THE RELATIVISTICALLY RIGID ROD

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ABSTRACT. A relativistically rigid rod is defined as one along which any disturbance travels with the fundamental speed c without dissipation of energy. A necessary consequence of this definition is that the rod should satisfy a certain 'equation of elasticity'. If this equation is satisfied, it is shown in accordance with relativistic mechanics that a disturbance of any amplitude is propagated along the (unstressed or stressed) rod with speed c and without dissipation. Consequently, the equation provides also a sufficient condition for rigidity in the sense defined.

1. *Introduction.* The term *rod* is here applied to a uniform one-dimensional body, and all motions, whether of translation or of deformation, are restricted to the single spatial dimension in which the body is taken to exist. Consequently the only stresses to be considered are longitudinal.

If desired, the work can be applied to a three-dimensional body if it be supposed capable of sustaining stress in only one direction and if all strains and motions be restricted to this direction. Such application is of no additional physical interest. However, it permits us to employ, when convenient, the mathematical treatment appropriate to a three-dimensional body including, for instance, the ordinary use of a stress-tensor. We hope in due course properly to extend the present work to the treatment of isotropic three-dimensional bodies.

In this paper we are concerned only with *special* relativity theory.

One of us has recently (4) considered the problem of a rigid rod in accordance with the theory of special relativity. He defined *a rigid rod as one along which disturbances are transmitted with speed c.* Here c is the fundamental speed of relativity theory, usually described as the speed of light, though this description happens to be irrelevant in the present paper.

With this definition the rod was shown to be not inextensible. But, since it possesses the maximum degree of rigidity consistent with the requirements of special relativity, the term 'rigid' is appropriate. We are concerned here only with longitudinal rigidity; nothing corresponding to 'flexural rigidity' makes its appearance in the present one-dimensional problems.

As usually developed, the theory of special relativity postulates a rigid rod as the fundamental measuring scale. Therefore it might be argued that there can be no meaning in the theory for a change of length of a rigid rod. As a consequence of the previous work referred to it is seen, however, that it is necessary to take as a measuring scale a rigid rod that is *at rest in an inertial frame and is unstressed.* If a similar rod is in uniform motion relative to the first, it is of course taken as providing an equivalent measuring scale in the inertial frame in which it is at rest. When we say that a rigid rod is extensible we mean that, if it is allowed to attain equilibrium in an inertial frame under an applied tension then its length as measured by an unstressed rod in the same

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frame differs from its length before the tension was applied. Also, if the rod is accelerated relative to an inertial frame it is in general thereby stressed, and its length, as measured by an observer in the inertial frame using an unstressed rigid scale at rest in his frame, depends upon the stress-distribution and so upon the manner in which the accelerating forces are applied to the moving rod. Indeed, the rod cannot be accelerated like a classical rigid body but only (roughly speaking) like a classical deformable body. The processes of stopping a moving rod, as discussed in the previous paper, illustrate this feature. It is seen that nothing in all this is inconsistent with the principles of special relativity theory or devoid of observational interpretation in accordance with the theory.

From this discussion it follows that, if a body is accelerated relative to an inertial frame, then an observer attached to the body cannot use a rigid rod unambiguously to measure lengths unless its method of employment is prescribed in considerable detail. For, suppose he holds two such rods side by side, holding them first by the same ends and then shifting his hold upon one of them to its other end. In general, he will observe a relative change of length. In the first case he is pushing both rods or pulling both, while in the second case he is pushing one rod and pulling the other, with the result that there is a difference in their states of strain. This remark is trivial in itself, but it appears to have interesting consequences in regard to the so-called 'principle of equivalence'. The latter will not be pursued here, though they provide one reason for the investigation. The remark is, too, an illustration of the significance of the 'drag-point' defined by Synge and Gardiner (5).

Although the extensibility of a rigid rod is compatible with special relativity theory, our discussion does carry the assumption that, *whenever a particular rigid rod is at rest in an inertial frame, then it has always the same 'length'*. More precisely, if two unstressed rigid rods initially at rest in such a frame have the same length as measured by each other, then they must always have the same length when they are again in this state, whatever different motions they may undergo during the interval. Otherwise a rigid rod could not be used in the usual way to define length in an inertial frame, and the theory of special relativity would have to be re-formulated.

