A Vector Solution for Navigation on a Great Ellipse

Michael A. Earle

Distance and course to steer along part of a great ellipse can be readily determined with the use of vector analysis and differential geometry. Although the resulting integral for distance is not amenable to direct solution, it can nevertheless be handled efficiently and quickly by available commercial mathematical software. In this paper, a method of computing distance along with course to steer is presented and is one that has been prepared with the syntax of a particular commercial mathematics package in mind. This approach was intended to appeal to the navigator who is interested in the mathematics of navigation, and who nowadays solves his problems with a portable computer.

KEY WORDS

1. Geodesy. 2. Mathematics. 3. Computing.

1. INTRODUCTION. The Earth is like an oblate spheroid, flattened slightly at the poles, and can be described to a very close approximation by an ellipse of small eccentricity rotated about its polar axis. Because the Earth is not a true sphere, a great circle becomes a great ellipse. Using the ellipsoidal model, better accuracy for calculated distances between two points can be attained. Position of a point on the surface of the Earth is specified in terms of the geographic latitude, sometimes known as the geodetic latitude plus a longitude. Geographic latitude is that normally indicated on navigation charts and derived from various position fixing methods such as satellite or celestial navigation. Among the published methods that address the Great Ellipse, a notable recent article[1] provides a rigorous treatment of the topic that involves trigonometric functions. In contrast, the work presented here embraces a vector-based solution that may offer some ease of implementation and perhaps find virtue in a training environment.

Each position on the Earth's surface can be considered as the terminus of a vector drawn from the centre to a point on the surface. Using the properties of vector analysis, course angles – and hence courses to steer between points – are conveniently established without recourse to the rules of spherical trigonometry. Differential geometry permits an expression for a differential element of an arc on the surface of the ellipsoid to be derived, which leads to an integral expression for distance between two points. Although direct solutions for the integral for distance are not apparent, this integral, to be discussed later, is amenable to solution using either numerical methods such as the cubic spline methods[2] or the standard features of mathematical software.[3]

2. GEOMETRY OF THE EARTH. Since the Earth is ellipsoidal, the polar axis is the minor axis and the major axis lies in the equatorial plane. The equatorial radius, a, is taken to be equal to $10800/\pi$, and thus has the value of a = 3437·747 miles, where the mile is equal to one minute of equatorial arc. This in turn is equivalent to

https://doi.org/10.1017/S0373463300008948 Published online by Cambridge University Press

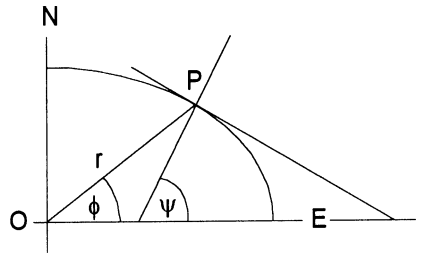


Figure 1.

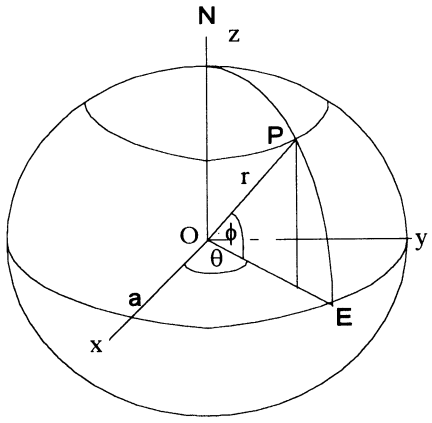


Figure 2.

1855·325 metres or 6087 feet. Eccentricity, e, for the ellipse is approximately equal to 0·082, though the value derived from the World Geodetic System of 1984 (WGS 84) is 0·08181919. Other values are sometimes justified; for example, that of the Bessel ellipsoid for which e = 0·081697.

