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NUMERICAL SOLUTION FOR STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY *G*-BROWNIAN MOTION UNDER LOCALLY LIPSCHITZ CONTINUOUS COEFFICIENTS

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Abstract

Since stochastic differential equations (SDEs) driven by *G*-Brownian motion are of great importance in modeling situations that incorporate ambiguity, it is essential to address efficient numerical schemes to approximate the solution of such equations. The stream of research related to the numerical solutions of *G*-SDEs under standard assumptions is to some extent well understood. In this note, we are interested in designing an implicit θ-Euler–Maruyama scheme to approximate the solution of *G*-SDEs under locally Lipschitz continuous coefficients. The convergence of the proposed scheme is established using the stopping time technique. In addition, we investigate the exponentially/quasi-surely asymptotic stability property of the scheme.

Keywords: *G*-SDEs; *G*-Brownian motion; locally Lipschitz; θ-Euler–Maruyama method; convergence; asymptotic stability

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1. Introduction

The motivation of the *model uncertainty/ambiguity* topic stems from the fact that the presence of a family of probability measures, which are singular from each other, in studying super hedging price and risk measuring problems is inevitable. In fact, since a family of martingale probability measures leads to incompleteness, it is impossible to perfectly replicate hedges with certainty.

Faced with such circumstances, several recent efforts have been put into developing the theory of arbitrage-free pricing under ambiguity, where there is a lack of consensus among the probability assignments. The optimal super-replication price of a European contingent claim in the context of uncertainty with (non-dominated) martingale probability measures was acquired in [7]. By focusing on ambiguity about both drift and volatility in a continuous-time framework, [8, 9] formulated a model of utility that captures the decision-maker's concern under ambiguity. The reader can refer to [21, 22, 25] and the references therein for more detailed discussions.

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In order to study the problem of ambiguity, [23, 24] developed the G-expectation theory and G-Itô stochastic calculus, which systematically established a framework of time-consistent sublinear expectation.

Since the stochastic differential equations driven by G-Brownian motion (G-SDEs for short) have a pivotal role in modeling financial processes, they have attracted a great deal of attention from researchers. For instance, [2] is devoted to studying the G-conditional expectation of a discounted payoff related to a sufficiently smooth G-SDE, which is obtained by a generalization of a Feynmac–Kac formula under volatility uncertainty. Also, the property of continuous dependence on the parameters of stochastic integrals and solutions of G-SDEs was studied in [16]. The pathwise properties and homeomorphic ones with respect to the initial values for such equations were established in [10]. The authors in [6] proved that the stochastic stability of stochastic delay differential equations driven by G-Brownian motion is equivalent to that of the Euler–Maruyama scheme. In [1], a split-step backward Euler–Maryuama scheme was employed to approximate the solution and the pathwise convergence of the scheme was proved.

Most of the results in the literature are concerned with G-SDEs under standard assumptions. However, recently, some research has been dedicated to the G-SDEs which possess more complex coefficients. Both the existence and uniqueness of the solution of G-SDEs under integral-Lipschitz conditions were investigated in [4]. In [3], the existence and uniqueness of the solution of the Cox–Ingersoll–Ross process driven by G-Brownian motion was provided, and some of its properties, including the Markov property and its distribution, were indicated.

We might need to equip ourselves with more advanced instruments of G-stochastic calculus in order to study G-SDEs with less smoothness. For instance, the integrand space of the first definition of the G-Itô integral is restricted to stochastic processes that are continuous in ω, while it might be required to define the G-Itô integral for processes that are beyond the quasi-continuous space. In particular, this is relevant to the case of the G-Itô integral on $[0, \tau]$ where τ is a stopping time; stopping time is a delicate issue in the G-framework and may no longer be quasi-continuous unless some strong conditions are imposed—see [17] to more details.

To overcome this difficulty, [15] extended the definition of the G-Itô integral to the larger spaces $M^2_\star([0, T])$ and $M^2_\omega([0, T])$ that contain processes that are not necessarily quasi-continuous, and also we have $M^2_G([0, T]) \subset M^2_\star([0, T]) \subset M^2_\omega([0, T])$.

The existence and uniqueness of the solution of G-SDEs with local Lipschitz coefficients under a Lyapunov-type condition are dealt with in [13], which also studies the stability of the equation under any arbitrarily small perturbation of the initial value of the solution. They benefited from [15] to find the solution of the equation in the $M_\omega([0, T])$ space by a localization technique. Motivated by the aforementioned paper studying G-SDEs under locally Lipschitz continuous coefficients, we develop a numerical scheme for the solution of the equation and then prove its convergence and asymptotic stability properties.

To this end, numerical solutions of stochastic differential equations driven by classical Brownian motion, in which the coefficients do not obey the global Lipschitz condition, can be an inspiration for the G-SDE case. For instance, a numerical solution for SDEs with non-linear and non-Lipschitz coefficients is presented in [20], and its strong convergence and almost sure stability are proved. In [12], an SDE satisfying only a local Lipschitz property is considered. Then the strong convergence of the Euler–Maruyama scheme and also the boundedness of the pth moments of the exact and numerical solutions are investigated for some $p > 2$. Moreover, an Euler–Maruyama-type method in [19] is applied to the numerical solutions for stochastic differential delay equations with variable delay under the local Lipschitz condition.

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Besides Euler-type schemes, the so-called θ-*schemes* include both explicit and implicit methods governed by a parameter $\theta \in [0, 1]$. In [11], a stochastic variant of the theta method is employed to approximate a linear test SDE with multiplicative noise, and the mean-square and asymptotic stability properties are subsequently analyzed. Moreover, the authors in [5] apply stochastic theta methods to find random periodic solutions of stochastic differential equations under non-globally Lipschitz conditions. The split-step theta method, addressing a class of SDEs with piecewise continuous arguments and polynomial growth conditions, is investigated in [18]. Additional relevant studies can be found in [26, 27]. Inspired by these θ-schemes for SDEs, we adopt a similar approach to develop an approximation method for the G-SDE under study.

The rest of the paper is arranged as follows. Section 2 is devoted to the necessary preliminaries of G-expectation theory. In Section 3, we study the theoretical property of G-SDEs under the locally Lipschitz condition. In Section 4, we introduce an implicit continuous Euler–Maruyama scheme to approximate the solution of the associated equation and then the convergence and stability of the scheme are examined. Section 5 introduces two test problems to illustrate the theoretical findings.

2. Preliminaries

In this section we review some necessary preliminaries on G-Brownian motion calculus to be used throughout the paper. Let $\Omega := C_0^d(\mathbb{R}^+)$ be the space of all $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{t\in\mathbb{R}^+}$ with $\omega_0 = 0$ equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i}\Big[\Big(\max_{t\in[0,i]}|\omega_t^1 - \omega_t^2|\Big) \wedge 1\Big], \quad \omega^1, \omega^2 \in \Omega.$$

Let $t \in [0, \infty)$. Define $\Omega_t := \{\omega_{\cdot\wedge t} : \omega \in \Omega\}$ and consider the canonical process $B_t(\omega) = \omega_t$ for $\omega \in \Omega$. Set

$$\mathrm{Lip}(\Omega_t) := \{\varphi(B_{t_1\wedge t}, \ldots, B_{t_n\wedge t}) : \text{for all } n \in \mathbb{N},\ t_i \geq 0, \varphi \in C_{\mathrm{l.Lip}}(\mathbb{R}^n)\},$$

where $a \wedge b = \min\{a, b\}$ and we denote by $C_{\mathrm{l.Lip}}(\mathbb{R}^n)$ the linear space of functions φ satisfying the local Lipschitz condition $|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|$ for $x, y \in \mathbb{R}^n$, where the constant $C > 0$ and the integer $m \in \mathbb{N}$ depend on φ. Set $\mathrm{Lip}(\Omega) := \bigcup_n \mathrm{Lip}(\Omega_n)$, which reveals a linear space of real-valued functions defined on Ω and constructs a vector lattice. Let $\hat{\mathbb{E}}$ be a functional defined on $\mathrm{Lip}(\Omega)$ satisfying the monotonicity, preservation of constants, sub-additivity, and positive homogenity properties. The functional $\hat{\mathbb{E}}$ is called a *sublinear expectation* and the triple $(\Omega, \mathrm{Lip}(\Omega), \hat{\mathbb{E}})$ a *sublinear expectation space*. [24] constructs a sublinear expectation $\hat{\mathbb{E}}$ on $(\Omega, \mathrm{Lip}(\Omega))$ called the G-expectation. Note that $\mathrm{Lip}(\Omega)$ can be extended to $L_G^p(\Omega)$ for $p \geq 1$, which denotes the completion of $\mathrm{Lip}(\Omega)$ under $\|X\|_p := (\hat{\mathbb{E}}[|X|^p])^{1/p}$ for $X \in \mathrm{Lip}(\Omega)$.

