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A MARKOV-MODULATED GROWTH COLLAPSE MODEL

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We consider a growth collapse model in a random environment for which the input rates might depend on the state of an underlying irreducible Markov chain and at state change epochs there is a possible downward jump to a level that is a random fraction of the level just before the jump. The distributions of these jumps are allowed to depend on both the originating and target states. Under a very weak assumption we develop an explicit formula for the conditional moments (of all orders) of the time stationary distribution. We then consider special cases and show how to use this result to study a growth collapse process in which the times between collapses have a phase-type distribution.

1. INTRODUCTION

In this article we consider a Markov-modulated growth collapse model, represented by a continuous-time Markov process $\{(W_t, J_t)\}_{t\geq 0}$. Here, $J = \{J_t |\ t \geq 0\}$ is a right-continuous irreducible continuous-time Markov chain with state space $\{1, \ldots, K\}$, rate transition matrix $Q = (q_{ij})$ with $q_i = -q_{ii}$, and stationary distribution $\pi = (\pi_i)$. As long as J is in state i, the right-continuous process $W(t)$ increases at a linear rate $r_i \geq 0$. Whenever there is a state change from, say, i to j, the process is reduced to a fraction of what it was before the jump. This fraction is independent of the history and is distributed like some random variable X_{ij} satisfying $\mathbb{P}[0 \leq X_{ij} \leq 1] = 1$; Namely if at time t there is a state change from i to j, then W_t/W_{t-} is distributed like X_{ij}.

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Our primary concern is the derivation of formulas for the moments associated with (W_t, J_t) and its stationary version (W_*, J_*). The present article is divided as follows. In the second section we first give general (necessary and sufficient) conditions that ensure convergence of the process (W_t, J_t) to a stationary limit. In Section 3 we show how the moments of W_* can be calculated, after which we apply similar ideas to the derivation of the transient moments $\mathbb{E}(W_t^n)$ in Section 4. Finally, we describe the special case, where the time between the jumps has a phase-type distribution and the jumps are all of the same type.

The monograph [4] provides an overview and a toolbox for studying piecewise deterministic Markov processes (PDMPs), a class of stochastic models to which our process belongs. In [2], some results for a general type of a Markovian growth collapse model are given, including a Markov-modulated case different from the one investigated here. More general processes are considered in [6]. In [9] formulas for the nonmodulated case are given, including results also for noninteger and transient moments. Further results, mainly focusing on the stationary case, can also be found in [1,5,7,10]. The latter articles use growth collapse processes with multiplicative jumps to describe the window size process for congestion avoidance in the TCP data transmission protocol. Instead of looking at the continuous-time process W_t, one might also study the behavior of the embedded processes just before and immediately after a jump. Related to this approach are stochastic relations that have been investigated in [3,11]. In the recent article [8], criteria for stability and formulas for moments for stationary growth collapse models in a more general setting are developed.

2. SOME PRELIMINARIES

Consider the following.

Condition 1: $\exists i, j$ such that $q_{ij} > 0$ and $\mathbb{P}[X_{ij} = 1] < 1$.

THEOREM 1: *Under Condition* 1, *the process* (W_t, J_t) *has a well-defined time stationary distribution that is also the limiting distribution, independent of initial conditions.*

PROOF: According to Condition 1, there are i and j such that $q_{ij} > 0$ and $\mathbb{P}[X_{ij} = 1] < 1$. Consider a growth collapse process $(\overline{W}_t, J_t)_{t\geq 0}$ with the following properties. For $t = 0$, we have $\overline{W}_0 = W_0$ and the process increases with rate $r = \max\{r_1, \ldots, r_K\}$. The process jumps at times $(\overline{T}_n)_{n=1,2,\ldots}$, when J_t moves from state i to state j. At such a jump time $t = \overline{T}_n$, the process jumps from $\overline{W}_{t-}$ to $\overline{W}_t = X_n \cdot \overline{W}_{t-}$, where $X_n = W_t / W_{t-}$.

The sequence $(Y_n)_{n=1,2,\ldots}$, defined by $Y_n = \overline{W}_{\overline{T}_n-} - \overline{W}_{\overline{T}_{n-1}} = r(\overline{T}_n - \overline{T}_{n-1})$, is a sequence of independent and identically distributed (i.i.d.) random variables independent of $(X_n)_{n=1,2,\ldots}$, in fact having a phase-type distribution. It follows that $(X_n, Y_n)_{n=1,2,\ldots}$ is i.i.d. and hence stationary and ergodic and we can apply Corollary 1 in [8] to deduce the existence of a stationary/limiting distribution for $\overline{W}_t$.

