k of the regions $D_1, D_2, \ldots, D_n$ are distinguished.

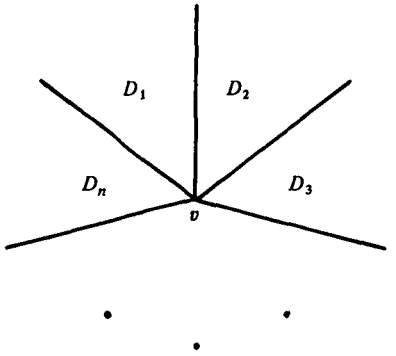


Fig. 6

For each positive integer n and integer k with $0 < k \leqslant n$ we let V^{kn} denote the total number of vertices v with $(d^*(v), d(v)) = (k, n)$.

2·1. *A formula concerning certain tesselations*

We consider tesselations where each *non*-distinguished region has boundary cycle of length 3.

The sum of the lengths of the boundary cycles of distinguished regions is $\Sigma_{k,n} kV^{kn}$, and so the sum of the lengths of the boundary cycles of all regions is $3(F-|\Theta|)+\Sigma_{k,n} kV^{kn}$. Thus we have

$$2E = 3(F-|\Theta|)+\sum_{k,n} kV^{kn}. \tag{1}$$

The sum of the valences of all the distinguished vertices is $\Sigma_{k,n} nV^{kn}$. If we suppose that the average valence of the non-distinguished vertices is $6+\alpha$, then we have

$$\begin{aligned} 2E &= (6+\alpha)\,V^0+\sum_{k,n} nV^{kn} \\ &= \alpha V^0+6(V-\sum_{k,n} V^{kn})+\sum_{k,n} nV^{kn} \\ &= \alpha V^0+6V+\sum_{k,n} (n-6)\,V^{kn}, \end{aligned}$$

and so

$$\frac{2E}{6} = V+\frac{\alpha}{6}V^0+\sum_{k,n}\frac{n-6}{6}V^{kn}. \tag{2}$$

https://doi.org/10.1017/S0305004100067499 Published online by Cambridge University Press

Dividing (1) by 3 and adding to (2), we obtain (using $V-E+F=2$):

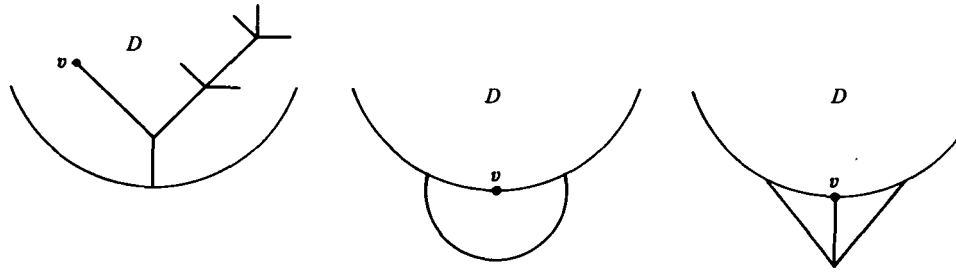
$$(2-|\Theta|)+\frac{\alpha}{6}V^0+\sum_{k,n}\frac{n+2k-6}{6}V^{kn}=0. \tag{3}$$

2·2. *The case of one distinguished region*

Suppose there is a single distinguished region D, and assume that each non-distinguished vertex has valence at least 6. Then the first term in (3) is positive, and the second term is non-negative, so we must have

$$\frac{n+2k-6}{6}V^{kn}<0$$

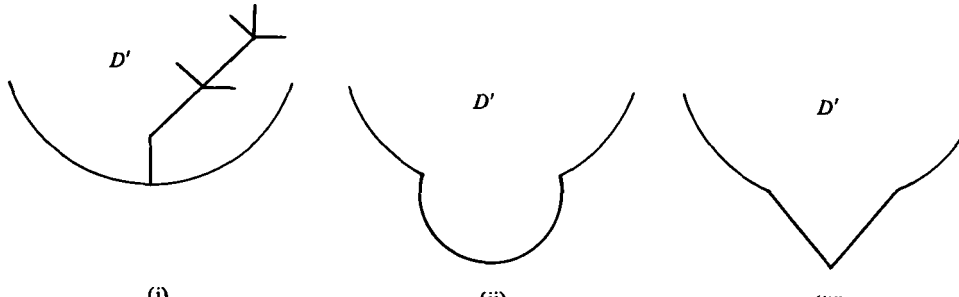
for some k,n. The only possibilities for (k,n) are $(1,1)$, $(1,2)$, $(1,3)$, so we must have a distinguished vertex v with $(d^*(v), d(v))$ one of $(1,1)$, $(1,2)$, $(1,3)$. The three cases are illustrated below.



(i) $(d^*(v), d(v)) = (1,1)$ (ii) $(d^*(v), d(v)) = (1,2)$ (iii) $(d^*(v), d(v)) = (1,3)$

Fig. 7

Now in each case we can modify the tesselation to obtain a new tesselation with a single distinguished region D' as follows (we will say that the new tesselation is obtained by *eliminating the vertex* v):



(i) (ii) (iii)

Fig. 8

This new tesselation still has the property that each non-distinguished vertex has valence at least 6. We can therefore repeat the process for this new tesselation, and so on. Eventually we will arrive at the trivial tesselation with one vertex, no edges, and a single (distinguished) region.

https://doi.org/10.1017/S0305004100067499 Published online by Cambridge University Press

