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Coding map for a contractive Markov system

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Abstract

We develop the theory of contractive Markov systems initiated in [**8**]. We construct a coding map for such systems and investigate some of its properties.

1. *Introduction*

In [**8**], we introduced a theory of *contractive Markov systems (CMSs)* which provides a unifying framework in so-called "fractal" geometry. It extends the known theory of *iterated function systems (IFSs) with place dependent probabilities* [**1**, **3**] in a way that it also covers *graph directed constructions* of "fractal" sets [**6**]. This theory also is a natural generalization of the theory of finite Markov chains.

A contractive Markov system with an average contracting rate $0 < a < 1$ is a family

$$\left(K_{i(e)}, w_e, p_e\right)_{e\in E}$$

(see Figure 1) where E is the set of edges of a directed (multi)graph (V, E, i, t) ($V := \{1, \ldots, N\}$ is the set of vertices of the directed (multi)graph (we do not exclude the case $N = 1$), $i : E \to V$ is a map indicating the initial vertex of each edge and $t : E \to V$ is a map indicating the terminal vertex of each edge), $K_1, K_2, \ldots, K_N$ is a partition of a metric space (K, d) into non-empty Borel subsets, $(w_e)_{e\in E}$ is a family of Borel measurable self-maps on the metric space such that $w_e(K_{i(e)}) \subset K_{t(e)}$ for all $e \in E$ and $(p_e)_{e\in E}$ is a family of Borel measurable probability functions on K (i.e. $p_e(x) \geqslant 0$ for all $e \in E$ and $\sum_{e\in E} p_e(x) = 1$ for all $x \in K$) (associated with the maps) such that each p_e is zero on the complement of $K_{i(e)}$ and the system satisfies the following *condition of a contractiveness on average*

$$\sum_{e\in E} p_e(x) d(w_e x, w_e y) \leqslant a d(x, y) \text{ for all } x, y \in K_i, \quad i = 1, \ldots, N. \tag{1·1}$$

This condition was discovered by R. Isaac [**4**] for the case $N = 1$.

Such a system determines a Markov operator U on the set of all bounded Borel measurable functions $\mathcal{L}^0(K)$ by

$$Uf := \sum_{e\in E} p_e f \circ w_e \text{ for all } f \in \mathcal{L}^0(K)$$

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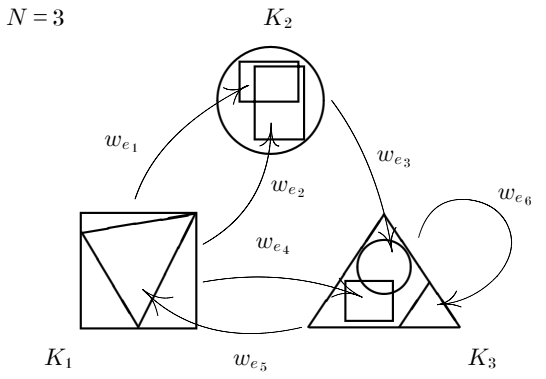


Fig. 1. A Markov system.

and its adjoint operator U^* on the set of all Borel probability measures $P(K)$ by

$$U^*\nu(f) := \int U(f)d\nu = \sum_{e\in E}\int_{K_{i(e)}} p_e f\circ w_e d\nu \quad \text{for all } f\in\mathcal{L}^0(K) \text{ and } \nu\in P(K).$$

Remark 1. Note that each map w_e and each probability p_e are required to be defined only on the corresponding vertex set $K_{i(e)}$. This is sufficient for the condition (1·1) and the definition of U^*. For the definition of U, we can consider each w_e to be extended on the whole space K arbitrarily and each p_e to be extended on K by zero.

Also, the situation applies where each vertex set K_i has its own metric d_i. In this case, one can set

$$d(x,y) = \begin{cases} d_i(x,y) & \text{if } x,y\in K_i \\ \infty & \text{otherwise} \end{cases}$$

and use the convention $0\times\infty = 0$.

We say $\mu\in P(K)$ is an invariant measure of the CMS if and only if $U^*\mu = \mu$. A Borel probability measure μ is called *attractive* measure of the CMS if

$$U^{*n}\nu \xrightarrow{w^*} \mu \quad \text{for all } \nu\in P(K),$$

where w^* means weak* convergence. Note that an attractive probability measure is a unique invariant probability measure of the CMS if U maps continuous functions on continuous functions.

It was shown in [**8**] that operator U^* inherits some properties from its trivial case of a finite Markov chain if the vertex sets $K_1,\dots,K_N$ form an open partition of the state space and the restrictions of the probability functions on their vertex sets are Dini-continuous and bounded away from zero. Namely, it has a unique invariant probability measure in an irreducible case and an attractive probability measure in an aperiodic case.

The coding map is an important tool in "fractal" geometry which allows one to represent a constructed set as an image of a code space under this map, that is, to code elements in this set by infinite sequences of elements of E. We shall denote $\Sigma := E^{\mathbb{Z}}$. We give some examples.

Example 1 (*decimal expansion*). Consider ten maps w_e, $e\in E := \{0,\dots,9\}$, on $([0,1],|.|)$ given by $w_e(x) := 1/10x + e/10$ for all $x\in[0,1]$. Obviously, for any family

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of probability functions p_e, $e \in E$, the family $([0,1], w_e, p_e)_{e\in E}$ is a CMS. Fix an $x \in [0,1]$. We define the coding map $F\colon \Sigma \to [0,1]$ by

$$F(\sigma) \coloneqq \lim_{m\to-\infty} w_{\sigma_0} \circ \cdots \circ w_{\sigma_m}(x) \quad \text{for all } \sigma \in \Sigma, \tag{1·2}$$

which is nothing else but the decimal expansion of real numbers from $[0,1]$.

Example 2 (*Cantor set*). Consider two maps w_e, $e \in E \coloneqq \{0,1\}$, on $([0,1], |.|)$ given by $w_e(x) \coloneqq 1/3x + e2/3$ for all $x \in [0,1]$. Again, for any family of probability functions p_e, $e \in E$, the family $([0,1], w_e, p_e)_{e\in E}$ is a CMS. Fix an $x \in [0,1]$. Then the coding map $F\colon \Sigma \to [0,1]$ given by the limit (1·2), for all $\sigma \in \Sigma$, is nothing else but the binary expansion of the Cantor set.

