Lattice dissections: plainer and less fancy

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Introduction

The title of this Article pays homage to Greg Frederickson's widely-known book on geometrical dissections, *Dissections: Plane & Fancy* [1]. I enjoyed the book so much, I built an honours geometry course around it – a terminal mathematics course for honours students needing a mathematics credit for their general education requirements. Not only is there significant geometry involved, but there is a fair amount of number theory relevant to finding infinite families of dissections of, for example, m squares to n squares.

However, although many dissections from one regular polygon to another are quite elegant, the mathematics involved in specifying the dissections is often formidable. As an example, for one of the simpler dissections – a triangle to a square (see Figure 1) – the segments of the pieces lying on the base of the triangle are in the ratios

$$\frac{3-\sqrt{4\sqrt{3}-3}}{2} : 1 : \frac{\sqrt{4\sqrt{3}-3}-1}{2}.$$

Not exactly elementary.

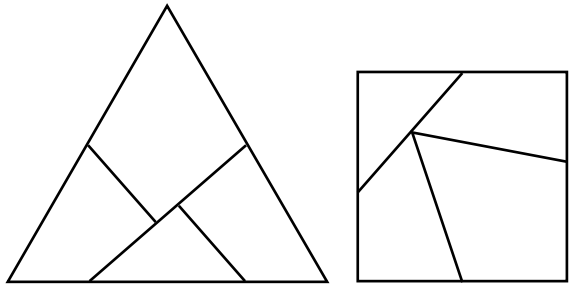


FIGURE 1: Dissection of triangle to square

Since this was a course designed for non-mathematics majors, I felt the need to create some more accessible dissections. As a result, I experimented with dissections based on square and triangular grids. This way, students could work out dissection problems on ordinary graph paper. I created hundreds of puzzles, and share a few here. These *lattice dissections* have the property that the vertices of all the pieces of the dissections lie on lattice points of a grid.

Two octagons to one

It is impossible for the vertices of a regular octagon all to have integer coordinates in the usual Cartesian plane. But it is easy to draw octagons on

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graph paper — they will be irregular, but it is possible to maximise their symmetry, as shown in Figure 2.

Frederickson references Frese's dissection of two regular octagons (of the same size) to one, so that the larger has side lengths $\sqrt{2}$ times as long as the smaller side lengths. Eight pieces are necessary [1, p. 188].

However, if we draw slightly less regular octagons, we can achieve a 2-to-1 dissection in just four pieces. Moreover, none of the pieces need to be rotated or reflected. Again, see Figure 2.

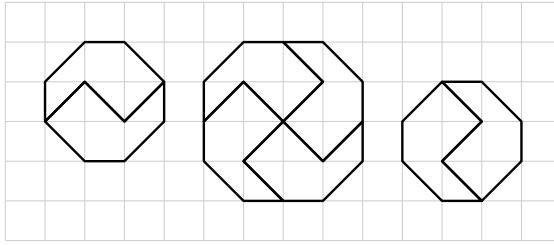


FIGURE 2: Two octagons to one

We now look at an interesting infinite family of 2-to-1 octagon dissections, illustrated in Figure 3. Note that the side lengths of the larger octagon are $\sqrt{2}$ times the lengths of the smaller octagons; as a result, if the vertices of the larger octagon are to be lattice points, the larger octagon must be rotated 45°.

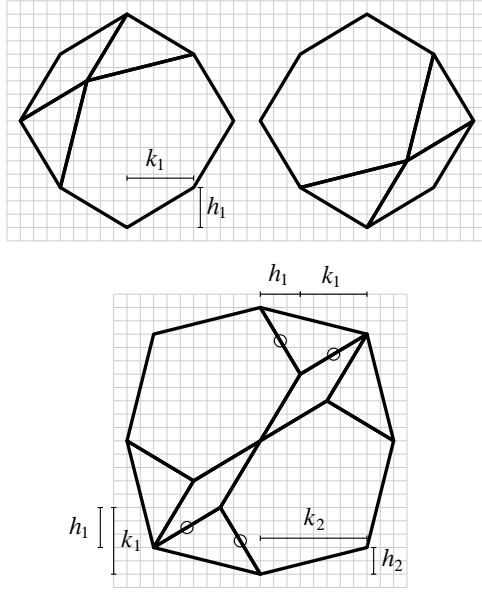


FIGURE 3: Dissection of two irregular octagons to one

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We note that the smaller octagons may be thought of as squares of side length $2k_1$ together with eight right triangles whose legs have lengths h_1 and k_1. This particular dissection only works for convex octagons, so we require that $h_1 < k_1$. The larger octagon may be similarly imagined as being described by parameters h_2 and k_2. We have the simple relationships

