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Nets of conics

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Segre's classification of pencils of quadrics is well known, and appears in standard texts such as (1), (2). By contrast, other linear systems of quadrics do not lend themselves to a similar exhaustive treatment. Here we discuss the simplest case, that of nets of conics. The result is intrinsically interesting, and involves some pleasant geometry. As well as deriving a list of types, and enumerating their properties, we study the elementary geometrical properties of the partition of the space Ω of nets into equivalence classes (or strata). We work over the field $\mathbb{R}$ of real numbers, after performing preliminary calculations over $\mathbb{C}$.

We shall use the language of classical projective geometry. In this introductory section I review some well known facts which will be used in the sequel.

Consider non-zero quadratic forms on a vector space W of dimension 3 over $\mathbb{R}$ or $\mathbb{C}$: our usual notation for such a form is

$$S_1 \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy. \tag{1}$$

Over $\mathbb{C}$, we can change coordinates to reduce S_1 to

$$x^2+y^2+z^2, \quad x^2+y^2 \quad \text{or} \quad x^2;$$

we refer to it correspondingly as non-singular, a line-pair, or a repeated line, though strictly this terminology refers to the locus $S = 0$ in the projective plane $P(W)$. In the real case, there are 5 possibilities: $x^2+y^2 \pm z^2$, $x^2 \pm y^2$, x^2 (up to scalars).

The space of forms has dimension 6: a 2-dimensional subspace is called a pencil. Our usual notation is

$$S = \lambda S_1 + \mu S_2,$$

where S_1, S_2 are as above. The discriminant of a form is the determinant of its matrix. Thus the discriminant of S is a homogeneous cubic in λ and μ. There are 5 types of pencils (over $\mathbb{C}$) where this does not vanish identically (regular) and 3 types where it is zero (singular). In the real case, the corresponding numbers are 9 and 4. See Table 0.

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Table 0

Notation	S_1	S_2	Discriminant	Number of real forms
[1 1 1]	x^2+y^2	y^2+z^2	$\lambda\mu(\lambda+\mu)$	3
[2 1]	$2xy$	y^2+z^2	$-\lambda^2\mu$	2
[(1 1) 1]	x^2+y^2	z^2	$\lambda^2\mu$	2
[3]	y^2+2xz	$2yz$	$-\lambda^3$	1
[(2 1)]	$2xy+z^2$	y^2	$-\lambda^3$	1
[; 1 ;]	$2xy$	$2yz$	0	1
[1 1 ; ; 1]	x^2	y^2	0	2
[2 ; ; 1]	$2xy$	y^2	0	1

We see by inspection that a conic corresponding to a simple root of the discriminant is always a line-pair (but not conversely). The only type of pencil with discriminant a perfect cube and the root corresponding to a line-pair is (3).

The 6-dimensional space $F(W)$ of quadratic forms on W is dual to the space $F(W^*)$, where W^* is dual to W. We use dual coordinates (X, Y, Z) in W^* to those in W. If two forms are orthogonal under the dual pairing, the corresponding conics in $P(W)$, $P(W^*)$ are called apolar. Orthogonality gives a bijection between r-dimensional subspaces of $F(W)$ and $(6-r)$-dimensional subspaces of $F(W^*)$. As we have just described the classification of subspaces with $r \leqslant 2$, and that for $r \geqslant 4$ follows by duality, only the case $r = 3$ (nets) remains. Moreover, each net of conics in $P(W)$ determines a dual (apolar) net in $P(W^*)$, and we shall make systematic use of this in our notation.

We are now ready to begin. We shall return later to considering nets as 3-dimensional subspaces of $F(W)$, or as points in the Grassmannian Ω of all such, and represent them simply as

$$\lambda S_1 + \mu S_2 + \nu S_3,$$

where $S_1, S_2, S_3 \in F(W)$ are linearly independent. We can – and shall – regard (λ, μ, ν) also as coordinates in a vector space V – or rather its projective space $P(V)$ – essentially V is the subspace of $F(W)$.

Classification: case of irreducible discriminant. The discriminant Δ is a cubic form in λ, μ, ν defining a plane cubic curve. If Δ is irreducible, so is this curve, which thus has a non-singular point M of inflexion. In the real case, we can choose M to be real. Each line through M meets the curve twice more, and the locus of the harmonic conjugate of M with respect to this point-pair is a line, $\mu = 0$ say. If we take the inflexional tangent as $\nu = 0$, then Δ must be of the form

$$\Delta = -c_1\mu^2\nu + c_2\lambda^3 + c_3\lambda^2\nu + c_4\lambda\nu^2 + c_5\nu^3,$$

where, since Δ is irreducible, c_1 and c_2 are non-zero. Setting $\nu' = c_1\nu$, $\lambda' = c_2^{\frac{1}{3}}(\lambda + c_3\nu/3c_2)$ reduces to $c_1 = c_2 = 1$, $c_3 = 0$, our *normal form*

$$\Delta = -\mu^2\nu + \lambda^3 + p\lambda\nu^2 + q\nu^3. \tag{2}$$

The cubic is non-singular, unless $\lambda^3 + p\lambda\nu^2 + q\nu^3 = 0$ has a repeated root, which then yields a singular point on $\mu = 0$: a node if the multiplicity is 2 and a cusp if it is 3.

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Further reduction of coordinates permits only $(p, q) \to (t^4 p, t^6 q)$. We thus have the unique invariant $j = 27q^2/(4p^3 + 27q^2)$.

The above is valid both in the complex and in the real case. In the real case, if $j < 0$ (more generally, including the case $q = 0$, if $4p^3 + 27q^2 < 0$), the cubic meets $\mu = 0$ in 3 real points, so has 2 real components; if $j > 0$, we have 1 intersection and 1 connected component. For $j = \infty$, we can set $p = -3t^2$, $q = 2t^3$. The nodal tangents are $3t(\lambda - t\nu)^2 = \mu^2$ so are real (a crunode) if t (or equivalently, q) is positive and conjugate complex (an acnode) for $t < 0$: for $t = 0$ we have a cusp.