Example 3. Consider two maps w_e, $e \in E \coloneqq \{0,1\}$, on $(\mathbb{R}, |.|)$ given by $w_0(x) \coloneqq 1/2x$ and $w_1(x) \coloneqq -2x + 3$ for all $x \in \mathbb{R}$ with constant probability functions $p_0 \coloneqq 3/4$ and $p_1 \coloneqq 1/4$. A simple calculation shows that $([0,1], w_e, p_e)_{e\in E}$ defines a CMS with an average contraction rate $7/8$. Fix an $x \in \mathbb{R}$. It was shown in [**2**] that the coding map $F\colon \Sigma \to \mathbb{R}$ given by the limit (1·2) exists for P-a.e. $\sigma \in \Sigma$ and is independent of the choice of x modulo a P-zero set where P is a Bernoulli measure on Σ given by $P([e_1, e_2, \ldots, e_k]) = p_{e_1}p_{e_2}\cdots p_{e_k}$ for every cylinder set $[e_1, \ldots, e_k] \subset \Sigma$.

For a CMS $(K_{i(e)}, w_e, p_e)_{e\in E}$ with several vertex sets (as in Figure 1), we are going to define the coding map as follows. For each i, fix $x_i \in K_i$ and set

$$F(\sigma) \coloneqq \lim_{m\to-\infty} w_{\sigma_0} \circ \cdots \circ w_{\sigma_m}\big(x_{i(\sigma_m)}\big) \tag{1·3}$$

for $\sigma \in \Sigma$ if the limit exists. It is a simple exercise for the reader to show that $F(\sigma)$ exists for all $\sigma \in \Sigma_G \coloneqq \{\sigma \in \Sigma\colon t(\sigma_m) = i(\sigma_{m-1}) \forall m \in \mathbb{Z}\}$, it is independent of the choice of x_i's and F is Hölder-continuous if all maps $w_e|_{K_{i(e)}}$ are contractions (see [**5**] for the case $N = 1$).

We would like to illustrate it by the following example, which is associated with the notion of g-measures (see e.g. [**7**]).

Example 4. Let $G \coloneqq (V, E, i, t)$ be a finite irreducible directed (multi)graph. Let $\Sigma_G^- \coloneqq \{(\ldots, \sigma_{-1}, \sigma_0)\colon \sigma_m \in E \text{ and } t(\sigma_m) = i(\sigma_{m-1})\ \forall m \in \mathbb{Z} \setminus \mathbb{N}\}$ (*one-sided subshift of finite type* associated with G) endowed with the metric $d(\sigma, \sigma') \coloneqq 2^k$ where k is the smallest integer with $\sigma_i = \sigma_i'$ for all $k < i \leqslant 0$. Let g be a positive, Dini-continuous function on Σ_G such that

$$\sum_{y\in T^{-1}(\{x\})} g(y) = 1 \quad \text{for all } x \in \Sigma_G$$

where T is the right shift map on Σ_G^-. Set $K_i \coloneqq \{\sigma \in \Sigma_G^- : t(\sigma_0) = i\}$ for every $i \in V$ and, for every $e \in E$,

$$w_e(\sigma) \coloneqq (\ldots, \sigma_{-1}, \sigma_0, e),\ p_e(\sigma) \coloneqq g(\ldots, \sigma_{-1}, \sigma_0, e) \quad \text{for all } \sigma \in K_{i(e)}.$$

Obviously, maps $(w_e)_{e\in E}$ are contractions. Therefore, $(K_{i(e)}, w_e, p_e)_{e\in E}$ defines a CMS. In this example, the coding map $F\colon \Sigma_G \to \Sigma_G^-$ given by (1·3) is nothing else but the natural projection.

In this paper, we are concerned with the question whether limit (1·3) exists almost everywhere and is independent of the choice of x_i up to a set of measure zero with

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respect to a natural measure if the maps w_e are contractive only on average and the probabilities $p_e|_{K_{i(e)}}$ are place-dependent (see the next example). We show that this is true under some conditions on the probability functions.

Example 5. Let $\mathbb{R}^2$ be normed by $\|.\|_1$. Let $K_1 := \{(x,y) \in \mathbb{R}^2 : \ y \geqslant 1/2\}$ and $K_2 := \{(x,y) \in \mathbb{R}^2 : \ y \leqslant -1/2\}$. Consider the following maps on $\mathbb{R}^2$:

$$w_1\begin{pmatrix}x\\y\end{pmatrix} := \begin{pmatrix}-\frac{1}{2}x - 1\\ -\frac{3}{2}y + \frac{1}{4}\end{pmatrix}, \quad w_2\begin{pmatrix}x\\y\end{pmatrix} := \begin{pmatrix}-\frac{3}{2}x + 1\\ \frac{1}{4}y + \frac{3}{8}\end{pmatrix} \text{ and}$$

$$w_3\begin{pmatrix}x\\y\end{pmatrix} := \begin{pmatrix}-\frac{1}{2}|x| + 1\\ -\frac{3}{4}y + \frac{1}{8}\end{pmatrix}$$

with probability functions

$$p_1\begin{pmatrix}x\\y\end{pmatrix} := \left(\frac{1}{15}\sin^2\|(x,y)\|_1 + \frac{53}{105}\right)1_{K_1}(x,y),$$

$$p_2\begin{pmatrix}x\\y\end{pmatrix} := \left(\frac{1}{15}\cos^2\|(x,y)\|_1 + \frac{3}{7}\right)1_{K_1}(x,y) \text{ and } p_3 := 1_{K_2}.$$

A simple calculation shows that $(K_{i(e)}, w_e, p_e)_{e\in\{1,2,3\}}$, where $i(1) = 1$, $i(2) = 1$ and $i(3) = 2$, defines a CMS with an average contracting rate $209/210$. (By theorem 2 in [**8**], it has a unique (attractive) invariant probability measure.)

Our proof consists of two parts. In the first part, we construct a suitable outer measure on the code space and show that, for every CMS, limit (1·3) exists almost everywhere with respect to this outer measure and is independent of the choice of x_i's up to a set of measure zero. It involves some measure-theoretic technique which seems to be new. In the second part, we show that the natural measure on the code space, which we call the *generalized Markov measure*, is absolutely continuous with respect to the outer measure if the restrictions of probability functions on their vertex sets are Dini-continuous and bounded away from zero. This implies the desired result.

The coding map which we construct here opens a way for various applications. In [**9**], we compute the Kolmogorov–Sinai entropy $h_M(S)$ of the generalized Markov shift associated with the CMS (see Definition 6) using the coding map. We show that

$$h_M(S) = \sum_{e\in E}\int_{K_{i(e)}} p_e \log p_e \, d\mu,$$

where μ is a unique invariant Borel probability measure of the CMS. In [**10**], we prove an ergodic theorem for CMSs using the coding map. In [**11**], we show that the generalized Markov measure is a unique equilibrium state with respect to an energy function the construction of which involves the coding map.