Figure 1 is a section of the ellipsoid through a meridian and the polar axis. A point on the surface such as P in Figure 1 is located at a geocentric latitude ϕ above the equatorial plane O–E. Also at P, a line perpendicular to the tangent horizon meets the equatorial plane at the angle of the geographic latitude ψ. The two angles ϕ and ψ differ by a small but significant amount and are related by:

$$\tan \phi = (1-e^2) \tan \psi \tag{1}$$

A coordinate description of the point P on the surface of the ellipsoid is shown in Figure 2. Here, the radius, r, is also the magnitude of the position vector OP elevated to the geocentric latitude ϕ, while a is the radius in the equatorial plane containing OE. In addition, the x-axis coincides with the point where equator and prime meridian cross, and from which east and west directions of longitude θ are determined. Easterly longitude directions are $0 < \theta < +\pi$ indicating that they are taken as positive angles, while those from $-\pi < \theta < 0$ are westerly and are taken to be negative ones.

At any latitude, the radius r is expressed as a function $r(\phi)$ of the geocentric latitude, eccentricity and equatorial radius and is given by

$$r(\phi) = a\,[(1-e^2)/(1-e^2 \cos^2(\phi))]^{1/2} \tag{2}$$

https://doi.org/10.1017/S0373463300008948 Published online by Cambridge University Press

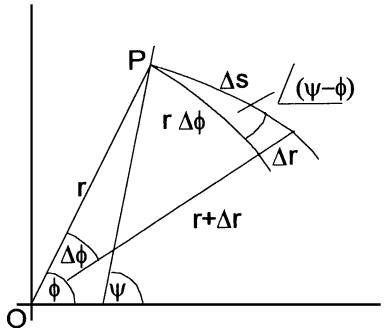


Figure 3.

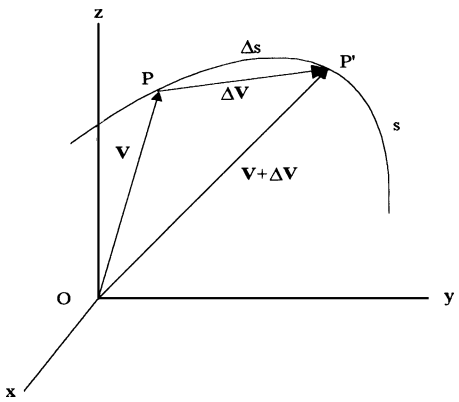


Figure 4.

However, it is convenient to normalise the radius such that a = 1 and reintroduce the numerical value later where required. Thus the normalised radius becomes

$$r(\phi) = [(1-e^2)/(1-e^2\cos^2(\phi))]^{1/2} \tag{3}$$

Latitude ϕ and longitude θ are expressed as direction N or S and E or W respectively. When used in conjunction with vectors and the scheme of Figure 2, it is appropriate to assign signs or direction coefficients for these according to the following:

$$N = 1 \quad S = -1 \quad E = 1 \quad W = -1$$

This will ensure that angles have the proper sense when using mathematical software.

On the surface of the spherical Earth, the length of an elemental arc on a meridian is $ds = a\,d\phi$ and on a parallel of latitude it is $ds = a\cos(\phi)\,d\theta$. On the surface of an ellipsoidal Earth however, the elemental arc on a parallel is $r(\phi)\cos(\phi)\,d\theta$ and the arc length on a meridian is also necessarily different. The geometry of the ellipsoid arc segment on a meridian in the vicinity of the point P in Figure 1 is shown in Figure 3. From this figure, we see that $(\Delta s)^2 = r^2(\Delta\phi)^2 + (\Delta r)^2$ which in the limit as $\Delta s \to ds$, converges upon $(ds)^2 = r^2(d\phi)^2 + (dr)^2$ which, from the geometry of Figure 3, is the same as $(ds)^2 = r^2\sec^2(\psi-\phi)(d\phi)^2$.