Suppose now that the net $\lambda S_1 + \mu S_2 + \nu S_3$ has discriminant Δ as in (2). The pencil $\nu = 0$ then has discriminant λ^3. Now the point M ($\lambda = \nu = 0$), being non-singular on $\Delta = 0$, corresponds to a line-pair. For a general line through M defines a pencil, and M has multiplicity one on its intersection with $\Delta = 0$. The pencil $\nu = 0$ is thus of type (3) and by changing coordinates (x, y, z) we may reduce it to the form

$$\lambda(2xz + y^2) + \mu(2yz).$$

Now take S_3 as in (1). We find that for Δ (or rather, $-\Delta$) to have the form (2) we must have $a = -1, f = h = 0, b = -2g$, and the cubic is then $(\lambda^2 + 2g\lambda\nu + (c + g^2)\nu^2)(\lambda - 2g\nu)$. Thus the assignment of a net with discriminant Δ singles out one of the three roots of the cubic.

This can be invariantly interpreted as follows. By choosing an inflexion M as base point, we obtain a natural group structure on the set of non-singular points of $\Delta = 0$. If we work over $\mathbb{C}$, in the non-singular case, this is a torus, and the three points on $\mu = 0$ are those of order 2 in the group structure. Now points of order 2 on elliptic curves correspond to double coverings. Here, each point of $\Delta = 0$ corresponds to a conic which is a pair of lines. I claim that these lines envelop a cubic Δ' (see (5) for this) and the double covering $\Delta' \to \Delta$ corresponds to our point of order 2.

For let $(\lambda_0, \mu_0, \nu_0)$ lie on $\Delta = 0$. Set

$$\xi = \nu_0 x - (\lambda_0 + g\nu_0) z, \quad \eta = \mu_0 z + (\lambda_0 - 2g\nu_0) y.$$

Then provided $\nu_0 \neq 0$, $\lambda_0 \neq 2g\nu_0$,

$$\begin{aligned}\lambda_0 S_1 + \mu_0 S_2 + \nu_0 S_3 &= \lambda_0(2xz + y^2) + \mu_0(2yz) + \nu_0(-x^2 - 2gy^2 + cz^2 + 2gzx) \\ &= -\nu_0^{-1}\xi^2 + (\lambda_0 - 2g\nu_0)^{-1}\eta^2 + z^2\left\{c\nu_0 + \frac{(\lambda + g\nu_0)^2}{\nu_0} - \frac{\mu_0^2}{\lambda_0 - 2g\nu_0}\right\}\end{aligned}$$

and the expression in braces vanishes since $\Delta(\lambda_0, \mu_0, \nu_0) = 0$. Thus our line-pair is

$$\nu_0 x - (\lambda_0 + g\nu_0) z = \pm (\lambda_0 \nu_0^{-1} - 2g)^{\frac{1}{2}} (\mu_0 z + (\lambda_0 - 2g\nu_0) y)$$

whence, after some reduction,

$$Y^2 Z = \tfrac{1}{2} c X^3 - 3gX^2 Z - \tfrac{1}{2} X Z^2$$

and the points of order 2 on $Y = 0$ are $[0, 0, 1]$ and $cX^2 - bgXZ - Z^2 = 0$. Here, the inflexion $[0, 1, 0]$ and point $[0, 0, 1]$ give the line pair yz corresponding to $\lambda = \nu = 0$ (the zero of our group): the other points of order 2 form the line-pair $-x^2 + 6gxz + cz^2$ corresponding to the preferred point of order 2 $(\lambda, \mu, \nu) = (2g, 0, 1)$. This proves our assertion.

We can now quickly complete our classification in the complex case. First, we have type A, with Δ non-singular. Next, if Δ is nodal and our cubic has a repeated root we

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must distinguish type B, where the repeated root is singled out, from type B^*, where the other is. Finally, we have type C, with Δ cuspidal, and a unique choice of root. In our canonical form, B is given by $c = -9g^2$ and B^* by $c = 0$; for C we have $g = c = 0$. Geometrically, we can consider the conic of the net which corresponds to the node of the cubic. In the case of B, this is a repeated line, but for B^* it is a line-pair.

The net of conic envelopes apolar to the given net is

$$\alpha(cX^2 + Z^2) + \beta(2XY) + 2\gamma(3gX^2 - Y^2 + XZ),$$

with discriminant $-\alpha\beta^2 + 2\gamma^3 - 12g\alpha\gamma^2 - 2c\alpha^2\gamma$, so the corresponding cubic is $\gamma(\gamma^2 - 6g\alpha\gamma - c\alpha^2)$ or, writing

$$\gamma' = \gamma - 2g\alpha \quad \text{it is} \quad (\gamma'^2 - 2g\alpha\gamma' - (c + 8g^2)\alpha^2)(\gamma' + 2g\alpha).$$

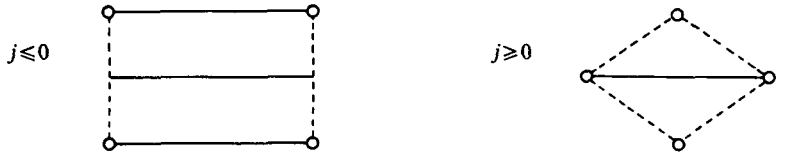
It is easy to deduce that this net has parameters

$$g^* = -g, \quad c^* = -c - 9g^2.$$

In particular, B^* is indeed dual to B.

For the real case, if the cubic just has one real root, it must be preferred: if there are three, they are distinguished by their order on the line $\mu = 0$ and any may be preferred. Thus A splits into four types. Each of B, B^* splits into two, as one may have an acnode or a crunode. Type C is already unique over $\mathbb{R}$.