We denote by $\mathcal{P}$ a weakly compact but possibly non-dominated family of probability measures that represents $\hat{\mathbb{E}}$ as $\hat{\mathbb{E}}[X] = \sup_{P\in\mathcal{P}} E_P[X]$ for $X \in \mathrm{Lip}(\Omega)$, where $\{E_P : P \in \mathcal{P}\}$ represents a subset of linear expectations. We denote by $\mathcal{B}(\Omega)$ the Borel σ-algebra of Ω. The *capacity* associated with $\mathcal{P}$ is denoted by $c(A) := \sup_{P\in\mathcal{P}} P(A)$, $A \in \mathcal{B}(\Omega)$. A set A is polar if $c(A) = 0$ and a property holds 'quasi-surely' (q.s.) if it holds outside a polar set.

Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a sublinear expectation space, i.e. $\mathcal{H}$ is a linear space of real-valued functions defined on Ω and constructs a vector lattice, and $\hat{\mathbb{E}}$ is a sublinear expectation on $\mathcal{H}$. The

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space $\mathcal{H}$ is considered as the space of 'random variables'. Moreover, we denote by $C_{\mathrm{b,Lip}}(\mathbb{R}^n)$ the space of bounded and Lipschitz continuous functions on $\mathbb{R}^n$.

Definition 2.1. A random variable $Y \in \mathcal{H}$ is said to be independent of another random vector $X \in \mathcal{H}$ under $\hat{\mathbb{E}}$ if, for each test function $\varphi \in C_{\mathrm{b,Lip}}(\mathbb{R}^2)$, $\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]|_{\{x=X\}}]$.

Definition 2.2. We say that a random variable X on the sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called (centered) G-normally distributed and denoted as $X \sim \mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$ with $\overline{\sigma}^2 = \hat{\mathbb{E}}[X^2]$ and $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-X^2]$ if $\hat{\mathbb{E}}[\varphi(aX + b\overline{X})] = \hat{\mathbb{E}}[\varphi(\sqrt{a^2 + b^2}X)]$, $\varphi \in C_{\mathrm{b,Lip}}(\mathbb{R})$, for any $a, b \geq 0$ and $\overline{X} \in \mathcal{H}$ independent of X such that $\hat{\mathbb{E}}[\varphi(\overline{X})] = \hat{\mathbb{E}}[\varphi(X)]$.

Remark 2.1. Let $X \sim \mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$. Then the distribution of X is characterized by the function $u(t, x) = \hat{\mathbb{E}}[\varphi(x + \sqrt{t}X)]$, $(t, x) \in [0, \infty) \times \mathbb{R}$, $\varphi \in C_{\mathrm{b,Lip}}(\mathbb{R})$, where u is the unique viscosity solution of the following non-linear parabolic partial differential equation (PDE), called G-heat equation:

$$\partial_t u - G(\partial_{xx} u) = 0, \quad u|_{t=0} = \varphi, \tag{2.1}$$

with the infinitesimal generator

$$G(a) = \frac{1}{2}(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-). \tag{2.2}$$

Since $u(1, 0) = \hat{\mathbb{E}}[\varphi(X)]$, solving the PDE (2.1) simulates the G-normal distribution.

Note that the space $C_{\mathrm{b,Lip}}(\mathbb{R})$ can be replaced by $C_{\mathrm{l.Lip}}(\mathbb{R})$ due to the assumptions on $\mathcal{H}$.

Definition 2.3. A one-dimensional stochastic process $B = (B_t)_{t \geq 0}$ on $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a G-Brownian motion if $B_{t_1}, \ldots, B_{t_n} \in \mathcal{H}$ for each $n \in \mathbb{N}$ and $0 \leq t_1 < \cdots < t_n < \infty$, and the following properties are satisfied:

(i) $B_0 = 0$.

(ii) For each $t, s \geq 0$, the increment $B_{t+s} - B_t$ is $\mathcal{N}(0; [\underline{\sigma}^2 s, \overline{\sigma}^2 s])$-distributed and is independent from $(B_{t_1}, \ldots, B_{t_n})$ for each $n \in \mathbb{N}$ and $0 \leq t_1 < \cdots < t_n \leq t$.

Let $\pi_t^n := \{0 = t_0^n < \cdots < t_{N_n}^n = t\}$, $n = 1, 2, \ldots$ be a sequence of partitions of $[0, t]$. Then the quadratic variation process of G-Brownian motion $(B_t)_{t \geq 0}$ is defined by

$$\langle B \rangle_t := \lim_{\mu(\pi_t^n) \to 0} \sum_{j=1}^{N_n - 1} \left(B_{t_{j+1}} - B_{t_j}\right)^2,$$

where $\mu(\pi_t^n) = \sup_{0 \leq i \leq N_n - 1} |t_{i+1}^n - t_i^n|$ and we assume that $\lim_{n \to \infty} \mu(\pi_T^n) = 0$. We can show that $\langle B \rangle_t = B_t^2 - 2 \int_0^t B_s \, \mathrm{d}B_s$ in $L_G^2(\Omega)$. The quadratic variation process $\langle B \rangle_t$ is an increasing process with $\langle B \rangle_0 = 0$ and increment $\langle B \rangle_{s+t} - \langle B \rangle_s$, is identically distributed with $\langle B \rangle_t$, and is independent from Ω_s for $s, t \geq 0$.

It is worth noting that the canonical process $B_t(\omega)$ defines a G-Brownian motion on $(\Omega, \mathrm{Lip}(\Omega), \hat{\mathbb{E}})$ by remembering that $\hat{\mathbb{E}}$ is the G-expectation on $\mathrm{Lip}(\Omega)$. The reader can refer to [23, 24] for more details of G-expectation theory, including the notation of conditional expectation $\hat{\mathbb{E}}[\cdot \mid \Omega_t]$.

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Lemma 2.1. *Let* $\overline{\sigma}^2 = \hat{\mathbb{E}}[B_1^2]$, $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-B_1^2]$, $0 \le z \le t < \infty$, *and* $n \ge 1$*; then*

$$\hat{\mathbb{E}}[(\langle B\rangle_t - \langle B\rangle_z)^n | \Omega_z] = \overline{\sigma}^{2n}(t-z)^n,$$

$$\hat{\mathbb{E}}[-(\langle B\rangle_t - \langle B\rangle_z)^n | \Omega_z] = -\underline{\sigma}^{2n}(t-z)^n,$$

$$\underline{\sigma}^2(t-z) \le \langle B\rangle_t - \langle B\rangle_z \le \overline{\sigma}^2(t-z) \quad q.s.$$

Lemma 2.2. (Borel–Cantelli.) *For a sequence of Borel sets* $(A_n)_{n\in\mathbb{N}}$ *with* $\sum_{n=1}^{\infty} c(A_n) < \infty$, $\limsup_{n\to\infty} A_n$ *is polar.*

We set up the following sets:

$L^0(\Omega)$: the space of all $\mathcal{B}(\Omega)$-measurable real functions;

$L^0(\Omega_t)$: the space of all $\mathcal{B}(\Omega_t)$-measurable real functions;

$B_{\mathrm{b}}(\Omega)$: the space of all bounded elements in $L^0(\Omega)$;

$B_{\mathrm{b}}(\Omega_t)$: the space of all bounded elements in $L^0(\Omega_t)$;

$C_{\mathrm{b}}(\Omega)$: all continuous elements in $B_{\mathrm{b}}(\Omega)$;

and, for $p > 0$,

$$\mathcal{L}^p := \{X \in L^0(\Omega) \colon \hat{\mathbb{E}}[|X|^p] = \sup_{P\in\mathcal{P}} E_P[|X|^p] < \infty\};$$

$$\mathcal{N} := \{X \in L^0(\Omega) \colon X = 0,\ c\text{-}q.s.\}.$$

Lemma 2.3 (Markov's inequality.) *Set* $\mathbb{L}^p := \mathcal{L}^p/\mathcal{N}$. *Let* $X \in \mathbb{L}^p$ *for* $p > 0$. *Then, for each* $\alpha > 0$, $c(\{|X| > \alpha\}) \le \hat{\mathbb{E}}[|X|^p]/\alpha^p$.

As indicated in the introduction, the G-Itô integral of a quasi-continuous process was first defined in [24] and the definition was extended to the larger spaces containing the stochastic process without assuming quasi-continuity in [15]. In the following, based on [14, 15], we briefly review this.

Let $p > 0$. We denote by $L_\star^p(\Omega)$ and $L_\star^p(\Omega_t)$ the completion of $B_{\mathrm{b}}(\Omega)$ and $B_{\mathrm{b}}(\Omega_t)$ under the norm $\|\cdot\|_p$, respectively. We can prove that $L_G^p(\Omega)$ is the space of the completion of $C_{\mathrm{b}}(\Omega)$ under the natural norm $\|\cdot\|_p$, and

$$L_G^p(\Omega) = \{X \in L^0(\Omega) \mid X \text{ has a q.c. version with } \lim_{n\to\infty} \hat{\mathbb{E}}[|X|^p I_{\{|X|>n\}}] = 0\}.$$

Also, $L_\star^p(\Omega) = \{X \in L^0(\Omega) \mid \lim_{n\to\infty} \hat{\mathbb{E}}[|X|^p I_{\{|X|>n\}}] = 0\}$. Clearly, $L_\star^p(\Omega) \supset L_G^p(\Omega)$.

