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On the binomial transforms of Apéry-like sequences

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Abstract. In his proof of the irrationality of $\zeta(3)$ and $\zeta(2)$, Apéry defined two integer sequences through 3-term recurrences, which are known as the famous Apéry numbers. Zagier, Almkvist–Zudilin, and Cooper successively introduced the other 13 sporadic sequences through variants of Apéry's 3-term recurrences. All of the 15 sporadic sequences are called Apéry-like sequences. Motivated by Gessel's congruences mod 24 for the Apéry numbers, we investigate congruences of the form $u_n \equiv \alpha^n \pmod{N_\alpha}$ $(\alpha \in \mathbb{Z}, N_\alpha \in \mathbb{N}^+)$ for all of the 15 Apéry-like sequences $\{u_n\}_{n\geq 0}$. Let N_α be the largest positive integer such that $u_n \equiv \alpha^n \pmod{N_\alpha}$ for all non-negative integers n. We determine the values of $\max\{N_\alpha | \alpha \in \mathbb{Z}\}$ for all of the 15 Apéry-like sequences $\{u_n\}_{n\geq 0}$. The binomial transforms of Apéry-like sequences provide us a unified approach to this type of congruences for Apéry-like sequences.

1 Introduction

1.1 Apéry-like sequences

In his ingenious proof of the irrationality of $\zeta(3)$ and $\zeta(2)$, Apéry [2] introduced two particular sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ through 3-term recurrences with given initial values:

$$(n+1)^3 a_{n+1} - (2n+1)(17n^2+17n+5)a_n + n^3 a_{n-1} = 0, \quad (a_0 = 1, a_1 = 5), \tag{1.1}$$

$$(n+1)^2 b_{n+1} - (11n^2+11n+3)b_n - n^2 b_{n-1} = 0, \quad (b_0 = 1, b_1 = 3). \tag{1.2}$$

Apéry also showed that the sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ possess explicit formulas:

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2,$$

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}.$$

These sequences are known as the famous Apéry numbers.

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Table 1: Apéry-like sequences of the second kind.

(A,B,λ)	Name	Other names	Formula
$(7,-8,2)$	**A**	Franel numbers	$u_n=\sum_{k=0}^{n}\binom{n}{k}^3$
$(9,27,3)$	**B**		$u_n=\sum_{k=0}^{n}(-1)^k 3^{n-3k}\binom{n}{3k}\binom{3k}{2k}\binom{2k}{k}$
$(10,9,3)$	**C**		$u_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}$
$(11,-1,3)$	**D**	Apéry numbers	$u_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}$
$(12,32,4)$	**E**		$u_n=\sum_{k=0}^{n}4^{n-2k}\binom{n}{2k}\binom{2k}{k}^2$
$(17,72,6)$	**F**		$u_n=\sum_{k=0}^{n}(-1)^k 8^{n-k}\binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}^3$

Let $F_{\mathbf{a}}(y)$ and $F_{\mathbf{b}}(y)$ be the generating functions for the sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$, respectively,

$$F_{\mathbf{a}}(y)=\sum_{n=0}^{\infty}a_n y^n,$$
$$F_{\mathbf{b}}(y)=\sum_{n=0}^{\infty}b_n y^n.$$

The recurrences (1.1) and (1.2) can be rewritten in forms of differential equations for the generating functions $F_{\mathbf{a}}(y)$ and $F_{\mathbf{b}}(y)$. In other words, $F_{\mathbf{a}}(y)$ is annihilated by the differential operator:

$$\mathcal{L}_{\mathbf{a}}=\theta_y^3-y(2\theta_y+1)(17\theta_y^2+17\theta_y+5)+y^2(\theta_y+1)^3,$$

and $F_{\mathbf{b}}(y)$ is annihilated by the differential operator:

$$\mathcal{L}_{\mathbf{b}}=\theta_y^2-y(11\theta_y^2+11\theta_y+3)-y^2(\theta_y+1)^2,$$

where $\theta_y=y\frac{d}{dy}$.

It is a surprise that the solutions of the recurrences (1.1) and (1.2) are integers. Motivated by the two interesting examples, Zagier [14] searched for triples $(A,B,\lambda)\in\mathbb{Z}^3$ such that the solution of the recurrence:

$$(n+1)^2u_{n+1}-(An^2+An+\lambda)u_n+Bn^2u_{n-1}=0,\quad (u_{-1}=0,u_0=1) \tag{1.3}$$

is an integer sequence $\{u_n\}_{n\geq 0}$. The generating function $F(y)=\sum_{n=0}^{\infty}u_n y^n$ for the sequence defined by (1.3) is annihilated by the differential operator:

$$\mathcal{L}_1=\theta_y^2-y(A\theta_y^2+A\theta_y+\lambda)+By^2(\theta_y+1)^2. \tag{1.4}$$

In Zagier's computer search, six sporadic solutions are found, the integrality of which are not explained easily. For $(A,B,\lambda)=(11,-1,3)$, we already know that there is a desired solution $\{b_n\}_{n\geq 0}$.

The six sporadic sequences are called Apéry-like sequences of the second kind, which are listed in Table 1 (see [9, p. 642]).

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Table 2: Apéry-like sequences of the first kind.