$$h_2 = k_1 - h_1, \qquad k_2 = k_1 + h_1, \tag{1}$$

which are also illustrated in Figure 3. At the top of the figure, the horizontal displacements of the segments indicated with a small circle are h_1 and k_1. But this is the same displacement as k_2, so that $k_2 = k_1 + h_1$. At the bottom of the figure, the vertical displacements of the segments indicated with a small circle are h_1 and k_1. The difference of these displacements is h_2, so that $h_2 = k_1 - h_1$.

While Figure 3 may be compelling, we provide a more rigorous justification that the dissection is in fact valid. The key geometrical observation is that the triangles in the dissection are in fact right isosceles triangles, which we will show follows directly from (1).

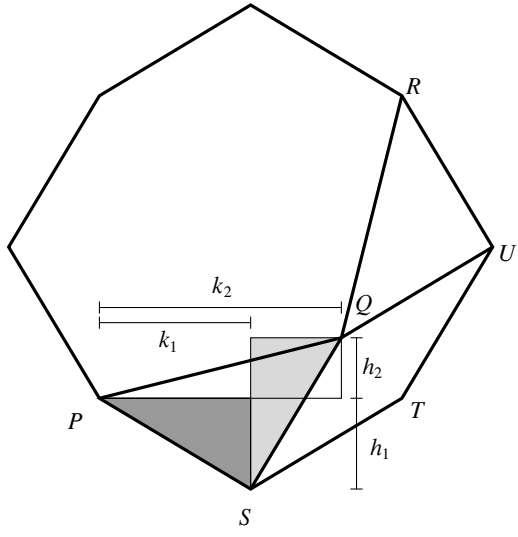


FIGURE 4: Illustrating a proof that the dissection is valid

Consider Figure 4, a smaller octagon with parameters h_1 and k_1. From P, move over k_2 units and up h_2 units to Q, and proceed to draw segments to R, S and U.

Now from (1), it follows that $h_1 + h_2 = k_1$ and $k_2 - k_1 = h_1$, so that the two shaded grey triangles are in fact congruent. This immediately implies that ΔPSQ is a right isosceles triangle, with the legs having length equal to the sides of the smaller octagon, and the hypotenuse being the same length as the sides of the larger octagon. This also implies that quadrilateral $STUQ$ is a rhombus.

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It is interesting to note that when h_1 and k_1 are chosen so that h_1/k_1 is a good approximation to $\sqrt{2} - 1 = \tan 22.5°$, this dissection closely approximates Frese's 2-to-1 dissection. So in a sense, this family of dissections generalises the dissection in the case that the octagons are regular.

Irregular dodecagon to square

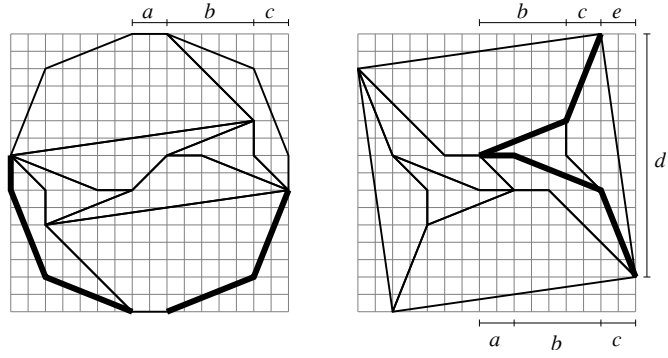


FIGURE 5: Irregular dodecagon to square

Going a step further, we now consider a family of dissections involving irregular dodecagons with square symmetry, as illustrated in Figure 5. We parameterise such dodecagons by three parameters, a, b and c, as indicated on the left of the figure; we also require that $b > c > 0$ so that the dodecagon is convex and does not degenerate to an octagon or a square. It is straightforward to see that the area of this dodecagon is

$$a^2 + 4(a + b)(c + b).$$

To dissect into a square with vertices on the grid, we seek d and e as shown in the right of Figure 5 such that

$$a^2 + 4(a + b)(c + b) = d^2 + e^2.$$

We now proceed to characterise the infinite family of dodecagon-to-square dissections illustrated in the diagram. One essential feature of this dissection is that the bold paths in the left of Figure 5, when the pieces are rearranged, actually meet near the centre of the square, as shown on the right. Projecting the top path on the right onto the top of the grid, and the bottom path onto the bottom of the grid, we see that we must have $b + c + e = a + b + c$, so that $a = e$.

