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STUDY OF INSTANT SYSTEM AVAILABILITY

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In this paper, we study the instant availability $A(t)$ of a repairable system by integral equation. We have proved initial monotonicity of availability, and derived lower bounds of $A(t)$ and average availability. The availabilities of two systems are compared.

1. INTRODUCTION

Consider a repairable system that has two states, namely "up" and "down". The state of the system is "up" if it is properly functioning, is "down" if it is under repair. Let binary random variable $X(t)$ denote the state of the system at time t, that is $X(t) = 1$ means that the system is in "up" state, and $X(t) = 0$ otherwise. Suppose that a new system with lifetime U_1 starts working at time $t = 0$ until it fails and it will be repaired at the time of failure taking time D_1. The first up period and down period consist the first cycle $[0, U_1 + D_1]$ of the operation of the system. At the end of the first cycle the second cycle begins, and so on and so forth. Throughout this article we will use U_i and D_i to denote the lifetime and repair time of the system in the ith cycle, $i \geq 1$.

Availability is a very important measure of system performance when the system is repairable and maintenance is implemented. Precisely, the probability that the system is in its "up" state at time t, that is, $A(t) \equiv \mathbb{P}\big(X(t) = 1\big)$ is called the availability of the system at time t. This is actually the instant (or point) system availability. Because of its complexity the so-called steady-state system availability has received a lot of attention. This quantity is defined as the limit of $A(t)$ as t goes to infinity provided the limit exists, that is, the steady-state system availability $A(\infty) = \lim_{t\to\infty} A(t)$. Some contributions to $A(\infty)$, among others, are the works of [4–8,11–14,16,18–21]. Only for few special cases the closed form expressions of $A(t)$ are derived, see for example, [10,23]. There are, to the authors' knowledge, no work regarding the behavior of the function $A(t)$ except for these special cases. This paper will focus on the behavior of the instant system availability $A(t)$ as a function of time t.

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Throughout this paper we will assume that $\{U_i, i \geq 1\}$ are i.i.d. with common CDF $F(t)$, $\{D_i, i \geq 1\}$ are i.i.d. with common CDF $G(t)$ respectively. That is, here we assume the repair is perfect and the repaired system is as good as new. If we interpret "replacement" as "repair", then the as-good-as-new assumption is reasonable. Moreover, we also assume that $\{U_i, i \geq 1\}$ and $\{D_i, i \geq 1\}$ are independent of each other. We further let $T_i = U_i + D_i, i \geq 1$ be the length of the ith cycle, and denote its CDF as $H(t)$. Obviously $H(t)$ is the convolution of $F(t)$ and $G(t)$ due to the assumed independence of U_i and D_i, that is, $H(t) = \int_0^t F(t-x)dG(x)$. In the case when both F and G have probability density function (pdf) f and g, then H also has pdf, say $h(t)$, which is the convolution of f and g $h(t) = \int_0^t f(t-x)g(x)dx = \int_0^t g(t-x)f(x)dx$ for any $t \geq 0$. For the simplicity of notation we will use U to denote an i.i.d. copy of those lifetimes U_i's. The repair time D is similarly defined. Certainly, U and D are assumed to be independent of each other.

Based on the i.i.d. assumption it is well known (see, e.g., page 101 of [3]) that the point availability $A(t)$ is the unique solution of the following renewal equation:

$$A(t) = \overline{F}(t) + \int_0^t A(t-s)\, dH(s) \tag{1}$$

where $\overline{F}(t) = 1 - F(t)$.

In Section 2, we will determine an interval in which $A(t)$ is decreasing. Some lower bounds of $A(t)$ and the average availability $\int_0^t A(x)\, dx/t$ are derived in Section 3. In Section 4, system availabilities of two systems are compared. And finally, in Section 5 several numerical examples are given for illustration.

2. INITIAL MONOTONICITY

In general, the positive real line $[0, \infty)$ can be partitioned into mutually disjoint intervals in such a way that the availability function $A(t)$ decreases or increases alternatively on these intervals. At time $t = 0$ the availability $A(0) = 1$ and thus it is easy to understand that $A(t)$ should decrease in a neighborhood of 0. Let $\hat{T}_0 \equiv \sup\{t > 0 : A(x) \text{ decreases in } x \in [0, t]\}$. The interval $[0, \hat{T}_0]$ is important since often the availability $A(\hat{T}_0) = \inf_{0 \leq t \leq \hat{T}_0} A(t)$ at time $\hat{T}_0$ is lower than the steady-state availability $A(\infty)$, which can be observed from examples given in Section 5, and so system operator should try to improve the system in order to avoid very low availability at the very beginning of system operation. It is not easy to obtain the exact value of $\hat{T}_0$ hence instead of $\hat{T}_0$ we will determine a subinterval of $[0, \hat{T}_0]$, say $[0,\ T_0]$, in which the system availability $A(t)$ is decreasing in t, and consequently one could get some idea about $A(\hat{T}_0)$ based on the value of $A(T_0)$.

