ON THE PARAMETRIC REPRESENTATION OF BOUNDED FUNCTIONS

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1. *Introduction.* Let $a_1, a_2, \ldots$ be a sequence of variables. Write

$$B_1 = -a_1,$$

$$(r+1)\,B_{r+1} = (-1)^{r+1}[a_{r+1} - B_1 a_r + B_2 a_{r-1} - \ldots + (-1)^r B_r a_1] \quad (r = 1, 2, \ldots).$$

We use the notation $B_n = B_n(a_1, \ldots, a_n)$ to indicate the variables on which B_n depends.

Let α be a variable. Write

$$\Delta_0(\alpha) = 1 - i\alpha \quad (i^2 = -1),$$

$$\Delta_1(a_1; \alpha) = \begin{vmatrix} 1 & i\alpha \\ B_1 + i\alpha & 1 \end{vmatrix},$$

$$\Delta_2(a_1, a_2; \alpha) = \begin{vmatrix} 1 & 0 & -i\alpha \\ B_1 & 1 - i\alpha & 0 \\ B_2 - i\alpha & B_1 & 1 \end{vmatrix},$$

and generally

$$\Delta_n(a_1, \ldots, a_n; \alpha) = \begin{vmatrix} 1 & 0 & \ldots & 0 & -(-1)^n i\alpha \\ B_1 & & & & 0 \\ \vdots & & \Delta_{n-2} & & \vdots \\ B_{n-1} & & & & 0 \\ B_n - (-1)^n i\alpha & B_{n-1} & \ldots & B_1 & 1 \end{vmatrix}.$$

Let $c_0, c_1, \ldots$ be a sequence of real numbers. For $n = 0, 1, 2, \ldots$, we define

$$\theta_n = \Delta_n(ic_1, 2ic_2, \ldots, nic_n; e^{-ic_0/2})\, e^{(n+1)ic_0/4} \quad (n \text{ odd}),$$

$$= \Delta_n(ic_1, 2ic_2, \ldots, nic_n; e^{-ic_0/2})\, e^{(n+1)ic_0/4 + i\pi/4} \quad (n \text{ even}).$$

We write θ_n' for the expression obtained on replacing $c_0, \ldots, c_n$ by $-c_0, \ldots, -c_n$, respectively, in θ_n. In a previous paper* we proved the following result:

* Verblunsky, *Proc. Camb. Phil. Soc.* 32 (1936), 201–11. This paper will be referred to as II. A knowledge of it is assumed.

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THEOREM A. *Given $k+1$ numbers $c_1, \ldots, c_k$, it is necessary and sufficient in order that there shall exist a function $g(x)$ in $(0, \pi)$, such that*

$$\text{Max}\,|g(x)| < 1; \quad c_\nu = \int_0^\pi g(x)\cos \nu x\,dx \quad (\nu = 0, \ldots, k),$$

that $-\pi < c_0 < \pi$, and that $\theta_1, \ldots, \theta_k, \theta'_1, \ldots, \theta'_{k'}$ shall be all positive.

Consider the class of functions $f(x)$, defined in $(0, \pi)$, which satisfy $|f(x)| \leqslant 1$. Each such function gives rise to a sequence

$$c_\nu = \int_0^\pi f(x)\cos \nu x\,dx \quad (\nu = 0, 1, \ldots).$$

The above class of functions defines a certain class B of sequences.

Let $\sigma_0, \sigma_1, \ldots$ be a sequence of independent real parameters, subject to the conditions

$$\sigma_\nu^2 \leqslant 1 \quad (\nu = 0, 1, \ldots).$$

The object of this paper is to show that there exists a sequence of functions p_n $(n = 1, 2, \ldots)$, where p_n is a polynomial in $n+2$ variables $t_1, t_2, x_1, \ldots, x_n$, with rational coefficients,

$$p_n \equiv p_n(t_1, t_2, x_1, \ldots, x_n),$$

such that the class B is identical with the class defined by

$$c_0 = \pi\sigma_0; \quad c_n = p_n(\cos \pi\sigma_0/4, \sin \pi\sigma_0/4, \sigma_1, \ldots, \sigma_n). \tag{i}$$

More precisely, every sequence (c_ν), obtained by substituting values of $\sigma_0, \sigma_1, \ldots$ in (i), is a sequence of class B; every sequence of class B can be obtained by such a substitution, and, subject to exceptions of a trivial kind, in only one way.