But if $a = e$, we must then have $4(a + b)(c + b) = d^2$, which may be rewritten as

$$(a - c)^2 + d^2 = (2b + a + c)^2.$$

Thus, we may use Pythagorean triples to generate suitable parameters for dissections of an irregular dodecagon to a square, still maintaining

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$b > c > 0$. Figure 5 illustrates the case when $a = c$, so that $d = 2(a + b)$. This corresponds to a degenerate Pythagorean triple, where one of the sides of the right triangle has length 0.

Dissections on a triangular grid

There is no reason to restrict our consideration to dissections on a square grid. There are some equally interesting dissections on a triangular grid, such as this dissection from an irregular dodecagon to an equilateral triangle.

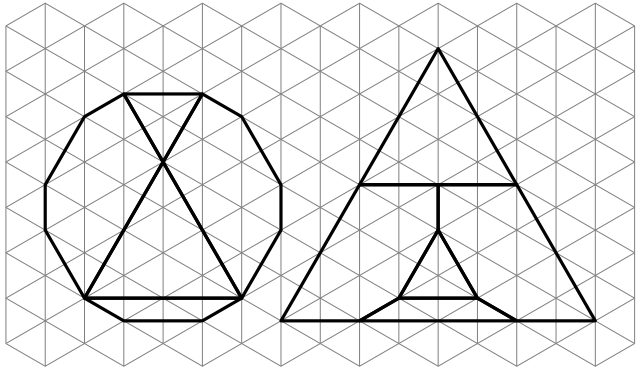


FIGURE 6: Irregular dodecagon to equilateral triangle

It is also possible to find infinite families of dissections, as illustrated by the dodecagon-to-rhombus dissection shown below.

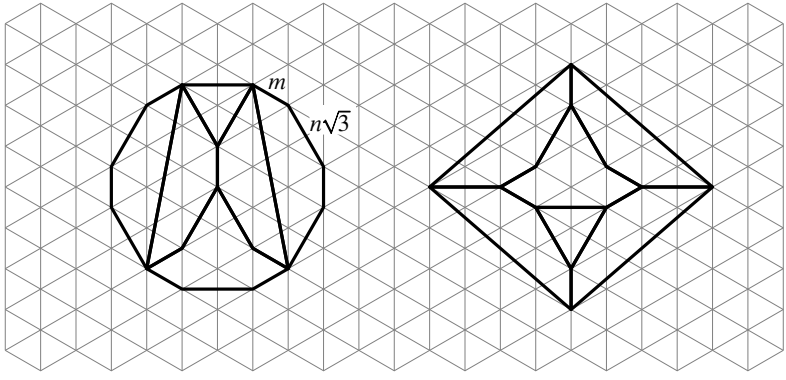


FIGURE 7: Irregular dodecagon to rhombus

We parameterise these dissections with m and n (see Figure 7) so that the dodecagon has sides whose lengths alternate between m and $n\sqrt{3}$. It is not difficult to see that the pieces actually do fit together, as so many of the angles are 60° and 150°.

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It is fairly straightforward to calculate the angles of the rhombus. In terms of the parameters m and n, we see that the diagonals of the rhombus are $3m + 3n$ (the vertical diagonal in Figure 7), and $(m + 3n)\sqrt{3}$ (the horizontal diagonal). Applying the usual trigonometric identities reveals that the cosines of the angles of the rhombus are given by

$$\pm\frac{m^2 - 3n^2}{2\left(m^2 + 3mn + 3n^2\right)}.$$

Note that when m/n is a good approximation to $\sqrt{3}$, the angles in the rhombus are very near 90°. The limiting case is in fact Lindgren's dissection of a regular dodecagon to a square (see [1, p. 107]). Thus, this family of dissections, in a sense, generalises Lindgren's dissection.

Conclusion

It is hoped that these few examples will encourage other dissection enthusiasts to consider lattice dissections. It would be an interesting project to see which other dissections using regular polygons find a generalisation on a grid, as illustrated in two of the cases above.

Finally, Frederickson has an extensive bibliography [1, pp. 293-301] of well over 200 references for those interested in learning more about geometrical dissections.

Acknowledgment

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Reference

1. Greg N. Frederickson, *Dissections: Plane & Fancy*, Cambridge University Press, New York (1997).

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