It was shown in effect in (4) that the condition is satisfied when it was remarked that a relativistically rigid rod is also perfectly elastic.

The previous work was, however, incomplete in regard to this feature. The object of the present paper is to state the definition of rigidity more rigorously and to derive several theorems. In a subsequent note, the results will be applied to the problem of a rotating rigid 'hoop'.

2. *Definition of rigid rod.* We shall adopt:

DEFINITION A. *A rigid rod is one* (*a*) *along which any disturbance travels with speed c for all possible amplitudes of the accompanying strain and for all possible states of stress existing before the application of the disturbing forces,* (*b*) *in which any change of configuration is reversible and takes place without dissipation of energy.*

The reasons for adopting this definition are contained in § 1 and in (4). Of course, it is not obvious that the definition is self-consistent or consistent with the principles of special relativity. The results about to be proved may be regarded as supplying

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the needed 'existence theorems'. We proceed by deriving a necessary property and then proving several results which establish its sufficiency for our requirements.

3. *Equation of elasticity.* We may now assert:

THEOREM 1. *The stress in any element of a given rigid rod depends only upon the strain in that element.*

From (4) and from Definition A (a), it follows that the rod is extensible and that the extension does depend upon the stress. The only quantities other than the strain upon which the stress could depend are the time-derivatives of the strain. But, just as in classical mechanics, a dependence upon the latter quantities would make it possible for one to stretch and re-compress the rod at different rates in such a way as to bring it back to its initial configuration with a net gain of work. Such a possibility is disallowed by Definition A (b). This establishes the theorem.

When the rod is unstressed and at rest in an inertial frame, let L_0 be its length and m_0 be the mass per unit length. When it is at rest under tension T, let L be the length and m the mass per unit length. We prove

THEOREM 2.
$$T = \tfrac{1}{2}m_0c^2[1-(L_0/L)^2], \tag{3·1}$$
$$m = \tfrac{1}{2}m_0[1+(L_0/L)^2]. \tag{3·2}$$

If the rod is moving initially with uniform velocity V relative to an inertial frame, and if the leading end is suddenly stopped then ((4), equation (7·1)) the whole rod in due course comes momentarily to rest with length L_1, where

$$L_1 = L_0\left(\frac{1-V/c}{1+V/c}\right)^{\frac{1}{2}}. \tag{3·3}$$

According to Theorem 1, the strain-energy depends only upon the initial and final states, not upon the mode of passage from one to the other. So the energy is equal to the work that would be done in a quasi-static process producing the same change of length in the rod. This is

$$-\int_{L_0}^{L_1} T\,dL, \tag{3·4}$$

where T is the tension when the length is L and, by Theorem 1, T varies only with L.

The kinetic energy lost by the rod is, by a standard formula of special relativity,

$$m_0L_0c^2[(1-V^2/c^2)^{-\frac{1}{2}}-1].$$

Using (3·3), this may be expressed in the form

$$\tfrac{1}{2}m_0c^2(L_0-L_1)^2/L_1. \tag{3·5}$$

According to the definition, there is no dissipation of energy and therefore the energies (3·4), (3·5) are equal. Their equality yields the formula (3·1).

It is easily verified that the same result follows also from the corresponding calculation for the case where the rear end of the rod is stopped, instead of the leading end.

By the well-known relativistic relation between mass and energy, the mass of the rod is increased by the amount of its strain-energy divided by c^2. As we have seen,

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the strain-energy is given by (3·5). Hence the total mass per unit length in the strained state is m_1, say, where

$$m_1 = \frac{1}{L_1}\left[m_0 L_0 + \tfrac{1}{2}m_0 \frac{(L_0 - L_1)^2}{L_1}\right] = \tfrac{1}{2}m_0\left[1 + \left(\frac{L_0}{L_1}\right)^2\right].$$

This establishes the formula (3·2).

