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Homoclinic bifurcations and the Floquet torus

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Abstract. In this paper we show that a C^r diffeomorphism having an invariant Floquet torus which is non-normally hyperbolic can be approximated, in the C^{r-3} topology, by another one which exhibits a homoclinic tangency.

1. *Introduction*

The homoclinic phenomena have an important place among the most important dynamical behaviours. Among these are the homoclinic tangencies (see below for a definition) and one of its major features is that a small perturbation of them creates very rich and non-equivalent dynamical behaviours. It was conjectured by Palis that the obstruction to stable dynamics is just the presence of homoclinic tangencies. More precisely: let $\mathrm{Diff}^r(M)$ be the set of C^r diffeomorphisms with $r \geq 1$ of a compact manifold M, then the set of diffeomorphisms which exhibit a homoclinic bifurcation is dense in the complement of the closure of the structurally stable diffeomorphisms. For a comprehensive treatment of this topic see [**PT**].

Among the non-stable dynamical behaviours are *the invariant tori with parallel quasi-periodic dynamics.* This dynamics is typical in the conservative systems, for another context as the dissipative system etc, see [**BHS, BHT**]. When an invariant torus is normally hyperbolic it persists under C^r perturbations from a geometrical point of view; but not dynamically, i.e. the perturbed torus dynamics is not the same as that of the original torus.

Our aim here is to verify Palis' conjecture in this context. We need to state the general context and give some definitions in order to state our main theorem.

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Let M be a smooth compact manifold and $H : M \to M$ a C^r diffeomorphism ($r > 3$), an *invariant torus with parallel dynamics* is a submanifold H-invariant $V \subset M$, such that $\exists \varphi : V \to \mathbb{T}^n$ a C^r diffeomorphism which satisfies

$$\varphi \circ H \circ \varphi^{-1}(x) = x + \omega \pmod{2\pi}$$

where $\omega \in \mathbb{R}^n$ and $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ is the n-dimensional torus. In such a case we will call $\omega_1, \ldots, \omega_n$, the components of ω, the (inner) frequencies of V. They are well defined except the integer multiples of 2π.

An invariant torus is *normally hyperbolic* if there exists a DH-invariant splitting $\mathrm{T}_{\mathbb{T}^n}M = N^s \oplus N^u \oplus \mathrm{T}\mathbb{T}^n$ such that for some Riemaniann metric $\|\cdot\|$ we have:

(a) $\sup \|DH|N^s_p\| < \inf m(DH|\mathrm{T}\mathbb{T}^n_p)$;

(b) $\sup \|DH|\mathrm{T}\mathbb{T}^n_p\| < \inf m(DH|N^u_p)$;

where the sup and inf are taken on $p \in \mathbb{T}^n$ and where $m(T) := \inf\{|Tv| : |v| = 1\}$, for a linear transformation T. (a) says to us that the contraction in N^s is sharper than the possible contractions in $\mathrm{T}\mathbb{T}^n$ and (b) says that the expansion in N^u is sharper than the possible expansions in $\mathrm{T}\mathbb{T}^n$.

We are interested in studying invariant tori with parallel dynamics which are not normally hyperbolic. In general, the dynamics in the normal direction is too complicated and almost nothing is known. We will restrict to where this dynamics is very simple. Let V be a torus with parallel dynamics, we say that V is a *Floquet torus* of H if there exist C^r coordinates $\tilde{\varphi} : U \to \mathbb{T}^n \times \mathbb{R}^m$ defined in a neighbourhood U of V, such that H in such coordinates takes the form

$$H(x, y) = (x + \omega + O(\|y\|), \Omega y + O(\|y\|^2)), \quad (x \in \mathbb{T}^n, y \in \mathbb{R}^m) \tag{1}$$

where $\Omega \in \mathrm{GL}(m, \mathbb{R})$ does not depend on x. In such a case we call Ω the *Floquet matrix* of V and to the arguments of the eigenvalues of modulus 1 of Ω, with non-negative imaginary parts, we will call *the central normal frequencies* of V. These frequencies are well defined modulus integer multiples of 2π. We observe, by the way, that the four definitions of normally hyperbolic given in [**HPS**] are equivalent for Floquet tori.

We point out that V a Floquet torus is non-normally hyperbolic iff Ω is not a hyperbolic matrix, and hence, iff V has a central normal frequency. When we only look at the geometry of the manifold V, these tori bifurcate in the following way. Let us assume that V is non-normally hyperbolic in the weakest way, then we have three cases:

(1) Ω has a unique eigenvalue of modulus 1 which is 1 and has multiplicity one.

(2) Ω has a unique eigenvalue of modulus 1 which is -1 and has multiplicity one.

(3) Ω has just two eigenvalues of modulus 1, being these non-real, and with multiplicity one.

Let H_μ be a generic family which unfolds H with $H_0 = H$ then:

(i) In case (1) V disappears for μ in one side of zero and for μ in the other side of zero we have two invariant tori. This case is known as the saddle-node case.

(ii) In case (2) there exists a torus which survives for all values of parameter μ and besides, for μ in one side of zero, we have two invariant tori which build a periodic torus of period two. This case is known as the flip case.

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(iii) In case (3) we also have that a torus survives for all values of parameter μ and for μ in one side of zero there appears a new invariant torus, but whose dimension is one more than that of the original torus. This case is known as the Hopf case.

Given $p \in M$ we define *stable manifold* (respectively *unstable*) of p, $W^s(p)$ (respectively $W^u(p)$), by

$$W^s(p) = \left\{ q \in M : \lim_{n\to\infty} \operatorname{dist}(H^n(q), H^n(p)) = 0 \right\}$$

$$\left(\text{respectively } W^u(p) = \left\{ q \in M : \lim_{n\to-\infty} \operatorname{dist}(H^n(q), H^n(p)) = 0 \right\}\right).$$

A *homoclinic point* associated to p is a point $q \in W^s(p) \cap W^u(p) \setminus \{p\}$. If p is a periodic hyperbolic point of H, then $W^s(p)$ and $W^u(p)$ are C^r immerse manifolds. We say that q is a *homoclinic tangency* if q is a homoclinic point associated with a hyperbolic periodic point p such that $\mathrm{T}_q W^s(p) + \mathrm{T}_q W^u(p) \neq \mathrm{T}_q M$.