The sequences $$\gamma_n = \int_0^\pi f(x)\sin nx\,dx \quad (n = 1, 2, \ldots),$$

where $f(x)$ is a function of the class mentioned above, form a class C which admits of a similar parametric representation. To find this representation, we may consider the class D of sequences

$$\delta_\nu = \int_0^1 h(x)\,x^\nu dx \quad (\nu = 0, 1, \ldots),$$

where $h(x)$ satisfies $|h(x)| \leqslant 1$. [Cf. II, § 1.] Just as the parametric representation of class B is obtained in this paper from Theorem A of II, so the representation of class D may be obtained from Theorem A of I.* The problem and the result are, presumably, somewhat simpler.

2. LEMMA 1. *Let $c_0, \ldots, c_{n-1}$, where $n \geqslant 2$, be numbers such that*

$$-\pi < c_0 < \pi; \quad \theta_1, \ldots, \theta_{n-2}, \theta'_1, \ldots, \theta'_{n-2} \text{ are all positive,} \tag{1}$$

$$\theta_{n-1} \geqslant 0, \quad \theta'_{n-1} \geqslant 0, \quad \theta_{n-1}\theta'_{n-1} = 0. \tag{2}$$

* Paper I is *Proc. Camb. Phil. Soc.* 32 (1936), 30–9.

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Then there is one and only one function f such that

$$|f| \leqslant 1; \qquad c_\nu = \int_0^\pi f(x) \cos \nu x \, dx \quad (\nu = 0, \ldots, n-1); \tag{3}$$

and hence, one and only one sequence (c_ν) of class B, of which $c_0, \ldots, c_{n-1}$ are the initial members. The $(n+1)$th member c_n of this sequence satisfies $\theta_n = 0$, $\theta_n' = 0$.

By II, Lemma 9, the condition (1) means that, for every trigonometric polynomial of degree not greater than $n-2$,

$$T_{n-2}(x) = \lambda_0 + \lambda_1 \cos x + \ldots + \lambda_{n-2} \cos (n-2)x,$$

we have
$$\left| \sum_0^{n-2} c_j \lambda_j \right| < \int_0^\pi |T_{n-2}(x)| \, dx.$$

The additional conditions $\theta_{n-1} \geqslant 0$, $\theta_{n-1}' \geqslant 0$, mean that, for every trigonometric polynomial $T_{n-1}(x)$ of degree not greater than $n-1$, we have

$$\left| \sum_0^{n-1} c_j \lambda_j \right| \leqslant \int_0^\pi |T_{n-1}(x)| \, dx.$$

This follows from the arguments of II. By II, Lemma 2, we have (3) with some function f.

If now $\theta_{n-1} = 0$, then, for some polynomial

$$T^0(x) = \lambda_0^0 + \ldots + \lambda_{n-1}^0 \cos (n-1)x,$$

we have
$$\sum_0^{n-1} c_j \lambda_j^0 = \int_0^\pi |T^0(x)| \, dx;$$

in which case, since
$$\sum_0^{n-1} c_j \lambda_j^0 = \int_0^\pi T^0(x) f(x) \, dx,$$

with $|f| \leqslant 1$, we must have $f = \operatorname{sgn} T^0(x)$. Similarly, if $\theta_{n-1}' = 0$, then, for some polynomial

$$T^1(x) = \lambda_0^1 + \ldots + \lambda_{n-1}^1 \cos (n-1)x,$$

we have
$$-\sum_0^{n-1} c_j \lambda_j^1 = \int_0^\pi |T^1(x)| \, dx,$$

so that we must have $f = -\operatorname{sgn} T^1(x)$. This proves the first part of the lemma.