3. VECTOR METHODS. Cartesian axes x, y, z (Figures 2 and 5) along which there are the unit vectors **i**, **j** and **k** (Figure 5) are used to establish coordinates of a

https://doi.org/10.1017/S0373463300008948 Published online by Cambridge University Press

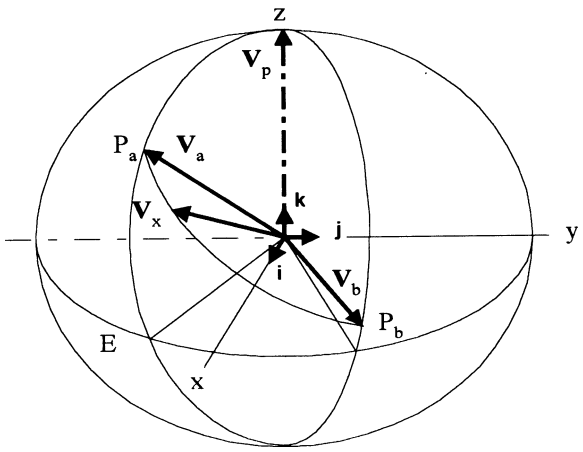


Figure 5.

point on the ellipsoid. We further define for later convenience a unit polar vector, namely

$$\mathbf{V}_p = N\mathbf{k} \tag{4}$$

Given the coordinates of point of departure P_a and of destination P_b (Figure 5), the equation of the great ellipse upon which the two points lie can be determined. This equation will permit the latitude of some other point on the great ellipse to be determined whose longitude has been specified. Thus a series of waypoints between departure and destination points can be established. From any such waypoint, distance to go or distance run plus course to steer can also be established.

Each point on the surface whose coordinates ϕ,θ are specified can be represented by a vector expressed here as a column matrix:

$$\mathbf{V}(\phi,\theta) = r(\phi)\cdot \begin{pmatrix} \cos(\phi)\cdot\cos(\theta) \\ \cos(\phi)\cdot\sin(\theta) \\ \sin(\phi) \end{pmatrix} \tag{5}$$

which has the same meaning as:

$$\mathbf{V}(\phi,\theta) = r(\phi)[\mathbf{i}\cos(\phi)\cos(\theta) + \mathbf{j}\cos(\phi)\sin(\theta) + \mathbf{k}\sin(\phi)] \tag{6}$$

but is better suited to use in commercial mathematical software.

An element of arc length Δs of an arc s lying on the surface of the ellipsoid between P and P′ is associated with a change $\Delta\mathbf{V}$ in the vector of position $\mathbf{V}$ as illustrated in Figure 4.

In the limit, as $\Delta\mathbf{V}\to 0$, $\Delta s/|\Delta\mathbf{V}| \to ds/|d\mathbf{V}| = 1$,

which implies that: $ds^2 = d\mathbf{V}\cdot d\mathbf{V}$, (7)

however: $d\mathbf{V} = \mathbf{v}_\phi d\phi + \mathbf{v}_\theta d\theta$, (8)

where: $\mathbf{v}_\phi = \partial\mathbf{V}/\partial\phi$ and $\mathbf{v}_\theta = \partial\mathbf{V}/\partial\theta$, (9)

are the vector partial derivatives of $\mathbf{V}$ along a meridian and a parallel of latitude respectively.

Therefore: $d\mathbf{V}\cdot d\mathbf{V} = (\mathbf{v}_\phi d\phi + \mathbf{v}_\theta d\theta)\cdot(\mathbf{v}_\phi d\phi + \mathbf{v}_\theta d\theta)$, (10)

and on expanding: $ds^2 = (\mathbf{v}_\phi\cdot\mathbf{v}_\phi)d\phi^2 + 2(\mathbf{v}_\phi\cdot\mathbf{v}_\theta)d\phi\, d\theta + (\mathbf{v}_\theta\cdot\mathbf{v}_\theta)d\theta^2$. (11)

https://doi.org/10.1017/S0373463300008948 Published online by Cambridge University Press