(a,b,c,d)	Name	Other names	Formula
$(7,3,81,0)$	(δ)	Almkvist-Zudilin numbers	$u_n=\sum_{k=0}^{n}(-1)^k 3^{n-3k}\binom{n}{3k}\binom{n+k}{k}\binom{3k}{2k}\times\binom{2k}{k}$
$(11,5,125,0)$	(η)		$u_n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\left(\binom{4n-5k-1}{3n}+\binom{4n-5k}{3n}\right)$
$(10,4,64,0)$	(α)	Domb numbers	$u_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k}$
$(12,4,16,0)$	(ε)		$u_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}^2$
$(9,3,-27,0)$	(ζ)		$u_n=\sum_{k=0}^{n}\sum_{l=0}^{n}\binom{n}{k}^2\binom{n}{l}\binom{k}{l}\binom{k+l}{n}$
$(17,5,1,0)$	(γ)	Apéry numbers	$u_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2$
$(13,4,-27,3)$	s_7		$u_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}\binom{2k}{n}$
$(6,2,-64,4)$	s_{10}	Yang-Zudilin numbers	$u_n=\sum_{k=0}^{n}\binom{n}{k}^4$
$(14,6,192,-12)$	s_{18}		$u_n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}\times\left(\binom{2n-3k-1}{n}+\binom{2n-3k}{n}\right)$

Almkvist and Zudilin [1] conducted similar searches related to $\{a_n\}_{n\geq 0}$. They searched for triples $(a,b,c)\in\mathbb{Z}^3$ such that the solution of the recurrence:

$$(n+1)^3u_{n+1}-(2n+1)(an^2+an+b)u_n+cn^3u_{n-1}=0,\quad (u_{-1}=0,u_0=1) \tag{1.5}$$

is an integer sequence $\{u_n\}_{n\geq 0}$. Six sporadic sequences $(\delta),(\eta),(\alpha),(\varepsilon),(\zeta)$ and (γ) were found, which include the desired sequence $\{a_n\}_{n\geq 0}$.

Cooper [6] added a parameter d to the recurrence (1.5), and studied a more general recurrence:

$$(n+1)^3u_{n+1}-(2n+1)(an^2+an+b)u_n+n(cn^2+d)u_{n-1}=0, \tag{1.6}$$

with $u_{-1}=0$ and $u_0=1$. Note that the case $d=0$ in (1.6) reduces to (1.5). The generating function $F(y)=\sum_{n=0}^{\infty}u_ny^n$ for the sequence defined by (1.6) is annihilated by the differential operator:

$$\mathcal{L}_2=\theta_y^3-y(2\theta_y+1)(a\theta_y^2+a\theta_y+b)+y^2\left(c(\theta_y+1)^3+d(\theta_y+1)\right). \tag{1.7}$$

In Cooper's search, three additional sporadic sequences were found, named s_7, s_{10}, and s_{18}.

The nine sporadic sequences are called Apéry-like sequences of the first kind, which are listed in Table 2 (see [9, pp. 642–643]).

1.2 Motivation

Chowla et al. [5] investigated congruence properties for the Apéry numbers $\{a_n\}_{n\geq 0}$, and conjectured that $a_{2n}\equiv 1 \pmod 8$, $a_{2n+1}\equiv 5 \pmod 8$, $a_{2n}\equiv 1 \pmod 3$ and $a_{2n+1}\equiv 2 \pmod 3$ for all non-negative integers n, all of which were proved by Gessel [8] in the forms:

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$$a_n \equiv 5^n \pmod 8, \tag{1.8}$$

$$a_n \equiv (-1)^n \pmod 3. \tag{1.9}$$

Since $(3, 8) = 1$, congruences (1.8) and (1.9) can be restated in a unified form:

$$a_n \equiv 5^n \pmod{24}, \tag{1.10}$$

for all non-negative integers n.

In 2016, Malik and Straub [11, Corollary 5.2] established the following three interesting results:

(1) If $\{u_n\}_{n\geq 0}$ is one of the 15 Apéry-like sequences, we have $u_n \equiv u_1 \pmod 2$ for all $n \geq 1$.
(2) If $\{u_n\}_{n\geq 0}$ is one of $\mathbf{C}, \mathbf{B}, \mathbf{F}, (\delta), (\alpha), (\varepsilon), (\zeta)$ and s_{18}, we have $u_n \equiv u_1 \pmod 3$ for all $n \geq 1$. If $\{u_n\}_{n\geq 0}$ is $\mathbf{A}$ or (γ), we have $u_n \equiv (-1)^n \pmod 3$ for all $n \geq 0$.
(3) If $\{u_n\}_{n\geq 0}$ is $\mathbf{D}$, we have $u_n \equiv 3^n \pmod 5$ for all $n \geq 0$. If $\{u_n\}_{n\geq 0}$ is (η), we have $u_n \equiv 0 \pmod 5$ for all $n \geq 1$.

From the above results due to Malik and Straub [11], we find that

$$b_n \equiv 3^n \pmod{10}, \tag{1.11}$$

for all non-negative integers n.

The two interesting examples (1.10) and (1.11) motivate us to find more congruences of this type for all of the 15 Apéry-like sequences $\{u_n\}_{n\geq 0}$.

For an Apéry-like sequence $\{u_n\}_{n\geq 0}$ and an integer α, let N_α be the largest positive integer such that

$$u_n \equiv \alpha^n \pmod{N_\alpha}, \tag{1.12}$$

for all non-negative integers n. Since all of the 15 Apéry-like sequences are not geometric sequences, we conclude that the set $\{d \in \mathbb{N} : \forall n \geq 0\ u_n \equiv \alpha^n \pmod d\}$ is bounded, and so N_α is well-defined.

The main purpose of the article is to determine the values of $\beta \in \mathbb{Z}$ such that $N_\beta = \max\{N_\alpha | \alpha \in \mathbb{Z}\}$ and the values of $\max\{N_\alpha | \alpha \in \mathbb{Z}\}$ for all of the 15 Apéry-like sequences $\{u_n\}_{n\geq 0}$. The binomial transforms of Apéry-like sequences provide us a unified approach to this type of congruence (1.12) for Apéry-like sequences.

We remark that Gessel proved (1.10) through Lucas congruences, and we shall reprove (1.10) by using the binomial transforms. The two approaches are essentially different.

1.3 Main results

For a sequence $\{u_n\}_{n\geq 0}$ with $u_0 = 1$ and a complex number x, the binomial transform of $\{u_n\}_{n\geq 0}$ is defined by

$$v_n(x) = \sum_{k=0}^{n} \binom{n}{k} (-x)^{n-k} u_k, \tag{1.13}$$

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for all non-negative integers n. Note that $v_n(x)$ is a polynomial of degree n with integer coefficients. Throughout the article, given a sequence $\{u_n\}_{n\geq 0}$, the polynomial sequence $\{v_n(x)\}_{n\geq 0}$ denotes its corresponding binomial transform (1.13).