Write $\delta = 1/(1+\epsilon)$, where $\epsilon > 0$. Since $c_0\delta, \ldots, c_n\delta$ are the initial constants of $f\delta$, whose maximum modulus is less than 1, we have, on applying Theorem A and letting $\epsilon \to 0$, that $\theta_n \geqslant 0$, $\theta_n' \geqslant 0$.

We observe that $\theta_n = \alpha_n - \theta_{n-2} c_n$, where α_n does not depend on c_n. Similarly, $\theta_n' = \alpha_n' + \theta_{n-2}' c_n$. If $\theta_n > 0$, then, writing $\theta_n(\gamma_n) = \alpha_n - \theta_{n-2}\gamma_n$, we have

$$\theta_n(\gamma_n) > 0 \quad \text{for} \quad |\gamma_n - c_n| < \eta,$$

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where $\eta > 0$. Writing $\theta_n'(\gamma_n) = \alpha_n' + \theta_{n-2}'\gamma_n$, we have two possibilities:

(*a*) If $\theta_n' > 0$, then $\theta_n'(\gamma_n) > 0$ for $|\gamma_n - c_n| < \eta'$, where $\eta' > 0$.

(*b*) If $\theta_n' = 0$, then $\theta_n'(\gamma_n) > 0$ for $\gamma_n > c_n$, since $\theta_{n-2}' > 0$.

In either case, $\theta_n(\gamma_n) > 0$, $\theta_n'(\gamma_n) > 0$ for $c_n < \gamma_n < c_n + \zeta$, for a suitable $\zeta > 0$.

Let γ_n have any assigned value in the open interval $(c_n, c_n + \zeta)$. On grounds of continuity, if for $c_0, \ldots, c_{n-1}, \gamma_n$ we substitute $c_0\delta, \ldots, c_{n-1}\delta, \gamma_n\delta$, respectively, then, for sufficiently small ϵ,

$$\theta_n(c_0\delta, \ldots, \gamma_n\delta) > 0, \quad \theta_n'(c_0\delta, \ldots, \gamma_n\delta) > 0.$$

But $$\theta_k(c_0\delta, \ldots, c_k\delta) > 0, \quad \theta_k'(c_0\delta, \ldots, c_k\delta) > 0 \quad (k = 1, \ldots, n-1).$$

Hence, by Theorem A, $c_0\delta, \ldots, \gamma_n\delta$ are the initial members of a sequence of class B. Letting $\epsilon \to 0$, we infer that $c_0, \ldots, \gamma_n$ are the $n+1$ initial members of such a sequence. But γ_n is any number within $(c_n, c_n + \zeta)$. This contradicts the first part of the lemma, in accordance with which there is only one sequence of class B beginning with $c_0, \ldots, c_{n-1}$. Hence $\theta_n = 0$. Similarly, $\theta_n' = 0$.

A sequence $c_0, c_1, \ldots$ will be said to be of class B_0 if

$$-\pi < c_0 < \pi; \quad \theta_n > 0, \ \theta_n' > 0 \quad (n = 1, 2, \ldots).$$

3. *Parametric representation of* B_0. Let $\rho_0, \rho_1, \ldots$ denote a sequence of independent real parameters subject to

$$\rho_\nu^2 < 1 \quad (\nu = 0, 1, \ldots).$$

For any sequence of class B_0, we can write

$$c_0 = \rho_0 \pi.$$

This is the parametric representation of c_0; but in what follows we shall continue to write c_0 for $\rho_0\pi$.

We next consider c_1. We must have $\theta_1 > 0$, $\theta_1' > 0$. Now $\theta_1 = 2\cos\frac{1}{2}c_0 - c_1$.