To understand the general types, think of the non-singular cubic as quotient of $\mathbb{C}$ by a lattice (i.e. parametrized by elliptic functions) and real points as those fixed by the map induced by complex conjugation. As this fixes the lattice, this must be of one of two types (we draw a period parallelogram):



Two components One component

We can likewise draw the four types of double covering (denote by a cross the corresponding element of order 2): and also introduce our notation for them (Fig. 0).

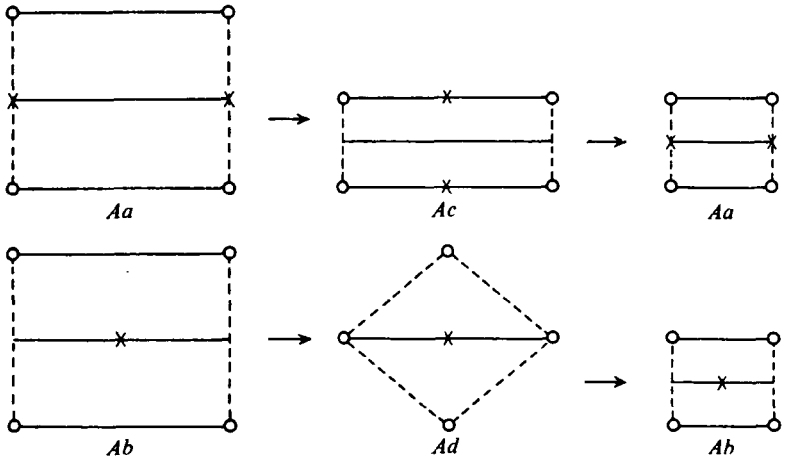


Fig. 0

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*Types Ba, B*a (acnode), Bc, B*c (crunode), C (cusp).* These cases occur in the family parametrized by (c, g) as shown in Fig. 1. We also illustrate the types of net, by drawing the cubic in the (λ, μ, ν) plane corresponding to degenerate conics, and shading those complementary regions corresponding to conics with no real points. The boundaries of these regions correspond to conjugate complex pairs of lines: other parts of $\Delta = 0$ to real line-pairs. We place an asterisk at points corresponding to repeated lines, and (for type A) a circle at the preferred point of order 2. See Fig. 2.

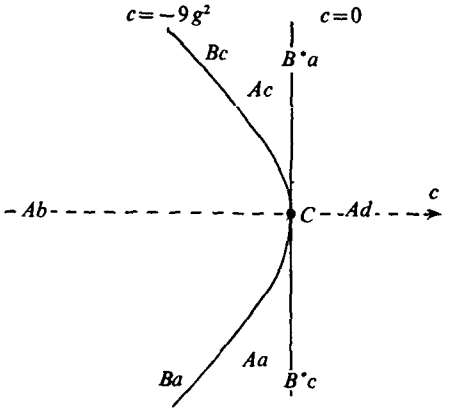


Fig. 1. $(c, g) \sim (ct^2, gt)$ for $t > 0$. The dual type to (c, g) is that of $(-9g^2-c, -g)$.

Classification: case of reducible discriminant. If Δ factorizes, it has a linear factor. Both in this case, and when Δ vanishes identically, we thus have a subpencil, say $\nu = 0$, consisting entirely of singular conics. We take this pencil as in Table 0.

If $S_1 = 2xz$, $S_2 = 2yz$ we can take S_3 of the form $f(x, y) + cz^2$. A suitable linear transformation in x and y (and recombination of S_1, S_2) reduces f to x^2, $2xy$ or 0: we may clearly suppose $c = 0$ or 1.

If $S_1 = x^2$, $S_2 = 2xy$, then if the coefficient of z^2 in S_3 is non-zero we can complete the square, reducing S_3 to $z^2 + f(x, y)$. If S_3 has the form $f(x, y) + 2z(\alpha x + \beta y)$ with $\beta \neq 0$, we can set $y' = \alpha x + \beta y$; if $\beta = 0$, $\alpha \neq 0$ multiply by α^{-1}. We thus have $f(x, y)$ added to z^2, $2yz$, $2xz$ or 0; now changing S_3 mod S_1 and S_2 we reduce f to by^2, and multiplying the variables by scalars we reduce b to 0 or 1.

If $S_1 = x^2$, $S_2 = y^2$, and the coefficient of z^2 in S_3 is non-zero, we complete the square, reducing S_3 eventually to $z^2 + 2xy$ or z^2. If z^2 does not appear but z does, a suitable substitution $z' = z + ax + by$ will get rid of the coefficient of xy; changing S_3 mod S_1 and S_2 now gets rid of the terms in x^2, y^2, $S_3 = 2z(\alpha x + \beta y)$; now by scaling the coordinates we can reduce α, β to 0 or 1.

This produces a finite list of cases, some of which are easily seen to be isomorphic to others. Sorting this out, we arrive at the list shown in Table 1. The names have been chosen so that an asterisk denotes the dual type.

The geometry of the degenerate nets I, I^* is essentially trivial. For the others, since a repeated line can occur only at a double point of $\Delta = 0$, it is easy to check for

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Table 1

S_1	S_2	S_3	Δ	Name
x^2	y^2	z^2+2xy	$\nu(\lambda\mu-\nu^2)$	D
$2xz$	$2yz$	z^2+2xy	$\nu(2\lambda\mu-\nu^2)$	D^*
x^2	y^2	z^2	$\lambda\mu\nu$	E
$2yz$	$2zx$	$2xy$	$2\lambda\mu\nu$	E^*
x^2	y^2	$2z(x+y)$	$-(\lambda+\mu)\,\nu^2$	F
$2xz$	$2yz$	x^2+z^2	$-\mu^2\nu$	F
x^2	$2xy$	y^2+z^2	$\nu(\lambda\nu-\mu^2)$	F^*
x^2	y^2	$2yz$	$-\lambda\nu^2$	G
x^2	$2xy$	$2yz$	$-\lambda\nu^2$	G^*
x^2	$2xy$	y^2+2xz	$-\nu^3$	H
x^2	$2xy$	y^2	0	I
$2xz$	$2yz$	z^2	0	I^*

Table 2

Type ...	A	B	B^*	C	D	D^*	E	E^*	F	F^*	G	G^*	H
Degrees of freedom	9	8	8	7	7	7	6	6	6	6	5	5	4
No. of double points of $\Delta = 0$	0	1	1	1	2	2	3	3	∞	1	∞	∞	∞
No. of repeated lines	0	1	0	1	2	0	3	0	2	1	2	1	1

existence (Table 2). Moreover, an immediate consequence of duality is that the repeated lines of a net are precisely the base points of the dual net.