LEMMA 1: *Assume that both $f(\cdot)$ and $h(\cdot)$ are continuous on $(0, \infty)$. Denote $T_0 \equiv \inf\{t > 0 :\ f(t) = h(t)\}$. Suppose $T_0 > 0$. Then $f(t) > h(t),\ \forall t \in (0, T_0)$.*

PROOF: Due to the assumption $T_0 > 0$, it must be true that $f(t) - h(t)$ does not change sign in the interval $(0, T_0)$.

The sign of $f(t) - h(t)$ in $(0, T_0)$ must be positive due to the fact that U is stochastically dominated by $U + D$. ■

LEMMA 2: *If $f(\cdot)$ is continuous on $(0, \infty)$, $g(\cdot)$ is bounded on any finite interval not including 0 and $f, g \in L^1(0, \infty)$, then the convolution $h \equiv f * g$ is continuous on $(0, \infty)$.*

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PROOF: See the Appendix. ■

COROLLARY 1: *Assume the following*

(a) $f(\cdot)$ is continuous, $g(\cdot)$ is bounded on any finite interval not including 0;

(b) $T_0 > 0$, where T_0 is defined in Lemma 1.

Then $f(t) > h(t),\ \forall t \in (0, T_0)$.

PROOF: Based on Lemma 2, the function $h(t)$ is continuous on $(0, \infty)$ so the desired result follows from Lemma 1. ■

LEMMA 3: *Let function $u(t)$ satisfy the renewal equation*

$$u(t) = c(t) + \int_0^t u(t-s)\, dK(s) \quad \forall t \geq 0$$

where $c(t)$ is a continuous function on $[0, \infty)$ and $K(t)$ is a distribution function on $[0, \infty)$. Then $u(t)$ is the unique solution of the renewal equation and is given by

$$u(t) = (c * M)(t) = \int_0^t c(t-s)\, dM(s),$$

where $M(t)$ is the renewal function associated with $K(\cdot)$ and given as $M(t) = \sum_{n=0}^{\infty} K^{(n)}(t)$, where $K^{(n)}(t)$ is the n-fold convolution of function $K(t)$.

PROOF: See, for example, Theorem 4.1 on page 184 of [17]. ■

THEOREM 1: *Suppose that $F(t)$ and $G(t)$ have pdf $f(t)$ and $g(t)$ that satisfy condition (a) of Corollary 1 and $h(t) = (f*g)(t) \leq f(t), \forall t \in [0,\ T_0]$. Then $A(t)$ is decreasing in $t \in [0,\ T_0]$.*

PROOF: Note that $A(t)$ is the unique solution of the renewal equation (1) and thus $A(t)$ is of bounded variation according to Theorem 7.4 in [9], so $A(t)$ is differentiable. Denote the derivative of $A(t)$ as $a(t) = A'(t)$. As a matter of fact, $a(t)$ is the unique solution of the equation

$$a(t) = [h(t) - f(t)] + \int_0^t a(t-s)h(s)\, ds.$$

Note that from our assumption $c(t) \equiv h(t) - f(t) \leq 0, \forall t \in [0,\ T_0]$ and thus by Lemma 3 it follows that

$$a(t) = (c * M)(t) = \int_0^t c(t-s)\, dM(s) \leq 0, \forall t \in [0,\ T_0]$$

where $M(t) = \sum_{n=0}^{\infty} H^{(n)}(t)$ is the renewal function associated with $H(t) = (F * G)(t)$. This certainly implies that $A(t)$ decreases in $t \in [0,\ T_0]$. ■

COROLLARY 2: *If there exists $T_0^* \in (0, T_0]$ such that $0 < h(t) < f(t),\ \forall 0 < t < T_0^*$, then it is true that $a(t) < 0, \forall 0 < t < T_0$, that is, $A(t)$ strictly decreases in $t \in [0, T_0]$.*

Remark 1: A sufficient condition for the existence of $T_0 > 0$ in Theorem 1 is $f(0) > 0$ and $h(0) = 0$. The following condition ensures $h(0) = 0$: if there exists $t^* > 0$ such that $f(\cdot)$ is

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continuous on $[0, t^*]$, then $h(0) = 0$ (right derivative of $H(t)$ at zero). This sufficiency can be seen as follows: For any $t \in [0, t^*]$

$$0 \le h(t) = \int_0^t f(t-s)g(s)\,ds \le \max_{0\le x\le t} f(x) \int_0^t g(s)\,ds.$$

Obviously $\lim_{t\to 0^+} \int_0^t g(s)ds = 0$ since pdf $g \in L^1(0,\infty)$. Hence, $\lim_{t\to 0^+} h(t) = 0$.

3. SOME LOWER BOUNDS FOR AVAILABILITY

3.1. Low Bounds for Instant Availability

The instant availability $A(t)$ has a trivial lower bound $\bar{F}(t)$. This can be seen from Eq. (1) directly. The following are two non-trivial lower bounds of $A(t)$.