2. *Construction with respect to an outer measure*

Let $(K_{i(e)}, w_e, p_e)_{e\in E}$ be a contractive Markov system with an average contracting rate $0 < a < 1$ and an invariant Borel probability measure μ. We assume metric space (K, d) to be complete, the set of edges E to be finite. We do not pose any conditions on the directed graph.

Let $\Sigma := \{(\ldots, \sigma_{-1}, \sigma_0, \sigma_1, \ldots) : e_i \in E \ \forall i \in \mathbb{Z}\}$. We shall denote by d' the metric on Σ defined by $d'(\sigma, \sigma') := (1/2)^k$ where k is the largest integer with $\sigma_i = \sigma'_i$ for all $|i| < k$.

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Let S be the left shift map on Σ. We call the set ${}_m[e_m, \dots, e_n] := \{\sigma \in \Sigma\colon\ \sigma_i = e_i$ for all $m \leqslant i \leqslant n\}$, for $m \leqslant n \in \mathbb{Z}$, a *cylinder*. Denote by $\mathcal{A}$ the finite σ-algebra generated by the zero time partition $\{{}_0[e]\colon e \in E\}$ of Σ, and define, for each integer $m \leqslant 1$,

$$\mathcal{A}_m := \bigvee_{i=m}^{+\infty} S^{-i}\mathcal{A},$$

which is the smallest σ-algebra containing all finite σ-algebras $\bigvee_{i=m}^{n} S^{-i}\mathcal{A}$, $n \geqslant m$. Let $x \in K$. For each integer $m \leqslant 1$ let P_x^m be the probability measure on the σ-algebra $\mathcal{A}_m$ given by

$$P_x^m({}_m[e_m, \dots, e_n]) = p_{e_m}(x)p_{e_{m+1}}\big(w_{e_m}(x)\big) \cdots p_{e_n}\big(w_{e_{n-1}} \circ \cdots \circ w_{e_m}(x)\big)$$

for all cylinder sets ${}_m[e_m, \dots, e_n]$, $n \geqslant m$.

Lemma 1. *Let $m \leqslant 1$ and $A \in \mathcal{A}_m$. Then $x \mapsto P_x^m(A)$ is a Borel measurable function on K.*

Proof. Set

$$\mathcal{D} := \{A \in \mathcal{A}_m : K \ni x \longmapsto P_x^m(A) \text{ Borel measurable}\}.$$

Then, by definition of P_x^m, $\mathcal{D}$ contains all cylinders of the form ${}_m[e_m, \dots, e_n], n \geqslant m$, which generate $\mathcal{A}_m$. Furthermore, obviously it holds true that

$$\Sigma \in \mathcal{D},$$
$$A \in \mathcal{D} \Rightarrow \Sigma \backslash A \in \mathcal{D}$$

and, for any pairwise disjoint family $(A_n)_{n\in\mathbb{N}} \subset \mathcal{D}$,

$$\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{D},$$

i.e. $\mathcal{D}$ is a Dynkin-system. Hence, $\mathcal{D}$ contains the Dynkin-system which is generated by the cylinders. Since the set of the cylinders is $\bigcap$-stable, it follows that $\mathcal{D} = \mathcal{A}_m$.

Definition 1. Let $\nu \in P(K)$. We call a probability measure $\Phi_m(\nu)$ on $(\Sigma, \mathcal{A}_m)$ given by

$$\Phi_m(\nu)(A) := \int P_x^m(A)\, d\nu(x),\ A \in \mathcal{A}_m,$$

the m-lift of ν.

Definition 2. Set

$$\mathcal{C}(B) := \left\{(A_m)_{m=0}^{-\infty} : A_m \in \mathcal{A}_m\ \forall m \text{ and } B \subset \bigcup_{m=0}^{-\infty} A_m\right\}$$

for $B \subset \Sigma$. Let $\nu \in P(K)$. We call a set function given by

$$\Phi(\nu)(B) := \inf\left\{\sum_{m=0}^{-\infty} \Phi_m(\nu)(A_m) : (A_m)_{m\leqslant 0} \in \mathcal{C}(B)\right\}, B \subset \Sigma,$$

the lift of ν.

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LEMMA 2. *Let $\nu, \lambda \in P(K)$. Then*

(i) *$\Phi(\nu)$ is an outer measure on Σ.*

(ii) *If $\Phi_m(\nu) \ll \Phi_m(\lambda)$ for all $m \leqslant 0$, then for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\Phi(\lambda)(B) < \delta \;\Rightarrow\; \Phi(\nu)(B) < \epsilon \text{ for all } B \subset \Sigma.$$

Proof. It is obvious that $\Phi(\nu)(\varnothing) = 0$.

Let $B_1 \subset B_2 \subset \Sigma$. Then $\mathcal{C}(B_1) \supset \mathcal{C}(B_2)$ and therefore

$$\Phi(\nu)(B_1) \leqslant \Phi(\nu)(B_2).$$

Now, we show

$$\Phi(\nu)\left(\bigcup_{i=1}^{\infty} B_i\right) \leqslant \sum_{i=1}^{\infty} \Phi(\nu)(B_i)$$

for all $B_i \subset \Sigma$, $i \in \mathbb{N}$. We can assume that the right-hand side is finite. Let $\epsilon > 0$. Then for every $i \in \mathbb{N}$ there exists $(A_{im})_{m \leqslant 0} \in \mathcal{C}(B_i)$ such that

$$\Phi(\nu)(B_i) > \sum_{m=0}^{-\infty} \Phi_m(\nu)(A_{im}) - \epsilon 2^{-i}.$$

Since $(\bigcup_{i=1}^{\infty} A_{im})_{m \leqslant 0} \in \mathcal{C}(\bigcup_{i=1}^{\infty} B_i)$, it follows that

$$\begin{aligned}
\Phi(\nu)\left(\bigcup_{i=1}^{\infty} B_i\right) &\leqslant \sum_{m=0}^{-\infty} \Phi_m(\nu)\left(\bigcup_{i=1}^{\infty} A_{im}\right) \\
&\leqslant \sum_{m=0}^{-\infty} \sum_{i=1}^{\infty} \Phi_m(\nu)(A_{im}) \\
&\leqslant \sum_{i=1}^{\infty} \Phi(\nu)(B_i) + \epsilon,
\end{aligned}$$

as desired.

