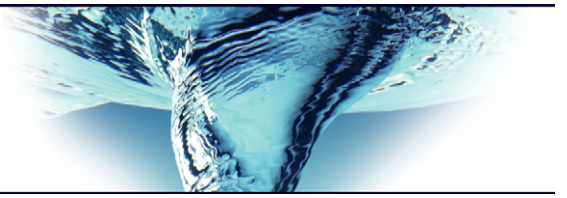
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Reflections on dissipation associated with thermal convection

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Buoyancy-driven convection is modelled using the Navier–Stokes and entropy equations. It is first shown that the coefficient of heat capacity at constant pressure, c_p, must in general depend explicitly on pressure (i.e. is not a function of temperature alone) in order to resolve a dissipation inconsistency. It is shown that energy dissipation in a statistically steady state is the time-averaged volume integral of $-\mathrm{D}P/\mathrm{D}t$ and not that of $-\alpha T(\mathrm{D}P/\mathrm{D}t)$. Secondly, in the framework of the anelastic equations derived with respect to the adiabatic reference state, we obtain a condition when the anelastic liquid approximation can be made, $\gamma - 1 \ll 1$, independent of the dissipation number.

Key words: convection, general fluid mechanics, geophysical and geological flows

1. The general expression for dissipation

Convection in the geophysical context is often associated with compressible effects (as early as in the work of Carnot 1824) and simultaneously with a significant energy dissipation (see Backus 1975), of the same order of magnitude as the heat flux. In the case of convection in the Earth's mantle, it is important to determine the distribution of dissipation as it is strongly coupled with effective viscosity and plays a role in the structure of convection (see Bercovici 1996; Tackley 1996), while the amount of dissipation in the Earth's core is a direct measure of the energy available to the geodynamo (see Buffett 2002; Christensen & Tilgner 2004). In this paper we shall be interested specifically in dissipation in 'liquids', defined as those fluids with a small product αT compared to unity, where α is the coefficient of thermal expansion at constant pressure and T is the thermodynamic temperature. This product is unity for perfect and semi-perfect gases, but is usually very small for liquids, of order 0.05 both in the outer core of the Earth and in the mantle, which are hence both usually considered as 'liquids' from the perspective of convection.

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Let us consider a simple configuration of a region of Newtonian fluid with no internal heat production, bounded with solid walls or stress-free boundaries and prescribed temperatures or heat fluxes within a uniform, constant gravity field $\boldsymbol{g}$. It might be a simple Rayleigh–Bénard configuration with an imposed temperature difference. It might also be more complex; we simply require that the boundary exerts no work on the fluid region and that a statistically steady state can be reached so that the net heat input is zero. Density ρ, velocity $\boldsymbol{u}$, pressure P, temperature T and entropy s satisfy the continuity, Navier–Stokes and entropy (or heat transfer) equations

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0, \tag{1.1}$$

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} P + \rho \boldsymbol{g} + \boldsymbol{\nabla} \cdot \boldsymbol{\tau}, \tag{1.2}$$

$$\rho T \frac{\mathrm{D}s}{\mathrm{D}t} = \dot{\boldsymbol{\epsilon}} : \boldsymbol{\tau} - \boldsymbol{\nabla} \cdot \boldsymbol{\phi}, \tag{1.3}$$

where $\boldsymbol{\phi} = -k \boldsymbol{\nabla} T$ is the conduction heat flux (with thermal conductivity k), $\boldsymbol{\tau}$ is the stress tensor and $\dot{\boldsymbol{\epsilon}}$ is the tensor of the rate of deformation. Rewriting (1.3) using the thermodynamic relation $T\,\mathrm{d}s = c_p\,\mathrm{d}T - (\alpha T/\rho)\,\mathrm{d}P$ (the well-known equation (A 6) re-derived in the Appendix), with c_p the specific heat capacity at constant pressure, leads to

$$\rho c_p \frac{\mathrm{D}T}{\mathrm{D}t} - \alpha T \frac{\mathrm{D}P}{\mathrm{D}t} = \dot{\boldsymbol{\epsilon}} : \boldsymbol{\tau} - \boldsymbol{\nabla} \cdot \boldsymbol{\phi}. \tag{1.4}$$