For a positive integer M, let $\mathrm{rad}(M)$ denote the radical of M:

$$\mathrm{rad}(M) = \prod_{\substack{p|M \\ p \text{ prime}}} p.$$

For an Apéry-like sequence $\{u_n\}_{n\geq 0}$ and an integer α, let

$$M_\alpha = \begin{cases} (v_1(\alpha), v_2(\alpha), v_3(\alpha)) & \text{if } \{u_n\}_{n\geq 0} \text{ is of the second kind,} \\ (v_1(\alpha), v_2(\alpha), v_3(\alpha), v_4(\alpha)) & \text{if } \{u_n\}_{n\geq 0} \text{ is of the first kind,} \end{cases} \tag{1.14}$$

where $(n_1, n_2, \cdots, n_r)$ denotes the greatest positive common divisor of integers $n_1, n_2, \ldots, n_r$.

The main results of the article consist of the following two theorems.

Theorem 1.1 *Let $\{u_n\}_{n\geq 0}$ be one of the* 15 *Apéry-like sequences and α be an integer. For all non-negative integers n, we have*

$$u_n \equiv \alpha^n \pmod{\mathrm{rad}(M_\alpha)}. \tag{1.15}$$

Theorem 1.2 *Let $\{u_n\}_{n\geq 0}$ be one of the* 15 *Apéry-like sequences. For all integers α, N_α divides N_{u_1} and*

$$N_{u_1} = M_{u_1} = \begin{cases} (v_2(u_1), v_3(u_1)) & \text{if } \{u_n\}_{n\geq 0} \text{ is of the second kind,} \\ (v_2(u_1), v_3(u_1), v_4(u_1)) & \text{if } \{u_n\}_{n\geq 0} \text{ is of the first kind,} \end{cases} \tag{1.16}$$

where N_α is defined by (1.12).

Remark From Theorem 1.2, we deduce that

$$N_{u_1} = \max\{N_\alpha | \alpha \in \mathbb{Z}\}.$$

In particular, the congruence $u_n \equiv u_1^n \pmod{N_{u_1}}$ includes all of the congruences $u_n \equiv \alpha^n \pmod{N_\alpha}$ as special cases for all integers α.

1.4 Values of N_{u_1}

By (1.16), we can determine the values of N_{u_1} for all of the 15 Apéry-like sequences. The values of u_1 and N_{u_1} for the Apéry-like sequences of the second kind are listed in Table 3.

The values of u_1 and N_{u_1} for the Apéry-like sequences of the first kind are listed in Table 4.

From Tables 3 and 4, we show the novel fact that Gessel congruence (1.10) and Malik-Straub congruence (1.11) are already optimal.

1.5 Structure of the article

In Section 2, we investigate the binomial transforms of Apéry-like sequences, which are important ingredients in the proof of Theorems 1.1 and 1.2. We first establish a key

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Table 3: Values of u_1 and N_{u_1}.

Name	Other names	Formula	u_1	N_{u_1}
A	Franel numbers	$u_n = \sum_{k=0}^{n} \binom{n}{k}^3$	2	6
B		$u_n = \sum_{k=0}^{n} (-1)^k 3^{n-3k} \binom{n}{3k}\binom{3k}{2k}\binom{2k}{k}$	3	6
C		$u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$	3	6
D	Apéry numbers	$u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$	3	10
E		$u_n = \sum_{k=0}^{n} 4^{n-2k} \binom{n}{2k}\binom{2k}{k}^2$	4	4
F		$u_n = \sum_{k=0}^{n} (-1)^k 8^{n-k} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j}^3$	6	6

Table 4: Values of u_1 and N_{u_1}.

Name	Other names	Formula	u_1	N_{u_1}
(δ)	Almkvist-Zudilin numbers	$u_n = \sum_{k=0}^{n} (-1)^k 3^{n-3k} \binom{n}{3k}\binom{n+k}{k}\binom{3k}{2k}\binom{2k}{k}$	3	24
(η)		$u_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n}\right)$	10	10
(α)	Domb numbers	$u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}\binom{2n-2k}{n-k}$	4	12
(ε)		$u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}^2$	4	24
(ζ)		$u_n = \sum_{k=0}^{n} \sum_{l=0}^{n} \binom{n}{k}^2 \binom{n}{l}\binom{k}{l}\binom{k+l}{n}$	3	6
(γ)	Apéry numbers	$u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$	5	24
s_7		$u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}\binom{2k}{n}$	4	8
s_{10}	Yang-Zudilin numbers	$u_n = \sum_{k=0}^{n} \binom{n}{k}^4$	2	2
s_{18}		$u_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}\left(\binom{2n-3k-1}{n} + \binom{2n-3k}{n}\right)$	12	12

equivalent condition for (1.12) through binomial transforms of Apéry-like sequences. Secondly, we prove that all of the binomial transforms of Apéry-like sequences satisfy the Gauss congruences through constant term sequences. Thirdly, we compute the generating functions for the binomial transforms of Apéry-like sequences, and the annihilating differential operators for these generating functions. Section 3 is devoted to the proof Theorem 1.1. In Section 4, we establish an auxiliary result and prove Theorem 1.2 in detail.

2 Binomial transforms

2.1 A key equivalent condition

Lemma 2.1 *Let α be an integer and N be a positive integer. For all non-negative integers n,*

$$u_n \equiv \alpha^n \pmod{N},$$

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if and only if for all positive integers n,

$$v_n(\alpha) \equiv 0 \pmod{N}.$$

Proof If $u_n \equiv \alpha^n \pmod{N}$ for all non-negative integers n, then

$$v_n(\alpha) \equiv \alpha^n \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} = 0 \pmod{N},$$

for all positive integers n.