We now consider the classification in the real case. For type D, the two repeated lines may be real or complex conjugate. In the former case, the argument above allows us to reduce over $\mathbb{R}$ to the above normal form (type Da). In the latter, we may take the singular pencil $\nu = 0$ as $\lambda(x^2-y^2)+\mu(2xy)$. We can then reduce S_3 to ax^2+cz^2 with $a, c \neq 0$, and hence to $x^2 \pm z^2$ giving two further types Db, Dc. Dually, type D^* determines 3 types D^*a, D^*b, D^*c in the real case.

For type E also, the 3 repeated lines may be all real or one real and two complex conjugate. This determines the net: Ea (as above) and Eb, say $\lambda(x^2-y^2)+2\mu xy+\nu z^2$. Similarly for F, if the two repeated lines are real (case Fa) we have equations as above otherwise (case Fb) we obtain $\lambda(x^2-y^2)+2\mu xy+2\nu yz$. We have dual types E^*a, E^*b, F^*a, F^*b.

Finally, for type G both repeated lines must be real (just one occurs at the triple point of $\Delta = 0$) whence follows easily that G (and G^*) has a unique real form. So do H (conics having 3-point contact with a given conic at a given – hence real – point). I (line-pairs with a given – hence real – vertex), and I^*. We thus have 28 types altogether.

Again, we illustrate (Fig. 3) these types of net, and show incidentally that they are indeed all inequivalent. The descriptions here are as in Fig. 2, except that for types Fa and G, the segment joining the two asterisks corresponds to pairs of conjugate complex lines, and that a repeated line is indicated by drawing 2 (or 3) lines close together.

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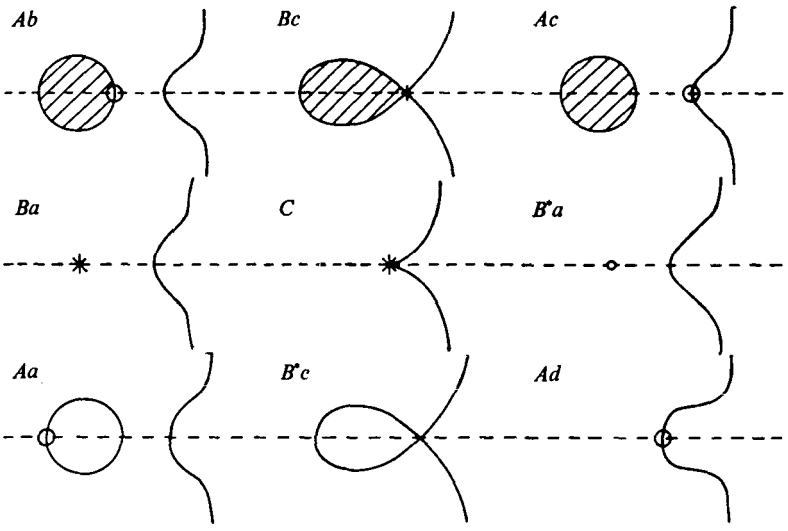


Fig. 2

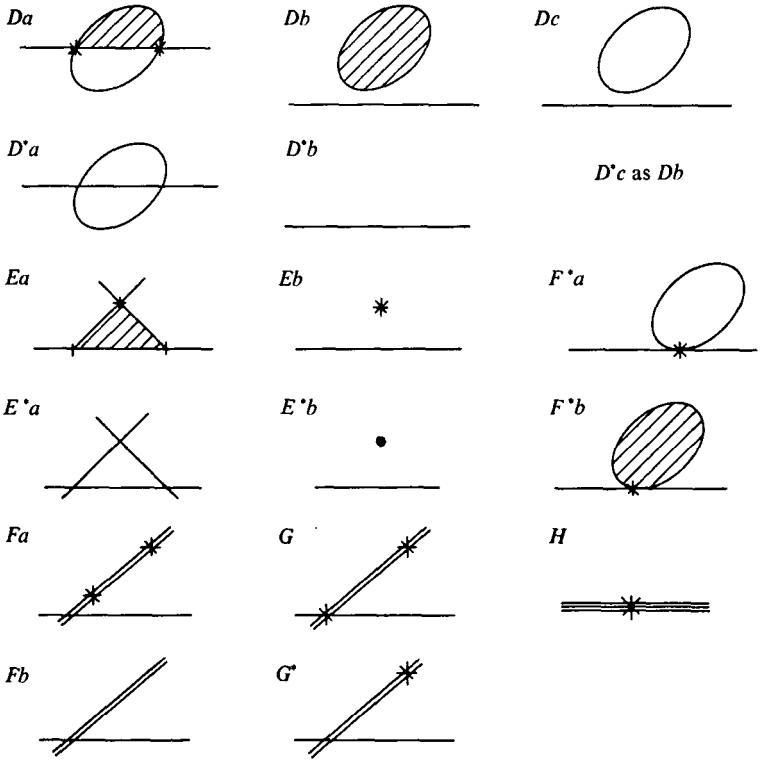


Fig. 3

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We close this section with the observation that a net includes a conic with no real points if and only if *all* conics in the dual net are either non-singular with real points or pairs of real lines. The direct implication can be seen as follows. Taking coordinates so that the conic with no real points is $X^2+Y^2+Z^2$, we see that any conic of the dual net has matrix with trace zero. As the matrix is real, symmetric and non-zero, it then has at least one positive eigenvalue and at least one negative eigenvalue. This implies our assertion.