A classical expression for the energy dissipation is obtained by integration of (1.4) over the whole fluid domain, under the tentative assumption that c_p can be taken as uniform, and by taking its time-average (see Hewitt, McKenzie & Weiss 1975; McKenzie & Jarvis 1980):

$$\langle \dot{\boldsymbol{\epsilon}} : \boldsymbol{\tau} \rangle = -\left\langle \alpha T \frac{\mathrm{D}P}{\mathrm{D}t} \right\rangle, \tag{1.5}$$

where $\langle \cdot \rangle$ denotes time-averaged volume integral. However, the integration of the dot product of the Navier–Stokes equation (1.2) with velocity $\boldsymbol{u}$, assuming no work is done through the boundary, leads to

$$\langle \dot{\boldsymbol{\epsilon}} : \boldsymbol{\tau} \rangle = -\langle \boldsymbol{u} \cdot \boldsymbol{\nabla} P \rangle = -\left\langle \frac{\mathrm{D}P}{\mathrm{D}t} \right\rangle, \tag{1.6}$$

which is very different from (1.5) for liquids as $\alpha T \ll 1$. The inconsistency is resolved when it is recognized that c_p is generally not uniform nor a function of temperature only. As $c_p = \partial H/\partial T|_P$ by definition, with H the enthalpy, the term $\rho c_p(\mathrm{D}T/\mathrm{D}t)$ in (1.4) can be integrated as follows:

$$\left\langle \rho c_p \frac{\mathrm{D}T}{\mathrm{D}t} \right\rangle = \left\langle \rho \frac{\mathrm{D}H}{\mathrm{D}t} \right\rangle - \left\langle \rho \left.\frac{\partial H}{\partial P}\right|_T \frac{\mathrm{D}P}{\mathrm{D}t} \right\rangle. \tag{1.7}$$

The first term of the right-hand side is always zero (this would be true for any function of state, not just for H) in the statistically steady case. The second term is expressed using (A 7) of the Appendix, so that

$$\left\langle \rho c_p \frac{\mathrm{D}T}{\mathrm{D}t} \right\rangle = -\left\langle (1 - \alpha T) \frac{\mathrm{D}P}{\mathrm{D}t} \right\rangle. \tag{1.8}$$

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By making the hypothesis of a uniform c_p, an important term has been forgotten, thus invalidating expression (1.5). The correct averaging of the heat equation, just like the balance of mechanical energy, leads to (1.6). What we have done is to show that it does not make sense to assume c_p constant when $\alpha T \ll 1$. Equation (A 7) imposes that H depends on pressure; that dependence cannot be made as weak as one wishes and has consequences for the term of the dissipation budget involving c_p treated in (1.7).

Incidentally, we may just write two other exact expressions for dissipation, at least as fundamental as (1.6), obtained from the integration of (1.4) where entropy variations are not expanded in terms of pressure and temperature variations (see Verhoogen 1980, p. 82):

$$\langle \boldsymbol{\tau} : \dot{\boldsymbol{\epsilon}} \rangle = \left\langle \rho T \frac{\mathrm{D}s}{\mathrm{D}t} \right\rangle = - \left\langle \rho s \frac{\mathrm{D}T}{\mathrm{D}t} \right\rangle . \tag{1.9}$$

2. Dissipation in the anelastic approximation (AA)

Equation (1.6) will now be used, within the anelastic approximation (AA) (see Ogura & Phillips 1961), to determine a condition when the anelastic liquid approximation (ALA) (see Braginsky & Roberts 1995; Schubert, Turcotte & Olson 2001) can be made. We assume that convection is sufficiently vigorous to drive the fluid close to an iso-entropy state, commonly called 'adiabatic'. All quantities will be expanded around their depth-dependent adiabatic reference state, denoted with the subscript 0, while fluctuations are indicated by primes. The reference state is dependent on height only and is defined such that the entropy s_0 is uniform and that the hydrostatic equation is satisfied, $\partial P_0/\partial z = -\rho_0 g$, where z is the vertical upward coordinate. Equation (A 6) shows that, when entropy is vigorously mixed, a relation develops between pressure and temperature variations. In addition, when hydrostatics is considered, this leads to the so-called 'adiabatic gradient'

$$\frac{\mathrm{d}T_0}{\mathrm{d}z} = -\frac{\alpha_0 g T_0}{c_{p0}}. \tag{2.1}$$