Let $T \in \mathbb{R}^+$. We define the following type of simple process:

$$M_{\mathrm{b},0}(0,T) = \Bigg\{ \eta \mid \eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j,t_{j+1})}(t)$$

$$\text{for all } N \in \mathbb{N},\ 0 = t_0 < \cdots < t_N = T,\ \xi_j \in B_{\mathrm{b}}\big(\Omega_{t_j}\big),\ j = 0,\ldots,N-1 \Bigg\}.$$

This allows us to define Itô's integral with respect to G-Brownian motion and its quadratic variation process for functions belonging to $M_{\mathrm{b},0}(0,T)$. From now, we assume that $d = 1$.

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Now, for $\eta \in M_{\mathrm{b},0}(0,T)$, we define

$$I(\eta) = \int_0^T \eta_t(\omega)\,\mathrm{d}B_t := \sum_{j=0}^{N-1} \xi_j(\omega)\big(B_{t_{j+1}} - B_{t_j}\big),$$

$$J(\eta) = \int_0^T \eta_t(\omega)\,\mathrm{d}\langle B\rangle_t := \sum_{j=0}^{N-1} \xi_j(\omega)\big(\langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j}\big).$$

Let $p \geq 1$. We denote by $M_*^p(0,T)$ the completion of $M_{\mathrm{b},0}(0,T)$ under the norm

$$\|\eta\|_{M_*^p(0,T)} := \left\{\hat{\mathbb{E}}\left[\int_0^T |\eta_t|^p\,\mathrm{d}t\right]\right\}^{1/p}. \tag{2.3}$$

The mapping $I\colon M^{b,0}(0,T) \to L_\star^2(\Omega_T)$ is a linear continuous mapping and thus can be continuously extended to $I\colon M_\star^2(0,T) \to L_\star^2(\Omega_T)$. Moreover, for $\eta \in M_\star^2(0,T)$,

$$\hat{\mathbb{E}}\left[\int_0^T \eta_v\,\mathrm{d}B_v\right] = 0. \tag{2.4}$$

Similarly, J can be continuously extended to $J\colon M_\star^1(0,T) \to L_\star^1(\Omega_T)$. Note that $M_\star^p(0,T) \supset M_\star^q(0,T)$ for $p < q$. We can show that $\int_0^t \eta_s\,\mathrm{d}B_s$ is continuous in t quasi-surely for $0 \leq t \leq T$, and also $\int_0^\cdot \eta_s\,\mathrm{d}B_s \in M_\star^2(0,T)$ for any $\eta \in M_\star^2(0,T)$.

Proposition 2.1. *Let $p \geq 1$, X, $\eta \in M_\star^p([0,T])$ with η bounded; then $X\eta \in M_\star^p([0,T])$.*

Analogously to $M_{\mathrm{b},0}(0,T)$, we define $\bar{M}_{\mathrm{b},0}(0,T)$ as the collection of simple processes where $\xi_j \in L_G^p(\Omega_{t_j})$ for $p \geq 1$ and $M_G^p(0,T)$, as the completion of $\bar{M}_{\mathrm{b},0}(0,T)$ under the norm $\|\eta\|_{M_G^p(0,T)}$, which is defined similarly to $\|\cdot\|_{M_*^p(0,T)}$ in (2.3).

Definition 2.4. Let $p > 0$ be fixed. A stochastic process $(\eta_t)_{t\geq 0}$ is said to be in $M_\omega^p(0,T)$ if there exists a sequence of increasing stopping times $\{\sigma_m\}_{m=1}^\infty$ with $\sigma_m \uparrow T$ quasi-surely such that $\eta I_{[0,\sigma_m]} \in M_\star^p(0,T)$ and $\int_0^T |\eta_s|^p\,\mathrm{d}s < \infty$, quasi-surely.

Let $\mathcal{F}_t := \mathcal{B}(\Omega_t)$ for $t \geq 0$. A map $\tau\colon \Omega \to [0,T]$ is said to be a stopping time if $\{\tau \leq t\} \in \mathcal{F}_t$. For each stopping time τ, $p \geq 1$, and $\eta \in M_\star^p(0,T)$, using Proposition 2.1 we can prove that $I_{[0,\tau]}(\cdot)\eta \in M_\star^p(0,T) \subseteq M_\omega^p(0,T)$. So, the integral $\int_0^t \eta_s\,\mathrm{d}B_s$ is well-defined and we have

$$\int_0^t I_{[0,\tau]}(s)\eta_s\,\mathrm{d}B_s = \int_0^{t\wedge\tau} \eta_s\,\mathrm{d}B_s \quad q.s. \tag{2.5}$$

Lemma 2.4 ([13].) *Let $\eta \in M_\omega^p([0,T])$. Then, for $p > 0$, there exists $C_p > 0$ depending only on p such that*

$$\hat{\mathbb{E}}\left[\sup_{0\leq s\leq t}\left|\int_0^s \eta_v\,\mathrm{d}B_v\right|^p\right] \leq C_p\overline{\sigma}^{p/2}\hat{\mathbb{E}}\left[\left(\int_0^t |\eta_v|^2\,\mathrm{d}v\right)^{p/2}\right],$$

where $\hat{\mathbb{E}}[B_1^2] = \overline{\sigma}^2$.

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3. The *G*-SDE under locally Lipschitz conditions

In this section we consider the following type of one-dimensional SDE driven by *G*-Brownian motion:

$$X_t = x_0 + \int_0^t f(X_s)\,\mathrm{d}s + \int_0^t g(X_s)\,\mathrm{d}\langle B\rangle_s + \int_0^t \sigma(X_s)\,\mathrm{d}B_s, \quad t \in [0, T], \tag{3.1}$$

where $x_0 \in \mathbb{R}$ and $f(x), g(x), \sigma(x) \in M_G^p([0, T]) \subset M_\star^p([0, T])$ for any $x \in \mathbb{R}$ and some $p \geq 2$. Also, for any $R > 0$, the coefficients $f, g, \sigma : \mathbb{R} \to \mathbb{R}$ satisfy the assumption

$$|b(x) - b(y)| \leq K_R|x - y| \tag{3.2}$$

for any $x, y \in \mathbb{R}$ with $|x| \vee |y| \leq R$ and some positive constant K_R. Note that function b is taken from set $\{f, g, \sigma\}$. Moreover,

$$xf(x) + xg(x) + |\sigma(x)|^2 \leq K(1 + |x|^2) \tag{3.3}$$

for any $x \in \mathbb{R}$ and some positive constant K.

Definition 3.1. Let $p \geq 2$. The process $(X_t)_{t\in[0,T]} \in M_\omega^p([0, T]; \mathbb{R})$ is said to be the solution of G-SDE (3.1) if it satisfies the equation and has t-continuous paths on [0, T].

It is well known that *G*-SDEs under globally Lipschitz coefficients possess a unique solution. Now, based on [13, Theorem 4.5 and Example 4.7], we present following theorem which deals with the locally Lipschitz coefficients case.

Theorem 3.1. *Under the assumptions (3.2) and (3.3), G-SDEs has a unique solution.*

In the next section we deal with the stability property of the trivial and approximation solutions.

3.1. Stability of the *G*-SDE

In this section we review the concept of stability of the trivial solution $X \equiv 0$ of the *G*-SDE (3.1). To this end, we first assume that

$$f(0) = 0, \qquad g(0) = 0, \qquad \sigma(0) = 0. \tag{3.4}$$

Definition 3.2. Let $X^{s,x}$ be the solution of G-SDE (3.1) starting at s and $X(s) = x$, $x \in \mathbb{R}$. Let $p > 0$. If there exist some positive constants C and λ such that $\hat{\mathbb{E}}[|X_t^{s,x}|^p] \leq C|x_0|^p \mathrm{e}^{-\lambda(t-s)}$, then the trivial solution is called exponentially *p*-stable. In particular, when $p = 2$, the trivial solution is said to be asymptotically exponentially stable in mean square. Moreover, the trivial solution of the G-SDE is said to be quasi-surely exponentially stable if there exists a positive $\eta > 0$ such that $\limsup_{t\to\infty}\{\log(|X_t^{s,x}|)/t\} < -\eta$ q.s.

We now present [13, Corollary 5.6] for the one-dimensional case.

Theorem 3.2. *Let $p > 0$. Assume that conditions (3.2) and (3.4) hold, and there exist constants $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ with $0 \leq \alpha_2 < \alpha_3$ such that, for $x \in \mathbb{R}$,*

$$xf(x) + G(2xg(x) + \sigma^2(x)) \leq \alpha_1 x^2, \tag{3.5}$$

where the function G is defined by (2.2) and

$$\alpha_2^2 \leq \overline{\sigma}^2\sigma^2(x) \leq \alpha_3^2 \tag{3.6}$$

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for any $x \in \mathbb{R}$. Then, the trivial solution of the G-SDE (3.1) is exponentially p-stable if $\alpha_1 < 0$ and $p < 2 + 2|\alpha_1|/\alpha_3^2$, or $0 \le \alpha_1 \le \alpha_2^2$ and $p < 2 - 2\alpha_1/\alpha_2^2$.

Remark 3.1. In addition to conditions (3.2) and (3.4), let the G-SDE (3.1) satisfy the condition $xf(x) \vee |xg(x)| \vee \sigma^2(x) \le K|x|^2$ for some $K > 0$ and any $x \in \mathbb{R}$. If the trivial solution of the equation is exponentially p-stable, $p > 0$, then it is quasi-surely exponentially stable. See [13, Theorem 5.8].