Hence $$c_1 < 2\cos\tfrac{1}{2}c_0, \quad c_1 > -2\cos\tfrac{1}{2}c_0.$$

Thus $$c_1 = \rho_1 . 2\cos\tfrac{1}{2}c_0. \tag{4}$$

Also $\theta_1 = 2\cos\frac{1}{2}c_0(1-\rho_1)$; and since $\theta_0 = 2\cos\frac{1}{4}(c_0 + \pi)$, we have

$$\theta_1 = \theta_0 . 2\cos\tfrac{1}{4}(c_0 - \pi)(1 - \rho_0).$$

We next consider c_2. We require $\theta_2 > 0$, $\theta_2' > 0$. We find on using (4) that

$$\theta_2 = 2\cos\tfrac{1}{4}(c_0 + \pi)[2\cos\tfrac{1}{2}c_0(1-\rho_1^2) - \rho_1^2\sin c_0 - c_2].$$

Since $-\pi < c_0 < \pi$, both $\cos\frac{1}{4}(c_0 + \pi)$ and $\cos\frac{1}{4}(-c_0 + \pi)$ are positive. We therefore require that

$$c_2 < 2\cos\tfrac{1}{2}c_0(1-\rho_1^2) - \rho_1^2\sin c_0; \quad c_2 < -2\cos\tfrac{1}{2}c_0(1-\rho_1^2) - \rho_1^2\sin c_0.$$

Hence $$c_2 = -\rho_1^2\sin c_0 + \rho_2(1-\rho_1^2)2\cos\tfrac{1}{2}c_0, \tag{5}$$

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and $$\theta_2 = 2\cos\tfrac{1}{4}(c_0+\pi)\,2\cos\tfrac{1}{2}c_0\,(1-\rho_2)\,(1-\rho_1^2),$$

i.e. $$\theta_2 = \theta_1 . 2\cos\tfrac{1}{4}(c_0+\pi)\,(1+\rho_1)\,(1-\rho_2).$$

Next consider c_3. We require that $\theta_3 > 0$, $\theta_3' > 0$. If we write

$$A = -2\cos\tfrac{1}{2}c_0\,[\tfrac{1}{3}\rho_1^3(2\cos c_0 - 1) + \rho_1\rho_2(1-\rho_1^2)(\rho_2 + 2\sin\tfrac{1}{2}c_0)],$$

we find, on using (4) and (5), that

$$\theta_3 = \theta_1\,[(1-\rho_1^2)\,(1-\rho_2^2)\,2\cos\tfrac{1}{2}c_0 + A - c_3].$$

We require, in the first place, that

$$c_3 < A + (1-\rho_1^2)\,(1-\rho_2^2)\,2\cos\tfrac{1}{2}c_0. \tag{6}$$

We further require the inequality to be satisfied, which is obtained by replacing $c_0, \ldots, c_3$ by $-c_0, \ldots, -c_3$, in (6). This substitution changes ρ_1, ρ_2 to $-\rho_1$, $-\rho_2$ respectively, and A to $-A$. Our second condition is thus

$$c_3 > A - (1-\rho_1^2)\,(1-\rho_2^2)\,2\cos\tfrac{1}{2}c_0.$$

We can therefore write

$$c_3 = A + \rho_3(1-\rho_1^2)\,(1-\rho_2^2)\,2\cos\tfrac{1}{2}c_0,$$

and then $$\theta_3 = \theta_1 . 2\cos\tfrac{1}{2}c_0\,(1-\rho_1^2)\,(1-\rho_2^2)\,(1-\rho_3),$$

i.e. $$\theta_3 = \theta_2 . 2\cos\tfrac{1}{4}(c_0-\pi)\,(1-\rho_1)\,(1+\rho_2)\,(1-\rho_3).$$

4. We now have

THEOREM B. *There exists a sequence $q_2, q_3, \ldots$ of functions, where*

$$q_r = q_r(t_1, t_2, x_1, \ldots, x_{r-1})$$

is a polynomial in $r+1$ variables, with rational coefficients, such that the class B_0 is identical with the class defined by

$$-\pi < c_0 < \pi,$$

$$c_1 = \rho_1 . 2\cos\tfrac{1}{2}c_0,$$

$$c_2 = q_2(\cos\tfrac{1}{4}c_0, \sin\tfrac{1}{4}c_0, \rho_1) + \rho_2(1-\rho_1^2)\,2\cos\tfrac{1}{2}c_0,$$

$$c_3 = q_3(\cos\tfrac{1}{4}c_0, \sin\tfrac{1}{4}c_0, \rho_1, \rho_2) + \rho_3(1-\rho_1^2)\,(1-\rho_2^2)\,2\cos\tfrac{1}{2}c_0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$c_n = q_n(\cos\tfrac{1}{4}c_0, \sin\tfrac{1}{4}c_0, \rho_1, \ldots, \rho_{n-1}) + \rho_n(1-\rho_1^2)\ldots(1-\rho_{n-1}^2)\,2\cos\tfrac{1}{2}c_0.$$