THEOREM 2: *Suppose that $F(t)$ and $G(t)$ have pdf $f(t)$ and $g(t)$, respectively. Let $T_0 > 0$ be such that $h(t) < f(t), \forall t \in (0, T_0)$. Then for any $t \in [0,\ T_0]$ it holds that*

$$A(t) \ge \frac{\overline{F}(t)}{\overline{H}(t)}.$$

PROOF: For any $t \in [0,\ T_0]$ it follows from the renewal Eq. (1) that

$$\begin{aligned} A(t) &= \overline{F}(t) + \int_0^t A(t-s)\,dH(s) \\ &\ge \overline{F}(t) + \int_0^t A(t)\,dH(s) = \overline{F}(t) + A(t)H(t), \end{aligned}$$

where the inequality holds by Theorem 1. The desired result thus follows. ■

Next result gives a lower bound of $A(t)$ on interval $[0,\ T_1]$ for any given $T_1 > 0$.

THEOREM 3: *For any $T_1 \ge 0$ the following inequality is always true:*

$$A(t) \ge \min_{0\le x\le T_1} \frac{\overline{F}(x)}{\overline{H}(x)}, \quad \forall 0 \le t \le T_1.$$

PROOF: $A(t)$ is the unique solution of the renewal Eq. (1). Moreover, constant function $u(t) = c \equiv \min_{0\le x\le T_1} \overline{F}(x)/\overline{H}(x)$ is the unique solution of the equation

$$u(t) = c\overline{H}(t) + \int_0^t u(t-s)\,dH(s).$$

Denote $\Delta(t) = A(t) - c$. Then obviously, $\Delta(t)$ is the unique solution of the equation

$$\Delta(t) = \left[\overline{F}(t) - c\overline{H}(t)\right] + \int_0^t \Delta(t-s)\,dH(s)$$

and $\overline{F}(t) - c\overline{H}(t) \ge 0,\ \forall 0 \le t \le T_1$. From these, by Lemma 3, it follows that $\Delta(t) \ge 0, \forall 0 \le t \le T_1$. That is

$$A(t) \ge \min_{0\le x\le T_1} \frac{\overline{F}(x)}{\overline{H}(x)}, \quad \forall 0 \le t \le T_1.$$

■

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Remark 2: We briefly compare the two bounds given in Theorems 2 and 3 as follows: the lower bound in Theorem 3 applies for any interval $[0,\ T_1]$ and it does not require F and G to have density functions, while Theorem 2 is restricted to the fixed interval $[0,\ T_0]$ and assumes both F and G have density functions.

However, if time t is restricted to the interval $[0,\ T_0]$, it is then obviously true that Theorem 2 provides a better bound than Theorem 3 does. Of course, in the case when $\overline{F}(x)/\overline{H}(x)$ decreases in $x \in (0, T_0)$ where T_0 is that one assumed in Theorem 2, the results of Theorems 2 and 3 are equivalent. This is particularly true if the hazard rate function of U is greater than or equal to that of $U + D$ all the time.

Remark 3: Another lower bound $A(t) \geq 1 - \mathbb{E}(D)\big(\sup_{0\leq x\leq t} \lambda_F(x)\big) \equiv L_{AJ}$ for any $t \geq 0$ was given in Proposition 34 on page 107 of [3].

Although in general it is difficult to compare bound L_{AJ} with our bounds in Theorems 2 and 3, but an obvious drawback of L_{AJ} is that this lower bounds could be either zero or even negative. For instance, if U follows gamma distribution with shape parameter smaller than one, then $\lambda_F(0) \equiv \lim_{t\to 0^+} \lambda_F(t) = \infty$ and so $\sup_{0\leq x\leq t} \lambda_F(x) = \infty$ which yields $L_{AJ} = -\infty$.

Remark 4: In practice, it is common that the repair time of a system is much shorter than the lifetime of the system. For instance, there may exist a constant $p \geq 1$ such that $\overline{G}(x) \leq \big(\overline{F}(x)\big)^p,\ \forall x \geq 0$. In this case, it can be shown by Theorem 2 that for any fixed $T_1 > 0$,

$$A(t) \geq \min_{0\leq x\leq T_1} \frac{\overline{F}(x)}{1 + pF(x)\left[1 - \big(\overline{F}(x)\big)^p\right]}, \quad \forall 0 \leq t \leq T_1.$$

To see this we note that

$$\begin{aligned}
\bar{H}(x) &= 1 - \int_0^x G(x-s)\, dF(s) = \bar{F}(x) + \int_0^x [1 - G(x-s)]\, dF(s) \\
&\leq \bar{F}(x) + \int_0^x \big(\bar{F}(x-s)\big)^p\, dF(s) = 1 + p\int_0^x F(x-y)\,\big(\bar{F}(y)\big)^{p-1} f(y)\, dy \\
&\leq 1 + p\int_0^x F(x)\,\big(\bar{F}(y)\big)^{p-1}\, dF(y) = 1 + pF(x)\left[1 - (\bar{F}(x))^p\right].
\end{aligned}$$