The result (3·1) was stated in (4), equation (9·9), but the derivation was not given. It is interesting that it should be similar in form to a formula for 'incompressibility' given by Clark (2) for a three-dimensional medium. The present derivation is, however, entirely different from Clark's; the interpretation of Clark's formula from the present viewpoint will be considered in subsequent work on the three-dimensional problem.

Equation (3·1) is now seen to be a necessary consequence of Definition A. We proceed to show that it is also a sufficient condition for the definition to be satisfied. That is to say, we shall now adopt

DEFINITION B. *A rigid rod is one for which formula* (3·1) *is valid.*

Then, from the ensuing results we shall infer the equivalence of the two definitions.

4. *Transmission of impulses: infinitesimal amplitudes.*

THEOREM 3. *Longitudinal vibrations and disturbances of infinitesimal amplitude are propagated in the unstressed rod with speed c.*

Writing $L = L_0 + \Delta L$, (3·1) gives

$$T = m_0 c^2 \Delta L / L_0 + O[(\Delta L / L_0)^2].$$

Hence for small ΔL the rod obeys Hooke's law with modulus λ, where

$$\lambda = m_0 c^2. \tag{4·1}$$

For vibrations with infinitesimal amplitude both the velocities of the particles of the rod and the associated stresses are also infinitesimal. In this case there is, therefore, no difference between relativistic and classical mechanics. Hence the wave-velocity has the classical value $(\lambda/m_0)^{\frac{1}{2}}$. With the value of λ in (4·1) this gives velocity c; there being consequently no dispersion, the group-velocity is also c.

THEOREM 4. *If the rod is stretched to any given tension T_1, it transmits longitudinal vibrations and disturbances of infinitesimal amplitude with speed c.*

The appropriate relativistic equation of motion is well known (3). For one-dimensional motion under no body-force it takes the form

$$\left(1 - \frac{u^2}{c^2}\right)^{-1}\left(m - \frac{T}{c^2}\right)\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) - \frac{\partial T}{\partial x} - \frac{u}{c^2}\frac{\partial T}{\partial t} = 0, \tag{4·2}$$

where T is the tension and u is the velocity of the material at position x. If the motion is infinitesimal, this reduces to

$$\left(m_1 - \frac{T_1}{c^2}\right)\frac{\partial u}{\partial t} - \frac{\partial T}{\partial x} = 0, \tag{4·3}$$

where T_1 is the given tension and m_1 the corresponding mass per unit length.

Let L_1 be the length of the whole rod when under tension T_1. Let the origin of x be taken at one end of the rod, and let x_0 be the distance of the particle at x from this end before stretching. Since the motion is infinitesimal, we have therefore

$$\frac{x}{x_0} = \frac{L_1}{L_0}, \quad \text{giving} \quad \frac{dx}{dx_0} = \frac{L_1}{L_0}. \tag{4·4}$$

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Let ξ be the (infinitesimal) displacement at time t of the particle at x from its equilibrium position. The element $(x, x+dx)$ is then displaced to

$$(x+\xi, x+dx+\xi+(\partial\xi/\partial x)\,dx)$$

so that it becomes of length $(1+\partial\xi/dx)\,dx$, its unstretched length being dx_0. By (3·1) the tension in this element is accordingly

$$T = \tfrac{1}{2}m_0c^2\left\{1-\left[\frac{dx_0}{(1+\partial\xi/\partial x)\,dx}\right]^2\right\} = T_1 + m_0c^2(L_0/L_1)^2\,\partial\xi/\partial x$$

to the first order in ξ, using (4·4). This gives

$$\frac{\partial T}{\partial x} = m_0c^2\left(\frac{L_0}{L_1}\right)^2\frac{\partial^2\xi}{\partial x^2}. \tag{4·5}$$

From (3·1) and (3·2) we obtain

$$m_1 - T_1/c^2 = m_0(L_0/L_1)^2. \tag{4·6}$$

Noting that $u = \partial\xi/\partial t$ and using (4·5) and (4·6) the equation of motion (4·3) becomes

$$\partial^2\xi/\partial t^2 - c^2\,\partial^2\xi/\partial x^2 = 0.$$

Since this is the equation of wave-propagation with speed c the theorem is proved.