If $v_n(\alpha) \equiv 0 \pmod{N}$ for all positive integers n, then for all non-negative integers n,

$$u_n = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} v_k(\alpha) \equiv \alpha^n \pmod{N},$$

where we have used the binomial inversion formula in the first step. ∎

2.2 Gauss congruences

A sequence $\{u_n\}_{n\geq 1}$ is said to satisfy the Gauss congruences if $u_{np^k} \equiv u_{np^{k-1}} \pmod{p^k}$ for all primes p and all positive integers n, k. A sequence $\{u_n\}_{n\geq 1}$ is called the constant term sequence of a Laurent polynomial $\Lambda \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ if u_n is the constant term of Λ^n for every $n \geq 1$. It is well-known that Gauss congruences hold for the constant term sequence of a integral Laurent polynomial (see, for instance, [4, Lemma 3.1]).

Recently, Gorodetsky [9] showed that all of the 15 Apéry-like sequences are constant term sequences.

Lemma 2.2 *(See [9, Theorem 1.1].) Let $\{u_n\}_{n\geq 0}$ be one of the 15 Apéry-like sequences. Then $\{u_n\}_{n\geq 0}$ is a constant term sequence of a Laurent polynomial with integer coefficients in 2 or 3 variables.*

Theorem 2.3 *Let $\{u_n\}_{n\geq 0}$ be one of the 15 Apéry-like sequences and α be an integer. Then the corresponding binomial transform $\{v_n(\alpha)\}_{n\geq 0}$ satisfies the Gauss congruences:*

$$v_{np^k}(\alpha) \equiv v_{np^{k-1}}(\alpha) \pmod{p^k}, \tag{2.1}$$

for all primes p and all positive integers n, k.

Proof It suffices to show that the binomial transform $\{v_n(\alpha)\}_{n\geq 0}$ is a constant term sequence of a Laurent polynomial with integer coefficients.

For a Laurent series $f(x_1, \ldots, x_k)$, let $CT(f(x_1, \ldots, x_k))$ denote the constant term of the Laurent series. By Lemma 2.2, for all non-negative integers n,

$$u_n = CT(\Lambda^n),$$

where Λ is a Laurent polynomial with integer coefficients in 2 or 3 variables. It follows that for all non-negative integers n,

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$$\begin{aligned} CT\left((\Lambda-\alpha)^n\right) &= CT\left(\sum_{k=0}^{n}\binom{n}{k}(-\alpha)^{n-k}\Lambda^k\right) \\ &= \sum_{k=0}^{n}\binom{n}{k}(-\alpha)^{n-k}CT\left(\Lambda^k\right) \\ &= \sum_{k=0}^{n}\binom{n}{k}(-\alpha)^{n-k}u_k \\ &= v_n(\alpha). \end{aligned}$$

We conclude that the binomial transform $\{v_n(\alpha)\}_{n\geq 0}$ is a constant term sequence of the integral Laurent polynomial $\Lambda-\alpha$. ∎

2.3 Differential operators

Lemma 2.4 *For a sequence $\{u_n\}_{n\geq 0}$ and a complex number x, let $F(y)=\sum_{n=0}^{\infty}u_n y^n$ and $G(z)=\sum_{n=0}^{\infty}v_n(x)z^n$. Then*

$$G(z)=\frac{1}{1+xz}F\left(\frac{z}{1+xz}\right). \tag{2.2}$$

Proof By (1.13), we have

$$\begin{aligned} G(z) &= \sum_{n=0}^{\infty}z^n\sum_{k=0}^{n}\binom{n}{k}(-x)^{n-k}u_k \\ &= \sum_{k=0}^{\infty}\frac{u_k}{(-x)^k}\sum_{n=k}^{\infty}\binom{n}{k}(-xz)^n \\ &= \sum_{k=0}^{\infty}u_k z^k\sum_{n=0}^{\infty}\binom{n+k}{k}(-xz)^n. \end{aligned} \tag{2.3}$$

Note that

$$\sum_{n=0}^{\infty}\binom{n+k}{k}(-xz)^n=(1+xz)^{-k-1}. \tag{2.4}$$

Combining (2.3) and (2.4), we complete the proof of (2.2). ∎

Theorem 2.5 *For an Apéry-like sequence $\{u_n\}_{n\geq 0}$ and a complex number x, let $F(y)=\sum_{n=0}^{\infty}u_n y^n$ and $G(z)=\sum_{n=0}^{\infty}v_n(x)z^n$.*

If $F(y)$ is annihilated by the differential operator (1.4), then $G(z)$ is annihilated by the differential operator:

$$\begin{aligned} &\theta_z^2+z\left((3x-A)\theta_z^2+(3x-A)\theta_z+x-\lambda\right)-z^2(\theta_z+1)^2(2Ax-3x^2-B) \\ &\quad -z^3x(\theta_z+1)(\theta_z+2)(Ax-x^2-B). \end{aligned} \tag{2.5}$$

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If $F(y)$ is annihilated by the differential operator (1.7), then $G(z)$ is annihilated by the differential operator:

$$\begin{aligned}&\theta_z^3 - z(2\theta_z+1)((a-2x)\theta_z^2+(a-2x)\theta_z+b-x)\\&\quad - z^2(\theta_z+1)((6ax-6x^2-c)\theta_z^2+(12ax-12x^2-2c)\theta_z\\&\quad + 6ax+2bx-7x^2-c-d)\\&\quad - z^3x(\theta_z+1)(\theta_z+2)(2\theta_z+3)(3ax-2x^2-c)\\&\quad - z^4x^2(\theta_z+1)(\theta_z+2)(\theta_z+3)(2ax-x^2-c).\end{aligned} \tag{2.6}$$