These values make

$$\left.\begin{aligned} \theta_n/\theta_{n-1} &= 2\cos\tfrac{1}{4}(c_0+\pi)\,(1+\rho_1)\,(1-\rho_2)\ldots(1-\rho_n) \quad (n \text{ even}) \\ &= 2\cos\tfrac{1}{4}(c_0-\pi)\,(1-\rho_1)\,(1+\rho_2)\ldots(1-\rho_n) \quad (n \text{ odd}) \end{aligned}\right\} (n = 1, 2, \ldots).$$

These formulae have been proved for $n = 1, 2, 3$ in § 3. We may suppose that they have been established for suffixes less than k. We then show that they hold for $n = k$. We suppose that k is even. The proof when k is odd is similar. We may suppose that $k > 3$.

If θ_k^0 denotes what θ_k becomes when c_k is replaced by zero, then $\theta_k = \theta_k^0 - \theta_{k-2}c_k$. Now θ_k^0 is $e^{i\pi/4}$ times a polynomial in $\cos\tfrac{1}{2}c_0$, $\sin\tfrac{1}{2}c_0$, $c_1, \ldots, c_{k-1}$, with rational

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complex coefficients. Hence $\sqrt{2}\,\theta_k^0$ is a polynomial in $\cos\frac{1}{2}c_0$, $\sin\frac{1}{2}c_0$, $c_1, \ldots, c_{k-1}$ with rational real coefficients. Using the formulae for $c_1, \ldots, c_{k-1}$, we have

$$\sqrt{2}\,\theta_k^0 = J_k(\cos\tfrac{1}{2}c_0, \sin\tfrac{1}{2}c_0, \rho_1, \ldots, \rho_{k-1}),$$

where J_k is a polynomial with rational real coefficients in the variables indicated.

The condition $\theta_k > 0$ is $\qquad c_k < \theta_k^0/\theta_{k-2}$,

i.e. $$c_k < J_k/\sqrt{2}\,\theta_{k-2}. \tag{7}$$

Now c_k must satisfy $-\pi < c_k < \pi$. Hence the second member in (7) must be bounded for permissible $c_0, \rho_1, \ldots, \rho_{k-1}$; for the only other condition to be satisfied, viz. $\theta_k' > 0$, merely assigns a lower bound to c_k.

Our inductive hypothesis tells us that

$$\begin{aligned}\theta_{k-2} &= 2\cos\tfrac{1}{4}(c_0+\pi)(2\cos\tfrac{1}{2}c_0)^{\frac{k-2}{2}}(1-\rho_1)^{i_1}(1+\rho_1)^{i_2}\ldots(1-\rho_{k-2})\\ &= 2\cos\tfrac{1}{4}(c_0+\pi)(2\cos\tfrac{1}{2}c_0)^{\frac{k-2}{2}}\,\Pi,\end{aligned}$$

where $i_1, i_2, \ldots$ are positive integers. We must have $J_k = \Pi . H_k$, where H_k is a polynomial in $\cos\frac{1}{2}c_0$, $\sin\frac{1}{2}c_0$, $\rho_1, \ldots, \rho_{k-1}$, with rational coefficients. Otherwise, by choosing $\rho_1, \rho_2, \ldots$ sufficiently near to ± 1, the second member in (7) could be made arbitrarily large in absolute value.