Suppose the claim in (ii) is not true. Then there exist $\epsilon > 0$ and a sequence of subsets $(B_n)_{n \in \mathbb{N}}$ such that

$$\Phi(\lambda)(B_n) < 2^{-n} \text{ and } \Phi(\nu)(B_n) \geqslant \epsilon \text{ for all } n \in \mathbb{N}.$$

Then for every $n \in \mathbb{N}$ there exist $(A_{nm})_{m \leqslant 0} \in \mathcal{C}(B_n)$ such that

$$\sum_{m=0}^{-\infty} \Phi_m(\lambda)(A_{nm}) < 2^{-n}.$$

Set

$$D_m := \bigcap_{k=1}^{\infty} \bigcup_{n \geqslant k} A_{nm} \text{ for each } m \leqslant 0.$$

Then, for each $m \leqslant 0$,

$$\Phi_m(\lambda)(D_m) \leqslant \Phi_m(\lambda)\left(\bigcup_{n \geqslant k} A_{nm}\right) \leqslant \sum_{n \geqslant k} 2^{-n}$$

for all $k \geqslant 0$. Hence $\Phi_m(\lambda)(D_m) = 0$ for all $m \leqslant 0$. Since $\Phi_m(\nu) \ll \Phi_m(\lambda)$ for all $m \leqslant 0$,

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this implies that

$$0 = \Phi_m(\nu)(D_m) = \lim_{k\to\infty} \Phi_m(\nu)\left(\bigcup_{n\geqslant k} A_{nm}\right) \geqslant \limsup_{k\to\infty} \Phi_m(\nu)(A_{km})$$

for all $m \leqslant 0$. Hence

$$\limsup_{k\to\infty} \Phi(\nu)(B_k) \leqslant \limsup_{k\to\infty} \sum_{m=0}^{-\infty} \Phi_m(\nu)(A_{km}) = 0,$$

which is a contradiction.

We use further the following notation.

Notation 1. Fix $x_i \in K_i$ for each $i \in \{1,\dots,N\}$ and set

$$P^m_{x_1\dots x_N} := \Phi_m\left(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}\right) \text{ and } P_{x_1\dots x_N} := \Phi\left(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}\right)$$

for every $m \in \mathbb{Z}\backslash\mathbb{N}$, where δ_x denotes the Dirac probability measure concentrated at x. Then

$$P^m_{x_1\dots x_N}({}_m[e_m,\dots,e_n]) = \frac{1}{N} p_{e_m}\left(x_{i(e_m)}\right) p_{e_{m+1}}\left(w_{e_m} x_{i(e_m)}\right)\cdots p_{e_n}\left(w_{e_{n-1}}\circ\cdots\circ w_{e_m} x_{i(e_m)}\right)$$

for every cylinder set ${}_m[e_m,\dots,e_n]$.

Now, for every $m \leqslant 0$ and $n \geqslant m$ define a random variable

$$\begin{aligned} Y^{x_1\dots x_N}_{mn} : \Sigma &\longrightarrow K \\ \sigma &\longmapsto w_{\sigma_n}\circ w_{\sigma_{n-1}}\circ\cdots\circ w_{\sigma_m}\left(x_{i(\sigma_m)}\right) \end{aligned}$$

with respect to the measure $P^m_{x_1\dots x_N}$.

Now, we are going to prove the main lemma which enables us to define the coding map. The proof of it involves a more general version of Borel–Cantelli argument than that which was used by Barnsley and Elton in [**2**] (they considered the case $N = 1$ with constant probabilities). Their key point, the reversion of the order of finite sequences of the maps, does not work here because $P^m_{x_1\dots x_N}({}_m[e_m,e_{m+1},\dots,e_n]) \neq P^m_{x_1\dots x_N}({}_m[e_n,e_{n-1},\dots,e_m])$ in general. That is why we first needed to construct the outer measure $P_{x_1\dots x_N}$. The kind of Borel–Cantelli argument which we apply here seems to be new in its generality.

Lemma 3. *Let $x_i, y_i \in K_i$ for each $1 \leqslant i \leqslant N$.*

(i)
$$\lim_{m\to-\infty} d\left(Y^{x_1\dots x_N}_{m0}, Y^{y_1\dots y_N}_{m0}\right) = 0 \ P_{x_1\dots x_N}\text{-a.e.},$$

(ii)
$$F_{x_1\dots x_N} := \lim_{m\to-\infty} Y^{x_1\dots x_N}_{m0} \text{ exists } P_{x_1\dots x_N}\text{-a.e.},$$

and by (i) $F_{x_1\dots x_N} = F_{y_1\dots y_N}$ $P_{x_1\dots x_N}$-*a.e.*

(iii) *There exists a sequence of closed subsets $Q_1 \subset Q_2 \subset \cdots \subset \Sigma$ with $\sum_{k=1}^\infty P_{x_1\dots x_N}(\Sigma\backslash Q_k) < \infty$ such that $F_{x_1\dots x_N}|_{Q_k}$ is locally Hölder-continuous with the same Hölder-constants for all $k \in \mathbb{N}$, i.e. there exist $\alpha, C > 0$ such that for every k there exists $\delta_k > 0$ such that*

$$\sigma,\sigma' \in Q_k \text{ with } d'(\sigma,\sigma') \leqslant \delta_k \Longrightarrow d\left(F_{x_1\dots x_N}(\sigma), F_{x_1\dots x_N}(\sigma')\right) \leqslant C d'(\sigma,\sigma')^\alpha.$$

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Proof. Applying the average contractiveness condition $-m+1$ times gives

$$\begin{aligned}\sum_{e_m,\dots,e_0} &\frac{1}{N} p_{e_m}\big(x_{i(e_m)}\big)\cdots p_{e_0}\big(w_{e_{-1}}\circ\cdots\circ w_{e_m}\big(x_{i(e_m)}\big)\big)\\ &\times d\big(w_{e_0}\circ\cdots\circ w_{e_m}\big(x_{i(e_m)}\big), w_{e_0}\circ\cdots\circ w_{e_m}\big(y_{i(e_m)}\big)\big)\\ &\leqslant a^{-m+1}\frac{1}{N}\sum_{i=1}^{N} d(x_i,y_i),\end{aligned}$$

i.e.

$$\int d\big(Y_{m0}^{x_1\dots x_N}, Y_{m0}^{y_1\dots y_N}\big)\, dP_{x_1\dots x_N}^m \leqslant a^{-m+1}\frac{1}{N}\sum_{i=1}^{N} d(x_i,y_i).$$

So, by the Markov inequality,

$$P_{x_1\dots x_N}^m\left(d\big(Y_{mn}^{x_1\dots x_N}, Y_{mn}^{y_1\dots y_N}\big) > a^{\frac{-m+1}{2}}\frac{1}{N}\sum_{i=1}^{N} d(x_i,y_i)\right) \leqslant a^{\frac{-m+1}{2}}.$$