From this we see that

$$\frac{\bar{F}(x)}{\bar{H}(x)} \geq \frac{\bar{F}(x)}{1 + pF(x)[1 - (\bar{F}(x))^p]}, \quad \forall x \geq 0.$$

Therefore, according to Theorem 3, we have

$$A(t) \geq \min_{0\leq x\leq T_1} \frac{\bar{F}(x)}{1 + pF(x)\left[1 - \big(\bar{F}(x)\big)^p\right]}, \quad \forall 0 \leq t \leq T_1.$$

3.2. Lower Bounds for Average Availability

The following result gives a lower bound for the average availability $\int_0^t A(x)dx/t$.

THEOREM 4: *Suppose that $A(t)$ is decreasing in $[0,\ T_0^*]$. Then for any $t \in (0, T_0^*]$,*

$$\frac{1}{t}\int_0^t A(x)\, dx \geq \frac{\int_0^t \overline{F}(s)\, ds}{t\overline{H}(t) + \int_0^t s\, dH(s)}.$$

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PROOF: From the renewal Eq. (1) we can obtain

$$\int_0^t A(x)\,dx = \int_0^t \overline{F}(x)\,dx + \int_0^t dx\left(\int_0^x A(x-s)\,dH(s)\right)$$
$$= \int_0^t \overline{F}(x)\,dx + \int_0^t \left(\int_0^{t-s} A(x)\,dx\right) dH(s).$$

Let $u(t) = \int_0^t A(x)dx$. It follows that $u(t)$ is the unique solution of the renewal equation

$$u(t) = \int_0^t \overline{F}(x)\,dx + \int_0^t u(t-s)\,dH(s). \tag{2}$$

It is easy to see that $u(t)/t$ is decreasing in $t \in (0, T_0]$ since $A(t)$ is. From the renewal Eq. (2), we have

$$\frac{u(t)}{t} = \frac{\int_0^t \overline{F}(x)\,dx}{t} + \frac{1}{t}\int_0^t u(t-s)\,dH(s) \geq \frac{\int_0^t \overline{F}(x)\,dx}{t} + \frac{1}{t}\int_0^t \frac{u(t)}{t}(t-s)\,dH(s)$$
$$= \frac{\int_0^t \overline{F}(x)\,dx}{t} + \frac{u(t)}{t}H(t) - \frac{u(t)}{t}\cdot\frac{\int_0^t s\,dH(s)}{t}. \tag{3}$$

Therefore,

$$\frac{u(t)}{t}\left[1 - H(t) + \frac{\int_0^t s\,dH(s)}{t}\right] \geq \frac{\int_0^t \overline{F}(x)\,dx}{t}$$
$$\frac{u(t)}{t} \geq \frac{\int_0^t \overline{F}(x)\,dx/t}{\overline{H}(t) + \int_0^t s\,dH(s)/t}.$$

This ends the proof. ■

COROLLARY 3: *Under the same conditions in Theorem 4 if further* $Var(U+D) < \infty$, *then*

$$\frac{1}{t}\int_0^t A(x)\,dx \geq \frac{\int_0^t \overline{F}(s)\,ds}{t\overline{H}(t) + \left(\mathbb{E}(U+D)^2\right)^{1/2}}, \quad \forall t \in (0, T_0).$$

PROOF: From (3) we have

$$\frac{u(t)}{t} \geq \frac{\int_0^t \overline{F}(x)\,dx}{t} + \frac{u(t)}{t}H(t) - \frac{u(t)}{t}\cdot\frac{\int_0^t s\,dH(s)}{t}.$$

By the Schwarz inequality it follows that

$$\int_0^t s\,dH(s) \leq \left(\int_0^t s^2\,dH(s)\right)^{1/2}\left(\int_0^t dH(s)\right)^{1/2} \leq \left(\mathbb{E}(U+D)^2\right)^{1/2}.$$

Thus,

$$\frac{u(t)}{t} \geq \frac{\int_0^t \overline{F}(x)\,dx}{t} + \frac{u(t)}{t}H(t) - \frac{u(t)}{t}\cdot\frac{(\mathbb{E}(U+D)^2)^{1/2}}{t}$$

and the desired lower bound follows. ■

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Another lower bound of the average availability is given below.