Now we can state our main result.

THEOREM. *Let $H : M \to M$ be a C^r diffeomorphism which has a non-normally hyperbolic invariant Floquet torus $V \subset M$. Then for all* $\mathrm{Diff}^{r-3}(M)$ *neighbourhood $\mathcal{N}$ of H there exists $\bar{H} \in \mathcal{N}$ such that $\bar{H}$ exhibits a homoclinic tangency.*

By the definition of a Floquet torus, a tubular neighbourhood of V is always orientable. An interesting problem is to know if the conclusion of the theorem is true in a more general setting, for example for a non-normally hyperbolic torus with parallel dynamics. The following corollary gives a positive answer to this problem when the dimension of the torus is greater than one, without assuming anything about the normal behaviour.

COROLLARY. *Let $H : M \to M$ be a C^r diffeomorphism which has an invariant torus $V \subset M$ of dimension $n \geq 2$ with parallel dynamics. Then for all neighbourhoods $\mathcal{N} \subset \mathrm{Diff}^{r-3}(M)$ of H there exists $\bar{H} \in \mathcal{N}$ such that $\bar{H}$ exhibits a homoclinic tangency.*

Proof. Let $\hat{H} = H|_V$, in conveniently choosen coordinates $\hat{H}(x) = x + \omega \pmod{2\pi}$. Take $\tilde{H}$ near enough $\hat{H}$ with the following form: $\tilde{H}(x) = x + \omega' \pmod{2\pi}$, with $\omega' = (\omega_1, \omega_2', \ldots, \omega_n')$ near enough to $\omega = (\omega_1, \ldots, \omega_n)$, such that $\omega_2', \ldots, \omega_n'$ are rationally dependent with respect to ω_1. Then $\tilde{H}$ has a non-normally hyperbolic invariant circle with parallel dynamics, so there exists $\bar{H}$ near enough to $\tilde{H}$ exhibiting a homoclinic tangency, $\bar{H}$ can be extended to M in such a way that $\bar{H}$ is near enough to H. □

In [**BBH**] the bifurcation of a Floquet torus is considered, taking into account the dynamics on the torus. To do that, they consider the unfolding of $(n+1)$-parameter families $H_{\boldsymbol{\mu}}$ where at $\boldsymbol{\mu} = \boldsymbol{\mu}_0$, $H_{\boldsymbol{\mu}_0}$ has a non-normally hyperbolic Floquet torus. An interesting problem, not addressed here, is to get parameters $\boldsymbol{\mu}$ near enough to $\boldsymbol{\mu}_0$ such that $H_{\boldsymbol{\mu}}$ exhibits homoclinic tangencies. Moreover, one can ask: what can be concluded about the dynamical properties of these $(n+1)$-parameter families $H_{\boldsymbol{\mu}}$? This would interprete the results in quasi-periodic bifurcation theory.

Now we give an idea of the theorem's proof. We consider the simplest case when $n = 1$, $m = 1$ and we are in a saddle-node bifurcation case. The result is obtained by using several perturbations. First, using normal forms, we get that the derivative along the torus can be written as $\Omega = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ with $a > 0$ and where the first direction is the direction along

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the torus and the second is the normal direction. As the torus dynamics is quasiperiodic it can be perturbed to get a saddle-node periodic orbit. Ω turns the normal direction on the right and this allows us to perturb in such a way to create a weak contraction in a transversal direction to the torus. Also, the strong stable foliation $\mathcal{F}^{ss}$ associated with this weak contraction makes an angle with the tangent direction to the torus that is as small as we please. Then we can break the torus to create a criticality between the unstable manifold of the saddle-node with $\mathcal{F}^{ss}$. Now the result follows from [**NPT**].

This paper is organized as follows. In §2 we prove some results about normal forms which allows us to give the definitions of bifurcation tori of saddle-node, flip and Hopf type. In §3 we consider the case when V is one dimensional and prove some results about the behaviour of certain perturbations of H along the normal direction of V. Finally, in §4 we give the proof of the theorem.

2. *Preliminaries*

In this section we prove some results about normal forms which we will use in the proof of the theorem and to define several types of Floquet tori.

Through this section we are going to consider Floquet tori so we can assume from now on that $H : \mathbb{T}^n \times \mathbb{R}^m \to \mathbb{T}^n \times \mathbb{R}^m$ is given by

$$H(x, y) = (x + \omega + a(x)y + O(\|y\|^2), \Omega y + O(\|y\|^2)). \tag{2}$$

We say that $\mathbb{T}^n = \mathbb{T}^n \times \{0\}$ is *quasi-periodic* if

$$\langle k, \omega\rangle + 2\pi k_0 \neq 0, \quad \forall k \in \mathbb{Z}^n \setminus \{0\}\ \forall k_0 \in \mathbb{Z}$$

where $\langle k, \omega\rangle = \sum_{i=1}^n k_i\omega_i$. $\mathbb{T}^n$ will be a *diophantine torus* if there exist $\beta \geq 0$ and $C > 0$, such that

$$|\langle k, \omega\rangle + 2\pi k_0| \geq C|k|^{-n-\beta}, \quad \forall k \in \mathbb{Z}^n \setminus \{0\}\ \forall k_0 \in \mathbb{Z} \tag{3}$$

with $|k| = \max_{1\leq i\leq n} |k_i|$.

If $\mathbb{T}^n$ is a non-normally hyperbolic Floquet torus then it has a central normal frequency τ. We say that $\mathbb{T}^n$ satisfies a *mixed diophantine condition* if there exist $C > 0$ and $\beta \geq 0$ such that

$$|\langle k, \omega\rangle + l\tau + 2\pi k_0| > C|k|^{-n-\beta} \tag{4}$$

$\forall k \in \mathbb{Z}^n \setminus \{0\}$, $\forall k_0 \in \mathbb{Z}$ and $\forall l \in \mathbb{Z}$ with $|l| < 2$.