4. A time discretization scheme

In this section we set up a so-called θ-*Euler–Maruyama* method (θ-EM for short) to approximate the true solution of the G-SDE (3.1); we then prove that it converges to the exact solution.

Let $n \in \mathbb{N}$. We consider a time discretization of the interval $[0, T]$ as $0 = t_0^n < \cdots < t_{N_n}^n = T$, with the time steps $\Delta_i = t_{i+1}^n - t_i^n$ for $i = 0, \ldots, N_n - 1$. We denote by $\widetilde{Y}$ the numerical solution θ-EM obtained by the scheme

$$\widetilde{Y}_{i+1} = \widetilde{Y}_i + (1-\theta) f(\widetilde{Y}_i)\Delta_i + \theta f(\widetilde{Y}_{i+1})\Delta_i + g(\widetilde{Y}_i)\,\Delta\langle B\rangle_i + \sigma(\widetilde{Y}_i)\,\Delta B_i \tag{4.1}$$

for $i = 0, 1, \ldots, N_n - 1$, where $\theta \in [0, 1]$, $\widetilde{Y}_0 = x$, $\Delta\langle B\rangle_i = \langle B\rangle_{t_{i+1}} - \langle B\rangle_{t_i}$, and $\Delta B_i = B_{t_{i+1}} - B_{t_i}$. Note that $\theta \in [0, 1]$ controls the degree of implicitness of the scheme.

Let $t \in [0, T]$. Now, we define a continuous-time approximation of the scheme (4.1) as

$$\begin{aligned}\widetilde{Y}^n(t) = \widetilde{Y}_0 &+ (1-\theta)\int_0^t f(\widetilde{Y}_{\lfloor s\rfloor})\,\mathrm{d}s + \theta\int_0^t f(\widetilde{Y}_{\lceil s\rceil})\,\mathrm{d}s \\ &+ \int_0^t g(\widetilde{Y}_{\lfloor s\rfloor})\,\mathrm{d}\langle B\rangle_s + \int_0^t \sigma(\widetilde{Y}_{\lfloor s\rfloor})\,\mathrm{d}B(s),\end{aligned} \tag{4.2}$$

where $\lfloor s\rfloor := i$ and $\lceil s\rceil := i+1$ for $t_i \le s < t_{i+1}$; we set $\lfloor N_n\rfloor := N_n$ and $\lceil N_n\rceil := N_n$ as a convenience. Note that the process $(\widetilde{Y}^n(t))_{\{t\in[0,T]\}} \in M_\omega^p([0, T]; \mathbb{R})$.

In the following we investigate the convergence of the discretization scheme (4.2) in the pth mean for $p \ge 1$. To this end, inspired by Lemma 2.1, we present following remark.

Remark 4.1. Let $p > 0$ and $\eta \in M_\omega^p([0, T]; \mathbb{R})$. Then

$$\hat{\mathbb{E}}\left[\sup_{0\le s\le t}\left|\int_0^s \eta_v\,\mathrm{d}\langle B\rangle_v\right|^p\right] \le \overline{\sigma}^{2p} t^{p-1}\hat{\mathbb{E}}\left[\int_0^t |\eta_v|^p\,\mathrm{d}v\right]$$

for any $t \in [0, T]$.

We assume that for $p \ge 2$, there exists a constant A such that

$$\hat{\mathbb{E}}(\sup_{0\le t\le T}|X(t)|^p) \vee \hat{\mathbb{E}}(\sup_{0\le t\le T}|\widetilde{Y}^n(t)|^p) \le A. \tag{4.3}$$

This assumption does not pose a limitation. For example, the relation (4.6) stated in Theorem 4.2 ensures that (4.3) holds true.

Now, we present Young's inequality before investigating the convergence of the numerical scheme.

Lemma 4.1 (Young's inequality.) *For $1/r + 1/q = 1$,*

$$ab \le \frac{\delta}{r}a^r + \frac{1}{q\delta^{q/r}}b^q \quad \text{for all } a, b, \delta > 0.$$

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Now let $R > 0$. Set $\tau_R := \inf\{t \geq 0 \colon |\widetilde{Y}^n(t)| \geq R\}$, $\rho_R := \inf\{t \geq 0 \colon |X(t)| \geq R\}$, and $\gamma_R := \tau_R \wedge \rho_R$.

Lemma 4.2. *Assume that $R > 0$ and $t \in [t_i, t_{i+1}] \subseteq [0, T \wedge \gamma_R]$ for some $i \in \{0, \ldots, N_n - 1\}$. Let assumption (3.2) hold. Then, there exists a positive constant $\bar{c}$ depending on K_R, C_2, $\overline{\sigma}$, T, A, $f(0)$, $g(0)$, and $h(0)$ such that*

$$\hat{\mathbb{E}}[|\widetilde{Y}^n(t) - \widetilde{Y}_i|^2] \leq \bar{c}(t - t_i),$$

$$\hat{\mathbb{E}}[|\widetilde{Y}^n(t) - \widetilde{Y}_{i+1}|^2] \leq \bar{c}(t_{i+1} - t).$$

Proof. According to (4.2), we can write

$$\widetilde{Y}^n(t) = \widetilde{Y}_i + \theta \int_{t_i}^t f(\widetilde{Y}_{\lfloor s \rfloor})\,\mathrm{d}s + (1-\theta)\int_{t_i}^t f(\widetilde{Y}_{\lceil s \rceil})\,\mathrm{d}s + \int_{t_i}^t g(\widetilde{Y}_{\lfloor s \rfloor})\,\mathrm{d}\langle B\rangle_s + \int_{t_i}^t \sigma(\widetilde{Y}_{\lfloor s \rfloor})\,\mathrm{d}B(s).$$

Using Hölder's inequality, Lemma 2.4, and Remark 4.1 results in

$$\begin{aligned}\hat{\mathbb{E}}[|\widetilde{Y}^n(t) - \widetilde{Y}_i|^2] &\leq 4\theta^2(t - t_i)\int_{t_i}^t \hat{\mathbb{E}}[|f(\widetilde{Y}_{\lfloor s \rfloor})|^2]\,\mathrm{d}s \\ &\quad + 4(1-\theta)^2(t - t_i)\int_{t_i}^t \hat{\mathbb{E}}[|f(\widetilde{Y}_{\lceil s \rceil})|^2]\,\mathrm{d}s \\ &\quad + 4\overline{\sigma}^4(t - t_i)\int_{t_i}^t \hat{\mathbb{E}}[|g(\widetilde{Y}_{\lfloor s \rfloor})|^2]\,\mathrm{d}s + 4C_2\overline{\sigma}\int_{t_i}^t \hat{\mathbb{E}}[|\sigma(\widetilde{Y}_{\lfloor s \rfloor})|^2]\,\mathrm{d}s.\end{aligned}$$

Since $[t, t_{i+1}] \subseteq [0, T \wedge \gamma_R]$, by the locally Lipschitz property (3.2) and assumption (4.3), we can see that

$$\hat{\mathbb{E}}[|b(\widetilde{Y}_{\lfloor s \rfloor})|^2] \leq 2|b(0)|^2 + 2K_R^2\hat{\mathbb{E}}(\sup_{0\leq s\leq T}|\widetilde{Y}^n(s)|^2) \leq 2|b(0)|^2 + 2K_R^2 A$$

for $b \in \{f, g, h\}$. Then, $\hat{\mathbb{E}}[|\widetilde{Y}^n(t) - \widetilde{Y}_i|^2] \leq 8(K_R^2 A + M_{f,g,h})(T(2 + \overline{\sigma}^4) + C_2\overline{\sigma})(t - t_i)$, where $M_{f,g,h} = \max\{|f(0)|^2, |g(0)|^2, |h(0)|^2\}$ and the second assertion follows. □

Theorem 4.1. *Let $n \in \mathbb{N}$ and set $e_n(t) := X(t) - \widetilde{Y}^n(t)$ for $t \in [0, T]$. Then the continuous scheme (4.2) converges to the true solution of (3.1) in the sense*

$$\lim_{h_n \to 0} \hat{\mathbb{E}}(\sup_{0\leq t\leq T}|e_n(t)|^2) = 0,$$

where $h_n = \sup_{0\leq i\leq N_n - 1}|t_{i+1}^n - t_i^n|$.