THEOREM 5: *Suppose that* $Var(U) < \infty$ *and* $Var(D) < \infty$. *For any given* $T_2 > 0$, *we have*

$$\frac{1}{t}\int_0^t A(x)\,dx \geq \min_{0\leq x\leq T_2} \frac{\int_0^x \overline{F}(s)\,ds}{\int_0^x \overline{H}(s)\,ds}, \quad \forall 0 \leq t \leq T_2.$$

PROOF: Let $c = \min_{0\leq x\leq T_2} \int_0^x \overline{F}(s)\,ds / \int_0^x \overline{H}(s)\,ds$. It is obvious that

$$ct\overline{H}(t) + c\int_0^t s\,dH(s) + \int_0^t c\cdot(t-s)\,dH(s) = ct.$$

On the other hand, the assumption $Var(U) < \infty$ and $Var(D) < \infty$ imply that $t\overline{H}(t)$ is bounded on $[0,\infty)$ and $\int_0^\infty x dH(x) < \infty$. Thus, $v(t) = ct$ is the unique solution of the renewal equation

$$v(t) = \left[ct\overline{H}(t) + c\int_0^t x dH(x)\right] + \int_0^t v(t-s)\,dH(s).$$

Let $\Delta(t) = \int_0^t A(x)\,dx - ct$. Then $\Delta(t)$ is the unique solution of the renewal equation

$$\Delta(t) = \left[\int_0^t \overline{F}(x)\,dx - ct\overline{H}(t) - c\int_0^t x dH(x)\,dx\right] + \int_0^t \Delta(t-s)\,dH(s).$$

We have

$$\begin{aligned}\int_0^t \overline{F}(x)\,dx - ct\overline{H}(t) - c\int_0^t x dH(x) &= \int_0^t \overline{F}(x)\,dx - c\int_0^t \overline{H}(x)\,dx \\ &= \int_0^t \overline{H}(x)\,dx\left[\frac{\int_0^t \overline{F}(x)\,dx}{\int_0^t \overline{H}(x)\,dx} - c\right] \geq 0, \quad \forall t,\ 0 \leq t \leq T_2.\end{aligned}$$

Now, according to Lemma 3 we are readily to obtain $\Delta(t) \geq 0, \forall t \in [0,\ T_2]$. That is, $\forall 0 \leq t \leq T_2$, $\int_0^t A(x)dx - ct \geq 0$. ■

Remark 5: It can be shown that the lower bound in Theorem 5 is greater than that in Theorem 3. To see this we let $T_1 = T_2 = T^*$ and show it with additional assumption that $\overline{F}(x)/\overline{H}(x)$ is continuous on $[0,\ T^*]$. With this assumption we can suppose there exists $x^* \in [0,\ T^*]$ such that $\overline{F}(x^*)/\overline{H}(x^*) = \min_{0\leq x\leq T^*} \overline{F}(x)/\overline{H}(x)$. Clearly for any $x \in [0,\ T^*]$,

$$\int_0^x \overline{H}(s)\overline{H}(x^*)\left[\frac{\overline{F}(s)}{\overline{H}(s)} - \frac{\overline{F}(x^*)}{\overline{H}(x^*)}\right] ds \geq 0.$$

It implies

$$\overline{H}(x^*)\int_0^x \overline{F}(s)\,ds \geq \overline{F}(x^*)\int_0^x \overline{H}(s)\,ds$$

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or

$$\frac{\int_0^x \overline{F}(s)\,ds}{\int_0^x \overline{H}(s)\,ds} \geq \frac{\overline{F}(x^*)}{\overline{H}(x^*)}, \quad \forall x \in [0,\ T^*].$$

Therefore,

$$\min_{0\leq x\leq T^*} \frac{\int_0^x \overline{F}(s)\,ds}{\int_0^x \overline{H}(s)\,ds} \geq \frac{\overline{F}(x^*)}{\overline{H}(x^*)} = \min_{0\leq x\leq T^*} \frac{\overline{F}(x)}{\overline{H}(x)}.$$

The claimed inequality holds without the additional assumption that $\overline{F}(x)/\overline{H}(x)$ is continuous on $[0,\ T^*]$, but the proof is much longer and hence is omitted.

4. COMPARISON OF SYSTEM AVAILABILITIES

In the two theorems below we compare the availabilities of two systems, say S and S^*. The lifetime and the repair time, etc. of S^* are denoted by the same notation as the counterparts of S with an asterisk.

THEOREM 6: *Suppose* $U + D =_{st} U^* + D^*$ *and* $U \leq_{st} U^*$, *here* $\leq_{st}$ *is the usual stochastic ordering. Then* $A(t) \leq A^*(t),\ \forall t \geq 0$.

PROOF: From $U + D =_{st} U^* + D^*$ we have $H(t) = H^*(t), \forall t \geq 0$. It holds that

$$A(t) = \overline{F}(t) + \int_0^t \overline{F}(t-s)\,dM(s),$$

where $M(s) = \sum_{j=1}^{\infty} H^{(j)}(s)$, $H^{(j)}$ is the k-fold convolution of H.