Since the vector partial derivatives of $\mathbf{V}$ namely $\mathbf{v}_\phi = \partial\mathbf{V}/\partial\phi$ and $\mathbf{v}_\theta = \partial\mathbf{V}/\partial\theta$ are orthogonal, the middle term on the right hand side in Equation 11 must vanish leaving

$$ds^2 = (\mathbf{v}_\phi \cdot \mathbf{v}_\phi)\, d\phi^2 + (\mathbf{v}_\theta \cdot \mathbf{v}_\theta)\, d\theta^2 \tag{12}$$

or

$$ds^2 = |\mathbf{v}_\phi|^2\, d\phi^2 + |\mathbf{v}_\theta|^2\, d\theta^2. \tag{13}$$

Writing Equation 5 or equivalently Equation 6 as $\mathbf{V} = r\ \upsilon$ meaning $\mathbf{V}(\phi,\theta) = r(\phi)\ \upsilon(\phi,\theta)$ and then putting $\upsilon'_\phi = \partial\upsilon/\partial\phi$ we get:

$$\mathbf{v}_\phi = r\,\upsilon'_\phi + \upsilon\, dr/d\phi, \tag{14}$$

thus

$$|\mathbf{v}_\phi|^2 = \mathbf{v}_\phi \cdot \mathbf{v}_\phi = (r\,\upsilon'_\phi + \upsilon\, dr/d\phi)\cdot(r\,\upsilon'_\phi + \upsilon\, dr/d\phi), \tag{15}$$

and on expanding further:

$$|\mathbf{v}_\phi|^2 = r^2(\upsilon'_\phi \cdot \upsilon'_\phi) + 2\, r\, dr/d\phi\, (\upsilon \cdot \upsilon'_\phi) + ((\upsilon \cdot \upsilon)(dr/d\phi)^2, \tag{16}$$

in which $\upsilon'_\phi \cdot \upsilon'_\phi = \upsilon \cdot \upsilon = 1$ and $\upsilon \cdot \upsilon'_\phi = 0$ thereby reducing Equation 16 to:

$$|\mathbf{v}_\phi|^2 = r^2 + (dr/d\phi)^2. \tag{17}$$

Similarly with $\upsilon'_\phi = \partial\upsilon/\partial\theta$ we can show that $|\mathbf{v}_\phi|^2 = r^2\ \cos^2(\theta)$ (18)

Substituting Equations 17 and 18 into Equation 13 gives:

$$ds^2 = (r^2 + (dr/d\phi)^2)\, d\phi^2 + r^2 \cos^2(\phi)\, d\theta^2 \tag{19}$$

Referring to Figure 3, we discern that the first term on the right side of Equation 19 is equivalent to $r^2\ \sec^2(\psi - \phi)$, which allows Equation 19 to be re written as:

$$ds^2 = r^2 \sec^2(\psi - \phi)\, d\phi^2 + r^2 \cos^2(\phi)\, d\theta^2, \tag{20}$$

or:

$$ds^2 = [r^2 \sec^2(\psi - \phi)\, (d\phi/d\theta)^2 + r^2 \cos^2(\phi)]\, d\theta^2,$$

or as:

$$ds^2 = r^2 \cos^2(\theta)\, [r^2\ \sec^2(\psi\text{-}\theta) \sec^2(\phi)\, (d\phi/d\theta)^2 + 1]\, d\theta^2. \tag{21}$$

In Equation 20, we note that the first term on the right side corresponds to the square of an element of arc on a meridian at constant longitude as illustrated in Figure 3, while the second term is the square of an element of arc on a parallel of latitude. Their sum (Equation 20) is then the square of the length of any element of arc on the surface. This is a general equation that requires a knowledge of the functional relationship between ϕ and θ and also requires that the derivative $d\phi/d\theta$ exists.