Proof Let

$$y = \frac{z}{1+xz}. \tag{2.7}$$

Then (2.2) can be rewritten as

$$F(y) = (1+xz)G(z). \tag{2.8}$$

Differentiating both sides of (2.8) with respect to z, we obtain

$$\theta_y F(y) = \left((1+xz)^2\theta_z + xz(1+xz)\right)G(z). \tag{2.9}$$

Differentiating both sides of (2.9) with respect to z, we arrive at

$$\theta_y^2 F(y) = \left((1+xz)^3\theta_z^2 + 3xz(1+xz)^2\theta_z + xz(1+xz)(1+2xz)\right)G(z). \tag{2.10}$$

Differentiating both sides of (2.10) with respect to z, we get

$$\begin{aligned}\theta_y^3 F(y) &= ((1+xz)^4\theta_z^3 + 6xz(1+xz)^3\theta_z^2\\&\quad + xz(1+xz)^2(11xz+4)\theta_z + xz(1+xz)(6(xz)^2+6xz+1))G(z).\end{aligned} \tag{2.11}$$

Finally, substituting (2.7)–(2.11) into the annihilating differential operators (1.4) and (1.7) for $F(y)$, after simplifications, we are led to the annihilating differential operators (2.5) and (2.6) for $G(z)$. ∎

From the annihilating differential operators (2.5) and (2.6) for $G(z)$, we immediately get the recurrences for the binomial transforms $\{v_n(\alpha)\}_{n\geq 0}$ of Apéry-like sequences $\{u_n\}_{n\geq 0}$.

Corollary 2.6 *Let $\{u_n\}_{n\geq 0}$ be an Apéry-like sequence and α be an integer. If $\{u_n\}_{n\geq 0}$ is of the second kind, then $\{v_n(\alpha)\}_{n\geq 0}$ satisfies a recurrence in the form:*

$$n^2 v_n(\alpha) + f_1(n)v_{n-1}(\alpha) + f_2(n)v_{n-2}(\alpha) + f_3(n)v_{n-3}(\alpha) = 0, \tag{2.12}$$

where $f_i(x)$ is a polynomial of degree 2 with integer coefficients for $i = 1, 2, 3$.

If $\{u_n\}_{n\geq 0}$ is of the first kind, then $\{v_n(\alpha)\}_{n\geq 0}$ satisfies a recurrence in the form:

$$n^3 v_n(\alpha) + g_1(n)v_{n-1}(\alpha) + g_2(n)v_{n-2}(\alpha) + g_3(n)v_{n-3}(\alpha) + g_4(n)v_{n-4}(\alpha) = 0, \tag{2.13}$$

where $g_i(x)$ is a polynomial of degree 3 with integer coefficients for $i = 1, 2, 3, 4$.

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3 Proof of Theorem 1.1

We shall prove the case that $\{u_n\}_{n\geq 0}$ is an Apéry-like sequence of the second kind. The proof for the Apéry-like sequences of the first kind runs analogously, and we omit the details.

Assume that $\{u_n\}_{n\geq 0}$ is an Apéry-like sequence of the second kind. We shall prove that for all non-negative integers n,

$$u_n \equiv \alpha^n \pmod{\operatorname{rad}((v_1(\alpha), v_2(\alpha), v_3(\alpha)))},$$

which is equivalent, by Lemma 2.1, to

$$v_n(\alpha) \equiv 0 \pmod{\operatorname{rad}((v_1(\alpha), v_2(\alpha), v_3(\alpha)))}, \tag{3.1}$$

for all positive integers n.

Since $\operatorname{rad}((v_1(\alpha), v_2(\alpha), v_3(\alpha)))$ is square-free, let $\operatorname{rad}((v_1(\alpha), v_2(\alpha), v_3(\alpha))) = p_1 p_2 \dots p_s$ where $p_1, p_2, \dots, p_s$ are distinct primes.

It is clear that (3.1) holds for $n = 1, 2, 3$. Let $i \in \{1, 2, \dots, s\}$. If p_i divides n, then, by (2.1), we have

$$v_n(\alpha) \equiv v_{n/p_i}(\alpha) \pmod{p_i}. \tag{3.2}$$

If p_i does not divide n, then $1/n^2$ is invertible modulo p_i. It follows from (2.12) that

$$v_n(\alpha) \equiv c_1 v_{n-1}(\alpha) + c_2 v_{n-2}(\alpha) + c_3 v_{n-3}(\alpha) \pmod{p_i}, \tag{3.3}$$

where c_1, c_2, c_3 are elements of $\mathbb{Z}/p_i\mathbb{Z}$.

Using (3.2), (3.3) and the induction on n, we obtain

$$v_n(\alpha) \equiv 0 \pmod{p_i},$$

for all positive integers n.

Since $p_1, p_2, \dots, p_s$ are distinct primes, we have

$$v_n(\alpha) \equiv 0 \pmod{p_1 p_2 \dots p_s},$$

for all positive integers n. This completes the proof of (3.1).

4 Proof of Theorem 1.2

4.1 An auxiliary result

Lemma 4.1 *Let c be an integer and $f(x)$ be a polynomial with integer coefficients. For all integers α, we have*

$$(-\alpha + c, f(\alpha)) = (-\alpha + c, f(c)). \tag{4.1}$$

Proof Assume that $f(x) = a_s x^s + a_{s-1}x^{s-1} + \dots + a_1 x + a_0 \in \mathbb{Z}[x]$. Note that

$$f(x) - f(c) = a_s(x^s - c^s) + a_{s-1}(x^{s-1} - c^{s-1}) + \dots + a_1(x - c). \tag{4.2}$$

We rewrite (4.2) in the form:

$$f(x) = (-x + c)g(x) + f(c),$$

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where $g(x)$ is a polynomial with integer coefficients. It follows that

$$(-\alpha+c, f(\alpha)) = (-\alpha+c, (-\alpha+c)g(\alpha)+f(c)) = (-\alpha+c, f(c)),$$

as desired. ∎

4.2 Proof of Theorem 1.2

Assume that $\{u_n\}_{n\geq 0}$ is an Apéry-like sequence of the second kind. Using the fact $v_1(\alpha) = -\alpha + u_1$ and (4.1), we have

$$\begin{aligned} M_\alpha &= (v_1(\alpha), v_2(\alpha), v_3(\alpha)) \\ &= ((v_1(\alpha), v_2(\alpha)), (v_1(\alpha), v_3(\alpha))) \\ &= ((-\alpha+u_1, v_2(\alpha)), (-\alpha+u_1, v_3(\alpha))) \\ &= ((-\alpha+u_1, v_2(u_1)), (-\alpha+u_1, v_3(u_1))) \\ &= (-\alpha+u_1, v_2(u_1), v_3(u_1)) \\ &= (-\alpha+u_1, (v_2(u_1), v_3(u_1))), \end{aligned}$$

and so $M_{u_1} = (v_2(u_1), v_3(u_1))$ and M_α divides M_{u_1} for all integers α.