Theorem 3 is, of course, the special case for $T_1 = 0$. It should be noted that the result of Theorem 4 does depend essentially upon the occurrence of the factor $(m - T/c^2)$ in (4·2) in place of the simple factor m that would occur in the classical equation of motion.

The remarkable property possessed by the formula (3·1) of yielding the speed of propagation c independently of the tension provides a further illustration of the fact that we have the closest possible analogue to the classical rigid rod. For we expect this analogue to yield the fundamental speed c *wherever* we should have infinite speed in the classical case. This is what we are now finding to hold good.

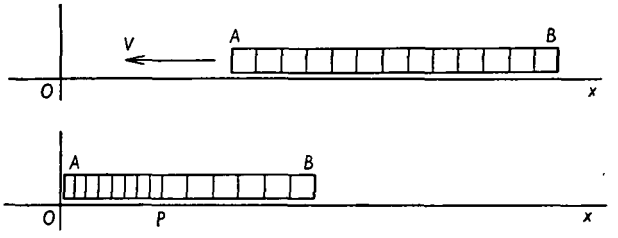
5. *Transmission of impulses: finite amplitudes.* We now consider the propagation along the rod of a disturbance of finite amplitude. Any such disturbance may be regarded as being generated by constraining one or both ends of the rod to move in a particular manner. We consider first the simplest case. The rod being initially at rest in an inertial frame, one end is constrained, from some particular instant, to move with given uniform speed V. This is the case of the rod being struck a 'sledge-hammer' blow at one end. It is physically equivalent to the case of the rod, initially moving with uniform velocity V in an inertial frame S, running into a stationary barrier. This is the form in which it is convenient to discuss the problem.

It may be remarked that there is nothing inconsistent with our assumptions in treating the barrier as unyielding. For the barrier could be composed of material for which the parameter m_0 is arbitrarily large. Alternatively, we could suppose that the rod collides with an exactly similar rod moving with speed V in the opposite direction. By symmetry, the only possibility is that the colliding ends are brought instantaneously to rest.

Let t, x be time and distance measured in frame S. Let the barrier be fixed at the origin O of the x-axis in S and let the rod AB be moving with uniform velocity $-V$ in such a way that the leading end A reaches O and is stopped there at time $t = 0$.

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It is taken that the following process will ensue. At time $t = 0$ a disturbance starts travelling from O along the rod with some uniform speed v relative to S. It will reach a point P at distance $x\,(>0)$ at time $t = x/v$. At this instant the material in PB has not yet been 'informed' of the collision and is therefore still moving with its initial velocity $-V$. The material in OP is at rest and is in a state of uniform compression producing (negative) tension T. This material will remain in equilibrium under this tension until the disturbance reaches the end B of the rod. After that, the (negative) tension will push B in the positive x-direction and the process will be reversed. When the return impulse has reached the end A, the rod will be moving as a whole with velocity $+V$; that is to say, it will have rebounded as in a perfectly elastic collision. Here, however, we are concerned only with the first stage of the process.



This is merely a description of the classical process of elastic impact (with no dissipation). But there is nothing in it that does not carry over into special relativity theory. Of course, it assumes that there is no dispersion, i.e. that the impulse does not 'spread' as it traverses the rod. This assumption is justified by the fact that the law of elasticity (3·1) contains no parameter that could produce dispersion. It is verified by Theorems 3 and 4 in cases of infinitesimal disturbances.

We now treat the process quantitatively and prove

THEOREM 5. *The disturbance produced in stopping a moving free rigid rod traverses it with speed c.*

The particle of the rod at P at time $t = x/v$ has travelled a distance Vx/v after time $t = 0$. Hence, at $t = 0$ the portion of the rod OP had length $x(1 + V/v)$ as measured in S. Taking account of the FitzGerald contraction corresponding to the fact that it was then moving with speed V relative to S, its proper length was $\beta x(1 + V/v)$, where $\beta = (1 - V^2/c^2)^{-\frac{1}{2}}$. Therefore, as each element of the rod is brought to rest, it becomes compressed in the ratio

$$1 : \beta(1 + V/v). \tag{5·1}$$

Hence, using (3·1), the tension in the portion OP is

$$T = \tfrac{1}{2}m_0c^2[1 - \beta^2(1 + V/v)^2], \tag{5·2}$$

this being, of course, negative in such circumstances.