The main purpose of the anelastic approximation is to eliminate sound waves by replacing the general continuity equation (1.1), by its zeroth-order expansion

$$\boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{u}) = 0. \tag{2.2}$$

The first-order expansion of the entropy equation (1.3) is

$$\rho_0 T_0 \frac{\mathrm{D}s'}{\mathrm{D}t} = \dot{\boldsymbol{\epsilon}} : \boldsymbol{\tau} - \boldsymbol{\nabla} \cdot \left(\boldsymbol{\phi}_0 + \boldsymbol{\phi}' \right) . \tag{2.3}$$

Writing the first term in a conservative form introduces an advection term for the adiabatic temperature profile which is expressed using the adiabatic gradient (first term on the right-hand side of the following equation)

$$\rho_0 \frac{\mathrm{D}\left(T_0 s'\right)}{\mathrm{D}t} = -\frac{\alpha_0 \rho_0 T_0 g}{c_{p0}} u_z s' + \dot{\boldsymbol{\epsilon}} : \boldsymbol{\tau} - \boldsymbol{\nabla} \cdot \left(\boldsymbol{\phi}_0 + \boldsymbol{\phi}' \right) . \tag{2.4}$$

The derivation of the anelastic momentum equation is then based on the expansion of density fluctuations in terms of entropy and pressure fluctuations

$$\rho' = \left. \frac{\partial \rho_0}{\partial s} \right|_P s' + \left. \frac{\partial \rho_0}{\partial P} \right|_s P'. \tag{2.5}$$

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Using the Maxwell relation associated with (A 3) and the expression for the adiabatic temperature gradient obtained from (A 6) for the first term, and hydrostatics for the second, one obtains

$$\rho' = -\frac{\alpha_0 \rho_0 T_0}{c_{p0}} s' - \frac{1}{\rho_0 g}\frac{\partial \rho_0}{\partial z} P'. \tag{2.6}$$

Substituting in the Navier–Stokes equation (1.2) and selecting the lowest-order terms leads to

$$\rho_0 \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} P' + \frac{\alpha_0 \rho_0 T_0 g}{c_{p0}} s' \hat{\boldsymbol{e}}_z + \frac{1}{\rho_0}\frac{\partial \rho_0}{\partial z} P' \hat{\boldsymbol{e}}_z + \boldsymbol{\nabla} \cdot \boldsymbol{\tau}, \tag{2.7}$$

where $\hat{\boldsymbol{e}}_z$ is the vertical unit vector. The first and third terms on the right-hand side can be put together into a single conservative term

$$\rho_0 \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\rho_0 \boldsymbol{\nabla}\left(\frac{P'}{\rho_0}\right) + \frac{\alpha_0 \rho_0 T_0 g}{c_{p0}} s' \hat{\boldsymbol{e}}_z + \boldsymbol{\nabla} \cdot \boldsymbol{\tau}. \tag{2.8}$$

This form of the dynamic equation was derived by Braginsky & Roberts (1995). From this point, we can derive important consequences in terms of energy dissipation. In (2.8) the work of $-\rho_0 \boldsymbol{\nabla}(P'/\rho_0)$ integrated over the volume is zero. Therefore the mechanical work done by the first ($-\boldsymbol{\nabla} P'$) and third ($P' \boldsymbol{\nabla} \rho_0/\rho_0$) terms of (2.7) is opposite to each other. As the expression for the dissipation $-\langle \mathrm{D}P/\mathrm{D}t \rangle$ becomes $-\langle \mathrm{D}P'/\mathrm{D}t \rangle = -\langle \boldsymbol{u} \cdot \boldsymbol{\nabla} P' \rangle$, we have

$$\langle \dot{\boldsymbol{\epsilon}} : \boldsymbol{\tau} \rangle = -\left\langle \frac{1}{\rho_0}\frac{\partial \rho_0}{\partial z} u_z P' \right\rangle = \left\langle \frac{\rho_0 g}{K_{s0}} u_z P' \right\rangle, \tag{2.9}$$

where K_s is the incompressibility at constant entropy. Introducing $c = \sqrt{K_s/\rho}$, the celerity of sound waves, one can finally express the energy dissipation in terms of the correlation of velocity and pressure fluctuations

$$\langle \dot{\boldsymbol{\epsilon}} : \boldsymbol{\tau} \rangle = \left\langle \frac{g}{c_0^2} u_z P' \right\rangle. \tag{2.10}$$