Proof. For any $t \in [0, T]$, we have that

$$\hat{\mathbb{E}}(\sup_{0\leq v\leq t}|e_n(v)|^2) \leq \hat{\mathbb{E}}(\sup_{0\leq v\leq t}|e_n(v)|^2\mathbf{1}_{\{\gamma_R > T\}}) + \hat{\mathbb{E}}(\sup_{0\leq v\leq t}|e_n(v)|^2\mathbf{1}_{\{\gamma_R \leq T\}}).$$

Applying Lemma 4.1 for the second term by setting $r = p/2$ and $q = p/(p-2)$ for $p > 2$, we arrive at

$$\hat{\mathbb{E}}(\sup_{0\leq v\leq t}|e_n(v)|^2) \leq \hat{\mathbb{E}}(\sup_{0\leq v\leq t}|e_n(v)|^2\mathbf{1}_{\{\gamma_R > T\}}) + \frac{2\delta}{p}\hat{\mathbb{E}}(\sup_{0\leq v\leq t}|e_n(v)|^p) + \frac{1 - (2/p)}{\delta^{2/(p-2)}}\hat{\mathbb{E}}\big(\mathbf{1}_{\{\gamma_R \leq T\}}\big).$$

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According to (4.3), we see that $\hat{\mathbb{E}}(\sup_{0\le v\le t}|e_n(v)|^p)\le 2^pA$ and

$$\begin{aligned}\hat{\mathbb{E}}\big(\mathbf{1}_{[\gamma_R\le T]}\big)&\le c(\tau_R\le T)+c(\rho_R\le T)\\&=\hat{\mathbb{E}}\bigg(\mathbf{1}_{\{\tau_R\le T\}}\frac{|\widetilde{Y}^n(\tau_R)|^p}{R^p}\bigg)+\hat{\mathbb{E}}\bigg(\mathbf{1}_{\{\rho_R\le T\}}\frac{|X(\rho_R)|^p}{R^p}\bigg)\\&\le\frac{1}{R^p}\big(\hat{\mathbb{E}}(\sup_{0\le t\le T}|\widetilde{Y}^n(t)|^p)+\hat{\mathbb{E}}(\sup_{0\le t\le T}|X(t)|^p)\big)\le\frac{2A}{R^p}.\end{aligned}$$

Finally,

$$\hat{\mathbb{E}}(\sup_{0\le v\le t}|e_n(v)|^2)\le\hat{\mathbb{E}}(\sup_{0\le v\le t}|e_n(v\wedge\gamma_R)|^2)+A\bigg(\frac{2^{p+1}\delta}{p}+\frac{2(1-(2/p))}{R^p\delta^{2/(p-2)}}\bigg).\tag{4.4}$$

Note that by the time evolution of X and $\widetilde{Y}^n$ in (3.1) and (4.2),

$$\begin{aligned}e_n(v\wedge\gamma_R)=\theta\int_0^{v\wedge\gamma_R}f(X(s))\,\mathrm{d}s+(1-\theta)\int_0^{v\wedge\gamma_R}f(X(s))\,\mathrm{d}s\\+\int_0^{v\wedge\gamma_R}g(X(s))\,\mathrm{d}\langle B\rangle_s+\int_0^{v\wedge\gamma_R}\sigma(X(s))\,\mathrm{d}B(s)\\-\theta\int_0^{v\wedge\gamma_R}f(\widetilde{Y}_{\lfloor s\rfloor})\,\mathrm{d}s-(1-\theta)\int_0^{v\wedge\gamma_R}f(\widetilde{Y}_{\lceil s\rceil})\,\mathrm{d}s\\-\int_0^{v\wedge\gamma_R}g(\widetilde{Y}_{\lfloor s\rfloor})\,\mathrm{d}\langle B\rangle_s-\int_0^{v\wedge\gamma_R}\sigma(\widetilde{Y}_{\lfloor s\rfloor})\,\mathrm{d}B(s)\end{aligned}$$

for any $v\le t$. Due to $(a+b+c+d)^2\le 4(a^2+b^2+c^2+d^2)$ and by remembering (2.5), we have

$$\begin{aligned}\hat{\mathbb{E}}(\sup_{0\le v\le t}|e_n(v\wedge\gamma_R)|^2)&\le 4\theta^2\,\hat{\mathbb{E}}\bigg(\sup_{0\le v\le t}\bigg|\int_0^v(f(X(s))-f(\widetilde{Y}_{\lfloor s\rfloor}))\mathbf{1}_{[0,v\wedge\gamma_R]}(s)\,\mathrm{d}s\bigg|^2\bigg)\\&+4(1-\theta)^2\,\hat{\mathbb{E}}\bigg(\sup_{0\le v\le t}\bigg|\int_0^v(f(X(s))-f(\widetilde{Y}_{\lceil s\rceil}))\mathbf{1}_{[0,v\wedge\gamma_R]}(s)\,\mathrm{d}s\bigg|^2\bigg)\\&+4\hat{\mathbb{E}}\bigg(\sup_{0\le v\le t}\bigg|\int_0^v(g(X(s))-g(\widetilde{Y}_{\lfloor s\rfloor}))\mathbf{1}_{[0,v\wedge\gamma_R]}(s)\,\mathrm{d}\langle B\rangle_s\bigg|^2\bigg)\\&+4\hat{\mathbb{E}}\bigg(\sup_{0\le v\le t}\bigg|\int_0^v(\sigma(X(s))-\sigma(\widetilde{Y}_{\lfloor s\rfloor}))1_{[0,v\wedge\gamma_R]}(s)\,\mathrm{d}B(s)\bigg|^2\bigg).\end{aligned}$$

Remebering that $\theta\in[0,1]$, the Hölder inequality, Lemma 2.4, and Remark 4.1 lead to

$$\begin{aligned}\hat{\mathbb{E}}(\sup_{0\le v\le t}|e_n(v\wedge\gamma_R)|^2)\le 4\bigg(t\int_0^t\hat{\mathbb{E}}(|(f(X(s))-f(\widetilde{Y}_{\lfloor s\rfloor}))\mathbf{1}_{[0,t\wedge\gamma_R]}(s)|^2)\,\mathrm{d}s\\+t\int_0^t\hat{\mathbb{E}}(|(f(X(s))-f(\widetilde{Y}_{\lceil s\rceil}))\mathbf{1}_{[0,t\wedge\gamma_R]}(s)|^2)\,\mathrm{d}s\\+t\overline{\sigma}^4\int_0^t\hat{\mathbb{E}}(|(g(X(s))-g(\widetilde{Y}_{\lfloor s\rfloor}))\mathbf{1}_{[0,t\wedge\gamma_R]}(s)|^2)\,\mathrm{d}s\\+C_2\overline{\sigma}\int_0^t\hat{\mathbb{E}}(|(\sigma(X(s))-\sigma(\widetilde{Y}_{\lfloor s\rfloor}))\mathbf{1}_{[0,t\wedge\gamma_R]}(s)|^2)\,\mathrm{d}s\bigg).\end{aligned}$$

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Due to the local Lipschitz condition (3.2), we get

$$\begin{aligned}
\hat{\mathbb{E}}(\sup_{0\le v\le t}|e_n(v\wedge\gamma_R)|^2) &\le M_R\int_0^t \hat{\mathbb{E}}[|X(s)-\widetilde{Y}_{\lfloor s\rfloor}|^2\mathbf{1}_{[0,t\wedge\gamma_R]}(s)]\,\mathrm{d}s\\
&\quad + 4TK_R^2\int_0^t \hat{\mathbb{E}}[|X(s)-\widetilde{Y}_{\lceil s\rceil}|^2\mathbf{1}_{[0,t\wedge\gamma_R]}(s)]\,\mathrm{d}s\\
&\le \big(2M_R+8TK_R^2\big)\int_0^t \hat{\mathbb{E}}[|X(s)-\widetilde{Y}^n(s)|^2\mathbf{1}_{[0,t\wedge\gamma_R]}(s)]\,\mathrm{d}s\\
&\quad + 2M_R\int_0^t \hat{\mathbb{E}}[|\widetilde{Y}^n(s)-\widetilde{Y}_{\lfloor s\rfloor}|^2\mathbf{1}_{[0,t\wedge\gamma_R]}(s)]\,\mathrm{d}s\\
&\quad + 8TK_R^2\int_0^t \hat{\mathbb{E}}[|\widetilde{Y}^n(s)-\widetilde{Y}_{\lceil s\rceil}|^2\mathbf{1}_{[0,t\wedge\gamma_R]}(s)]\,\mathrm{d}s\\
&\le (2M_R+8TK_R^2)\int_0^t \hat{\mathbb{E}}[|e_n(s\wedge\gamma_R)|^2]\,\mathrm{d}s\\
&\quad + 2M_R\sum_{i=0}^{N_n-1}\int_{t_i}^{t_{i+1}} \hat{\mathbb{E}}[|\widetilde{Y}^n(s)-\widetilde{Y}_i|^2\mathbf{1}_{[0,t\wedge\gamma_R]}(s)]\,\mathrm{d}s\\
&\quad + 8TK_R^2\sum_{i=0}^{N_n-1}\int_{t_i}^{t_{i+1}} \hat{\mathbb{E}}[|\widetilde{Y}^n(s)-\widetilde{Y}_{i+1}|^2\mathbf{1}_{[0,t\wedge\gamma_R]}(s)]\,\mathrm{d}s,
\end{aligned}$$

where $M_R=4K_R^2(T+T\overline{\sigma}^4+C_2\overline{\sigma})$. By Lemma 4.2, we get

$$\begin{aligned}
\hat{\mathbb{E}}(\sup_{0\le v\le t}|e_n(v\wedge\gamma_R)|^2) &\le \big(2M_R+8TK_R^2\big)\int_0^t \hat{\mathbb{E}}(|e_n(s\wedge\gamma_R)|^2)\,\mathrm{d}s\\
&\quad + 2M_R\bar{c}\sum_{i=0}^{N_n-1}\int_{t_i}^{t_{i+1}}(s-t_i)\,\mathrm{d}s + 8TK_R^2\bar{c}\sum_{i=0}^{N_n-1}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)\,\mathrm{d}s\\
&\le \big(2M_R+8TK_R^2\big)\int_0^t \hat{\mathbb{E}}(|e_n(s\wedge\gamma_R)|^2)\,\mathrm{d}s\\
&\quad + \bar{c}\big(M_R+4TK_R^2\big)\sum_{i=0}^{N_n-1}{\Delta t_i}^2.
\end{aligned}$$