We are interested in the case when $\mathbb{T}^n$ is non-normally hyperbolic in the weakest possible way. This happens in the following three different ways:

(1) 1 is the unique eigenvalue, of modulus 1, of Ω and has multiplicity one,
(2) -1 is the unique eigenvalue of modulus 1, of Ω and has multiplicity one,
(3) Ω has only two non-real eigenvalues of modulus 1, each one having multiplicity one.

In any of these cases we can restrict H to the central manifold W^c of $\mathbb{T}^n$. As $\mathbb{T}^n$ is a Floquet torus, we get that W^c is diffeomorphic to $\mathbb{T}^n \times \mathbb{R}$ in the first two cases and to $\mathbb{T}^n \times \mathbb{R}^2$ in the last case. We observe that $\mathbb{T}^n$ continues to be a Floquet torus for $H|_{W^c}$ so we can take coordinates on W^c in such a way that $H|_{W^c}$ looks like (2). We can assume from now on that $m = 1$ in the first two cases and that $m = 2$ in the last case.

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Let λ be the eigenvalue of Ω with argument $\tau \in [0, \pi]$, i.e. $\lambda = e^{i\tau}$. We assume that ω satisfies the mixed diophantine condition (4).

Summing up: we assume that $H : \mathbb{T}^n \times \mathbb{R}^m \to \mathbb{T}^n \times \mathbb{R}^m$ is given by (2) with $m = 1, 2$; $\Omega = \pm 1$ if $m = 1$ and Ω has two eigenvalues with non-zero imaginary parts if $m = 2$, and ω satisfies (4). Remark that if $\Omega = 1$ then (4) is the diophantine condition (3).

Now we are going to present the normal forms for these kinds of diffeomorphisms in each of the above three cases. Firstly we recall a well-known lemma.

LEMMA 1. *Let $h : \mathbb{T}^n \to \mathbb{C}$ be a C^r function. Let $\lambda = e^{i\tau}$ and $\omega \in \mathbb{R}^n$ both satisfying a diophantine condition (4). Then there exists a $C^{r-\theta}$ function, $\varphi : \mathbb{T}^n \to \mathbb{C}$, such that*

$$\varphi(x) - \lambda\varphi(x + \omega) = \tilde{h} - h(x) \tag{5}$$

where $\theta = n + 1 + [\beta]$ and

$$\tilde{h} = \begin{cases} \displaystyle\int_{\mathbb{T}^n} h(x)\, dx, & \text{if } \lambda = 1 \\ 0, & \text{in any other case.} \end{cases}$$

In addition, if h just takes real values and $\lambda \in \mathbb{R}$ then φ can be taken as a real-valued function.

Proof. See [**H**, Annexe]. □

The following three propositions present the normal forms. The first is used in all of the above described cases. Denote by $M_{m\times n}(\mathbb{R})$ the set of $m \times n$ matrices with real coefficients.

PROPOSITION 1. *Let H be a C^r map as above. There exists a change of coordinates $C^{r-\theta}$ in a neighbourhood of $\mathbb{T}^n$ such that, in such coordinates*

$$H(x, y) = (x + \omega + \hat{a}y + O(\|y\|^2), \Omega y + O(\|y\|^2)),$$

where $\hat{a} \in M_{n\times m}(\mathbb{R})$, and θ is given by Lemma 1. Moreover, if $\Omega \neq 1$, i.e. we are in the second and the third cases, we can take these coordinates such that $\hat{a} = 0$.

Proof. To begin, we assume that $m = 1$. Let

$$\hat{a} = \begin{cases} \displaystyle\int_{\mathbb{T}^n} a(x)\, \mathrm{d}x, & \text{if } \Omega = 1 \\ 0, & \text{if } \Omega \neq 1 \end{cases}$$

then by Lemma 1 there exists $\varphi : \mathbb{T}^n \to M_{n\times m}(\mathbb{R})$, such that

$$\varphi(x) - \varphi(x + \omega)\Omega = \hat{a} - a(x), \quad \forall x \in \mathbb{T}^n \tag{6}$$

where φ is $C^{r-\theta}$. Hence, if we define the change of coordinates

$$\Phi(x, y) = (x + \varphi(x)y, y)$$

we have that

$$\Phi^{-1} \circ H \circ \Phi(x, y) = (x + \omega + \hat{a}y + O(\|y\|^2), \Omega y + O(\|y\|^2)).$$

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If $m = 2$ then we begin making a change of coordinates $(x, y) \to (x, \Xi y)$, with Ξ linear, such that $\Xi^{-1}\Omega\Xi y = \lambda y$ if we identify $\mathbb{R}^2$ with $\mathbb{C}$. With this identification $M_{n\times 2} \simeq \mathbb{C}^n$, equation (6) becomes

$$\varphi(x) - \bar{\lambda}\varphi(x + \omega) = \hat{a} - a(x).$$

This equation can be solved again using Lemma 1. In this case $\hat{a} = 0$ and Φ is defined in the same way. □

PROPOSITION 2. *Let H be a C^r map as above with $m = 1$. Given $1 < N < (r - \theta)/\theta$, there exists a $C^{r-(N+1)\theta}$ change of coordinates such that H in these coordinates has the following form*

$$H(x, y) = (x + \omega + \hat{a}y + O(y^2), \Omega y + b_2 y^2 + \cdots + b_N y^N + O(y^{N+1})).$$

Moreover, if $\Omega = -1$ then such coordinates can be taken in such a way that $\hat{a} = 0$ and $b_{2k} = 0$ for $k = 1, \ldots, [N/2]$.