However, energy dissipation must also be equal to the work done by the second term on the right-hand side of (2.7), in agreement with the expression for dissipation associated with (2.4) and the general expression (1.9),

$$\langle \dot{\boldsymbol{\epsilon}} : \boldsymbol{\tau} \rangle = \left\langle \frac{\alpha_0 \rho_0 T_0 g}{c_{p0}} u_z s' \right\rangle, \tag{2.11}$$

now in terms of the correlation between velocity and entropy fluctuations. Expanding entropy fluctuations in terms of temperature and pressure fluctuations, using (A 6), leads to

$$\langle \dot{\boldsymbol{\epsilon}} : \boldsymbol{\tau} \rangle = \left\langle \alpha_0 \rho_0 g u_z T' \right\rangle - \left\langle \frac{\alpha_0^2 T_0 g}{c_{p0}} u_z P' \right\rangle. \tag{2.12}$$

3. Energy validation of the anelastic liquid approximation (ALA)

Comparing (2.10) and (2.12) shows the relative importance of the pressure and the temperature terms in (2.12). The coefficient within the pressure/velocity correlation

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term can be written

$$\frac{\alpha_0^2 T_0 g}{c_{p0}} = \frac{\alpha_0^2 c_0^2 T_0}{c_{p0}} \frac{g}{c_0^2} = (\gamma_0 - 1)\frac{g}{c_0^2}, \tag{3.1}$$

where $\gamma = c_p/c_v$ is the ratio of specific heat capacities at constant pressure and at constant volume. The second equality in (3.1) can be retrieved from the general Mayer relation $c_p - c_v = -(T/\rho^2)\,(\partial P/\partial T)_{|\rho}\,(\partial \rho/\partial T)_{|P}$. In the case of a constant value of γ, independent of temperature and pressure (one must here verify that there exist equations of state with constant ratio of specific heat coefficients: this is indeed the case for the class of ideal gases, with arbitrary polytropic index), (3.1) and (2.10) lead immediately to the fraction of pressure/velocity correlations to dissipation in (2.12)

$$\left\langle \frac{\alpha_0^2 T_0 g}{c_{p0}} u_z P' \right\rangle = (\gamma_0 - 1)\,\langle \dot{\boldsymbol{\epsilon}} : \boldsymbol{\tau} \rangle\,. \tag{3.2}$$

It is hence safe from the point of view of energy dissipation to express entropy fluctuations s' (see (2.11)) in terms of temperature fluctuations only T' (see (3.2)) in (2.8) under the condition

$$\gamma - 1 = \frac{c^2 \alpha^2 T}{c_p} \ll 1. \tag{3.3}$$

While that condition was obtained rigorously only in the case of constant γ, we expect it to apply in the general case. Furthermore, provided there are reasonable correlations between u_z and P' as well as between u_z and T', one can expect the condition (3.3) to be the correct condition of application of the anelastic liquid approximation. It then follows that, while entropy fluctuations are replaced by temperature fluctuations in (2.8), $c_{p0}T'$ must be substituted for $T_0 s'$ in (2.4) to ensure a consistent energy balance. This leads to the following anelastic liquid approximation equations, derived from the anelastic approximation (2.2), (2.4) and (2.8):

$$\boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{u}) = 0, \tag{3.4}$$

$$\rho_0 \frac{\mathrm{D}\left(c_{p0} T'\right)}{\mathrm{D}t} = -\alpha_0 \rho_0 g u_z T' + \dot{\boldsymbol{\epsilon}} : \boldsymbol{\tau} - \boldsymbol{\nabla} \cdot (\boldsymbol{\phi}_0 + \boldsymbol{\phi}'), \tag{3.5}$$

$$\rho_0 \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\rho_0 \boldsymbol{\nabla}\left(\frac{P'}{\rho_0}\right) + \alpha_0 \rho_0 g T' \hat{\boldsymbol{e}}_z + \boldsymbol{\nabla} \cdot \boldsymbol{\tau}. \tag{3.6}$$

Let us mention here, for completeness, that a so-called truncated anelastic liquid approximation (TALA) has been used (for instance in Tan & Gurnis 2007 and Jarvis & McKenzie 1980) in which the contribution of pressure fluctuations to density fluctuations is also neglected in (2.6). As a consequence, the term $-\rho_0 \boldsymbol{\nabla}\,(P'/\rho_0)$ in (3.6) above is replaced by $-\boldsymbol{\nabla} P'$. However, as pointed out in Leng & Zhong (2008), this change introduces an imbalance between energy dissipation calculated from the dynamical equation and heat dissipation in the thermal equation. Such a TALA formulation should be avoided when the calculation of energy dissipation is an issue.