Note that, for simplicity, the points t_i^n have been replaced by t_i for $i=0,1,\ldots,N_n$. Since $\sum_{j=0}^{N_n-1}\Delta t_j=T$, $\sum_{j=0}^{N_n-1}{\Delta t_j}^2\le h_nT$ and, using Grönwall's lemma, we get

$$\hat{\mathbb{E}}(\sup_{0\le v\le t}|e_n(v\wedge\gamma_R)|^p)\le \bar{c}m_RT\mathrm{e}^{2m_RT}h_n,$$

where $m_R=M_R+4TK_R^2$. Now, we return to (4.4). We have

$$\hat{\mathbb{E}}(\sup_{0\le v\le t}|e_n(v)|^2)\le \bar{c}m_RT\mathrm{e}^{2m_RT}h_n + A\bigg(\frac{2^{p+1}\delta}{p}+\frac{2(1-(2/p))}{R^p\delta^{2/(p-2)}}\bigg), \tag{4.5}$$

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and we can choose the step sizes and the parameters δ and R such that

$$\bar{c}m_R T\mathrm{e}^{2m_R T}h_n < \frac{\varepsilon}{3}, \qquad \frac{2^{p+1}\delta A}{p} < \frac{\varepsilon}{3}, \qquad \frac{2(1-(2/p))A}{R^p\delta^{2/(p-2)}} < \frac{\varepsilon}{3};$$

then the desired assertion follows. □

Remark 4.2. In light of (4.5), taking $\delta = h_n$ and $R = h_n^{-1/(p-2)}$ yields the estimate

$$\hat{\mathbb{E}}(\sup_{0\le v\le t}|e_n(v)|^2) \le ch_n$$

for some positive constant c. This implies that the proposed scheme achieves a global mean-square convergence order proportional to $h_n^{1/2}$.

4.1. Numerical stability

In this section we aim to inspect the numerical stability characteristics of the scheme (4.1) in the exponentially mean-square and quasi-sure senses as the value of T approaches infinity. It is important to note that the approximation is not limited to a bounded interval. To this end, we start with the following definitions.

Definition 4.1. A numerical scheme $\{\widetilde{Y}_i\}_{i\in\mathbb{N}}$ applied for approximating G-SDE (3.1) with the initial value x_0 is called exponentially p-stable if there exist positive constants C, λ, and Δ_0 such that, for any $i\in\mathbb{N}$, $\Delta_i\in(0,\Delta_0)$ and $\hat{\mathbb{E}}[|\widetilde{Y}_i|^p] \le C\hat{\mathbb{E}}[|x_0|^p]\mathrm{e}^{-\lambda i\Delta^*}$ for some $\Delta^*\in(0,\Delta_0)$.

Definition 4.2. A numerical scheme $\{\widetilde{Y}_i\}_{i\in\mathbb{N}}$ applied for approximating G-SDE (3.1) with the initial value x_0 is called quasi-surely exponentially stable if there exist positive constants λ and Δ_0 such that, for any $i\in\mathbb{N}$, $\Delta_i\in(0,\Delta_0)$ and $\limsup_{i\to\infty}\{\log(|\widetilde{Y}_i|)/i\Delta_i\} < -\lambda$ q.s.

Theorem 4.2. *Let assumption (3.2) and the necessary assumptions in Theorem 3.2 hold. If*

$$|f(x)|\vee|g(x)| \le \hat{c}|x| \tag{4.6}$$

for any $x\in\mathbb{R}$ and some $\hat{c}>0$, and

$$2\alpha_1 < -\left(k\frac{\alpha_3^2}{\overline{\sigma}} + \overline{\sigma}(1+\hat{c}^2)\right), \tag{4.7}$$

where k is a sufficiently large positive constant and

$$\begin{cases} 16\hat{c}(\beta - 4\hat{c}(1+\overline{\sigma}^4)) < -c_1 - c_2 - c_3, \\ -16\hat{c}(\beta + 4\hat{c}(1+\overline{\sigma}^4)) < -c_1 - c_2 + c_3, \end{cases} \tag{4.8}$$

where $c_1 = (1-2\alpha_1)^2$, $c_2 = 64\hat{c}^2$, $c_3 = 16\hat{c}(1-2\alpha_1)$, and

$$\beta = 4C_2\frac{\alpha_3^2}{\overline{\sigma}} + (1+\hat{c}^2)\overline{\sigma}^2 + 2\alpha_1.$$

Then the proposed scheme (4.1) is conditionally asymptotically mean-square stable.

Proof. Taking the square of both sides of (4.1) yields

$$\begin{aligned}\widetilde{Y}_{i+1}^2 \le{}& \widetilde{Y}_i^2 + 4\big((1-\theta)^2 f^2(\widetilde{Y}_i)\Delta_i^2 + \theta^2 f^2(\widetilde{Y}_{i+1})\Delta_i^2 + g^2(\widetilde{Y}_i)(\Delta\langle B\rangle_i)^2 + \sigma^2(\widetilde{Y}_i)(\Delta B_i)^2\big) \\ &+ 2(1-\theta)\widetilde{Y}_i f(\widetilde{Y}_i)\Delta_i + 2\theta\widetilde{Y}_i f(\widetilde{Y}_{i+1})\Delta_i + 2\widetilde{Y}_i g(\widetilde{Y}_i)\Delta\langle B\rangle_i + 2\widetilde{Y}_i\sigma(\widetilde{Y}_i)\Delta B_i \end{aligned} \tag{4.9}$$

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for $i=0,1,\ldots,N_n-1$. By adding and subtracting $2(1-\theta)G(2\widetilde{Y}_i g(\widetilde{Y}_i)+\sigma^2(\widetilde{Y}_i))\Delta_i$ on the right-hand side of (4.9), we have

$$\begin{aligned}\widetilde{Y}_{i+1}^2 \le \widetilde{Y}_i^2 &+ 4\big((1-\theta)^2 g^2(\widetilde{Y}_i)\Delta_i^2 + \theta^2 f^2(\widetilde{Y}_{i+1})\Delta_i^2 + g^2(\widetilde{Y}_i)(\Delta\langle B\rangle_i)^2 + \sigma^2(\widetilde{Y}_i)(\Delta B_i)^2\big)\\ &+ I_1 + I_2 + 2\widetilde{Y}_i\sigma(\widetilde{Y}_i)\Delta B_i,\end{aligned} \tag{4.10}$$

where

$$I_1 = 2(1-\theta)\big(\widetilde{Y}_i f(\widetilde{Y}_i) + G(2\widetilde{Y}_i g(\widetilde{Y}_i)+\sigma^2(\widetilde{Y}_i))\big)\Delta_i,$$

$$I_2 = -2(1-\theta)G(2\widetilde{Y}_i g(\widetilde{Y}_i)+\sigma^2(\widetilde{Y}_i))\Delta_i + 2\theta\widetilde{Y}_i f(\widetilde{Y}_{i+1})\Delta_i + 2\widetilde{Y}_i g(\widetilde{Y}_i)\Delta\langle B\rangle_i.$$

By (3.5) we get $I_1 \le 2(1-\theta)\alpha_1\widetilde{Y}_i^2\Delta_i$. By considering the two cases $2\widetilde{Y}_i g(\widetilde{Y}_i)+\sigma^2(\widetilde{Y}_i)>0$ and $2\widetilde{Y}_i g(\widetilde{Y}_i)+\sigma^2(\widetilde{Y}_i)<0$ and relation (4.6), and using the inequality $2ab \le a^2+b^2$, we get $I_2 \le ((1+\hat{c}^2)\overline{\sigma}^2+\theta)\widetilde{Y}_i^2\Delta_i + \theta\hat{c}^2\widetilde{Y}_{i+1}^2\Delta_i$. By applying $\hat{\mathbb{E}}$ on both sides of (4.10), we get

$$\begin{aligned}&\hat{\mathbb{E}}\big[\widetilde{Y}_{i+1}^2\big]\\ &\le \hat{\mathbb{E}}\big[\widetilde{Y}_i^2\big]\\ &\quad + 4\big((1-\theta)^2\hat{\mathbb{E}}[g^2(\widetilde{Y}_i)]\Delta_i^2 + \theta^2\hat{\mathbb{E}}[f^2(\widetilde{Y}_{i+1})]\Delta_i^2 + \hat{\mathbb{E}}[g^2(\widetilde{Y}_i)(\Delta\langle B\rangle_i)^2] + \hat{\mathbb{E}}[\sigma^2(\widetilde{Y}_i)(\Delta B_i)^2]\big)\\ &\quad + 2(1-\theta)\alpha_1\hat{\mathbb{E}}\big[\widetilde{Y}_i^2\big]\Delta_i + ((1+\hat{c}^2)\overline{\sigma}^2+\theta)\hat{\mathbb{E}}\big[\widetilde{Y}_i^2\big]\Delta_i + \theta\hat{c}^2\hat{\mathbb{E}}\big[\widetilde{Y}_{i+1}^2\big]\Delta_i + 2\hat{\mathbb{E}}[\widetilde{Y}_i\sigma(\widetilde{Y}_i)\,\Delta B_i].\end{aligned}$$