Proof. By Proposition 1 we can write H in $C^{r-\theta}$ coordinates as

$$H(x, y) = (x + \omega + \hat{a}y + O(y^2), \Omega y + b_2(x) y^2 + \cdots + b_N(x) y^N + O(y^{N+1})).$$

We observe that given $2 \le i \le N$ and letting

$$\hat{b}_i = \begin{cases} 0, & \text{if } \Omega = -1 \text{ and } i \text{ is even} \\ \displaystyle\int_{\mathbb{T}^n} b_i(x)\,\mathrm{d}x, & \text{if not} \end{cases}$$

we have that there exists a $C^{r-2\theta}$ map $\varphi : \mathbb{T}^n \to \mathbb{R}$ such that

$$\Omega\varphi(x) - \Omega^i\varphi(x + \omega) = \hat{b}_i - b_i(x), \quad \forall x \in \mathbb{T}^n.$$

Now we define the $C^{r-2\theta}$ change of coordinates

$$\Phi(x, y) = (x, y + \varphi(x) y^i)$$

and get that

$$\begin{aligned}\Phi^{-1} \circ H \circ \Phi(x, y) = (x + \omega + \hat{a}y + O(y^2), \Omega y + b_2(x) y^2 + \cdots + b_{i-1}(x) y^{i-1} \\ + \hat{b}_i y^i + \tilde{b}_{i+1}(x) y^{i+1} + \cdots + \tilde{b}_N(x) y^N + O(y^{N+1}))\end{aligned}$$

where $\tilde{b}_j$ is not necessarily equal to b_j. So making N changes of coordinates of this type we get the thesis of the proposition. □

Now we address the above proposition in the case $m = 2$. For the sake of simplicity we identify $\mathbb{R}^2$ with $\mathbb{C}$. Using this identification and from Proposition 1 we get in $C^{r-\theta}$ coordinates that

$$H(x, y) = (x + \omega + O(|z|^2), \lambda z + B_2(x, z) + \cdots + B_N(x, z) + O(|z|^{N+1}))$$

where $\lambda \in \mathbb{C}$ is one of the eigenvalues of Ω and $B_i(x, z) = \sum_{j=0}^{i} \beta_j^i(x) z^{i-j} \bar{z}^j$, with $\beta_j^i : \mathbb{T}^n \to \mathbb{C}$.

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PROPOSITION 3. *With the above conditions there exists a* $C^{r-(N+1)\theta}$ *change of coordinates which brings* H *into the following form:*

$$H(x,z) = (x+\omega+O(|z|^2), \lambda z + \hat{\beta}_3 z|z|^2 + \hat{\beta}_5 z|z|^4 + \cdots + \hat{\beta}_{2[(N-1)/2]+1} z|z|^{2[(N-1)/2]} + O(|z|^{N+1})).$$

Proof. The proof of this proposition follows the same line as that of Proposition 1. We define

$$\hat{\beta}^i_j = \begin{cases} \displaystyle\int_{\mathbb{T}^n} \beta^i_j(x)\,\mathrm{d}x, & \text{if } i \text{ is odd and } j = \frac{i-1}{2} \\ 0, & \text{in any other case} \end{cases}$$

then by Lemma 1 there exist $C^{r-2\theta}$ maps $\varphi_j : \mathbb{T}^n \to \mathbb{C}$, $\forall j = 0, \ldots, i$, such that

$$\varphi_j(x) - \lambda^{i-j-1}\bar{\lambda}^j \varphi_j(x+\omega) = \hat{\beta}^i_j - \beta^i_j(x)$$

$\forall x \in \mathbb{T}^n$ and $\forall j = 0, \ldots, i$. Let $\Phi(x,z)$ be the change of coordinates

$$\Phi(x,z) = (x, z + \varphi(x,z))$$

where $\varphi(x,z) = \sum_{j=0}^{i} \varphi_j(x) z^{i-j}\bar{z}^j$, and put

$$\hat{\beta}_i = \begin{cases} \hat{\beta}^i_{(i-1)/2}, & \text{if } i \text{ is odd} \\ 0, & \text{if } i \text{ is even} \end{cases}$$

then

$$\Phi^{-1} \circ H \circ \Phi(x,z) = (x+\omega+O(|z|^2), \lambda z + B_2(x,z) + \cdots + B_{i-1}(x,z) + \hat{\beta}_i z|z|^{i-1} + \tilde{B}_{i+1}(x,z) + \cdots + \tilde{B}_N(x,z) + O(|z|^{N+1}))$$

where $\tilde{B}_k(x,.)$ is an homogeneous polynomial of degree k in the variables z and $\bar{z}$. In general, $\tilde{B}_k(x,.)$ does not coincide with $B_k(x,.)$. In the same way as in Proposition 1 this allows us to conclude the proof. □

Now we will give the definitions of the different types of bifurcation tori. Let $\mathbb{T}^n$ be a Floquet torus, then we get the following.

Saddle-node. We say that $\mathbb{T}^n$ is a *quasi-periodic saddle-node bifurcation torus* if Ω has a unique eigenvalue of modulus 1, which is equal to 1 and its multiplicity is equal to one. In addition to this, it happens that $\hat{a} \neq 0$ and $b_2 \neq 0$, where $\hat{a}$ and b_2 are given by Proposition 2.

Flip. We say that $\mathbb{T}^n$ is a *quasi-periodic flip bifurcation torus* if Ω has a unique eigenvalue of modulus 1, which is equal to -1 and its multiplicity is equal to one. In addition, $b_3 \neq 0$, where b_3 is given by Proposition 2 (we are assuming that $\hat{a} = 0$ and $b_2 = 0$ in that proposition).

Hopf. Finally, we say that $\mathbb{T}^n$ is a *quasi-periodic Hopf bifurcation torus* f if Ω has two eigenvalues of modulus 1, λ and $\overline{\lambda}$, that are non-real and have multiplicity equal to 1. In addition, $\hat{\beta}_3 \neq 0$, where $\hat{\beta}_3$ is given by Proposition 3.

In [**BBH**] the unfolding of these kinds of bifurcation tori in the case of vector fields is discussed.