Set $A_m := \{\sigma\in\Sigma\colon d(Y_{m0}^{x_1\dots x_N}(\sigma), Y_{m0}^{y_1\dots y_N}(\sigma)) > a^{\frac{-m+1}{2}}\frac{1}{N}\sum_{i=1}^{N} d(x_i,y_i)\}$ and

$$A := \bigcap_{l=0}^{-\infty}\bigcup_{m=l}^{-\infty} A_m.$$

Then

$$P_{x_1\dots x_N}(A) \leqslant P_{x_1\dots x_N}\left(\bigcup_{m=l}^{-\infty} A_m\right) \leqslant \sum_{m=l}^{-\infty} P_{x_1\dots x_N}^m(A_m) \leqslant \sum_{m=l}^{-\infty} a^{\frac{-m+1}{2}},$$

since $(\varnothing,\dots,\varnothing,A_l,A_{l-1},\dots)\in\mathcal{C}(\bigcup_{m=l}^{-\infty} A_m)$ for all $l\leqslant 0$. Hence $P_{x_1\dots x_N}(A)=0$ and for every $\sigma\in\Sigma\backslash A$

$$d\left(Y_{m0}^{x_1\dots x_N}(\sigma), Y_{m0}^{y_1\dots y_N}(\sigma)\right) \leqslant a^{\frac{-m+1}{2}}\frac{1}{N}\sum_{i=1}^{N} d(x_i,y_i)$$

for all m except finitely many. This implies (i).

Now, for part (ii) set $C := \max_{e\in E} d(x_{t(e)}, w_e(x_{i(e)}))$. Then applying the average contractiveness condition $-m+1$ times reveals that

$$\begin{aligned}\int d\left(Y_{m0}^{x_1\dots x_N}, Y_{(m-1)0}^{x_1\dots x_N}\right) dP_{x_1\dots x_N}^m &\leqslant a^{-m+1}\frac{1}{N}\sum_{e\in E} p_e\big(x_{i(e)}\big)\, d\big(x_{t(e)}, w_e\big(x_{i(e)}\big)\big)\\ &\leqslant a^{-m+1}C.\end{aligned}$$

So, by the Markov inequality,

$$P_{x_1\dots x_N}^m\left(d\big(Y_{m0}^{x_1\dots x_N}, Y_{(m-1)0}^{x_1\dots x_N}\big) > a^{\frac{-m+1}{2}}C\right) \leqslant a^{\frac{-m+1}{2}}.$$

Set $B_m := \{\sigma\in\Sigma\colon d(Y_{m0}^{x_1\dots x_N}(\sigma), Y_{(m-1)0}^{x_1\dots x_N}(\sigma)) > a^{\frac{-m+1}{2}}C\}$ and

$$B := \bigcap_{l=0}^{-\infty}\bigcup_{m=l}^{-\infty} B_m.$$

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Then

$$P_{x_1\dots x_N}(B) \leqslant P_{x_1\dots x_N}\left(\bigcup_{m=l}^{-\infty} B_m\right) \leqslant \sum_{m=l}^{-\infty} P^m_{x_1\dots x_N}(B_m) \leqslant \sum_{m=l}^{-\infty} a^{\frac{-m+1}{2}},$$

since $(\varnothing,\dots,\varnothing,B_l,B_{l-1},\dots)\in\mathcal{C}(\bigcup_{m=l}^{-\infty}B_m)$ for all $l\leqslant 0$. Hence $P_{x_1\dots x_N}(B)=0$ and, for every $\sigma\in\Sigma\setminus B$,

$$\sum_{m=0}^{-\infty} d\big(Y^{x_1\dots x_N}_{m0}(\sigma), Y^{x_1\dots x_N}_{(m-1)0}(\sigma)\big) < \infty.$$

This implies that $(Y^{x_1\dots x_N}_{m0}(\sigma))_{m\in\mathbb{Z}\setminus\mathbb{N}}$ is a Cauchy sequence for $P_{x_1\dots x_N}$-a.e. $\sigma\in\Sigma$, and so $F_{x_1\dots x_N} := \lim_{m\to-\infty} Y^{x_1\dots x_N}_{m0}$ exists $P_{x_1\dots x_N}$-a.e.

(iii) Let

$$Q_l := \Sigma\Big\setminus \bigcup_{m=l}^{-\infty} B_m \text{ for all } l\in\mathbb{Z}\setminus\mathbb{N}.$$

Then $Q_l\subset Q_{l-1}$ for all $l\in\mathbb{Z}\setminus\mathbb{N}$. Since every $Y^{x_1\dots x_N}_{m0}$ is continuous, B_m is open for all $m\in\mathbb{Z}\setminus\mathbb{N}$, and so, every Q_l is closed. By the above,

$$\sum_{l=0}^{-\infty} P_{x_1\dots x_N}(\Sigma\setminus Q_l) = \sum_{l=0}^{-\infty}\sum_{m=l}^{-\infty} a^{\frac{-m+1}{2}} < \infty.$$

Now, let $\sigma,\sigma'\in Q_l$ for some l. Then

$$d\big(Y^{x_1\dots x_N}_{m0}(\sigma), Y^{x_1\dots x_N}_{(m-1)0}(\sigma)\big) \leqslant a^{\frac{-m+1}{2}} C \text{ for all } m\leqslant l.$$

Therefore, by the triangle inequality,

$$d\big(Y^{x_1\dots x_N}_{m0}(\sigma), Y^{x_1\dots x_N}_{(m-k)0}(\sigma)\big) \leqslant \sum_{i=m}^{-\infty} a^{\frac{-i+1}{2}} C = C\frac{1}{1-\sqrt{a}}\, a^{\frac{|m-1|}{2}} \text{ for all } m\leqslant l,\ k\geqslant 1.$$

Hence

$$d\left(Y^{x_1\dots x_N}_{m0}(\sigma), F_{x_1\dots x_N}(\sigma)\right) \leqslant C\frac{1}{1-\sqrt{a}}\, a^{\frac{|m-1|}{2}} \text{ for all } m\leqslant l.$$

Analogously,

$$d\left(Y^{x_1\dots x_N}_{m0}(\sigma'), F_{x_1\dots x_N}(\sigma')\right) \leqslant C\frac{1}{1-\sqrt{a}}\, a^{\frac{|m-1|}{2}} \text{ for all } m\leqslant l.$$

Now, let $\delta_l=(1/2)^{|l-1|}$, $\alpha := \log\sqrt{a}/\log(1/2)$ and $d'(\sigma,\sigma')=(1/2)^{|m-1|}$ for some $m\leqslant l$. Then $Y^{x_1\dots x_N}_{m0}(\sigma')=Y^{x_1\dots x_N}_{m0}(\sigma)$. Therefore,

$$d\left(F_{x_1\dots x_N}(\sigma), F_{x_1\dots x_N}(\sigma')\right) \leqslant \frac{2C}{1-\sqrt{a}}\, a^{\frac{|m-1|}{2}} = \frac{2C}{1-\sqrt{a}}\, d'(\sigma,\sigma')^{\alpha}.$$

Definition 3. We call the map

$$F_{x_1\dots x_N}:\Sigma\longrightarrow K,$$

which is defined $P_{x_1\dots x_N}$-a.e. by Lemma 3, the *coding map* of the CMS.