Similarly,

$$A^*(t) = \overline{F^*}(t) + \int_0^t \overline{F^*}(t-s)\,dM^*(s).$$

However, $M^*(s) = \sum_{j=1}^{\infty} H^{*(j)}(s) = \sum_{j=1}^{\infty} H^{(j)}(s) = M(s)$, so

$$\begin{aligned} A^*(t) &= \overline{F^*}(t) + \int_0^t \overline{F^*}(t-s)\,dM(s) \\ &\geq \overline{F}(t) + \int_0^t \overline{F}(t-s)\,dM(s) = A(t), \quad \forall t \geq 0, \end{aligned}$$

since $U \leq_{st} U^*$ implies $\overline{F}(t) \leq \overline{F^*}(t),\ \forall t \geq 0$. ■

THEOREM 7: *Let* $T > 0$ *be such that* $\overline{F}(t) \leq \overline{F^*}(t)$ *and* $h(t) \leq h^*(t)$ *for any* $t \in [0,\ T)$. *Then* $A(t) \leq A^*(t), \forall t \in [0,\ T]$.

PROOF: Following the Picard's approximation method (see, for instance, Theorem 7.2 of [9]), for any $t \in (0, T)$ we define

$$A_0(t) = \overline{F}(t) \quad \text{and} \quad A_i(t) = \overline{F}(t) + \int_0^t A_{i-1}(s)h(t-s)\,ds, \quad 1 \leq i \leq n+1.$$

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We define $A_0^*(t), \ldots, A_n^*(t), A_{n+1}^*(t)$ in the same manner. Picard's method ensures that $A(t) = \lim_{n\to\infty} A_n(t)$ and $A^*(t) = \lim_{n\to\infty} A_n^*(t)$ for any $t \in (0, T)$. On the other hand, it is easy to show that $A_n(t) \le A_n^*(t), \forall t \in (0, T)$ by induction.

Therefore, $A(t) = \lim_{n\to\infty} A_n(t) \le \lim_{n\to\infty} A_n^*(t) = A^*(t), \forall t \in [0,\ T]$. ■

5. EXAMPLES

In this section, we will show some examples by solving the renewal equation (1). We test three different types of distribution, that is, Gamma, Log-normal, and Weibull distributions. We will plot a vertical line at $t = T_0$ defined in Theorem 1 in each figure associated with these examples. All these figures clearly indicate that the interval determined by Theorem 1, in which $A(t)$ decreases, is very sharp.

EXAMPLE 1: Suppose $U \sim \ln\mathcal{N}(\mu, \sigma)$ and $D \sim Weibull(k, \lambda)$ with density functions

$$f(t) = \frac{1}{t\sigma\sqrt{2\pi}} e^{-\frac{(\ln t-\mu)^2}{2\sigma^2}} \quad \text{and} \quad g(t) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}.$$

We choose $\mu = 2, \sigma = 0.25$, and $k = 2, \lambda = 3$ in this example. In Figure 1, the curve of $A(t)$ obtained from numerical solution is depicted. This example demonstrates that the result of Theorem 1 is very sharp. In fact, in this example the value of $\hat{T}_0$ is between 8.608 and 8.618, while the value of T_0 from Theorem 1 is between 8.598 and 8.608 which is just slightly smaller than $\hat{T}_0$.

EXAMPLE 2: Let $U \sim Gamma(p, \alpha)$ and $D \sim \ln\mathcal{N}(\mu, \sigma)$, Choosing $p = 0.5, \alpha = 10, \mu = 2$ and $\sigma = 0.2$, we compare $A(t)$ with the lower bound $\overline{F}(t)/\overline{H}(t)$ given in Theorem 2. The functions $A(t)$ and $\overline{F}(t)/\overline{H}(t)$ are plotted in Figure 2. When $t < T_0 = 5.8000$, they are indistinguishable.