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3. *Propositions*

In this section we give some results which we will use in §4. We consider here C^r diffeomorphisms $H : \mathbb{T}^1 \times \mathbb{R}^m \to \mathbb{T}^1 \times \mathbb{R}^m$, such that has $\mathbb{T}^1$ as a non-normally hyperbolic Floquet torus, i.e.

$$H(x, y) = (x + \omega + f(x, y), \Omega y + g(x, y))$$

with f and g C^r functions such that $f(x, 0) = 0$ and $g(x, y) = O(\|y\|^2)$. Moreover, we assume that either $m = 1$ or $m = 2$ depending on if Ω has as eigenvalues ± 1 or if the eigenvalues are $\lambda = e^{i\tau}, \bar{\lambda}$ with $\tau \in 2\pi\mathbb{Q} \setminus \pi\mathbb{Z}$, see §2.

In what follows we will consider perturbations H_δ of H which leaves $\mathbb{T}^1$ invariant. Let G be the line (respectively plane) bundle of $\mathbb{T}^1 \times \mathbb{R}^m$ restricted to $\mathbb{T}^1$ when $m = 1$ (respectively $m = 2$). We are interested in describing the dynamics which this perturbation H_δ induces on G. In these bundles we introduce the following metrics: if $V_1, V_2 \in G$ then $d(V_1, V_2) := \max_{v_1 \in V_1} \min_{v_2 \in V_2} \|v_1 - v_2\|$, where $\|v_1\| = \|v_2\| = 1$. We observe that if $m = 1$ then $G \simeq \mathbb{T}^2$ and if $m = 2$ then $G \simeq \mathbb{T}^1 \times G_P$, where G_P is the set of planes in $\mathbb{R}^3$.

The perturbations H_δ has the following form:

$$H_\delta(x, y) = (h_\delta + f(x, y), \Omega y + g(x, y)).$$

h_δ has the effect of inducing saddle-node-like dynamics on $\mathbb{T}^1$. For constructing h_δ we follow [**M**]. Let $\omega = 2\pi p/q$ and fix $x_0 = 0$ together with its orbit under $R_\omega(x) = x + \omega$ (mod 2π), $\{0, \omega, 2\omega, \ldots, (q-1)\omega\}$ (mod 2π). Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined as $\phi(x) = \cos(qx + \pi) + 1$, then

- $\phi(x + 2\pi) = \phi(x)$,
- $\phi^{-1}(0) \cap [0, 2\pi] = \{0, \omega, \ldots, (q-1)\omega\}$ (mod 2π),
- $\phi'(i\omega) > 0 \; \forall i = 0, 1, \ldots, q-1$.

Now we define h_δ as $h_\delta(x) := (x + \omega + \delta\phi(x))$ (mod 2π), see Figure 1.

LEMMA 2. *For $\delta > 0$ small enough, $\exists C(q)$ such that:*

(i) $\|h_\delta - R_\omega\|_{C^r} \le C(q)\delta$;

(ii) *the unique fixed points of h_δ^q are $\{0, \omega, 2\omega, \ldots, (q-1)\omega\}$ (mod 2π);*

(iii) $h'_\delta(h_\delta^i(x_0)) = 1$ *and* $h''_\delta(h_\delta^i(x_0) \neq 0, \; \forall i = 0, \ldots, q-1$;

(iv) *there exists an interval $I \subset [0, 2\pi]$ such that*

$$|h_\delta^q(x) - x| \ge \frac{\delta q M}{4}, \quad \forall x \in I,$$

where $M = \max_{x \in [0,1]} \phi$.

Proof. See [**M**, Lemma 1]. □

H_δ induces on G a map $\tilde{H}_\delta : G \to G$, which is given by

$$\tilde{H}_\delta(x, L) := (h_\delta^q(x), DH_\delta^q(x, 0)L).$$

Let $\mathbb{A}_\xi = \{(x, y) : x \in \mathbb{T}^1 \text{ and } \|y\| < \xi\}$, with $\xi > 0$.

Assume now that $m = 1$ so $\Omega = \pm 1$.

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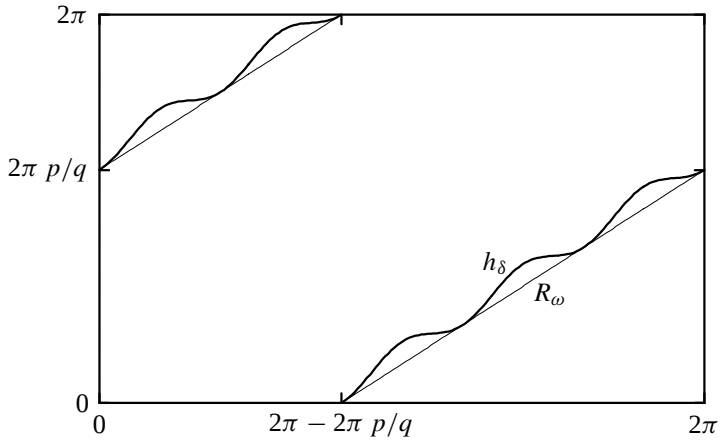


FIGURE 1. Functions h_δ and R_ω, with $p = 3$ and $q = 5$.

PROPOSITION 4. *If $\Omega = 1$ and $\exists \xi > 0$ such that $H_\delta^q|_{\mathbb{A}_\xi}$ twist the vertical to the right, then for $\delta \neq 0$ small enough we have that:*

(i) $\tilde{H}_\delta(\mathrm{T}\mathbb{T}^1) = \mathrm{T}\mathbb{T}^1$;

(ii) $W^s(\cup_{i=0}^{q-1}\mathrm{T}_{x_i}\mathbb{T}^1) = W^u(\cup_{i=0}^{q-1}\mathrm{T}_{x_i}\mathbb{T}^1) = G$, $(x_i = h_\delta^i(x_0))$;

(iii) $W^s(\mathrm{T}\mathbb{T}^1) = W^u(\mathrm{T}\mathbb{T}^1) = G$;

(iv) *if $\Lambda \subset G$ is an invariant set under $\tilde{H}_\delta$ and has elements of negative inclination then* $\Lambda = \mathrm{T}\mathbb{T}^1$.

Proof. This proposition has the same proof as that of [**M**, Proposition 4]. □

Remark. An analogous result can be obtained if we change $H_\delta^q|_{\mathbb{A}_\xi}$ by twisting the vertical on the left and changing the 'element of negative inclination' with the 'element of positive inclination' in (iv).