Alternative approaches. Since the subject matter of this paper is so eminently classical, it may be asked whether the results are not all well known. This appears not to be the case, principally since the production of a complete list of degenerate special cases was never considered a problem of interest. The underlying geometry has indeed been studied – e.g. by Bertini (0) – from the viewpoint of plane sections of the Grassmannian. While the identification of the list above with geometrically special positions of the plane is of interest, and yields new insight, I feel that with this approach it is harder to be certain of having a complete list.

It is also of interest that since I obtained these results, an alternative account (in the complex case) has been given by Iarrobino (unpublished, but see (6)).

From the classical works, one can note two further invariant methods of associating nets with cubic curves. First, each cubic curve $f(x,y,z)=0$ has a polar net

$$\lambda\,\partial f/\partial x+\mu\,\partial f/\partial y+\nu\,\partial f/\partial z=0$$

and in fact a net of type A comes from a unique non-singular curve. Second, a net $\lambda f_0+\mu f_1+\nu f_2=0$ determines a Jacobian curve $\partial(f_0,f_1,f_2)/\partial(x,y,z)=0$. This seems more interesting, since – by definition – it is the locus of points whose polar lines with respect to conics of the net have (at least) a common point. It thus includes vertices of singular members of the net, so is closely related to the discriminant cubic. Moreover, the polarity gives an involution on the Jacobian (in general) but (for example) if a point P has the same polar line l for all conics of the net, l must be a component of the Jacobian. Also, any base points belong to this locus: in fact they must be double points of it. Moreover, a line l belongs to the Jacobian envelope of the dual net if and only if there is a line l' conjugate to l for each conic of the dual. But this is equivalent to having $l.l'$ apolar to each conic of the dual, i.e. belonging to the given net. Thus the lines in line-pairs of the net envelope the Jacobian of the dual, which thus coincides with Δ' above.

I will now tabulate what happens for the various special types (the calculations proving these results are essentially trivial). First, we need a notation for types of cubic curve. I will write α for a non-singular curve (α_1 if connected and α_2 with two components), β for a nodal cubic (β_1 acnode, β_2 crunode), γ for cuspidal, δ for conic with chord (δ_1 real intersections, δ_2 not, δ_3 when the conic has no real points), ϵ for conic with tangent, ζ for triangle, η for three concurrent lines (ζ_1,η_1 if all lines real, ζ_2,η_2 if not), θ for line with repeated line and ι for line of multiplicity 3. For η, θ and ι, $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$ are not independent. Routine calculations now yield Tables 3 and 4.

To conclude the tabulation, we discuss invariants. The curve $y^2z=x^3+pxz^2+qz^3$

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has invariant $J = 27q^2/(4p^3 + 27q^2)$. In the real case, for each J there are two types of cubic, depending on the sign of q (if $q = J = 0$, of p). The net

$$\lambda(2xz + y^2) + \mu(2yz) + \nu(-x^2 - 2gy^2 + cz^2 + 2gxz)$$

has invariant $j = c/g^2$; in the real case, we also need the sign of g. Recall that $J \geqslant 0$ for α_1 and $J \leqslant 0$ for α_2; likewise $j > 0$ for Ad, $j < -9$ for Ab and if $0 > j > -9$ we have Aa if $g < 0$ and Ac if $g > 0$ (Fig. 1).

Then for a net with invariant j, both its discriminant and its Jacobian have $J = 27(j+1)^2/j(j+9)^2$.

Finally, a curve with invariant J has polar net with invariant $j = -9/J$.

Table 3

Curve ...	α_1	α_2	β_1	β_2	γ	δ_1	δ_2	δ_3	ϵ	ζ_1	ζ_2
Polar net	See below	Ad	B^*a	B^*c	G	D^*a	D^*b	D^*c	H	E^*a	E^*b

Curve ...	$0 \leqslant J < 1$	$J = 1, q > 0$	$J = 1, q < 0$	$J > 1, q > 0$	$J > 1, q < 0$
Polar net	Ab	Eb	Ea	Aa	Ac

Table 4

Net ...	Aa	Ab	Ac	Ad	Ba	Bc	B^*a	B^*c	C	
Discriminant	α_2	α_2	α_2	α_1	β_1	β_2	β_1	β_2	γ	
Jacobian	α_2	α_2	α_2	α_1	δ_2	δ_1	β_1	β_2	ϵ	
Net	Da	Db	Dc	D^*a	D^*b	D^*c	Ea	Eb	E^*a	E^*b
Discriminant	δ_1	δ_2	δ_2	δ_1	δ_3	δ_2	ζ_1	ζ_2	ζ_1	ζ_2
Jacobian	ζ_1	ζ_2	ζ_2	δ_1	δ_3	δ_2	ζ_1	ζ_2	ζ_1	ζ_2
Net	Fa	Fb	F^*a	F^*b	G	G^*	H	I	I^*	
Discriminant	θ	θ	ϵ	ϵ	θ	θ	ι	–	–	
Jacobian	η_1	η_2	θ	θ	θ	θ	ι	–	ι	

Stratification of the Grassmannian. Our nets can be regarded as 3-dimensional subspaces of the 6-dimensional space with basis $x^2, y^2, z^2, yz, zx, xy$; and hence as points of the 9-dimensional Grassmannian Ω of such subspaces. Our types can be regarded as a partition of Ω. We first observe that this partition into types, or strata, is a regular (i.e. Whitney) stratification. This follows, by a well known argument (see e.g. Lander (3)) from the result just obtained that the action of the general linear group GL_3 on x, y and z – which clearly respects our partition – is transitive on each stratum of dimension less than 9. We now propose to further investigate this stratification and, in particular to determine the closures of the strata.