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3. *Definition with respect to a generalized Markov measure*

Our next aim is to show that the coding map is defined almost everywhere with respect to any outer measure $\Phi(\nu)$ if each probability function $p_e|_{K_{i(e)}}$ is Dini-continuous and bounded away from zero by $\delta > 0$. For that, we only need to establish that $\Phi(\nu)$ is absolutely continuous with respect to $P_{x_1 \dots x_N}$ in this case.

Definition 4. We call a function $f\colon (X,d) \to \mathbb{R}$ *Dini-continuous* if and only if for some $c > 0$

$$\int_0^c \frac{\phi(t)}{t}\, dt < \infty$$

where ϕ is *the modulus of uniform continuity* of f, i.e.

$$\phi(t) := \sup\{|f(x) - f(y)| : d(x,y) \leqslant t,\ x, y \in X\}.$$

It is easily seen that the Dini-continuity is weaker than the Hölder and stronger than the uniform continuity. There is a well known characterization of the Dini-continuity, which will be useful later.

LEMMA 4. *Let $0 < c < 1$ and $b > 0$. A function f is Dini-continuous if and only if*

$$\sum_{n=0}^{\infty} \phi(bc^n) < \infty$$

where ϕ is the modulus of uniform continuity of f.

The proof is simple (e.g. see [**8**]).

Set $Z_{mn}^x(\sigma) := w_{\sigma_n} \circ \cdots \circ w_{\sigma_m}(x)$ for all $x \in K$, $m \leqslant n \in \mathbb{Z}$ and $\sigma \in \Sigma$.

LEMMA 5. *Let $x_i \in K_i$ for every $1 \leqslant i \leqslant N$ and $x \in K$. Let $i_0 \in \{1, \dots, N\}$ such that $x \in K_{i_0}$. Then for all integers $m \leqslant 0$ and for all $\epsilon > 0$ there exist $k \geqslant m$ and $B \in \mathcal{A}_m$ such that $P_x^m(B) < \epsilon$ and*

$$n \geqslant k \Rightarrow d\big(Z_{mn}^x(\sigma), Y_{mn}^{x_1 \dots x_N}(\sigma)\big) \leqslant a^{\frac{n-m+1}{2}} d(x, x_{i_0})$$

for all $\sigma \in \Sigma \setminus B$.

Proof. Fix $m \leqslant 0$. Applying the average contractiveness condition $n - m + 1$ times gives

$$\sum_{e_m, \dots, e_n} p_{e_m}(x) \cdots p_{e_n}\big(w_{e_{n-1}} \circ \cdots \circ w_{e_m}(x)\big) \times d\big(w_{e_n} \circ \cdots \circ w_{e_m}(x), w_{e_n} \circ \cdots \circ w_{e_m}\big(x_{i(e_m)}\big)\big) \leqslant a^{n-m+1} d\big(x, x_{i_0}\big),$$

i.e.

$$\int d\big(Z_{mn}^x, Y_{mn}^{x_1 \dots x_N}\big)\, dP_x^m \leqslant a^{n-m+1} d(x, x_{i_0}).$$

So, by the Markov inequality,

$$P_x^m\Big(d\big(Z_{mn}^x, Y_{mn}^{x_1 \dots x_N}\big) > a^{\frac{n-m+1}{2}} d(x, x_{i_0})\Big) \leqslant a^{\frac{n-m+1}{2}}.$$

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Set $A_{mn} := \{\sigma \in \Sigma\colon\ d(Z^x_{mn}(\sigma), Y^{x_1\dots x_N}_{mn}(\sigma)) > a^{\frac{n-m+1}{2}} d(x, x_{i_0})\}$ for $m \leqslant n$ and $B_k := \bigcup_{n\geqslant k}^{\infty} A_{mn}$ for $k \geqslant m$. Then

$$\sum_{n=m}^{\infty} P^m_x(A_{mn}) \leqslant \sum_{l=1}^{\infty} a^{\frac{l}{2}} < \infty.$$

Therefore

$$P^m_x\left(\bigcap_{k=m}^{\infty} B_k\right) = 0,$$

by the Borel–Cantelli argument. Hence, for all $\epsilon > 0$ there exists $k \geqslant m$ such that $P^m_x(B_k) < \epsilon$ and

$$n \geqslant k \Rightarrow d\big(Z^x_{mn}(\sigma), Y^{x_1\dots x_N}_{mn}(\sigma)\big) \leqslant a^{\frac{n-m+1}{2}} d(x, x_{i_0})$$

for all $\sigma \in \Sigma \setminus B_k$.

The next lemma is a generalization of lemma 3 in [**3**]. The proof of it is that of Elton.

Lemma 6. *Suppose that $p_e|_{K_{i(e)}}$ is Dini-continuous and there exists $\delta > 0$ such that $p_e|_{K_{i(e)}} \geqslant \delta$ for all $e \in E$. Let $x_i \in K_i$ for all $1 \leqslant i \leqslant N$ and $x \in K$. Then P^m_x is absolutely continuous with respect to $P^m_{x_1\dots x_N}$ for all $m \leqslant 0$.*

Proof. Fix $m \leqslant 0$. Let $A \in \mathcal{A}_m$ be such that $P^m_{x_1\dots x_N}(A) = 0$ and $\epsilon > 0$. We show $P^m_x(A) < \epsilon$.