Now consider the case $\Omega = -1$. In general, the last proposition is false. It is enough to take the case where $DH_\delta(x, 0) = \left(\begin{smallmatrix} 1 & a \\ 0 & -1 \end{smallmatrix}\right)$, with $a \neq 0$. What is true is that we always can perturb $H_\delta(x, y)$ in a neighbourhood of $\mathbb{T}^1$ in such a way that the last proposition is going on to be true.

PROPOSITION 5. *Let $\Omega = -1$ and $\xi \in \mathbb{R}^+$. Assume that $f(x, y) = O(y^2)$. There exists $C > 0$ such that for each ϵ and δ small enough $\exists H_{\delta,\epsilon} : \mathbb{A}_\xi \to \mathbb{T}^1 \times \mathbb{R}$ a C^r perturbation of H_δ such that:*

(i) $\|H_\delta|_{\mathbb{A}_\xi} - H_{\delta,\epsilon}|_{\mathbb{A}_\xi}\|_{C^r} \leq C\epsilon$;

(ii) $\tilde{H}_{\delta,\epsilon}(\mathrm{T}\mathbb{T}^1) = \mathrm{T}\mathbb{T}^1$;

(iii) $W^s(\cup_{i=0}^{q-1}\mathrm{T}_{x_i}\mathbb{T}^1) = W^u(\cup_{i=0}^{q-1}\mathrm{T}_{x_i}\mathbb{T}^1) = G$, $(x_i = h_\delta^i(x_0))$;

(iv) $W^s(\mathrm{T}\mathbb{T}^1) = W^u(\mathrm{T}\mathbb{T}^1) = G$;

(v) *if $\Lambda \subset G$ is an invariant subset under $\tilde{H}_{\delta,\epsilon}$ and has elements of negative inclination then* $\Lambda = \mathrm{T}\mathbb{T}^1$.

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Proof. Consider for $\epsilon > 0$

$$H_{\delta,\epsilon}(x, y) = (h_\delta(x) + \epsilon xy + f(x, y), -y + g(x, y))$$

then

$$DH_{\delta,\epsilon}(x, 0) = \begin{pmatrix} h'_\delta(x) & \epsilon x \\ 0 & -1 \end{pmatrix}$$

and so

$$DH^2_{\delta,\epsilon}(x, 0) = \begin{pmatrix} (h^2_\delta)'(x) & \epsilon x h'_\delta(h_\delta(x)) - \epsilon h_\delta(x) \\ 0 & 1 \end{pmatrix}.$$

Now $|\epsilon x h'_\delta(h_\delta(x)) - \epsilon h_\delta(x)| = |\epsilon x(1+\delta\phi'(h_\delta(x)) - \epsilon(x+\omega+\delta\phi(x))| \geq \epsilon|x\delta\phi'(h_\delta(x)) + \delta\phi(x) - \omega| \geq \epsilon(|\omega| - C'\delta x) > 0$ if δ is small, where C' is a constant which bounds the expression $x\phi'(h_\delta(x)) + \phi(x)$. Now we are in position to apply Proposition 4 to $H^2_{\delta,\epsilon}$ and from here get the conclusions of the proposition. □

We want to do now the same as in the above propositions, but with $m = 2$. To do that we first need the following.

LEMMA 3. *Let $\alpha \neq 0$, $\tau = 2\pi p'/q$ and*

$$A = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & \cos\tau & -\sin\tau \\ 0 & \sin\tau & \cos\tau \end{pmatrix} = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & & \\ 0 & & R_\tau \end{pmatrix}.$$

Then

$$A^q = \begin{pmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & & \\ 0 & & R^q_\tau = I \end{pmatrix}$$

with $\alpha_2 \neq 0$. Moreover, α_2 has the same sign as α.

Proof. We want to compute A^n. We observe that the A's have the form

$$A^n = \begin{pmatrix} 1 & \beta_n & \beta'_n \\ 0 & & \\ 0 & & R^n_\tau \end{pmatrix}.$$

It is easy to see that the recursive relationship between the coefficients β_n and β'_n is

$$\beta_n = \beta_{n-1} + \alpha\sin((n-1)\tau),$$
$$\beta'_n = \beta'_{n-1} + \alpha\cos((n-1)\tau),$$

and from here we get that

$$\beta_n = \beta_1 + \alpha(\sin((n-1)\tau) + \sin((n-2)\tau) + \cdots + \sin(\tau)),$$
$$\beta'_n = \beta'_1 + \alpha(\cos((n-1)\tau) + \cos((n-2)\tau) + \cdots + \cos(\tau)).$$

As $\beta_1 = 0$ and $\beta'_1 = \alpha$ it turns that

$$\beta'_n = \alpha(1 + \cos((n-1)\tau) + \cos((n-2)\tau) + \cdots + \cos(\tau))$$
$$= \alpha\frac{1 - \cos(\tau) + \cos(n\tau) - \cos((n-1)\tau)}{2(1 + \cos(\tau))}.$$

If $n = q$ then β'_n is different from zero and has the same sign as α. The same argument shows that the same happens to β_n. □

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Let

$$H(x, y) = (x + \omega + f(x, y), \Omega y + g(x, y))$$

where

$$\Omega = \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix}$$

and $\omega, \tau \in 2\pi\mathbb{Q}$, $\omega = 2\pi p/q$ and $\tau = 2\pi p'/q$. In addition, we assume that $f(x, y) = a_2 y_2 + O(\|y\|^2)$, with $a_2 > 0$, and $y = (y_1, y_2)$. Consider

$$H_\delta(x, y) = (h_\delta(x) + a_2 y_2 + O(\|y\|^2), \Omega y + g(x, y)),$$

we observe that

$$DH_0(x, 0) = \begin{pmatrix} 1 & 0 & a_2 \\ 0 & & \\ 0 & & \Omega \end{pmatrix}.$$

$DH_\delta^q(x, 0)$ induce a mapping $\tilde{H}_\delta : G \to G$, where G is the plane bundle on $\mathbb{T}^1$.