Since the stratification is regular, the closure of each stratum Y is a union of lower dimensional strata. To determine whether a given X is one of these, we choose a germ

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of submanifold T of Ω, transverse to X at some $P \in X$, and investigate whether T meets Y or indeed, which strata it does meet and how. We choose T as follows. Represent P by a triple (S_1, S_2, S_3) of quadratic forms in x, y, z. Other representatives are given by effecting linear substitutions in S_1, S_2, and S_3. Now the tangent space to the orbit under GL_3 is obtained by applying the operators $x\,\partial/\partial x, x\,\partial/\partial y, \ldots, z\,\partial/\partial z$ to the triple (S_1, S_2, S_3). Choose a basis $\{(A_1^i, A_2^i, A_3^i): 1 \leqslant i \leqslant l\}$ of triples of quadratic forms modulo (*a*) S_1, S_2, S_3 in any positions and (*b*) the above tangent space. Then a suitable T is the family of nets defined by $(S_1 + \Sigma_1^l A_1^i \alpha_i, S_2 + \Sigma A_2^i \alpha_i, S_3 + \Sigma A_3^i \alpha_i)$. Such a family will be referred to as an *unfolding* of P.

Taking the net $(2xz + y^2, 2yz, -x^2)$ of type C, an unfolding T is then found to be

$$(2xz + y^2, 2yz, -x^2 + cz^2 + 2g(xz - y^2)),$$

which was studied above, and its pattern of intersections with the strata determined (Fig. 1).

Next consider the stratum E (or Ea) – a simple example, since it is semi-stable in the sense of Mumford (4). An unfolding is

$$(x^2 + 2\alpha yz, y^2 + 2\beta zx, z^2 + 2\gamma xy)$$

and the type of this net is largely determined by $f = \alpha\beta\gamma$. Calculations show, in fact, that in general the type is given by

f...	$]-\infty, -\frac{1}{8}[$	$-\frac{1}{8}$	$]-\frac{1}{8}, 0[$	0	$]0, 1[$	1	$]1, \infty[$
Type...	Ad	B^*a	Ac	Bc	Ab	Ba	Aa

However, if two of α, β, γ vanish we have type Da. Note moreover that only values of f near 0 determine points close to our stratum. Thus Ea is in the closure of Ab, Ac, Bc and *Da only*.

These – and other calculations below – are completed as follows. Once we have a nodal cubic, see whether the node represents a line-pair or a repeated line, and whether it is a crunode or an acnode. The type is then determined by properties listed earlier (esp. Figs. 2, 3).

Now since the stratification is regular, T is transverse to all strata, at least near P. Thus, for example, Bc meets T in a submanifold of codimension 1, on the two sides of which we have types Ab and Ac.

We obtain an unfolding of Eb from the above by setting

$$x = x' + iy', \quad y = x' - iy', \quad z = z',$$

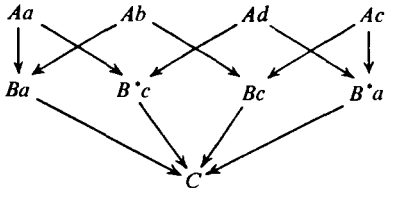
$$\alpha = \alpha' + i\beta', \quad \beta = \alpha' - i\beta', \quad \gamma = \tfrac{1}{2}\gamma',$$

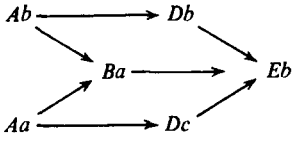
to obtain $(x + iy)^2 + 2(\alpha + i\beta)(x - iy)z$, its conjugate, and $z^2 + \gamma(x^2 + y^2)$ or equivalently, $x^2 - y^2 + 2(\alpha x + \beta y)z$, $2xy + 2(\beta x - \alpha y)z$ and $z^2 + \gamma(x^2 + y^2)$; with $f = \frac{1}{2}\gamma(\alpha^2 + \beta^2)$. Thus near 0, the strata other than A, given by $f = 0$, are the plane $\gamma = 0$ and line $\alpha = \beta = 0$.

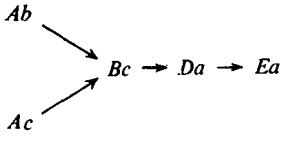
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Along the line we have type D and – by inspection – Db if $\gamma > 0$, Dc if $\gamma < 0$. On $\gamma = 0$ we have type Ba; now if $\gamma > 0$, Ab and if $\gamma < 0$, Aa.

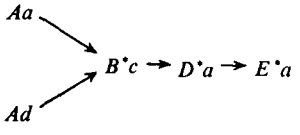
For any stratification, we can represent the closure relations by an ordered graph, with one vertex from each stratum, and an arrow $Y \to X$ if $X \subset \overline{Y}$. We refer to such an arrow as a *specialisation*. From the unfoldings so far, we have the graphs

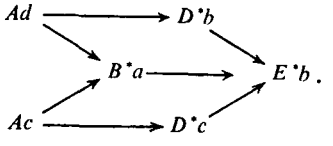






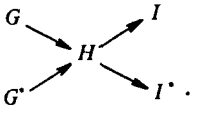
and dualizing yields





These are complete in the sense that for each stratum listed, we have indicated all those which specialize to it (arrows are taken to be transitive, and always proceed from a stratum to one of strictly lower dimension).