In order to establish an integral expression for Equation 21, it is necessary to replace functions containing ϕ with those containing ψ only. From Equation 1, on setting $\alpha = 1 - e^2$ and with the identity $\sec^2(x) = 1 + \tan^2(x)$, the following substitutions are established:

$$r^2 = (1 + \alpha^2 \tan^2(\psi))/(1 + \alpha \tan^2(\psi)),$$

$$\sec^2(\phi) = (1 + \alpha^2 \tan^2(\psi)),$$

$$\sec^2(\psi - \phi) = (1 + \tan^2(\psi))\,(1 + \alpha^2 \tan^2(\psi))/(1 + \alpha \tan^2(\psi))^2.$$

On making these substitutions into Equation 21 and restoring the value for the equatorial radius a, then with the understanding that $\tan(\psi)$ means the functional relationship $\tan(\psi(\theta))$ between geographic latitude and the longitude as in Equation 27, the integral for distance becomes:

$$s(\theta_1 - \theta_2) = a\left|\int_{\theta_1}^{\theta_2}\left[\frac{1}{1 + \alpha \tan^2(\psi)} \cdot \left(1 + \left(\frac{(1 + \tan^2(\psi)) \cdot (1 + \alpha^2 \tan^2(\Psi))^2}{(1 + \alpha \tan^2(\psi))^2}\right) \cdot \left(\frac{d\phi}{d\theta}\right)^2\right)\right]^{\frac{1}{2}} d\theta\right| \tag{22}$$

https://doi.org/10.1017/S0373463300008948 Published online by Cambridge University Press

where $d\phi/d\theta$ is the derivative $\phi'(\theta)$ from Equation 28.

The above integral for distance can be used, as previously noted, wherever the functional relationship between geographic or geocentric latitude and longitude is known as for example in the case of great ellipse or the loxodrome. This integral equation is very easily processed with the help of mathematical software. It should be noted however that it is not well behaved for tracks that pass close to the poles as the function becomes ill-conditioned there. No particular study of the practical limits of usage has been made; however, for all practical sailing latitudes it provides reliable results. When used for tracks that cross the 180° meridian, then because the magnitude of differences in $\theta_1 - \theta_2$ will exceed π or 180°, it becomes necessary to select the supplemental distance $s = (s(0, 2\pi) - s(\theta_1, \theta_2))$ using simple logic statements. As an aside, we note that along any parallel of latitude where $\phi' = 0$ including the equator, that Equation 22 reduces to $(a|\theta_2 - \theta_1|)/[1 + \alpha \tan^2(\psi(\theta))]^{1/2}$ as expected, which is equal to $r(\phi)\cos(\phi)|\theta_2 - \theta_1|$.

4. THE PATH OF THE GREAT ELLIPSE. A great ellipse is the path formed by all the points which lie at the intersection of the surface of the ellipsoidal Earth and a plane that passes through the centre. Given two known vectors of position $\mathbf{V}_a$ and $\mathbf{V}_b$ as illustrated in Figure 5, a third unknown vector $\mathbf{V}_x$ will lie in the same plane when their scalar triple product is zero.[4]

Under this condition $\mathbf{V}_x$, $\mathbf{V}_a$ and $\mathbf{V}_b$ must all lie on the great ellipse. The scalar triple product is:

$$\mathbf{V}_x \cdot \mathbf{V}_a \times \mathbf{V}_b = 0, \tag{23a}$$

and since the cross product $\mathbf{V}_a \times \mathbf{V}_b$ is implicitly known, we can put:

$$\begin{pmatrix} m \\ n \\ p \end{pmatrix} = \mathbf{V}_a \times \mathbf{V}_b, \tag{23b}$$

where: m, n, p are coefficients in the equation of the great ellipse.