For $\delta = 0$, because of Lemma 3, we have

$$DH_0^q(x, 0) = \begin{pmatrix} 1 & \tilde{a}_1 & \tilde{a}_2 \\ 0 & & \\ 0 & \Omega^{q'} = I & \end{pmatrix}$$

where $\tilde{a}_2 > 0$. Observe, also, that $h_0^q(x) = x$. Let P_0 be the plane spanned by the vectors $(1, 0, 0)$ and $(0, \tilde{a}_2, -\tilde{a}_1)$. Let

$$\Lambda_0 = \{(x, P) : x \in \mathbb{T}^1 \text{ and } (1, 0, 0) \in P\},$$

then $\tilde{H}_0(x, P) = (x, P)\ \forall (x, P) \in \Lambda_0$ and if $(x, P) \notin \Lambda_0$ then

$$\lim_{n \to \pm\infty} \tilde{H}_0^n(x, P) = (x, P_0).$$

For $\delta > 0$ the situation is more complicated. Λ_0 continues being an invariant set, but its elements are fixed points if its first coordinate is of the following way $j\omega \pmod{2\pi}$ for $j = 0, \ldots, q - 1$.

To simplify, we consider the sub-bundle G_1 of the line bundle on $\mathbb{T}^1$ defined by

$$G_1 = \{(x, L) : x \in \mathbb{T}^1 \text{ and } L \subset P_1\}$$

where P_1 is the plane spanned by the vectors $v_1 = (1, 0, 0)$ and $v_2 = (0, 0, 1)$. We denote $F : G_1 \to G_1$ as the mapping induced by DH_δ^q on G_1.

PROPOSITION 6. *For F as above we have that it is well defined and satisfies:*

(i) $F(\mathrm{T}\mathbb{T}^1) = \mathrm{T}\mathbb{T}^1$;

(ii) $W^s(\cup_{i=0}^{q-1} \mathrm{T}_{x_i}\mathbb{T}^1) = W^u(\cup_{i=0}^{q-1} \mathrm{T}_{x_i}\mathbb{T}^1) = G_1$ $(x_i = h_\delta^i(x_0))$;

(iii) $W^s(\mathrm{T}\mathbb{T}^1) = W^u(\mathrm{T}\mathbb{T}^1) = G_1$;

(iv) *if $\Lambda \subset G_1$ is an invariant set under F and has elements of negative inclination then* $\Lambda = \mathrm{T}\mathbb{T}^1$,

where G_1 has the orientation given by the ordered base $\{v_1, v_2\}$.

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Proof. For $\delta > 0$ small enough

$$DH_0^q(x,0) = \begin{pmatrix} (h_\delta^q)' & a_1(x,\delta) & a_2(x,\delta) \\ 0 & & \\ 0 & & \Omega^q = I \end{pmatrix}$$

with $a_1(x,\delta)$ and $a_2(x,\delta)$ different from zero. Moreover, $a_2(x,\delta) > 0\ \forall x \in \mathbb{T}^1$. Then $DH_\delta P_1 = P_1$ so F is well defined. Also, we have

$$DH_0^q(x,0)|_{P_1} = \begin{pmatrix} (h_\delta^q)' & a_2(x,\delta) \\ 0 & 1 \end{pmatrix}. \tag{7}$$

We observe that (7) implies that $DH_0^q(x,0)$ acts on G_1 as a mapping twisting the vertical and as $T\mathbb{T}^1 \subset G_1$ the proposition follows by applying the proof of Proposition 4. □

4. *Proof of the theorem*

Let $H : M \to M$ satisfying the hypotheses of the theorem, then we can assume that $H : \mathbb{T}^n \times \mathbb{R}^m \to \mathbb{T}^n \times \mathbb{R}^m$ is given by (2). If H has more than one normal central frequency we can perturb it, in the C^r topology, in such a way that the perturbed map $\tilde{H}$ has only one central frequency τ. Also, we can assume that τ satisfies the diophantine condition (4) since the set of $\omega \in \mathbb{T}^n$ which satisfies (4) is dense for τ fixed.

The next step is making $n = 1$ and $m = 1$ or 2. If $n > 1$ then we can perturb $\tilde{H}$ in order that $\mathbb{T}^n$ has an invariant circle $\mathcal{C}$ which is normally hyperbolic for $\tilde{H}|_{\mathbb{T}^n}$. In addition to this, we can also perform the perturbation in such a way that the normal component of $\tilde{H}$ is not changed and $\mathcal{C}$ has parallel dynamics with inner frequency ω_1. So $\mathcal{C}$ is a Floquet torus of dimension one that is non-normally hyperbolic. If we restrict $\tilde{H}$ to the central manifold $W^c_{\mathcal{C}}$ of $\mathcal{C}$, we have that $\tilde{H}|_{W^c_{\mathcal{C}}}$ has the form (2) with $n = 1$, $\omega = \omega_1$ and $m = 1$ (respectively $m = 2$) if $\tau = 0$ or π (respectively $0 < \tau < \pi$). Moreover, ω_1 and τ continue satisfying (4). So we can assume that $n = 1$ and $m = 1$ or 2 depending on τ.

For $\lambda \in \mathbb{C}$ let $\Im(\lambda)$ be the imaginary part of λ. By Proposition 1 and what was discussed at the beginning of §2 there exists a C^{r-3} diffeomorphism Φ defined from a neighbourhood of $\mathbb{T}^1$ in W^c to a neighbourhood of $\mathbb{T}^1$ in $\mathbb{T}^1 \times \mathbb{R}^m$, such that $\Phi \circ \tilde{H}|_{W^c} \circ \Phi^{-1}$ is given by:

$$H_{\mathrm{sn}}(x,y) = (x + \omega + ay + O(y^2),\ y + O(y^2)), \quad \text{if } m = 1 \text{ and } \lambda = 1 \tag{8}$$

$$H_{\mathrm{f}}(x,y) = (x + \omega + O(y^2),\ -y + O(y^2)), \quad \text{if } m = 1 \text{ and } \lambda = -1 \tag{9}$$

$$H_{\mathrm{h}}(x,y) = (x + \omega + O(\|y\|^2),\ \Omega y + O(\|y\|^2)), \quad \text{if } m = 2 \text{ and } \Im(\lambda) \neq 0 \tag{10}$$

with

$$\Omega = \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix}.$$

Each of these cases correspond, respectively, to whether $\mathbb{T}^1$ is a bifurcation torus of saddle-node, flip or Hopf type of $\tilde{H}$, see §2. We can assume without loss of generality that $a \neq 0$ in (8).