Applying Lemmas 2.1 and 2.4, the relations (2.4), (3.6), and (4.6) result in

$$\hat{\mathbb{E}}\big[\widetilde{Y}_{i+1}^2\big] \le (1+\gamma_i\Delta_i)\hat{\mathbb{E}}\big[\widetilde{Y}_i^2\big] + \xi_i\hat{\mathbb{E}}\big[\widetilde{Y}_{i+1}^2\big],$$

where

$$\begin{cases}\gamma_i = 4\hat{c}^2((1-\theta)^2+\overline{\sigma}^4)\Delta_i + (1-2\alpha_1)\theta + \beta,\\ \xi_i = \theta\hat{c}^2(4\theta\Delta_i+1)\Delta_i.\end{cases} \tag{4.11}$$

We select the partition of $[0,T]$ such that $\xi_i<1$, which results in $\Delta_i \le \kappa_1$, where

$$\kappa_1 = \frac{-\hat{c}+(\hat{c}^2+16)^{1/2}}{8\theta\hat{c}}.$$

Thus, we have that

$$\hat{\mathbb{E}}\big[\widetilde{Y}_{i+1}^2\big] \le \frac{1+\gamma_i\Delta_i}{1-\xi_i}\hat{\mathbb{E}}\big[\widetilde{Y}_i^2\big] = \left(1+\frac{\xi_i+\gamma_i\Delta_i}{1-\xi_i}\right)\hat{\mathbb{E}}\big[\widetilde{Y}_i^2\big]. \tag{4.12}$$

We set

$$-1 < \frac{\xi_i+\gamma_i\Delta_i}{1-\xi_i} < 0.$$

The condition $\xi_i+\gamma_i\Delta_i<0$ results in $\Delta_i<\kappa_2$, where

$$\kappa_2 = \frac{-(\beta+\theta(1+\hat{c}^2-2\alpha_1))}{4\hat{c}^2(\theta^2+(1-\theta)^2+\overline{\sigma}^4)},$$

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and we need $\beta + \theta(1 + \hat{c}^2 - 2\alpha_1) < 0$, which leads to

$$\theta < -\frac{\beta}{1 + \hat{c}^2 - 2\alpha_1}. \tag{4.13}$$

Note that, by (4.7), $\beta < 0$. Furthermore, the condition $\gamma_i \Delta_i > -1$ results in

$$4\hat{c}^2((1-\theta)^2 + \overline{\sigma}^4)\Delta_i^2 + ((1-2\alpha_1)\theta + \beta)\Delta_i + 1 > 0.$$

So we should have $((1-2\alpha_1)\theta + \beta)^2 < 16\hat{c}^2((1-\theta)^2 + \overline{\sigma}^4)^2$, and hence

$$\begin{cases} -4\hat{c}\theta^2 + \theta(1 - 2\alpha_1 + 8\hat{c}) + \beta - 4\hat{c}(1 + \overline{\sigma}^4) < 0, \\ 4\hat{c}\theta^2 + \theta(1 - 2\alpha_1 - 8\hat{c}) + \beta + 4\hat{c}(1 + \overline{\sigma}^4) > 0, \end{cases}$$

which is ensured by (4.8). Finally, by considering conditions (4.13), we set $\Delta^* = \min\{\kappa_1, \kappa_2\}$ and impose $\Delta_i \in (0, \Delta^*)$ for each $i \in \mathbb{N}$. Now return to (4.12). In a recursive manner, we have that

$$\hat{\mathbb{E}}[\widetilde{Y}_m^2] \le \prod_{j=0}^{m-1} \left(\frac{1 + \gamma_j \Delta_j}{1 - \xi_j} \right) \hat{\mathbb{E}}[\widetilde{Y}_0^2]$$

for $m \ge 1$, where

$$0 < \frac{1 + \gamma_j \Delta_j}{1 - \xi_j} < 1$$

for $j \in \{0, \dots, m-1\}$. Thus

$$\hat{\mathbb{E}}[\widetilde{Y}_m^2] \le \left(\sup_{\Delta \in (0, \Delta^*)} \frac{1 + \gamma(\Delta)\Delta}{1 - \xi(\Delta)} \right)^m \hat{\mathbb{E}}[\widetilde{Y}_0^2]. \tag{4.14}$$

Note that $\xi(\Delta)$ and $\gamma(\Delta)$ are defined similarly to (4.11) with the slight difference that Δ_i is replaced by Δ. We set

$$1 - \lambda_{\Delta^*}\hat{\Delta} := \sup_{\Delta \in (0, \Delta^*)} \frac{1 + \gamma(\Delta)\Delta}{1 - \xi(\Delta)},$$

where λ_{Δ^*} is a function of θ, $\hat{c}$, $\overline{\sigma}$, α_1, α_3, C_2, and $\hat{\Delta} \in (0, \Delta^*)$. It is clear that $\lambda_{\Delta^*} > 0$. By considering (4.14), we have that $\hat{\mathbb{E}}[\widetilde{Y}_m^2] \le (1 - \lambda_{\Delta^*}\hat{\Delta})^m \hat{\mathbb{E}}[\widetilde{Y}_0^2]$, and for sufficiently large m,

$$\hat{\mathbb{E}}[\widetilde{Y}_m^2] \le \mathrm{e}^{-\lambda_{\Delta^*} m \hat{\Delta}} \hat{\mathbb{E}}[\widetilde{Y}_0^2], \tag{4.15}$$

which completes the proof. □

Corollary 4.1. *Let the assumptions of Theorem 4.2 hold. Then the numerical scheme (4.1) is quasi-surely exponentially stable.*

Proof. By Theorem 4.2 and relation (4.15), we have

$$\hat{\mathbb{E}}[\widetilde{Y}_n^2] \le \mathrm{e}^{-\lambda_{\Delta^*} n \hat{\Delta}} \hat{\mathbb{E}}[\widetilde{Y}_0^2] \tag{4.16}$$

for sufficiently large $n \in \mathbb{N}$, where Δ^*, λ_{Δ^*}, and $\hat{\Delta}$ are positive constants. Now, let $\varepsilon > 0$ be such that $\frac{1}{2}(\lambda_{\Delta^*} - \varepsilon) > 0$. By Markov's inequality and using (4.16), we have

$$C(|\widetilde{Y}_n| > \mathrm{e}^{-(\lambda_{\Delta^*} - \varepsilon) n \hat{\Delta}/2}) \le \frac{\hat{\mathbb{E}}[\widetilde{Y}_n^2]}{\mathrm{e}^{-(\lambda_{\Delta^*} - \varepsilon) n \hat{\Delta}}} \le \mathrm{e}^{-\varepsilon n \hat{\Delta}} \hat{\mathbb{E}}[\widetilde{Y}_0^2].$$

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Since $\sum_{n=1}^{\infty} C(|\widetilde{Y}_n| > \mathrm{e}^{-(\lambda_{\Delta^*}-\varepsilon)n\hat{\Delta}/2}) < \infty$, owing to the Borel–Cantelli lemma we get

$$\limsup_{n\to\infty} |\widetilde{Y}_n| \le \mathrm{e}^{-(\lambda_{\Delta^*}-\varepsilon)n\hat{\Delta}/2}$$

quasi-surely. So, there exists $\bar{\Omega} \subset \Omega$ with $C(\bar{\Omega}) = 1$ such that, for all $\omega \in \bar{\Omega}$ and for any $\epsilon > 0$, there exists $n_0(\omega, \epsilon) \in \mathbb{N}$ and $|\widetilde{Y}_n(\omega)| \le \mathrm{e}^{-(\lambda_{\Delta^*}-\varepsilon)n\hat{\Delta}/2}$ for any $n \ge n_0(\omega, \epsilon)$, and then

$$\frac{\log(|\widetilde{Y}_n(\omega)|)}{n\hat{\Delta}} \le -\left(\frac{\lambda_{\Delta^*}-\varepsilon}{2}\right).$$

Ultimately, we have that

$$\limsup_{n\to\infty} \frac{\log(|\widetilde{Y}_n|)}{n\hat{\Delta}} \le -\left(\frac{\lambda_{\Delta^*}-\varepsilon}{2}\right) \quad q.s. \qquad \square$$

5. Simulation experiments

In this section we implement the proposed scheme (4.1) for two test problems satisfying conditions (3.2) and (3.3). For this, we need to generate the sample paths of G-Brownian motion. The distribution of G-Brownian motion is defined as $\hat{\mathbb{E}}[\varphi(B_\cdot)]$ for a suitable function φ. As a result, the properties of the G-Brownian motion distribution depend on the selected φ, and this selection must be uniquely specified whenever we simulate sample paths of the process. In [28], the authors focus on the test function $\varphi(x) := I_{\{x\le a\}}$ for some $a \in \mathbb{R}$, which leads to

$$\begin{aligned}\hat{\mathbb{E}}[\varphi(B_t)] = \hat{\mathbb{E}}[I_{\{B_t\le a\}}] &= \sup_{P\in\mathcal{P}} E^P[I_{B_t\le a}] \\ &= \sup_{P\in\mathcal{P}} P(B_t \le a) \\ &= \sup_{P\in\mathcal{P}} F^P_{B_t}(a).\end{aligned}$$

They initially generate a sample path of the G-Brownian motion at discrete points and subsequently the increments of G-Brownian motion are derived accordingly. Note that all probability measures $P \in \mathcal{P}$ are incorporated in the proposed technique, and that the obtained results correspond to the *Heaviside function* φ.