The equation for the great ellipse is now contained in the expression:

$$\mathbf{V}_x \cdot \begin{pmatrix} m \\ n \\ p \end{pmatrix} = 0, \tag{23c}$$

which upon expansion and removal of common factors yields the condition:

$$m\cos(\phi)\cos(\theta) + n\cos(\phi)\sin(\theta) + p\sin(\phi) = 0. \tag{24}$$

Setting $\lambda = m/p$ and $\mu = n/p$, then re-arranging and writing as a function we arrive at:

$$\tan(\phi(\theta)) = -[\lambda\cos(\theta) + \mu\sin(\theta)], \tag{25}$$

for which the first derivative as required for Equation 21 is:

$$\phi'(\theta) = \{[\lambda\cos(\theta) - \mu\sin(\theta)]/[1 + \tan^2(\phi(\theta))]\}. \tag{26}$$

Since geographic latitudes are called for, the last two expressions are converted to the following using Equation 1:

$$\tan(\psi(\theta)) = -[\lambda\cos(\theta) + \mu\sin(\theta)]/\alpha, \tag{27}$$

$$\phi'(\theta) = \{[\lambda\cos(\theta) - \mu\sin(\theta)]/[1 + \alpha^2\tan^2(\psi(\theta))]\}. \tag{28}$$

https://doi.org/10.1017/S0373463300008948 Published online by Cambridge University Press

Equation 27 provides the geographic latitude for any given value of longitude on a great ellipse containing the two points P_a and P_b. Consequently it can be used to fix the latitudes of waypoints whose longitudes have been previously specified. This equation can also be used in an iterative root-finding process to determine the longitudes for given latitudes. A procedure such as this has application to the determination of longitudes at which pre-determined limiting latitudes will be reached. Such a root-finding process is usually a standard feature of commercial mathematical software.

Compared to other approaches, there is no requirement here to establish the location of either the vertexes or nodes for the great ellipse in order to accomplish course or distance calculations. But on occasion, the navigator will want to know for reasons of safety, whether the highest latitude or vertex ψ_v reached by his track is safe and at what longitude it will occur. Vertexes can be found from the coefficients λ and μ using the following:

$$\tan(\psi_v) = (\lambda^2 + \mu^2)^{1/2}/\alpha,$$

and will occur at a longitude θ_v found on solving Equation 27 for θ_v from:

$$\tan(\psi_v) + [\lambda \cos(\theta_v) + \mu \sin(\theta_v)]/\alpha = 0.$$

5. COURSE ANGLE AND COURSE TO STEER. A course on a great ellipse going from point P_a to P_b starts with an initial angle on the spherical triangle of C_a. Travelling in the opposite direction from P_b to P_a, the initial angle is C_b. At each waypoint between P_a and P_b, the course angle changes. If the course angle is maintained constant at each waypoint, the resulting path becomes a loxodrome spiralling towards a pole. In order to maintain a track on the great ellipse, steering directions are trimmed at each waypoint. Vector methods allow course angles to be computed very easily without recourse to the rules of spherical trigonometry, and as already noted, matrix representations of vectors (for example, Equation 5) featured in commercial mathematical software can be efficiently manipulated to yield desired results.

The internal angle C_a at P_a between the plane of the great ellipse and the plane of the meridian plane passing through P_a is contained in the function $C_a = C(\mathbf{V}_a, \mathbf{V}_p, \mathbf{V}_b)$, where:

$$C(\mathbf{V}_a, \mathbf{V}_p, \mathbf{V}_b) = \arccos\left[\frac{(\mathbf{V}_a \times \mathbf{V}_p)\cdot(\mathbf{V}_a \times \mathbf{V}_b)}{|\mathbf{V}_a \times \mathbf{V}_p|\cdot|\mathbf{V}_a \times \mathbf{V}_b|}\right], \tag{29}$$

and where $\mathbf{V}_p$ is the polar vector defined earlier. To find the course angle at P_b, it is only necessary to reverse the order of data entry for the latitudes and longitudes of the two positions. For example, we compute:

$$C_a = C(\mathbf{V}_a, \mathbf{V}_p, \mathbf{V}_b) \text{ or } C_b = C(\mathbf{V}_b, \mathbf{V}_p, \mathbf{V}_a)$$

according to the direction established when data is entered. As the course progresses to a waypoint P_x represented by a vector V_x, the revised course angle going towards P_b is $C_x = C(\mathbf{V}_x, \mathbf{V}_p, \mathbf{V}_b)$.