Let $i_0 \in \{1, \dots, N\}$ such that $x \in K_{i_0}$. By Lemma 5, there exists $n_\epsilon \geqslant m$ and $B \in \mathcal{A}_m$ such that $P^m_x(B) < \epsilon/2$ and

$$n \geqslant n_\epsilon \Rightarrow d\big(Z^x_{mn}(\sigma), Y^{x_1\dots x_N}_{mn}(\sigma)\big) \leqslant a^{\frac{n-m+1}{2}} d(x, x_{i_0})$$

for all $\sigma \in \Sigma\setminus B$. Let ϕ_e be the modulus of uniform continuity of $p_e|_{K_{i(e)}}$ for each $e \in E$ and $\phi := \max_{e\in E} \phi_e$. Since each $p_e|_{K_{i(e)}}$ is Dini-continuous, by Lemma 4, we can choose $l \geqslant n_\epsilon$ such that $\sum_{k=l+1}^{\infty} \phi(a^{(k-m)/2} d(x, x_{i_0})) < \delta/2$. Set

$$Q_n := \begin{cases} \left\{\sigma \in \Sigma : d\big(Z^x_{mk}(\sigma), Y^{x_1\dots x_N}_{mk}(\sigma)\big) \leqslant a^{\frac{k-m+1}{2}} d(x, x_{i_0})\ \forall l \leqslant k \leqslant n\right\}, & \text{if } l \leqslant n \\ \Sigma, \text{ else} \end{cases}$$

and $Q := \bigcap_{n\geqslant m} Q_n$. Then $\Sigma\setminus B \subset Q$ and therefore $P_x(\Sigma\setminus Q) < \epsilon/2$. Now, for $\sigma \in Q_n$, if $l \leqslant n$ and $(\sigma_m, \dots, \sigma_n)$ is a path of the digraph starting in i_0, then

$$\begin{aligned}
&p_{\sigma_m}(x) \cdots p_{\sigma_n}\left(w_{\sigma_{n-1}} \circ \cdots \circ w_{\sigma_m} x\right) \\
&\quad \leqslant p_{\sigma_m}\left(x_{i(\sigma_m)}\right) \cdots p_{\sigma_n}\left(w_{\sigma_{n-1}} \circ \cdots \circ w_{\sigma_m} x_{i(\sigma_m)}\right) \left(\frac{1-\delta}{\delta}\right)^{l-m+1} \\
&\qquad \times \prod_{k=l+1}^{n} \left[1 + \frac{p_{\sigma_k}\left(w_{\sigma_{k-1}} \circ \cdots \circ w_{\sigma_m} x\right) - p_{\sigma_k}\left(w_{\sigma_{k-1}} \circ \cdots \circ w_{\sigma_m} x_{i(\sigma_m)}\right)}{p_{\sigma_k}\left(w_{\sigma_{k-1}} \circ \cdots \circ w_{\sigma_m} x_{i(\sigma_m)}\right)}\right] \\
&\quad \leqslant p_{\sigma_m}\left(x_{i(\sigma_m)}\right) \cdots p_{\sigma_n}\left(w_{\sigma_{n-1}} \circ \cdots \circ w_{\sigma_m} x_{i(\sigma_m)}\right) \left(\frac{1-\delta}{\delta}\right)^{l-m+1} \\
&\qquad \times \prod_{k=l+1}^{n} \left[1 + \frac{\phi\left(a^{\frac{k-m}{2}} d(x, x_{i_0})\right)}{\delta}\right].
\end{aligned}$$

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Since $\prod_{k=l+1}^{n}[1+\phi(a^{(k-m)/2}\,d(x,x_{i_0}))/\delta]\leqslant 1+2\sum_{k=l+1}^{\infty}\phi(a^{(k-m)/2}d(x,x_{i_0}))/\delta\leqslant 2$, it follows that

$$\begin{aligned}&p_{\sigma_m}(x)\cdots p_{\sigma_n}\left(w_{\sigma_{n-1}}\circ\cdots\circ w_{\sigma_m}x\right)\\&\quad\leqslant 2N\left(\frac{1-\delta}{\delta}\right)^{l-m+1}\frac{1}{N}p_{\sigma_m}\left(x_{i(\sigma_m)}\right)\cdots p_{\sigma_n}\left(w_{\sigma_{n-1}}\circ\cdots\circ w_{\sigma_m}x_{i(\sigma_m)}\right).\end{aligned}$$

If $l>n$ or $(\sigma_m,\ldots,\sigma_n)$ is not a path of the digraph starting in i_0, then it holds trivially for any $\sigma\in\Sigma$.

Let $\triangle_m$ be the algebra every element of which is a finite union of cylinders of the form ${}_m[e_m,\ldots,e_n]$, $n\geqslant m$. By Caratheodory construction, there exists a sequence $(A_k)_{k\in\mathbb{N}}\subset\triangle_m$ such that $A\subset\bigcup_{k=1}^{\infty}A_k$ and

$$\sum_{k=1}^{\infty}P^m_{x_1\cdots x_N}(A_k)<\frac{\epsilon}{4N}\left(\frac{\delta}{1-\delta}\right)^{l-m+1}.$$

We can write every finite union $\bigcup_{k=1}^{n}A_k$ as a disjoint union $\biguplus_{k=1}^{m_n}C_k$ of cylinders which generate $\triangle_m$. Let $C_k={}_m[e_m,\ldots,e_n]$ with $m\leqslant n$. Then

$$\begin{aligned}P^m_x(Q\cap C_k)&\leqslant P^m_x(Q_n\cap C_k)\\&=\sum_{(\sigma_m,\ldots,\sigma_n):\sigma\in Q_n\cap C_k}p_{\sigma_m}(x)\cdots p_{\sigma_n}\left(w_{\sigma_{n-1}}\circ\cdots\circ w_{\sigma_m}x\right)\\&\leqslant 2N\left(\frac{1-\delta}{\delta}\right)^{l-m+1}\sum_{(\sigma_m,\ldots,\sigma_n):\sigma\in Q_n\cap C_k}\frac{1}{N}\\&\quad\times p_{\sigma_m}\left(x_{i(\sigma_m)}\right)\cdots p_{\sigma_n}\left(w_{\sigma_{n-1}}\circ\cdots\circ w_{\sigma_m}x_{i(\sigma_m)}\right)\\&\leqslant 2N\left(\frac{1-\delta}{\delta}\right)^{l-m+1}P^m_{x_1\ldots x_n}(C_k).\end{aligned}$$

Hence

$$\begin{aligned}P^m_x(A)&=P^m_x(Q\cap A)+P^m_x(A\setminus Q)\\&\leqslant\lim_{n\to\infty}P^m_x\left(\biguplus_{k=1}^{m_n}C_k\cap Q\right)+\frac{\epsilon}{2}\\&=\lim_{n\to\infty}\sum_{k=1}^{m_n}P^m_x(C_k\cap Q)+\frac{\epsilon}{2}\\&\leqslant 2N\lim_{n\to\infty}\left(\frac{1-\delta}{\delta}\right)^{l-m+1}\sum_{k=1}^{m_n}P^m_{x_1\ldots x_n}(C_k)+\frac{\epsilon}{2}\\&\leqslant 2N\left(\frac{1-\delta}{\delta}\right)^{l-m+1}\sum_{k=1}^{\infty}P^m_{x_1\ldots x_n}(A_k)+\frac{\epsilon}{2}\\&<\epsilon.\end{aligned}$$