Next note, for the lower strata, that we have specializations



For the family $\lambda x^2 + 2\mu xz + \nu\{(z+\alpha y)^2 + 2xy\}$ is of type H when $\alpha = 0$, but when $\alpha \neq 0$ can be rewritten as

$$\alpha^{-2}\nu(x+\alpha z+\alpha^2 y)^2 + (\lambda - \alpha^{-2}\nu)\,x^2 + 2(\mu - \alpha^{-1}\nu)\,xz,$$

which has type G. The family $\lambda x^2 + 2\mu xz + \nu(\beta z^2 + 2xy)$ has type H if $\beta \neq 0$, type I^* if $\beta = 0$. We will thus complete the diagram of specialisations if we unfold G, and show that any stratum of higher dimension $\to G$ or G^*, and hence $\to H$.

A simple calculation shows that

$$\lambda(x^2 + \alpha z^2) + \mu(y^2 + 2\beta xz + \gamma z^2) + \nu(2yz + 2\delta xz)$$

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is an unfolding of G. This has discriminant

$$\Delta = \lambda\mu(\alpha\lambda + \gamma\mu) - \lambda\nu^2 - \mu(\beta\mu + \delta\nu)^2.$$

We first check for double points on the curve $\Delta = 0$. These are values of (λ, μ, ν) where Δ vanishes with its partial derivatives: in particular,

$$\Delta_1 = 2\alpha\lambda\mu - \nu^2 + \gamma\mu^2$$

$$-\tfrac{1}{2}\Delta_3 = \lambda\nu + \delta\mu(\beta\mu + \delta\nu).$$

$$\delta\Delta - \tfrac{1}{2}(\beta\mu + \delta\nu)\,\Delta_3 = \lambda\mu(\alpha\delta\lambda + \gamma\delta\mu + \beta\nu) = \lambda\mu\rho, \text{ say.}$$

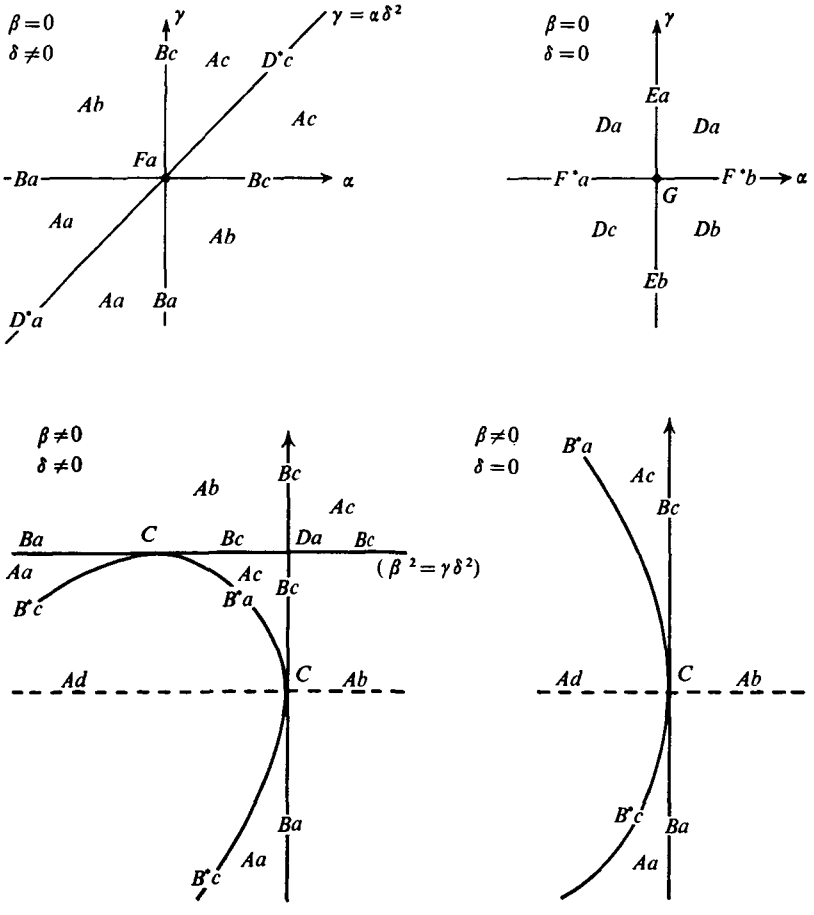


Fig. 4

For a node on $\mu = 0$, $\Delta_1 = 0$ implies $\nu = 0$. We have

P: $(1, 0, 0)$ is a node if and only if $\alpha = 0$.

For a node on $\lambda = 0$, $\mu \neq 0$; $\Delta = 0$ implies $\beta\mu + \delta\nu = 0$, and $\Delta_1 = 0$ shows $\nu^2 = \gamma\mu^2$. Hence $\beta^2 = \gamma\delta^2$.

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Q: $(0, 1, -\eta)$ is a node if and only if $\gamma = \eta^2$, $\beta = \eta\delta$.

For a node on $\nu = 0$ (with $\lambda, \mu \neq 0$), we must have $\Delta_1 = 0$ so $2\alpha\lambda + \gamma\mu = 0$ and $\rho = 0$, so either $\alpha\lambda + \gamma\mu = 0$ or $\delta = 0$. In the first case, $\alpha\lambda = \gamma\mu = 0$; now $\Delta = 0$ shows $\beta^2\mu^3 = 0$, so $\alpha = \beta = \gamma = 0$.

R: $\nu = 0$ is a repeated line if and only if $\alpha = \beta = \gamma = 0$.

If α and γ do not both vanish, $\delta = 0$.

S': $(-\gamma, 2\alpha, 0)$ is a node if $(\alpha, \gamma) \neq (0, 0)$, $\delta = 0$, $\gamma^2 + 4\alpha\beta^2 = 0$.