Our condition (3.3) is in contrast to a scaling law usually derived (see Spiegel & Veronis 1960; Braginsky & Roberts 1995; Anufriev, Jones & Soward 2005) from the anelastic liquid approximation momentum equation (3.6) expressing the typical magnitude of pressure fluctuations as a function of temperature fluctuations, $P' \sim \alpha_0 \rho_0 g L T'$, where L is the typical length scale of the fluid domain. This leads to

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the following relation:

$$\frac{\alpha_0^2 T_0 g}{c_{p0}} u_z P' \sim (\alpha_0 T_0) \frac{\alpha_0 g L}{c_{p0}} \left[\alpha_0 \rho_0 g u_z T'\right], \tag{3.7}$$

which would imply that the anelastic liquid approximation is valid when $(\alpha_0 T_0) D \ll 1$, where the dissipation number is $D = \alpha_0 g L / c_{p0}$. This heuristic scaling law (3.7) involving D, hence L and g, is not compatible with the exact relation (3.2) from which the condition (3.3) was obtained solely in terms of fluid properties. Actually, from (2.10) and (2.12), one can infer a more adequate scaling of the pressure fluctuations than $P' \sim \alpha_0 \rho_0 g L T'$, which is valid in the general anelastic approximation (liquid or not):

$$P' \sim K_{t0} \alpha_0 T', \tag{3.8}$$

where $K_t = K_s / \gamma$ is the incompressibility at constant temperature.

4. Discussion

Let us now discuss the implications of the results derived in this paper. Concerning the 'exact' model (1.1)–(1.3), we have shown that it is crucial to use thermodynamically consistent thermo-physical coefficients. Specifically, considering a fluid with constant heat capacity c_p and small product αT of temperature by thermal expansion coefficient leads to a severe inconsistency when determining the energy dissipation. Whatever αT, dissipation is always equal to $\langle -\mathrm{D}P/\mathrm{D}t \rangle$. Once compressibility effects have been taken into account in an anelastic model, no energy inconsistency can arise even in the case when constant uniform values for c_{p0} and other coefficients are used in (2.4) and (2.8). Using the correct expression for dissipation, within the model of the anelastic approximation, we have then been able to obtain the condition $\gamma - 1 \ll 1$ for the anelastic liquid approximation to be valid. That condition is only dependent on materials properties, not on gravity or length scale. Note that the condition (3.3) can be expressed with the Grüneisen parameter, $\Gamma = \alpha K_s / (\rho c_p)$ used in solid-state physics, as $c^2 \alpha^2 T / c_p = \gamma - 1 = \Gamma \alpha T$. The Grüneisen parameter is close to unity in the mantle and in the Earth's core and so is the ratio of heat capacities. This is the reason why the condition (3.3) is close to $\alpha T \ll 1$ in practice concerning the dynamics of the deep Earth. For the Earth's liquid outer core and for the mantle, the dimensionless number in (3.3) is of order 0.05 justifying the anelastic liquid approximation.

From the correct expression for dissipation (1.6), we have obtained (3.5) and (3.6), in the anelastic liquid approximation, which both lead to the following expression for dissipation:

$$\langle \dot{\boldsymbol{\epsilon}} : \boldsymbol{\tau} \rangle = \left\langle \alpha_0 \rho_0 g T' u_z \right\rangle. \tag{4.1}$$

Surprisingly, Hewitt *et al.* (1975) reach the same conclusion when starting from their expression (1.5):

$$\begin{aligned} \langle \dot{\boldsymbol{\epsilon}} : \boldsymbol{\tau} \rangle &= - \langle \alpha T \boldsymbol{u} \cdot \boldsymbol{\nabla} P \rangle \simeq - \langle \alpha T \rho_0 g u_z \rangle \\ &\simeq \left\langle \alpha_0 T' \rho_0 g u_z \right\rangle. \end{aligned} \tag{4.2}$$