2·3. *The case of two distinguished regions*

Now suppose that we have a tesselation with two distinguished regions, and again assume that each non-distinguished vertex has valence at least 6. If there are any distinguished vertices v with $(d^*(v), d(v))$ one of $(1,1)$, $(1,2)$, $(1,3)$ then we eliminate them, as above. We then obtain a tesselation with two distinguished regions D_0, D_1 which still has the property that each non-distinguished vertex has valence at least 6. Consider the formula (3) for this tesselation. All terms on the left-hand side are non-negative, and so all terms are zero. Hence each non-distinguished vertex has valence exactly 6, and if v is a distinguished vertex then $(d^*(v), d(v))$ is either $(2,2)$ or $(1,4)$. We deduce that the tesselation is made up of a sequence of 'layers'

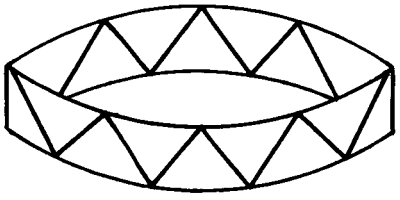


Fig. 9

fitted together as follows (the number of layers may be zero).

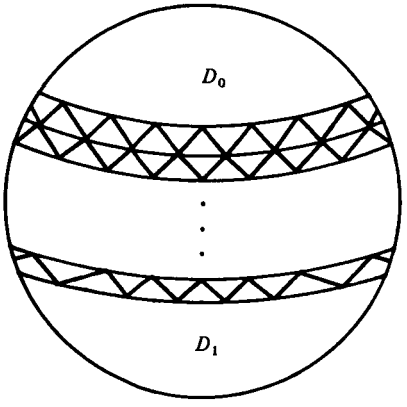


Fig. 10

By successively removing these layers we can then obtain a tesselation with no layers.

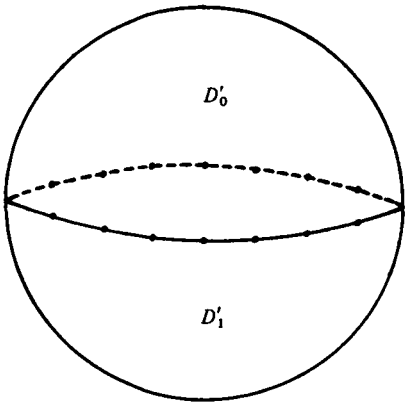


Fig. 11

https://doi.org/10.1017/S0305004100067499 Published online by Cambridge University Press

3. *Proof of Theorem*

Let $G = \langle \mathbf{x}; \mathbf{r} \rangle$, where the presentation satisfies $T(6)$.

Suppose W is a word on $\mathbf{x}$ which is equal to 1 in G. Then there is a reduced diagram $\mathbf{B}$ over $\langle \mathbf{x}; \mathbf{r} \rangle$ with a single distinguished region whose label is W. Since $\mathbf{B}$ is reduced each non-distinguished vertex of $\mathbf{B}$ has valence at least 6. Consider $\mathbf{B}_\Delta$ (see §1·2). The underlying sphere of this diagram satisfies the assumptions of §2·2, and so we deduce from §2·2 that W can be converted to the empty word by a finite sequence of operations (I)–(IV) in the triangulation of $\langle \mathbf{x}; \mathbf{r} \rangle$. Conversely, if W is a word which can be converted to the empty word by a finite sequence of operations in the triangulation of $\langle \mathbf{x}; \mathbf{r} \rangle$, then W is equal to 1 in the group G_Δ defined by the triangulation, and so $W = 1$ in G because of the isomorphism between G and G_Δ (see §1·2).

Now suppose W_0, W_1 are two words on $\mathbf{x}$ which are conjugate in G. If one of W_0, W_1 is equal to 1 then so is the other, so both can be converted to the empty word by a finite sequence of operations (I)–(IV). If neither W_0 nor W_1 is equal to 1 in G then there is a reduced diagram $\mathbf{C}$ with two distinguished regions labelled W_0, W_1^{-1}. Then the underlying sphere of the diagram $\mathbf{C}_\Delta$ satisfies the assumptions of §2·3, and so it follows from §2·3 that W_0 and W_1 can be converted to the same word by a finite sequence of operations (I)–(V) in the triangulation of $\langle \mathbf{x}; \mathbf{r} \rangle$. Conversely, if W_0, W_1 are words which can be converted to the same word by a finite sequence of operations (I)–(V) in the triangulation of $\langle \mathbf{x}; \mathbf{r} \rangle$, then W_0 and W_1 are conjugate in G_Δ, and hence in G.

REFERENCES

[1] P. Hill, S. J. Pride and A. D. Vella. On the $T(q)$-conditions of small cancellation theory. *Israel J. Math.* **52** (1985), 293–304.

[2] R. C. Lyndon and P. E. Schupp. *Combinatorial group theory* (Springer-Verlag, 1977).

[3] S. Meskin. Non-residually finite one-relator groups. *Trans. Amer. Math. Soc.* **164** (1972), 105–114.

[4] S. J. Pride. On Tits' conjecture and other questions concerning Artin and generalized Artin groups. *Invent. Math.* **86** (1986), 347–356.

[5] S. J. Pride. Star complexes, and the dependence problems for hyperbolic complexes. *Glasgow Math. J.* (To appear.)

[6] P. E. Schupp. A survey of small cancellation theory. In *Word Problems* (North-Holland, 1973).

[7] C. T. C. Wall (ed.). *Homological group theory* (Cambridge University Press, 1979).

[8] M. Edjvet and J. Howie. Star graphs, projective planes and free subgroups in small cancellation groups. Preprint (1987).

https://doi.org/10.1017/S0305004100067499 Published online by Cambridge University Press