Otherwise, $\lambda\mu\nu \neq 0$, so $\rho = 0$; also $\delta \neq 0$ (or else we recover case S'). Eliminating λ gives that

$$\sigma = 2\rho\mu - \delta\Delta_1 = \mu(\gamma\delta\mu + \beta\nu) + \nu(\beta\mu + \delta\nu)$$

$$\tau = \rho\nu + \tfrac{1}{2}\alpha\delta\Delta_3 = \nu(\gamma\delta\mu + \beta\nu) - \alpha\delta^2\mu(\beta\mu + \delta\nu)$$

vanish, hence also

$$\nu\sigma - \mu\tau = (\nu^2 + \alpha\delta^2\mu^2)(\beta\mu + \delta\nu).$$

If $\beta\mu + \delta\nu = 0$, $\Delta_3 = 0$ shows that λ or ν vanishes. So $\nu^2 + \alpha\delta^2\mu^2 = 0$, and now eliminating μ and ν from this with σ (or τ) yields $(\gamma\delta - \alpha\delta^3)^2 + 4\alpha\beta^2\delta^2 = 0$. We deduce

S: $(\epsilon\delta^2 - \beta, -\epsilon, \epsilon^2\delta)$ is a node if $\alpha = -\epsilon^2$, $\gamma = 2\beta\epsilon - \epsilon^2\delta^2$.

(This includes S' as the special case $\delta = 0$.) The only possible nodes arise as P, Q, R or S above.

Now P corresponds to the repeated line x^2. It is a crunode if $\gamma > 0$, an acnode if $\gamma < 0$. Q corresponds to the repeated line $(y - \eta z)^2$; it is a crunode if $\gamma + \alpha\delta^2 > 0$, an acnode if $\gamma + \alpha\delta^2 < 0$. S corresponds to $(\epsilon\delta^2 - \beta)(x - \epsilon z)^2 - \epsilon(y - \epsilon\delta z)^2$ which is a line pair unless $\epsilon = 0$ (i.e. $\alpha = 0$) or $\beta = \epsilon\delta^2$. It is a crunode if $\epsilon^2\delta^2 - \epsilon\beta > 0$, an acnode if $\epsilon^2\delta^2 - \epsilon\beta < 0$.

We can now easily determine which types occur in the unfolding (complex case):

A	$\alpha \neq 0$, $\beta^2 \neq \gamma\delta^2$, $(\gamma - \alpha\delta^2)^2 + 4\alpha\beta^2 \neq 0$		
B^*	$(\gamma - \alpha\delta^2)^2 + 4\alpha\beta^2 = 0$, $\alpha \neq 0$, $\beta \neq 0$, $\beta^2 \neq \gamma\delta^2$		
B	$\alpha = 0$, $\gamma \neq 0$, $\beta^2 \neq \gamma\delta^2$	or	$\beta^2 = \gamma\delta^2$, $\alpha \neq 0$, $\gamma + \alpha\delta^2 \neq 0$, $\delta \neq 0$
C	$\alpha = \gamma = 0$ $\beta \neq 0$	or	$\beta^2 = \gamma\delta^2 \neq 0$, $\gamma + \alpha\delta^2 = 0$
D	$\alpha = 0$, $\beta^2 = \gamma\delta^2 \neq 0$	or	$\beta = \delta = 0$, $\alpha\gamma \neq 0$
D^*	$\beta = 0$ $\gamma = \alpha\delta^2 \neq 0$		
E	$\alpha = \beta = \delta = 0$ $(\gamma \neq 0)$,	F	$\alpha = \beta = \gamma = 0$ $(\delta \neq 0)$
F^*	$\beta = \gamma = \delta = 0$ $(\alpha \neq 0)$.		

Type E^* does not occur.

The considerations above also determine the various types in the real case. We present the results graphically, drawing the intersections of the strata with planes where β and δ are constant. The only strata (of dimension > 5) which do NOT $\to G$ are D^*b, E^*a, E^*b and Fb.

If $\beta\delta \neq 0$, $(\gamma - \alpha\delta^2)^2 + 4\alpha\beta^2 = 0$ is a parabola, and $\alpha = 0$, $\beta^2 = \gamma\delta^2$ are both tangent lines to it. As δ tends to 0, the axis of the parabola swings up to the horizontal, and

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the tangent $\beta^2 = \gamma\delta^2$ disappears. As β tends to 0, the parabola shrinks to the repeated line $\gamma = \alpha\delta^2$ (Fig. 4).

Since this also yields unfoldings of Fa, F^*a and F^*b, we have completed our analysis. The diagram of specializations is made up as follows (I picture the following as rolled up into a cylinder):

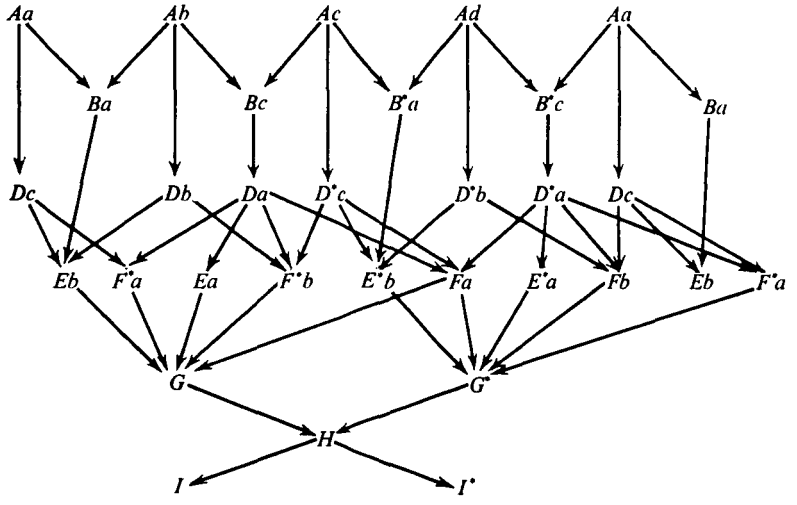
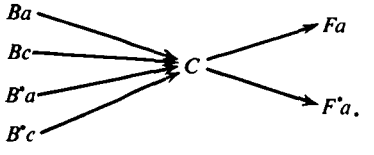


Fig. 5

together with the edges involving C:



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