The idea of the proof is to approximate H_σ by diffeomorphisms which exhibit a saddle-node bifurcation with criticalities and the use the well known result of [**NPT**] to approximate those by another which exhibit homoclinic tangencies. These perturbations

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will be performed in the C^{r-3} topology. For the first approximation we use the results of §3 and the following lemma whose proof can be found in [**N**].

LEMMA 4. *Let $\epsilon > 0$ and $r \in \mathbb{Z}^+$. $\exists C(r) > 0$, such that given $\delta > 0$ and a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ with $\|A - \mathrm{Id}\| \leq C(r)\delta^{r-1}\epsilon$, there exists a diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$, such that $\|\phi - \mathrm{Id}\|_{C^r} < \epsilon$, $\phi(0) = 0$, $D\phi(0) = A$ and $\phi(x) = x$ if $\|x\| \geq \delta$.*

We let

$$H_\sigma(x, y) = (x + \omega + f(x, y), Ay + g(x, y))$$

with $A = 1, -1,$ or Ω always that $\sigma =$ sn, f, or h (for the saddle-node, flip and Hopf cases, respectively). We consider two cases: the first one when $\sigma =$ sn or f, and the second one when $\sigma =$ h.

Saddle-node and Flip case. Here we first perturb of the following way

$$H_{\sigma,\epsilon_1}(x, y) = (x + \omega + \epsilon_1 + f(x, y), Ay + g(x, y)) \tag{11}$$

in order to get that $H_{\sigma,\epsilon_1}|\mathbb{T}^1$ is a rotation with rational angle $\bar{\omega} = \omega + \epsilon_1$, i.e. $\bar{\omega} = 2\pi p/q$ ($\sigma =$ sn or f).

Once we have a rational dynamics in $\mathbb{T}^1$, we consider for $\delta > 0$ and $\epsilon_2 > 0$,

$$\begin{cases} H_{\mathrm{sn},\epsilon_1,\delta,\lambda}(x, y) = (h_\delta(x) + f(x, y), \lambda y + g(x, y)) \\ H_{\mathrm{f},\epsilon_1,\delta,\lambda,\epsilon_2}(x, y) = (h_\delta(x) + \epsilon_2 xy + f(x, y), -\lambda y + g(x, y)) \end{cases}$$

where h_δ is as in Lemma 2 with $\omega = \bar{\omega}$.

In what follows we write these mappings as H_σ. When δ is positive, $H_\sigma|\mathbb{T}^1$ has saddle-node dynamics since $\lambda \neq 1$ ensures normal hyperbolicity. Hence there exists a strong stable foliation $\mathcal{F}^{ss}$ defined along $\mathbb{T}^1$. It happens that $\angle(\mathrm{T}_x\mathcal{F}^{ss}, \mathrm{T}_x\mathbb{T}^1) \to 0$ when $\lambda \to 1$ unless in the limit we would get an invariant set Λ of DH_σ different from $\mathrm{T}\mathbb{T}^1$ and this is forbidden either by Proposition 4 or by Proposition 5 if $\sigma =$ sn or $\sigma =$ f, respectively. For details of this argument see [**M**].

Now we fix $\epsilon_1, \epsilon_2,$ and λ, and we take $x \in \mathbb{T}^1$. Using Lemma 4 we can perturb in such a way that the new diffeomorphism at x has a criticality, that is to say $W^u(x_i)$ is tangent to $\mathcal{F}^{ss}$ at x. Generically, this criticality is quadratic and hence the result follows from [**NPT**].

Hopf case. Here we begin perturbing H_h as follows. First let ϵ_2 be given and put

$$\Omega' = \begin{pmatrix} \cos(\tau + \epsilon_2) & -\sin(\tau + \epsilon_2) \\ \sin(\tau + \epsilon_2) & \cos(\tau + \epsilon_2) \end{pmatrix}$$

then

$$H_{\mathrm{h},\epsilon_1,\epsilon_2,b}(x, y) = (x + \omega + \epsilon_1 + by_2 + f(x, y), \Omega' y + g(x, y)) \tag{12}$$

where ϵ_1, ϵ_2 and b are small enough, such that $\bar{\omega} = \omega + \epsilon_1$ and $\bar{\tau} = \tau + \epsilon_2$ are rational, i.e. $\bar{\omega} = 2\pi p/q$ and $\bar{\tau} = 2\pi p'/q$.

Now we introduce

$$H(x, y) = H_{\mathrm{h},\epsilon_1,\epsilon_2,b,\delta,\lambda}(x, y) = (h_\delta(x) + by_2 + f(x, y), \lambda\Omega' y + g(x, y)). \tag{13}$$

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In the same way, as in the above case, H leaves $\mathbb{T}^1$ invariant with its dynamics of saddle-node type. As $\mathbb{T}^1$ turns to be normally hyperbolic there exists a strong stable foliation $\mathcal{F}^{ss}$. Let P be the section of the plane bundle on $\mathbb{T}^1$ such that $P(x)$ is the plane spanned by the vectors $v_1 = (1, 0, 0)$ and $v_2 = (0, 0, 1)$. Then P is invariant under $\tilde{H}$ and therefore the same happens to Q where $Q(x) = P(x) \cap \mathcal{F}^{ss}(x)$. Also, we obtain that $\sup_{x\in\mathbb{T}^1} \angle(Q(x), \mathrm{T}_x\mathbb{T}^1) \to 0$ if $\lambda \to 1$. This follows from Proposition 6 with the same arguments as in the saddle-node case. Now applying Lemma 4 we get a criticality between $W^u(x_i)$ and $\mathcal{F}^{ss}$. □

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