This expression is indeed approximately correct for liquids and the apparently very different expressions (1.6) and (1.5) are nearly equal in the anelastic liquid approximation $\langle \mathrm{D}P/\mathrm{D}t \rangle \approx \langle \alpha T \mathrm{D}P/\mathrm{D}t \rangle$. This is due to the large cancellations in (1.6)

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which allow the same result to be reached when multiplied by a small term αT, with appropriate fluctuations (here $\alpha_0 T'$ to the leading order). We only expect significantly different values from expressions (1.6) and (1.5) when αT is neither too small nor too close to unity, where (1.6) and (1.5) become similar.

To conclude, we hope that we have contributed to a clarification of the expression for dissipation in a convective system and of the anelastic liquid approximation. We have not systematically investigated the consequences of these findings in the different possible fields of application: dynamics of the Earth's core and mantle, and dynamics of giant planets, ice planets, super-Earth exoplanets. Regarding the application of the anelastic liquid approximation for the terrestrial planets obeying a generic Murnaghan equation of state (see Murnaghan 1951), we have no indication that the new criterion (3.3), $\gamma - 1 \ll 1$, differs significantly from the classical criterion $\alpha TD \ll 1$. According to Murnaghan's equation of state, the coefficient of expansion α decreases strongly when density ρ is increased, $\alpha \sim \rho^{-n}$, while the coefficient of incompressibility increases with ρ, $K_s \sim \rho^n$, with a value of n around 3. We have considered the case of silicate planets of from one up to possibly ten Earth masses. The parameter $\gamma - 1$ is always less than 0.08, and that value decreases very quickly away from the surface to much lower values in the bulk of the mantle (decreasing also when planetary mass increases). Provided the general anelastic approximation can be applied to mantle dynamics, the anelastic liquid approximation is indeed well justified for the mantle dynamics of the Earth and even better justified for super-Earths.

Appendix. Useful thermodynamic relations

From the Gibbs equation expressed in terms of the Gibbs free energy G

$$\mathrm{d}G = S\,\mathrm{d}T - \frac{1}{\rho}\,\mathrm{d}P, \tag{A 1}$$

the following Maxwell relation is derived, expressing the partial derivative of entropy with respect to pressure at constant temperature:

$$\left.\frac{\partial S}{\partial P}\right|_T = -\left.\frac{\partial V}{\partial T}\right|_P = -\frac{\alpha}{\rho}. \tag{A 2}$$

The partial derivative of entropy with respect to temperature at constant pressure is obtained as follows. Two expression for $\mathrm{d}H$ are first written, one from Gibbs relation, the other from the definition of c_p:

$$\mathrm{d}H = T\,\mathrm{d}S + \frac{1}{\rho}\,\mathrm{d}P, \tag{A 3}$$

$$\mathrm{d}H = c_p\,\mathrm{d}T + \left.\frac{\partial H}{\partial P}\right|_T \mathrm{d}P. \tag{A 4}$$

The difference of these two expressions leads to

$$T\,\mathrm{d}S = c_p\,\mathrm{d}T + \left(\left.\frac{\partial H}{\partial P}\right|_T - \frac{1}{\rho}\right)\mathrm{d}P. \tag{A 5}$$

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This shows that the partial derivative of entropy with respect to temperature at constant pressure is c_p/T. Along with (A 2), this leads to

$$\mathrm{d}S = \frac{c_p}{T}\,\mathrm{d}T - \frac{\alpha}{\rho}\,\mathrm{d}P. \tag{A 6}$$

Comparing (A 5) and (A 6) leads to an important expression used in this paper:

$$\left.\frac{\partial H}{\partial P}\right|_T = \frac{1-\alpha T}{\rho}. \tag{A 7}$$

The partial derivative of this last equation with respect to temperature at constant pressure leads to expressions for the derivative of c_p with respect to pressure at constant temperature

$$\left.\frac{\partial c_p}{\partial P}\right|_T = \left.\frac{\partial}{\partial T}\left(\frac{1-\alpha T}{\rho}\right)\right|_P, \tag{A 8}$$

$$= -\frac{T}{\rho}\left[\alpha^2 + \left.\frac{\partial \alpha}{\partial T}\right|_P\right]. \tag{A 9}$$

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