The course angle provided by Equation 29 is not necessarily the course to steer. Regardless of whether P_a and P_b are in the northern or southern hemisphere, or indeed in different hemispheres, there is only one spherical triangle to be considered. Therefore the course to steer C_s at any waypoint is either $C_s = C_x$ or $C_s = (2\pi - C_x)$.

https://doi.org/10.1017/S0373463300008948 Published online by Cambridge University Press

Ambiguity is once more conveniently resolved with use of simple logic statements that take account of relative direction and the algebraic discontinuity at the 180° meridian.

6. COMMENTS ON EXPERIMENTAL RESULTS. Comparing results for distance obtained from the methods outlined here with the examples of Williams [2] [Tables 5.2 & 7.2], a good match with his results was obtained. The results also compare extremely well with Hairawa's data derived from a well-known correction method. Results are shown in Table 1 below, in which calculations were performed

Table 1. Comparison of results for distance (e = 0·081697).

Geodetic Latitude	Correction Method Williams – Table 5.2 & 7.2)	Cubic Spline 1° interval (Williams – Table 5.2)	This Method	Great Ellipse (Williams – Table 7.2)
10°	5877·33	5877·33	5877·33	5877·33
20°	5526·96	5526·96	5526·96	5526·96
30°	4991·21	4991·21	4991·21	4991·21
40°	4317·62	4317·61	4317·62	4317·62
50°	3546·71	3546·70	3546·72	3546·71
60°	2709·27	2709·26	2709·29	2709·29
70°	1828·05	1828·04	1828·06	1828·06
80°	920·29	920·28	920·30	920·30

at the latitudes shown in the first column for a track between two points separated by 100° of longitude.

Results are from a computational document that has been prepared[5] using the methods described. Comparison has also been made with examples found in Bowditch[6] with e = 0, which also agree with the results very closely, recognising of course that the examples in Reference 6 were worked using traditional methods rather than contemporary computer-aided methods which provide improved precision. One purpose of these tests was to confirm calculations for course directions particularly when crossing the 180° meridian.

7. CONCLUSIONS. A method of computing distance along a great ellipse has been described that allows the integral for distance to be computed directly using the built-in capabilities of commercial mathematical software. This obviates the need to write code in arcane computer languages. The method used here makes use of some basic vector analysis and differential geometry. Course angle and hence course to steer are directly calculated without recourse to the rules of spherical trigonometry and neither do the distance calculations require knowledge of the location of the vertex. Results have been verified against examples found in other sources with satisfactory results

ACKNOWLEDGMENTS

I am pleased to acknowledge the many insights gained in the preparation of this work through recent contact with Dr. Roy Williams FRIN.

https://doi.org/10.1017/S0373463300008948 Published online by Cambridge University Press

REFERENCES

[1] Walwyn, P. R. (1990). The great ellipse solution for distances and headings to steer between waypoints. This *Journal*, **52**, pp 421–424.
[2] Williams, R. (1998). *Geometry of Navigation*. Horwood Publishing, Chichester UK ISBN 1-898563-46-2.
[3] Mathcad 6.0 plus. Mathsoft Inc., Cambridge, Massachusetts 02142, U.S.A.
[4] Gibbs J. W. (1960). *Vector Analysis*. Dover Publications Inc., New York, U.S.A.
[5] Deposited in the astronomy and navigation section at www.mathsoft.com/appsindex.html and named gellip03.mcd.
[6] Bowditch, N. (1981). *The American Practical Navigator*. Defense Mapping Agency, Pub No.9 Vol.2, U.S.A.

https://doi.org/10.1017/S0373463300008948 Published online by Cambridge University Press