Next, if we write H_k in the form of a polynomial in $\cos\frac{1}{2}c_0$, $\sin\frac{1}{2}c_0$, we see that unless every term has the factor $(\cos\frac{1}{2}c_0)^{\frac{1}{2}k}$, we can, by choosing c_0 sufficiently near to π, make the second member of (7) arbitrarily large in absolute value. Hence

$$J_k = \Pi\,(2\cos\tfrac{1}{2}c_0)^{\frac{1}{2}k}\,G_k,$$

where G_k is a polynomial in $\cos\frac{1}{2}c_0$, $\sin\frac{1}{2}c_0$, $\rho_1, \ldots, \rho_{k-1}$, with rational coefficients. Thus

$$\begin{aligned}J_k/\sqrt{2}\,\theta_{k-2} &= \cos\tfrac{1}{2}c_0\,G_k/\sqrt{2}\cos\tfrac{1}{4}(c_0+\pi)\\ &= \cos\tfrac{1}{4}(c_0-\pi)\,G_k/\sqrt{2} = E_k,\end{aligned}$$

where E_k is a polynomial with rational coefficients in $\cos\frac{1}{4}c_0$, $\sin\frac{1}{4}c_0$, $\rho_1, \ldots, \rho_{k-1}$.

The condition $\theta_k > 0$ is therefore $c_k < E_k$. The condition $\theta_k' > 0$ is $c_k > -E_k'$, where E_k' is what E_k becomes when $c_0, \rho_1, \ldots, \rho_{k-1}$ are replaced by $-c_0, -\rho_1, \ldots, -\rho_{k-1}$, respectively. These conditions may be written

$$c_k < \tfrac{1}{2}(E_k - E_k') + \tfrac{1}{2}(E_k + E_k'); \quad c_k > \tfrac{1}{2}(E_k - E_k') - \tfrac{1}{2}(E_k + E_k').$$

Hence $$c_k = \tfrac{1}{2}(E_k - E_k') + \rho_k \cdot \tfrac{1}{2}(E_k + E_k').$$

To prove that this is the formula specified in Theorem B, we must show that

$$\tfrac{1}{2}(E_k + E_k') = 2\cos\tfrac{1}{2}c_0\,(1-\rho_1^2)\ldots(1-\rho_{k-1}^2). \tag{8}$$

Further, $$\frac{\theta_k}{\theta_{k-2}} = \frac{\theta_k^0}{\theta_{k-2}} - c_k = E_k - c_k = \tfrac{1}{2}(E_k + E_k')(1-\rho_k). \tag{9}$$

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By our inductive hypothesis,

$$\theta_{k-1}/\theta_{k-2} = 2\cos\tfrac{1}{4}(c_0-\pi)(1-\rho_1)\dots(1-\rho_{k-1}).$$

Thus, if (8) holds, we shall have

$$\theta_k/\theta_{k-1} = 2\cos\tfrac{1}{4}(c_0+\pi)(1+\rho_1)\dots(1-\rho_k),$$

and this will prove the theorem.

5. We have $\theta_k^0 = \Delta_k^0 \exp i\{\pi + c_0(k+1)\}/4,$

where

$$\Delta_j^0 = \begin{vmatrix} 1 & 0 & \dots & 0 & -i\alpha \\ B_1 & & & & 0 \\ \vdots & & \Delta_{k-2} & & \vdots \\ B_{k-1} & & & & 0 \\ B_k^0 - i\alpha & B_{k-1} & \dots & B_1 & 1 \end{vmatrix}.$$

Here, $\alpha = e^{-ic_0/2}$, $B_r = B_r(ic_1, \dots, ric_r)$ for $r = 1, \dots, k-1$, and

$$B_k^0 = B_k(ic_1, \dots, (k-1)ic_{k-1}, 0).$$

Now c_{k-1} occurs only in B_{k-1} and in B_k^0. We have

$$(k-1)B_{k-1} = -[(k-1)ic_{k-1} - \dots], \qquad kB_k^0 = [-B_1(k-1)ic_{k-1} + \dots],$$

so that B_{k-1} and B_k^0 are of the first degree in c_{k-1}. Hence θ_k^0 is of the second degree in c_{k-1}, and J_k (cf. § 4) is of the second degree in ρ_{k-1}.