The element of the rod coming to rest in time dt is of length $v\,dt$ as measured in its compressed state in S. Using (5·1), the proper mass of this element before coming to rest was dm, where

$$dm = m_0\beta(1 + V/v)\,v\,dt. \tag{5·3}$$

By the usual formula of special relativity, the momentum dp (in the negative x-direction) of this element relative to S was therefore

$$dp = \beta V\,dm = m_0\beta^2(v + V)\,V\,dt. \tag{5·4}$$

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The fundamental equation of motion, relativistic as well as classical, now yields

$$dp = -T\,dt. \tag{5·5}$$

For this expresses that the momentum destroyed in time dt is equal to the impulse of the force $-T$ acting upon the element during this time.

Using (5·3) and (5·4) in (5·5) we find

$$2\beta^2(v+V)\,V = c^2\beta^2(1+V/v)^2 - c^2,$$

which reduces to

$$(2vV+V^2)(v^2-c^2) = 0.$$

Since V is arbitrary and v is positive this gives uniquely

$$v = c \tag{5·6}$$

and the theorem is proved.

We note also that on putting $v = c$ in (5·1) we recover the value of the compression-ratio in (3·3).

We have now established the converse of the main result of (4) by showing that the law of elasticity (3·1) gives the speed c for the propagation of the disturbance in the process there considered. In that paper it was postulated that the speed is c and it was stated that this required the truth of formula (3·1) as a result of energy considerations. But it was not there shown generally that this formula necessarily yields the required speed of propagation.

We wish next to establish a similar result in the case of a rod that is initially stressed. It will be seen below why this result is needed.

In order to see the meaning of the problem, we may imagine the given rod to be initially stretched (or compressed) between massive end-supports. One of these can then be supposed suddenly set in motion with uniform velocity V relative to the other. We are concerned with the disturbance that then traverses the rod. Or, we may think of the given rod as being pinned at its ends, in a stretched (or compressed) condition, between two other rods identical with each other, and the whole system can then replace the single rod considered in Theorem 5. (In the latter realization of the problem, all three rods must be stressed, and we can regard the discussion as applying to any one of them. Three, rather than two, rods are mentioned merely for the sake of better realism: the reader might say that two would 'bow'!)

As before, we may suppose the system to be moving initially with velocity $-V$ relative to an inertial frame S and suppose that the effect to be studied is achieved by allowing the system to collide with a suitable barrier. So far as it is applicable, we employ the same notation as before.

We shall require

LEMMA. *If the material at some point P of a given rod has proper-mass m_1, per unit proper length and tension T_1, and if its velocity relative to an inertial frame* S *is V, then the momentum relative to* S *per unit length at P measured in* S *is*

$$\beta^2(m_1 - T_1/c^2)\,V. \tag{5·7}$$

This result is, in fact, well known. For the expression (5·7) is the relevant component (usually written T^{14}) of the stress-momentum-energy tensor appropriate to this case (1, 6). The only departure from the standard statement of the result is in

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our giving it in terms of a line-density instead of a volume-density. This is justified by the remark in §1 that we may, if desired, consider a suitable three-dimensional body instead of the one-dimensional rod; quantities relating to unit volume of the body would then correspond to those relating to unit length of the rod.

Although we take the result (5·7) itself from the standard treatment, it is of interest to give a physical interpretation. In relativity theory, momentum can be treated as energy-flux. Suppose for the moment that the velocities concerned are small. Then, if stressed material is in motion, the energy-flux across unit area of any geometrical surface drawn through the material is composed of the material flux (giving a term of the form mV) together with the work done by the material on one side of the boundary upon that on the other (giving a term of the form $-TV/c^2$, the factor $1/c^2$ being required merely to give the correct mass-units). If the velocity is not small, one factor of the form β will appear in order to take account of the relativistic 'change of mass with velocity', and another factor of the same form to take account of the FitzGerald contraction, since unit length in S corresponds to proper-length β. This interpretation may be taken as accounting physically for the several factors in (5·7), but the actual derivation can be given satisfactorily only by means of the tensor treatment to which reference has been made.