THEOREM 1. *Suppose that $p_e|_{K_{i(e)}}$ is Dini-continuous and there exists $\delta>0$ such that $p_e|_{K_{i(e)}}\geqslant\delta$ for all $e\in E$. Let $x_i,y_i\in K_i$ for all $1\leqslant i\leqslant N$ and $\nu\in P(K)$. Then*:

(i) *$F_{x_1\ldots x_N}$ is defined $\Phi(\nu)$-a.e;*

(ii) *$F_{x_1\ldots x_N}=F_{y_1\ldots y_N}$ $\Phi(\nu)$-a.e; and*

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(iii) *there exists a sequence of closed subsets* $Q_1 \subset Q_2 \subset \cdots \subset \Sigma$ *with*

$$\lim_{k\to\infty} \Phi(\nu)(\Sigma \backslash Q_k) = 0$$

such that all $F_{x_1\ldots x_N}|_{Q_k}$ *are locally Hölder-continuous with the same Hölder-constants.*

Proof. By Lemma 6, $\Phi_m(\nu)$ is absolutely continuous with respect to $P^m_{x_1\ldots x_N}$ for all $m \leqslant 0$. By Lemma 2 (ii), this implies that $\Phi(\nu)$ is absolutely continuous with respect to $P_{x_1\ldots x_N}$. The claim follows by Lemma 3.

Definition 5. We call

$$M := \Phi(\mu)$$

a *generalized Markov measure*, where μ is the invariant Borel probability measure of the CMS. Denote the Borel σ-algebra on Σ by $\mathcal{B}(\Sigma)$.

Proposition 1. *M is a shift invariant Borel probability measure on Σ with*

$$M({}_m[e_1,\ldots,e_k]) := \int p_{e_1}(x)p_{e_2}(w_{e_1}x)\cdots p_{e_k}\left(w_{e_{k-1}}\circ\cdots\circ w_{e_1}x\right) d\mu(x)$$

for every cylinder set ${}_m[e_1,\ldots,e_k] \subset \Sigma$.

Proof. First, define a set function $\bar{M}$ on all cylinders of Σ by

$$\bar{M}({}_m[e_m,\ldots,e_n]) := \Phi_m(\mu)({}_m[e_m,\ldots,e_n]).$$

We show that $\bar{M}$ extends uniquely to a shift invariant Borel probability measure on Σ and

$$\bar{M}|_{\mathcal{A}_m} = \Phi_m(\mu) \text{ for all } m \leqslant 0.$$

We only need to check that

$$\bar{M}({}_m[e_m,\ldots,e_n]) = \sum_{e_{n+1}} \bar{M}({}_m[e_m,\ldots,e_n,e_{n+1}])$$

and that

$$\bar{M}({}_m[e_m,\ldots,e_n]) = \sum_{e_{m-1}} \bar{M}({}_{m-1}[e_{m-1},e_m,\ldots,e_n]),$$

the rest follows by the standard extension argument. The first equation is obvious by the definition of P_x. For the second we need the invariance of μ.

$$\begin{aligned}
&\sum_{e_{m-1}} \bar{M}({}_{m-1}[e_{m-1},e_m,\ldots,e_n]) \\
&\quad= \sum_{e_{m-1}} \int p_{e_{m-1}}(x)p_{e_m}\left(w_{e_{m-1}}(x)\right)\cdots p_{e_n}\left(w_{e_{n-1}}\circ\cdots\circ w_{e_{m-1}}(x)\right) d\mu(x) \\
&\quad= U^*\mu\left(p_{e_m}\cdots p_{e_n}\circ w_{e_{n-1}}\circ\cdots\circ w_{e_m}\right) \\
&\quad= \mu\left(p_{e_m}\cdots p_{e_n}\circ w_{e_{n-1}}\circ\cdots\circ w_{e_m}\right) \\
&\quad= \bar{M}({}_m[e_m,\ldots,e_n]).
\end{aligned}$$

Now, we show that

$$\bar{M} = M|_{\mathcal{B}(\Sigma)}.$$

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Let $B \in \mathcal{B}(\Sigma)$ and $\epsilon > 0$. Since $\mathcal{B}(\Sigma)$ is the smallest σ-algebra containing all $\mathcal{A}_m$, $m \leqslant 0$, it follows that for every $(A_m)_{m\leqslant 0} \in \mathcal{C}(B)$

$$\bar{M}(B) \leqslant \bar{M}\left(\bigcup_{m\leqslant 0} A_m\right) \leqslant \sum_{m\leqslant 0} \bar{M}(A_m) = \sum_{m\leqslant 0} \Phi_m(\mu)(A_m).$$

Hence $\bar{M}(B) \leqslant M(B)$. On the other hand, let $\triangle$ be the algebra generated by all cylinders in Σ. Then every $A \in \triangle$ is also an element of some $\mathcal{A}_m$, $m \leqslant 0$, and therefore $\bar{M}(A) \geqslant M(A)$. By Caratheodory construction, there exists a sequence $(A_k)_{k\in\mathbb{N}} \subset \triangle$ such that $B \subset \bigcup_{k\in\mathbb{N}} A_k$ and

$$\bar{M}(B) > \sum_{k\in\mathbb{N}} \bar{M}(A_k) - \epsilon.$$

Therefore

$$\bar{M}(B) \geqslant \sum_{k\in\mathbb{N}} M(A_k) - \epsilon \geqslant M\left(\bigcup_{k\in\mathbb{N}} A_k\right) - \epsilon \geqslant M(B) - \epsilon.$$

Hence, $\bar{M}(B) = M(B)$.

Definition 6. We call the measure preserving transformation S of the probability space $(\Sigma, \mathcal{B}(\Sigma), M)$ a *generalized Markov shift*.

Now, we state explicitly the important special case of Theorem 1.

COROLLARY 1. *Suppose that $p_e|_{K_{i(e)}}$ is Dini-continuous and there exists $\delta > 0$ such that $p_e|_{K_{i(e)}} \geqslant \delta$ for all $e \in E$. Let $x_i, y_i \in K_i$ for all $1 \leqslant i \leqslant N$. Then*:

(i) *$F_{x_1\ldots x_N}$ is defined M-a.e*;

(ii) *$F_{x_1\ldots x_N} = F_{y_1\ldots y_N}$ M-a.e, and*

(iii) *there exists a sequence of closed subsets $Q_1 \subset Q_2 \subset \cdots \subset \Sigma$ with $\lim_{k\to\infty} M(Q_k) = 1$ such that all $F_{x_1\ldots x_N}|_{Q_k}$ are locally Hölder-continuous with the same Hölder-constants.*

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