We can write $\Delta_k^0 = -c_{k-1}^2 \Delta_{k-4} e^{-ic_0} + \Lambda_k,$

where Λ_k is of the first degree in c_{k-1}. Hence

$$\theta_k^0 = -c_{k-1}^2 \theta_{k+4} + M_k,$$

where M_k is of the first degree in ρ_{k-1}. If now we substitute the parametric value of c_{k-1}, we get

$$J_k = -\sqrt{2}\,\theta_{k-4}[\rho_{k-1}(1-\rho_1^2)\dots(1-\rho_{k-2}^2)\,2\cos\tfrac{1}{2}c_0]^2 + N_k,$$

where N_k is of the first degree in ρ_{k-1}. Hence (cf. § 4)

$$E_k = -\frac{\theta_{k-4}}{\theta_{k-2}}[\rho_{k-1}(1-\rho_1^2)\dots(1-\rho_{k-2}^2)\,2\cos\tfrac{1}{2}c_0]^2 + P_k,$$

where P_k is of the first degree in ρ_{k-1}. By our inductive hypothesis,

$$\theta_{k-2}/\theta_{k-4} = 2\cos\tfrac{1}{2}c_0(1-\rho_1^2)\dots(1-\rho_{k-3}^2)(1-\rho_{k-2}).$$

Hence $E_k = -2\cos\tfrac{1}{2}c_0\rho_{k-1}^2(1-\rho_1^2)\dots(1-\rho_{k-2}^2)(1+\rho_{k-2}) + P_k,$

and $$\tfrac{1}{2}(E_k + E_k') = -2\cos\tfrac{1}{2}c_0\rho_{k-1}^2(1-\rho_1^2)\dots(1-\rho_{k-2}^2) + \tfrac{1}{2}(P_k + P_k'), \qquad (10)$$

where P_k' is what P_k becomes on changing $c_0, \rho_1, \dots, \rho_{k-1}$ to $-c_0, -\rho_1, \dots, -\rho_{k-1}$ respectively.

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As $\rho_{k-1} \to \pm 1$, $\theta_{k-1} \to 0$. Hence, by the second part of Lemma 1, $\theta_k \to 0$. Thus θ_k is divisible by $1-\rho_{k-1}^2$, and, by (9), so is $\frac{1}{2}(E_k + E_k')$. Since $\frac{1}{2}(P_k + P_k')$ is of the first degree in ρ_{k-1}, a glance at (10) shows that

$$\tfrac{1}{2}(P_k + P_k') = 2\cos\tfrac{1}{2}c_0(1-\rho_1^2)\dots(1-\rho_{k-2}^2).$$

This, with (10), proves (8).

6. THEOREM C. *The class B is identical with the class defined by*

$$-\pi \leqslant c_0 \leqslant \pi, \tag{11}$$

$$\left.\begin{aligned} c_1 &= \sigma_1 . 2\cos\tfrac{1}{2}c_0, \\ c_2 &= q_2(\cos\tfrac{1}{4}c_0, \sin\tfrac{1}{4}c_0, \sigma_1) + \sigma_2(1-\sigma_1^2)\,2\cos\tfrac{1}{2}c_0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ c_n &= q_n(\cos\tfrac{1}{4}c_0, \sin\tfrac{1}{4}c_0, \sigma_1, \dots, \sigma_{n-1}) + \sigma_n(1-\sigma_1^2)\dots(1-\sigma_{n-1}^2)\,2\cos\tfrac{1}{2}c_0, \end{aligned}\right\} \tag{C}$$

where $q_2, \dots$ are the polynomials of Theorem B.

Let $c_0, c_1, \dots$ be a sequence of class B. Let $\epsilon > 0$ and $\delta = 1/(1+\epsilon)$. Then

$$\delta c_\nu = \int_0^\pi g(x)\cos\nu x\,dx \quad (\nu = 0, 1, \dots),$$

where $\operatorname{Max}|g| < 1$. By Theorem A, (δc_ν) is of class B_0. By Theorem B,

$$-\pi < \delta c_0 < \pi,$$

$$\delta c_n = q_n(\cos\tfrac{1}{4}\delta c_0, \sin\tfrac{1}{4}\delta c_0, \rho_1, \dots, \rho_{n-1}) + \rho_n \prod_1^{n-1}(1-\rho_1^2)\,2\cos\tfrac{1}{2}\delta c_0$$

for a unique set of ρ_n with $\rho_n^2 < 1$. We can find numbers σ_n, with $\sigma_n^2 \leqslant 1$, such that, when $\epsilon \to 0$ through a suitable sequence of values, $\rho_n \to \sigma_n$ for $n = 1, 2, \dots$. Then (c_ν) satisfies (11) and (C).