Example 5.1. The first example, which includes additive noise, is presented as follows:

$$\mathrm{d}X_t = -X_t\,\mathrm{d}t - \left(X_t + X_t^3\right)\mathrm{d}\langle B\rangle_t + 0.2\,\mathrm{d}B_t, \quad t \ge 0,\ X_0 = 1, \tag{5.1}$$

where $(B_t)_{t\ge0}$ is a G-Brownian motion with $\overline{\sigma}^2 = 1$ and $\underline{\sigma}^2 = 0.5$. As we see, the example satiafies the conditions (3.2) for $K_R = 1 + 3R^2$ for any $R \in \mathbb{R}$ such that $|x| \vee |y| \le R$, and also the condition (3.3) holds for $K = 0.04$. Hence, the equation possesses a unique solution.

Let $\{t_i\}_{i=0}^N$ be a uniform partition of the interval $[0, T]$, where the step size is defined by $h := t_{i+1} - t_i$ for each $i = 0, \ldots, N-1$. By applying the discretization scheme (4.1) to equation (5.1), we obtain

$$\widetilde{Y}_{i+1} = \frac{1}{1+\theta h}\left((1-(1-\theta)h)\widetilde{Y}_i - \left(\widetilde{Y}_i + \widetilde{Y}_i^3\right)\Delta\langle B\rangle_i + 0.2\Delta B_i\right), \quad i > 0,\ \theta \in [0, 1],$$

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TABLE 1. The average pathwise error at $T = 2$ computed over 100 trajectories for Example 5.1 with the parameters $\underline{\sigma}^2 = 0.5$ and $\overline{\sigma}^2 = 1$.

Step size	θ	ε
2^{-7}	0	0.1342
	$\frac{1}{2}$	0.0643
	1	0.0009
2^{-8}	0	0.1142
	$\frac{1}{2}$	0.0505
	1	0.0005
2^{-9}	0	0.0739
	$\frac{1}{2}$	0.0360
	1	0.0003

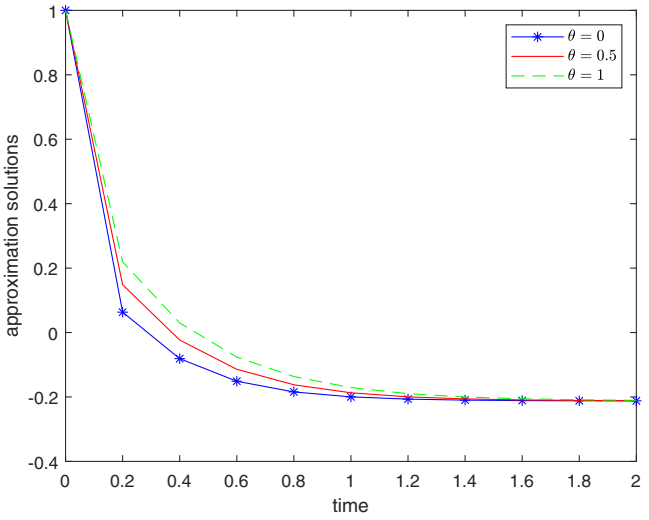


FIGURE 1. The sample paths of the θ-EM for equation (5.1) are presented with parameters $\underline{\sigma}^2 = 0.5$, $\overline{\sigma}^2 = 1$, and a step size of $h = 0.2$ over the interval $[0, 2]$.

with $\widetilde{Y}_0 = 0$. Because an exact closed-form solution is unavailable, we apply the scheme with $\theta = 1$ on a uniform fine discretization grid with a step size of 2^{-10}. We calculate

$$e(t, \omega_j) := |X(T, \omega_j) - \widetilde{Y}(T, \omega_j)|, \quad j = 1, \ldots, 100,$$

as an error estimate for the numerical method along each single trajectory. We denote by ε the average pathwise error calculated over 100 trajectories. Table 1 displays the values of ε corresponding to θ in the set $\{0, 0.5, 1\}$. As observed, the implicit scheme with $\theta = 1$ outperforms the semi-implicit scheme with $\theta = \frac{1}{2}$, which in turn is better than the case with $\theta = 0$. As expected, reducing the stepsize leads to improved results. Figure 1 illustrates three sample paths of the θ-EM equation (5.1).

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TABLE 2. The average pathwise error at $T = 3$ computed over 100 trajectories for Example 5.2 with the parameters $\underline{\sigma}^2 = 0.5$ and $\overline{\sigma}^2 = 1$.

Step size	θ	ε
2^{-4}	0	0.1570
	$\frac{1}{2}$	0.0391
	1	0.0340
2^{-5}	0	0.0403
	$\frac{1}{2}$	0.0176
	1	0.0199
2^{-6}	0	0.0189
	$\frac{1}{2}$	0.0101
	1	0.0111

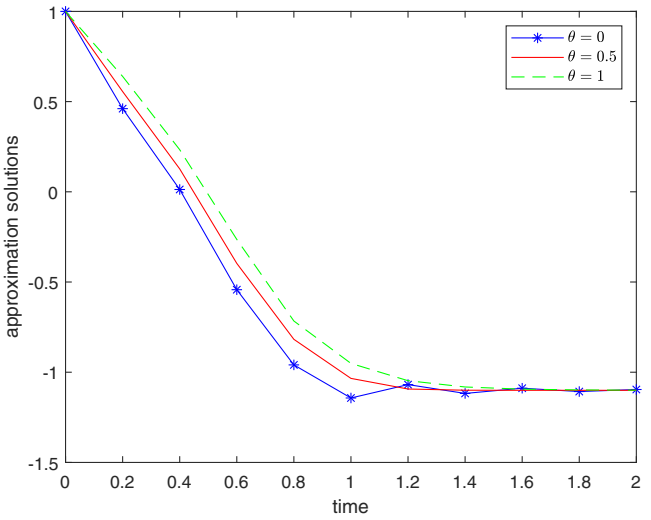


FIGURE 2. The sample paths of the θ-EM for equation (5.2) are presented with the parameters $\underline{\sigma}^2 = 0.5$, $\overline{\sigma}^2 = 1$, and a step size of $h = 0.2$ over the interval $[0, 2]$.

Example 5.2. The second example, featuring multiplicative noise, is as follows:

$$\mathrm{d}X_t = -X_t^5\,\mathrm{d}t + \mathrm{e}^{-|X_t|/2}\,\mathrm{d}B_t, \quad t \geq 0,\ X_0 = 1. \tag{5.2}$$

As we can see, the example satisfies the conditions (3.2) and (3.3) with parameters $K_R = 1 + 5R^4$ and $K = 1$. Similar to Example 5.1, considering the partition $\{t_i\}_{i=0}^N$ of $[0, T]$ and applying the scheme in (4.1) leads to

$$\widetilde{Y}_{i+1} + \theta h \widetilde{Y}_{i+1}^5 = \widetilde{Y}_i - (1-\theta)h\widetilde{Y}_i^5 + \mathrm{e}^{-|\widetilde{Y}_i|/2}\Delta B_i, \quad i \geq 0,\ \theta \in [0, 1].$$

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It is clear that $\widetilde{Y}_{t_{i+1}}$ is the root or zero of the real-valued function f_i with

$$f_i(x) := x + \theta h x^5 - c_i, \quad x \in \mathbb{R}, \tag{5.3}$$

where $c_i = \widetilde{Y}_i - (1-\theta)h\widetilde{Y}_i + \mathrm{e}^{-|\widetilde{Y}_i|/2}\Delta B_i$. Using the root-finding technique called the *Newton–Raphson method*, we solve the non-linear equation (5.3) to compute the numerical solutions at the specified points. We need to be mindful of rounding errors when solving non-linear equations numerically. Analogous to Table 1, the smaller step size yields more accurate results, as shown in Table 2. Moreover, Table 2 shows that the semi-implicit method with $\theta = \frac{1}{2}$ and the fully implicit method with $\theta = 1$ produce more satisfactory results than the explicit method with $\theta = 0$. However, for step sizes of 2^{-5} and 2^{-6}, the semi-implicit method performs slightly better than the fully implicit one, which could be attributed to the effects of rounding error. Figure 2 illustrates three sample paths of the θ-EM equation (5.2).

6. Conclusion

To the best of our knowledge, this is the first paper dedicated to numerical solution of the stochastic differential equations driven by G-Brownian motion under locally Lipschitz continuous coefficients. The research outlined designs an implicit continuous Euler–Maruyama scheme to discretize the G-SDE (3.1). Then, the convergence and exponential mean-square stability of the θ-EM scheme was investigated, which leads to quasi-sure exponential stability as well.

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Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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