Similarly, we can show that for an Apéry-like sequence $\{u_n\}_{n\geq 0}$ of the first kind,

$$M_\alpha = (-\alpha+u_1, (v_2(u_1), v_3(u_1), v_4(u_1))),$$

and so $M_{u_1} = (v_2(u_1), v_3(u_1), v_4(u_1))$ and M_α divides M_{u_1} for all integers α. It follows that for all integers α,

$$M_\alpha \big| M_{u_1}, \tag{4.3}$$

and

$$M_{u_1} = \begin{cases} (v_2(u_1), v_3(u_1)) & \text{if } \{u_n\}_{n\geq 0} \text{ is of the second kind,} \\ (v_2(u_1), v_3(u_1), v_4(u_1)) & \text{if } \{u_n\}_{n\geq 0} \text{ is of the first kind.} \end{cases}$$

By Lemma 2.1, we conclude that N_α is the largest positive integer such that

$$v_n(\alpha) \equiv 0 \pmod{N_\alpha}, \tag{4.4}$$

for all positive integers n. By (4.4) and the definition (1.14), we have

$$N_\alpha \big| M_\alpha, \tag{4.5}$$

for all integers α. Combining (4.3) and (4.5), we obtian

$$N_\alpha \big| M_{u_1}, \tag{4.6}$$

for all integers α. It remains to prove that

$$N_{u_1} = M_{u_1}. \tag{4.7}$$

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By (4.6), we have $N_{u_1}|M_{u_1}$. To prove (4.7), it suffices to show that $M_{u_1}|N_{u_1}$, which is equivalent to

$$u_n \equiv u_1^n \pmod{M_{u_1}}, \tag{4.8}$$

for all non-negative integers n. Next, we shall prove that (4.8) is true for all of the 15 Apéry-like sequences.

By (1.15), we have

$$u_n \equiv u_1^n \pmod{\mathrm{rad}(M_{u_1})}, \tag{4.9}$$

for all non-negative integers n. If M_{u_1} is square-free, then $M_{u_1} = \mathrm{rad}(M_{u_1})$, and so (4.8) obviously holds. It follows from the values of M_{u_1} in Tables 3 and 4 that (4.8) is true for Apéry-like sequences **A**, **B**, **C**, **D**, **F**, (η), (ζ) and s_{10}.

Next, we shall prove that (4.8) is also true for the remaining seven Apéry-like sequences. Let $M_{u_1}^*$ denote the square-free part in the prime factorization of M_{u_1}:

$$M_{u_1}^* = \prod_{\substack{p|M_{u_1},\, p^2 \nmid M_{u_1} \\ p \text{ prime}}} p.$$

Since $M_{u_1}^*$ divides $\mathrm{rad}(M_{u_1})$, by (4.9), we have

$$u_n \equiv u_1^n \pmod{M_{u_1}^*}, \tag{4.10}$$

for all non-negative integers n. Moreover, we have

$$(M_{u_1}^*, M_{u_1}/M_{u_1}^*) = 1. \tag{4.11}$$

To prove (4.8), by (4.10) and (4.11) it suffices to show that

$$u_n \equiv u_1^n \pmod{M_{u_1}/M_{u_1}^*}, \tag{4.12}$$

for all non-negative integers n.

For the remaining seven Apéry-like sequences, (4.12) reads

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \equiv 5^n \pmod 8, \tag{4.13}$$

$$\sum_{k=0}^{n} 4^{n-2k} \binom{n}{2k} \binom{2k}{k}^2 \equiv 4^n \pmod 4, \tag{4.14}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k} \equiv 4^n \pmod 4, \tag{4.15}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} \left(\binom{2n-3k-1}{n} + \binom{2n-3k}{n} \right) \equiv 12^n \pmod 4, \tag{4.16}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}^2 \equiv 4^n \pmod 8, \tag{4.17}$$

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$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}\binom{2k}{n} \equiv 4^n \pmod 8, \tag{4.18}$$

$$\sum_{k=0}^{n}(-1)^k 3^{n-3k}\binom{n}{3k}\binom{n+k}{k}\binom{3k}{2k}\binom{2k}{k} \equiv 3^n \pmod 8. \tag{4.19}$$

We remark that Gessel [8, Theorem 3(ii)] proved (4.13). Next, we shall prove (4.14)–(4.19) respectively.

Proof of (4.14) It suffices to show that for all positive integers n,

$$\sum_{k=0}^{n}4^{n-2k}\binom{n}{2k}\binom{2k}{k}^2 \equiv 0 \pmod 4. \tag{4.20}$$

If n is an odd positive integer, then (4.20) clearly holds. If n is an even positive integer, then

$$\sum_{k=0}^{n}4^{n-2k}\binom{n}{2k}\binom{2k}{k}^2 \equiv \binom{n}{n/2}^2 = 4\binom{n-1}{n/2-1}^2 \equiv 0 \pmod 4,$$

as desired. ∎

Proof of (4.15) It suffices to show that for all positive integers n,

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k} \equiv 0 \pmod 4.$$

Since $\binom{2k}{k} = 2\binom{2k-1}{k-1} \equiv 0 \pmod 2$ for all positive integers k, we have