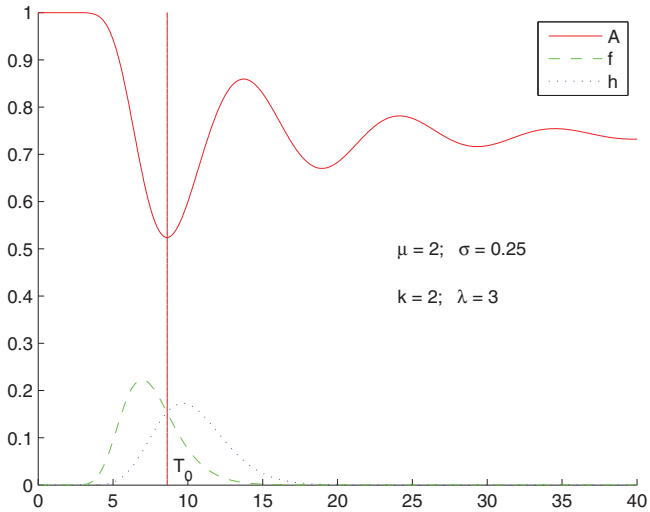


FIGURE 1. (Color online) Example 1: $U \sim Lognormal,\ D \sim Weibull$

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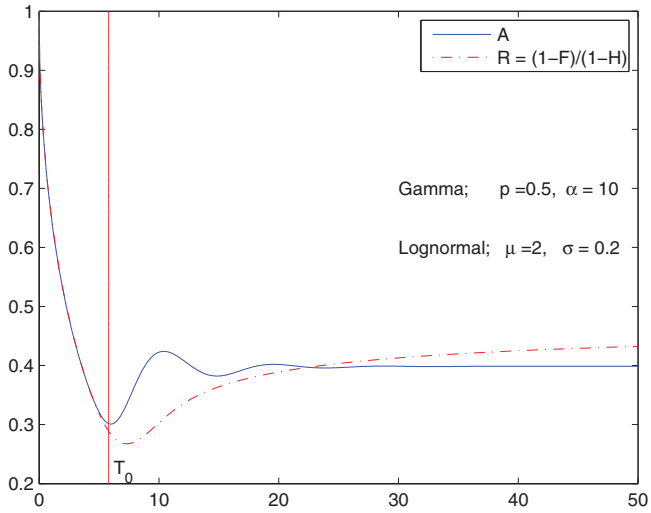


FIGURE 2. (Color online) Example 2: $U \sim Gamma,\ D \sim Lognormal$

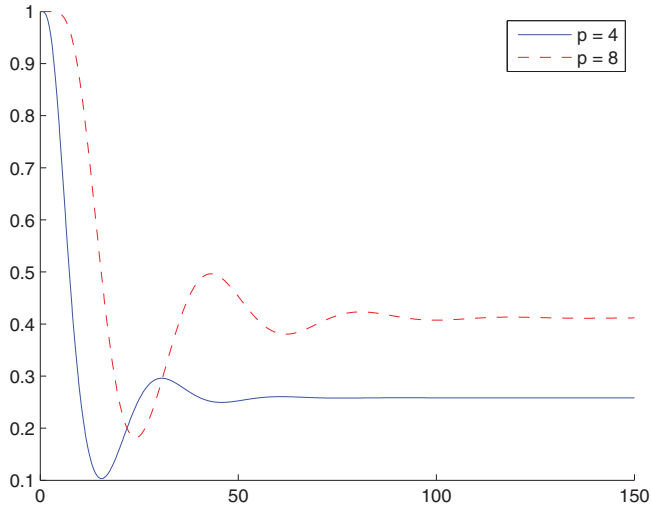


FIGURE 3. (Color online) Example 3: $U \sim Gamma, D \sim Lognormal$

EXAMPLE 3: Suppose that the lifetime U and repair time D have distributions F and G. Denote the availability of the system as $A_F(t)$ to emphasize its dependence on F. It is tempting to conjecture that $A_F(t)$ would increase if U is getting "longer and longer" in certain sense while G keeps unchanged. This is, however, not true in general as shown below.

Consider a system that has $U \sim Gamma(p, \alpha)$ and $D \sim \ln\mathcal{N}(\mu, \sigma)$ with density functions

$$f(t) = \frac{\alpha^p}{\Gamma(p)} e^{-\alpha t} t^{p-1} \quad \text{and} \quad g(t) = \frac{1}{t\sigma\sqrt{2\pi}} e^{-\frac{(\ln t - \mu)^2}{2\sigma^2}}.$$

From Figure 3 it is observed that the two curves of system availabilities corresponding to $p = 4$ and $p = 8$ cross each other twice even $Gamma(4, \alpha) \leq_{st} Gamma(8, \alpha)$. This example also indicates that the assumption of $D^* + U^* =_{st} U + D$ in Theorem 6 cannot be dropped.

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6. CONCLUSION

System availability, denoted as $A(t)$, of a repairable system is an important measure of its performance. Usually $A(t)$ does not have expression of closed-form so in the literature attention focuses on the steady-state availability $A(\infty)$. Different from the most works in the literature, this article studies various properties of the system availability $A(t)$ as a function of time t instead of the steady-state availability. An interval where $A(t)$ monotonically decreases is obtained, low bounds to $A(t)$ are derived, and some comparison results regarding availabilities of different systems are shown. Similar studies are conducted for the average availability. Several examples which validate the analytically derived properties of $A(t)$ are presented.

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APPENDIX A

A.1 Proof of Lemma 2

LEMMA A.1: *If $f(\cdot)$ is continuous on $(0,\infty)$, $g(\cdot)$ is bounded on any finite interval not including 0 and $f, g \in L^1(0,\infty)$, then the convolution $h \equiv f * g$ is continuous on $(0,\infty)$.*

PROOF: For $0 < t < \infty$, and $\Delta > 0$, we have

$$\begin{aligned} h(t) &= \int_0^t f(t-s)g(s)\,ds \\ &= \int_0^\Delta f(t-s)g(s)\,ds + \int_{t-\Delta}^t f(t-s)g(s)\,ds + \int_\Delta^{t-\Delta} f(t-s)g(s)\,ds \\ &= \int_0^\Delta f(t-s)g(s)\,ds + \int_0^\Delta f(x)g(t-x)\,dx + \int_\Delta^{t-\Delta} f(t-s)g(s)\,ds. \end{aligned}$$

Similarly, for fixed $0 < t^* < \infty$

$$h(t^*) = \int_0^\Delta f(t^*-s)g(s)\,ds + \int_0^\Delta f(x)g(t^*-x)\,dx + \int_\Delta^{t^*-\Delta} f(t^*-s)g(s)\,ds.$$

For any given $\epsilon > 0$, let $\Delta > 0$ be sufficiently small.