Conversely, let (c_ν) satisfy (11) and (C). If (ρ_n) is a sequence with $\rho_n^2 < 1$, and $|\rho_n - \sigma_n|$ sufficiently small, and if d_0 is sufficiently near to c_0 and satisfies $-\pi < d_0 < \pi$, then the sequence (d_ν), where

$$d_n = q_n(\cos\tfrac{1}{4}d_0, \sin\tfrac{1}{4}d_0, \rho_1, \dots \rho_{n-1}) + \rho_n \prod_1^{n-1}(1-\rho_i^2)\,2\cos\tfrac{1}{2}d_0,$$

is of class B_0 by Theorem B. By Theorem A, for each k there is a g_k such that

$$\operatorname{Max}|g_k| < 1, \quad d_\nu = \int_0^\pi g_k\cos\nu x\,dx \quad (\nu = 0, \dots, k).$$

By the principle of selection, there is a function g such that

$$\operatorname{Max}|g| \leqslant 1, \quad d_\nu = \int_0^\pi g\cos\nu x\,dx \quad (\nu = 0, 1, \dots).$$

Let $\rho_n \to \sigma_n$, so that $d_n \to c_n$; and let $d_0 \to c_0$. Again applying the principle of selection, we infer that (c_ν) is of class B.

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The last part of Theorem B gives us the values of θ_n, θ_n' for a sequence of class B_0. From Theorem C we can infer the values of θ_n, θ_n' for a sequence of class B The first few are

$$\theta_0 = 2\cos\tfrac{1}{4}(c_0+\pi),$$
$$\theta_1 = 2\cos\tfrac{1}{2}c_0(1-\sigma_1),$$
$$\theta_2 = 2\cos\tfrac{1}{4}(c_0+\pi)\,2\cos\tfrac{1}{2}c_0(1-\sigma_1^2)(1-\sigma_2),$$
$$\theta_3 = (2\cos\tfrac{1}{2}c_0)^2(1-\sigma_1)(1-\sigma_1^2)(1-\sigma_2^2)(1-\sigma_3).$$

A sequence of class B can be represented parametrically in essentially one way. To see this, let $c_0, c_1, \ldots$ be a sequence of class B. If $|c_0| = \pi$, then $c_n = 0$ for $n = 1, 2, \ldots$. It easily follows from (C) that the second members of those equations must vanish identically in $\sigma_1, \sigma_2, \ldots$.

If $|c_0| < \pi$, then the first equation in (C), which we denote by (C_1), gives the value of σ_1. If $|\sigma_1| = 1$, then $\theta_1\theta_1' = 0$. By Lemma 1, $c_2, c_3, \ldots$ are uniquely determined by c_0, c_1. Thus the second members of (C_2), (C_3), ... do not depend on $\sigma_2, \sigma_3, \ldots$; and we have

$$c_n = q_n(\cos\tfrac{1}{4}c_0, \sin\tfrac{1}{4}c_0, \sigma_1, 0, \ldots, 0) \quad (n = 2, 3, \ldots).$$

If $|\sigma_1| < 1$, then (C_2) gives the value of σ_2. If $|\sigma_2| = 1$, then $\theta_2\theta_2' = 0$. By Lemma 1, $c_3, c_4, \ldots$ are uniquely determined by c_0, c_1, c_2. Thus the second members of (C_3), (C_4), ... do not depend on $\sigma_3, \sigma_4, \ldots$; and we have

$$c_n = q_n(\cos\tfrac{1}{4}c_0, \sin\tfrac{1}{4}c_0, \sigma_1, \sigma_2, 0, \ldots, 0) \quad (n = 3, 4, \ldots).$$

If $|\sigma_2| < 1$, then (C_3) gives the value of σ_3; and so on.

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