It may also be noticed that the leading term in (4·3) is consistent with (5·7) for the case of a small velocity.

We are now in a position to prove

THEOREM 6. *The disturbance produced in stopping a moving stressed rigid rod traverses it with speed c.*

Let the initial tension be T_1. Then the initial proper-length L_1 is given by (3·1), i.e.

$$T_1 = \tfrac{1}{2}m_0c^2[1-(L_0/L_1)^2]. \tag{5·8}$$

The ratio $L_1 : L_0$ given by (5·8) is that by which every element of the rod is initially stretched (or compressed). Equation (5·1) now gives the further compression produced by the stopping process. Combining these factors, the tension in the portion OP of the rod is

$$T = \tfrac{1}{2}m_0c^2[1-\beta^2(1+V/v)^2(L_0/L_1)^2]. \tag{5·9}$$

Using (5·7) we have here in place of (5·4)

$$dp = \beta^2(m_1 - T_1/c^2)(v+V)\,V\,dt = m_0(L_0/L_1)^2\beta^2(v+V)\,V\,dt, \tag{5·10}$$

on using the value (5·8) for T_1 and the corresponding value from (3·2) for m_1. In place of (5·5) we must now write

$$dp = -(T-T_1)\,dt, \tag{5·11}$$

since $-(T-T_1)$ is here the resultant force acting upon the element of the rod concerned. From (5·8), (5·9), (5·11) we obtain

$$2(L_0/L_1)^2\beta^2(v+V)\,V = (L_0/L_1)^2[c^2\beta^2(1+V/v)^2 - c^2].$$

Since the factor $(L_0/L_1)^2$ cancels out, we are then left with exactly the same equation as before to determine v. As we have seen, this gives $v = c$, thus proving the theorem.

In regard to the proofs of Theorems 5 and 6 it should be remarked that v as defined was the speed of advance of the disturbance relative to the disturbed portion of the rod. Had the rod not been 'rigid', we should have found for v some value less than c. This would not then have been also the speed of advance relative to the undisturbed

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portion of the rod, though the latter could, of course, have then been evaluated using the standard Lorentz-velocity transformation. However, since in our case the velocity of advance has been shown to be the fundamental velocity c, it has this value whether it be taken relative to the disturbed or undisturbed material.

We may now formulate as the general conclusion of the work:

THEOREM 7. *Any disturbance of any amplitude is propagated in the rod with speed c.*

We shall now think of the rod as being initially at rest in an inertial frame.

Any disturbance generated at one end of the rod and traversing it in one direction is determined by the motion produced at that end. In Theorems 5 and 6 we have dealt with the case where that motion is uniform relative to the undisturbed part of the rod, and we have found that the resulting disturbance is propagated with speed c. As we have seen, it produces a uniform stress in the disturbed material. If after a certain time, a different uniform velocity is imparted to the end of the rod, its effect will therefore be propagated through the material already disturbed by the first motion again with speed c. This follows by application of Theorem 6. We can proceed in this manner for any discrete set of changes of velocity of the end of the rod. We can then consider the given motion of the end as the limit of such a set, and we conclude that the whole disturbance traverses the rod with speed c.

The admissibility of this argument here should be contrasted with its failure in the case of a classically compressible rod in which the speed of propagation does depend upon the stress. In the latter case, a single change of velocity of one end is propagated through the rod with some particular speed and produces a stress in the disturbed region. A second ensuing change of velocity must be propagated through this stressed region. But, owing to the stress, the speed is *different* for this second disturbance, and so on for any subsequent ones. Then the propagation of a general disturbance becomes a much more complicated matter than it is in our case, and no simple general conclusion can be reached. This remark applies with even more force to the case of a relativistic non-rigid rod.

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