Let $C(\Delta) = \max\{\max_{t^*-3\Delta/2 \le x \le t^*+3\Delta/2} f(x), \sup_{t^*-3\Delta/2 \le x \le t^*+3\Delta/2} g(x)\} < \infty$. It is obvious that $C = C(\Delta)$ is increasing as Δ increase.

$$\begin{aligned} s \in [0,\Delta] &\Rightarrow t^* - s \in [t^*-\Delta, t^*] \subset [t^*-3\Delta/2, t^*+3\Delta/2] \\ &\Rightarrow \int_0^\Delta f(t^*-s)g(s)\,ds \le C\int_0^\Delta g(s)\,ds \\ x \in [0,\Delta] &\Rightarrow t^* - x \in [t^*-\Delta, t^*] \\ &\Rightarrow \int_0^\Delta f(x)g(t^*-x)\,dx \le C\int_0^\Delta f(x)\,dx. \end{aligned}$$

If $|t - t^*| \le \Delta/2$, then

$$\begin{aligned} s \in [0,\Delta] &\Rightarrow t - s \in [t-\Delta, t] \subset [t^*-3\Delta/2, t^*+3\Delta/2] \\ &\Rightarrow \int_0^\Delta f(t^*-s)g(s)\,ds \le C\int_0^\Delta g(s)\,ds \\ x \in [0,\Delta] &\Rightarrow t - x \in [t-\Delta, t] \subset [t^*-3\Delta/2, t^*+3\Delta/2] \\ &\Rightarrow \int_0^\Delta f(x)g(t-x)\,dx \le C\int_0^\Delta f(x)\,dx. \end{aligned}$$

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Thus, if $|t - t^*| \le \Delta/2$, then

$$\begin{aligned}|h(t) - h(t^*)| &\le 2C\left[\int_0^{\Delta} g(x)\,dx + \int_0^{\Delta} f(x)\,dx\right] \\ &\quad + \left|\int_{\Delta}^{t-\Delta} f(t-s)g(s)\,ds - \int_{\Delta}^{t^*-\Delta} f(t^*-s)g(s)\,ds\right| \\ &\le 2C\left[\int_0^{\Delta} g(s)\,ds + \int_0^{\Delta} f(x)\,dx\right] \\ &\quad + \left|\int_0^{\infty} f(t-s)I_{[\Delta,t-\Delta]}(s)g(s)\,ds - \int_0^{\infty} f(t^*-s)I_{[\Delta,t^*-\Delta]}g(s)\,ds\right|. \qquad \textbf{(A.1)}\end{aligned}$$

Note that $s \in [\Delta, t-\Delta] \Rightarrow t - s \in [\Delta, t-\Delta]$

$$0 \le f(t-s)I_{[\Delta,t-\Delta)}(s) \le \max_{\Delta \le y \le t-\Delta} f(y) \le \max_{\Delta \le y \le t^*-\Delta/2} f(y) \equiv L(t^*, \Delta) < \infty,$$

$$\int_0^{\infty} L(t^*,\Delta)g(s)\,ds = L(t^*,\Delta)\int_0^{\infty} g(s)\,ds = L(t^*,\Delta) < \infty,$$

that is, $L(t^*, \Delta)g(s)$ is in $L^1(0, \infty)$. Note that

$$f(t-s)I_{[\Delta,t-\Delta)}(s) \to f(t^*-s)I_{\Delta,t^*-\Delta}(s), \quad \forall s \ge 0, \quad \text{as } t \to t^*.$$

By the Lebesgue's Dominated Convergence Theorem

$$\lim_{t \to t^*}\left|\int_{\Delta}^{t-\Delta} f(x-s)g(s)\,ds - \int_{\Delta}^{t^*-\Delta} f(t^*-\Delta)g(s)\,ds\right| = 0. \qquad \textbf{(A.2)}$$

From (A.1) and (A.2), we observe that

$$\overline{\lim_{t \to t^*}}\,|h(t) - h(t^*)| \le 2C(\Delta)\left[\int_0^{\Delta} g(s)\,ds + \int_0^{\Delta} f(x)\,dx\right].$$

Let $\Delta \downarrow 0$, we have $\overline{\lim}_{t \to t^*}\,|h(t) - h(t^*)| = 0$. ■

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