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k} \equiv 2\binom{2n}{n} \equiv 0 \pmod 4,$$

as desired. ∎

Proof of (4.16) It suffices to show that for all positive integers n,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}\left(\binom{2n-3k-1}{n}+\binom{2n-3k}{n}\right) \equiv 0 \pmod 4.$$

Since $\binom{2k}{k} \equiv 0 \pmod 2$ for all positive integers k, we have

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}\left(\binom{2n-3k-1}{n}+\binom{2n-3k}{n}\right)\\
&\equiv \binom{2n}{n}\left(\binom{2n-1}{n}+\binom{2n}{n}\right)+(-1)^n\binom{2n}{n}\left(\binom{-n-1}{n}+\binom{-n}{n}\right)\\
&= 2\binom{2n}{n}\left(\binom{2n-1}{n}+\binom{2n}{n}\right)\\
&\equiv 0 \pmod 4,
\end{aligned}$$

as desired. ∎

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Proof of (4.17) It is easy to check that (4.17) holds for $n = 0, 1, 2$. Let

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}^2.$$

It suffices to show that for all integers $n \geq 3$,

$$a_n \equiv 0 \pmod 8. \tag{4.21}$$

By the recurrence (1.6), we have

$$(2n+1)^3 a_{2n+1} - 4(4n+1)(12n^2+6n+1)a_{2n} + 128n^3 a_{2n-1} = 0.$$

It follows that

$$(4n^2+6n+1)a_{2n+1} + 4a_{2n} \equiv 0 \pmod 8,$$

and so

$$a_{2n+1} \equiv \frac{4}{4n^2+6n+1} a_{2n} \equiv 4a_{2n} \pmod 8. \tag{4.22}$$

On the other hand, we have

$$\begin{aligned} a_{2n} &= \sum_{k=0}^{2n} \binom{2n}{k}^2 \binom{2k}{2n}^2 \\ &= \sum_{k=0}^{n} \binom{2n}{2k}^2 \binom{4k}{2n}^2 + \sum_{k=0}^{n} \binom{2n}{2k+1}^2 \binom{2(2k+1)}{2n}^2 \\ &= \sum_{k=0}^{n} \binom{2n}{2k}^2 \binom{4k}{2n}^2 + 4n^2 \sum_{k=0}^{n} \frac{1}{(2k+1)^2} \binom{2n-1}{2k}^2 \binom{2(2k+1)}{2n}^2 \\ &\equiv \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}^2 + 4n^2 \sum_{k=0}^{n} \binom{2n-1}{2k}^2 \binom{2k+1}{n}^2 \pmod 8, \end{aligned} \tag{4.23}$$

where we have used the following congruence (see [10]):

$$\binom{2s}{2t} \equiv \binom{s}{t} \pmod 4. \tag{4.24}$$

Furthermore, for $n \geq 2$ we have

$$\begin{aligned} & n^2 \sum_{k=0}^{n} \binom{2n-1}{2k}^2 \binom{2k+1}{n}^2 \\ &= \sum_{k=0}^{n} (2k+1)^2 \binom{2(n-1)+1}{2k}^2 \binom{2k}{n-1}^2 \\ &\equiv \sum_{k=0}^{n-1} \binom{n-1}{k}^2 \binom{2k}{n-1}^2 \\ &= 4 \sum_{k=1}^{n-1} \binom{n-2}{k-1}^2 \binom{2k-1}{n-2}^2 \\ &\equiv 0 \pmod 2, \end{aligned} \tag{4.25}$$

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where we have used (4.24). Finally, combining (4.23) and (4.25), we arrive at

$$a_{2n} \equiv a_n \pmod 8, \tag{4.26}$$

for all integers $n \geq 2$.

By using the induction on n, (4.22) and (4.26), we complete the proof of (4.21). ∎

Proof of (4.18) It is easy to check that (4.18) holds for $n = 0, 1, 2, 3$. Let

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n}.$$

It suffices to show that for all integers $n \geq 4$,

$$b_n \equiv 0 \pmod 8. \tag{4.27}$$

By the recurrence (1.6), we have

$$(2n+1)^3 b_{2n+1} - 2(4n+1)(26n^2+13n+2)b_{2n} - 6n(6n-1)(6n+1)b_{2n-1} = 0.$$

It follows that

$$(4n^2+6n+1)b_{2n+1} + 2(2n^2+3n+2)b_{2n} + 6nb_{2n-1} \equiv 0 \pmod 8,$$

which can be rewritten in the form:

$$b_{2n+1} \equiv \beta_1 b_{2n} + \beta_2 b_{2n-1} \pmod 8, \tag{4.28}$$

where β_1, β_2 are elements of $\mathbb{Z}/8\mathbb{Z}$.

On the other hand, we have

$$\begin{aligned} b_{2n} &= \sum_{k=0}^{2n} \binom{2n}{k}^2 \binom{2n+k}{k} \binom{2k}{2n} \\ &= \sum_{k=0}^{n} \binom{2n}{2k}^2 \binom{2n+2k}{2k} \binom{4k}{2n} + \sum_{k=0}^{n} \binom{2n}{2k+1}^2 \binom{2n+2k+1}{2n} \binom{2(2k+1)}{2n} \\ &= \sum_{k=0}^{n} \binom{2n}{2k}^2 \binom{2n+2k}{2k} \binom{4k}{2n} \\ &\quad + 4n^2 \sum_{k=0}^{n} \frac{1}{(2k+1)^2} \binom{2n-1}{2k}^2 \binom{2n+2k+1}{2n} \binom{2(2k+1)}{2n} \\ &\equiv \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2n+2k}{2k} \binom{4k}{2n} \\ &\quad + 4n^2 \sum_{k=0}^{n} \binom{2n-1}{2k}^2 \binom{2n+2k+1}{2n} \binom{2(2k+1)}{2n} \pmod 8, \end{aligned} \tag{4.29}$$

where we have used (4.24).

Since for $n \geq 1$ and $k \geq 1$,

$$\binom{n}{k}\binom{4k}{2n} = 2\binom{n-1}{k-1}\binom{4k-1}{2n-1} \equiv 0 \pmod 2,$$

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and

$$\binom{n}{k}\binom{n+k}{k} = \binom{n+k}{2k}\binom{2k}{k} = 2\binom{n+k}{2k}\binom{2k-1}{k-1} \equiv 0 \pmod 2,$$

we have

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}^2\binom{2n+2k}{2k}\binom{4k}{2n} \\
&\equiv \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}\binom{4k}{2n} \\
&\equiv \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}\binom{2k}{n} \pmod 8, \qquad (4.30)
\end{aligned}$$

where we have also used (4.24).

Furthermore, for $n \geq 2$ we have

$$\begin{aligned}
&n^2\sum_{k=0}^{n}\binom{2n-1}{2k}^2\binom{2n+2k+1}{2n}\binom{2(2k+1)}{2n} \\
&= n^2\sum_{k=0}^{n}\binom{2(n-1)+1}{2k}^2\binom{2n+2k+1}{2n}\binom{2(2k+1)}{2n} \\
&\equiv n^2\sum_{k=0}^{n-1}\binom{n-1}{k}^2\binom{n+k}{n}\binom{2k+1}{n} \\
&= 2n\sum_{k=0}^{n-1}(2k+1)\binom{n-1}{k}\binom{n+k}{n}\binom{n-2}{k-1}\binom{2k-1}{n-2} \\
&\equiv 0 \pmod 2. \qquad (4.31)
\end{aligned}$$

Finally, combining (4.29)–(4.31) gives

$$b_{2n} \equiv b_n \pmod 8, \qquad (4.32)$$

for all integers $n \geq 2$.

By using the induction on n, (4.28) and (4.32), we complete the proof of (4.27). ■

Proof of (4.19) Let

$$c_n = \sum_{k=0}^{n}(-1)^k 3^{n-3k}\binom{n}{3k}\binom{n+k}{k}\binom{3k}{2k}\binom{2k}{k}.$$

Since $\binom{2k}{k} \equiv 0 \pmod 2$ for $k \geq 1$, we have

$$c_n \equiv 3^n\sum_{k=0}^{n}\binom{n}{3k}\binom{n+k}{k}\binom{3k}{2k}\binom{2k}{k} \pmod 8. \qquad (4.33)$$

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By the recurrence (1.6), we have

$$(2n+1)^3 c_{2n+1} - (4n+1)(28n^2+14n+3)c_{2n} + 648n^3 c_{2n-1} = 0.$$

It follows that

$$(4n^2+6n+1)c_{2n+1} + (4n^2+6n+5)c_{2n} \equiv 0 \pmod 8,$$

and so

$$\begin{aligned} c_{2n+1} &\equiv -\frac{4n^2+6n+5}{4n^2+6n+1}c_{2n} \\ &= -\left(1+\frac{4}{4n^2+6n+1}\right)c_{2n} \\ &\equiv 3c_{2n} \pmod 8. \end{aligned} \tag{4.34}$$

On the other hand, by (4.33), we have

$$\begin{aligned} c_{2n} &\equiv 3^{2n}\sum_{k=0}^{2n}\binom{2n}{3k}\binom{2n+k}{k}\binom{3k}{2k}\binom{2k}{k} \\ &= 3^{2n}\sum_{k=0}^{n}\binom{2n}{6k}\binom{2n+2k}{2k}\binom{6k}{4k}\binom{4k}{2k} \\ &\quad + 3^{2n}\sum_{k=0}^{n}\binom{2n}{3(2k+1)}\binom{2n+2k+1}{2k+1}\binom{3(2k+1)}{2(2k+1)}\binom{2(2k+1)}{2k+1} \pmod 8. \end{aligned} \tag{4.35}$$

Since $\binom{4k}{2k} \equiv 0 \pmod 2$ and $\binom{n}{3k}\binom{n+k}{k}\binom{3k}{2k} = \binom{2k}{k}\binom{4k}{2k}\binom{n+k}{4k} \equiv 0 \pmod 4$ for $k \geq 1$, we have

$$\begin{aligned} &3^{2n}\sum_{k=0}^{n}\binom{2n}{6k}\binom{2n+2k}{2k}\binom{6k}{4k}\binom{4k}{2k} \\ &\equiv 3^{2n}\sum_{k=0}^{n}\binom{n}{3k}\binom{n+k}{k}\binom{3k}{2k}\binom{4k}{2k} \\ &\equiv 3^{2n}\sum_{k=0}^{n}\binom{n}{3k}\binom{n+k}{k}\binom{3k}{2k}\binom{2k}{k} \\ &\equiv 3^n c_n \pmod 8, \end{aligned} \tag{4.36}$$

where we have used (4.24).

Furthermore, noting that for $k \geq 1$,

$$\binom{4k+1}{2k} = \frac{2(4k+1)}{2k-1}\binom{4k-1}{2k-2} \equiv 0 \pmod 2,$$

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we have

$$\begin{aligned}
&3^{2n}\sum_{k=0}^{n}\binom{2n}{3(2k+1)}\binom{2n+2k+1}{2k+1}\binom{3(2k+1)}{2(2k+1)}\binom{2(2k+1)}{2k+1}\\
&=3^{2n}4n\sum_{k=0}^{n}\frac{1}{6k+3}\binom{2n-1}{6k+2}\binom{2n+2k+1}{2k+1}\binom{6k+3}{4k+2}\binom{4k+1}{2k}\\
&\equiv 4n(n-1)\\
&\equiv 0 \pmod 8. \qquad (4.37)
\end{aligned}$$

Finally, combining (4.35)–(4.37) gives

$$c_{2n}\equiv 3^n c_n \pmod 8. \qquad (4.38)$$

By using the induction on n, (4.34) and (4.38), we complete the proof of (4.19).

We remark that one can also prove (4.14)–(4.19) through an alternative approach using automata (see, for instance, [3, 7, 12, 13]). ■

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