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Since $W_t \le \overline{W}_t$, it follows that the (Markov) process $\{(W_t, J_t) |\ t \ge 0\}$ is tight. It is easy to check that the difference between a process that starts with $W_0 = 0$ and that of a process that starts with $W_0 = x$ converges almost surely to zero as $t \to \infty$, since it is less than or equal to $x \prod_{n=1}^{N_{ij}(t)} X_n$, where $N_{ij}(t)$ is the number of (i,j) transitions that occurred by time t. Additionally, it is clear that any set of the form $[0, \epsilon]$ is accessible by $\overline{W}_t$ and, thus, any set of the form $[0, \epsilon] \times \{j\}$ is accessible by (W_t, J_t); thus, a unique stationary/limiting distribution for the process $\{(W_t, J_t) |\ t \ge 0\}$ exists. ■

Note that Condition 1 is the weakest possible, in the sense that as soon as it is not fulfilled (and $r_i > 0$ at least for one i), W_t will not jump at all and, hence, $W_t \to \infty$.

The extended generator (in the sense of [4]) of the Markov process (W, J) is of the form

$$\mathcal{A}f(x,i) = r_i f'(x,i) + \sum_{\substack{j=1 \\ j \ne i}}^{K} q_{ij} (\mathbb{E} f(xX_{ij}, j) - f(x,i))$$

with the convention that $X_{ii} \equiv 1$ and assuming that $f(\cdot,\cdot)$ is in the domain of $\mathcal{A}$. Bearing in mind that $\sum_{j=1}^{K} q_{ij} = 0$, we can write this as

$$\mathcal{A}f(x,i) = r_i f'(x,i) + \sum_{j=1}^{K} q_{ij} \mathbb{E} f(xX_{ij}, j).$$

THEOREM 2: *For every $a \ge 0$, the function $f(x,i) = c_i x^\alpha$ is in the domain of $\mathcal{A}$ and, thus, with $a_{ij}(\alpha) = \mathbb{E} X_{ij}^\alpha$, we have that*

$$\mathcal{A}f(x,i) = \alpha r_i c_i x^{\alpha-1} + x^\alpha \sum_{j=1}^{K} q_{ij} a_{ij}(\alpha) c_j. \tag{1}$$

If we let $A(\alpha) = a_{ij}(\alpha)$ and $A \circ B = (a_{ij} b_{ij})$ for any two matrices A and B and $D_r = \mathrm{diag}(r_1, \ldots, r_K)$ as well as $c = (c_i)$, then we can write the generator in vector form as

$$\mathcal{A}f(x) = \alpha x^{\alpha-1} D_r c + x^\alpha Q \circ A(\alpha) c, \tag{2}$$

where $f(x) = (f(x,1), \ldots, f(x,K))^T$ and $\mathcal{A}$ acts componentwise.

PROOF: Following the general theory of PDMPs (see [4]), a measurable function $f : [0,\infty) \times \mathbb{K} := \{1, \ldots, K\} \to \mathbb{R}$ belongs to the domain of $\mathcal{A}$ if $t \mapsto f(x + r_i t, i)$ is absolutely continuous for all $x \ge 0$ and $i \in \mathbb{K}$, and the integrability condition

$$\mathbb{E}\left[\sum_{k=1}^{N_t} \left| f(W_{T_k-}, J_{T_k-}) - f(W_{T_k}, J_{T_k}) \right| \Big| W_0 = w, J_0 = j \right] < \infty \tag{3}$$

holds for all w, j and all $t \ge 0$, where N_t denotes the number of jumps until time t and the T_i are the successive jump times.

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The function $t \mapsto c_i(x + r_i t)^\alpha$ is clearly absolutely continuous. Given that $W_0 = x$, we have $W_t \le x + rt$, with $r = \max\{r_1, \ldots, r_K\}$. Let $z(x,t)$ denote the left-hand side of (3); then

$$z(x,t) \le 2c\mathbb{E}\left[\sum_{k=1}^{N_t} W_{T_k-}^\alpha \Big| W_0 = w, J_0 = j\right] \le 2c(x+rt)^\alpha \mathbb{E}[N_t],$$

where $c = \max\{c_1, \ldots, c_K\}$. The last term is certainly finite for all $x \ge 0$ and all $t \ge 0$. ■

3. MOMENTS

We define the vector of moments $\xi^n = (\xi_1^n, \xi_2^n, \ldots, \xi_K^n)$ by $\xi_i^n = \mathbb{E}[W_*^n 1_{\{J^*=i\}}]$. Recall that the matrix A is given by $a_{ij}(\alpha) = \mathbb{E}X_{ij}^\alpha$.

LEMMA 1: *Let $D_q = \mathrm{diag}(q_1, \ldots, q_K)$. Under Condition 1, the matrix $Q \circ A(\alpha)$ is nonsingular for every positive α and $(-Q \circ A(\alpha))^{-1} \ge D_q^{-1}$.*

PROOF: Denoting $p_{ij} = q_{ij}/q_i$ for $i \ne j$, $p_{ii} = 0$, then $Q = D_q(P - I)$ where I is the identity matrix. Additionally, $Q \circ A(\alpha) = D_q(P \circ A(\alpha) - I)$, as $a_{ii}(\alpha) = 1$ and $p_{ii} = 0$. For a fixed α, let us define a new Markovian transition matrix with the extra absorbing state σ as follows:

$$\tilde{p}_{ij} = \begin{cases} p_{ij}a_{ij}(\alpha), & 1 \le i,j \le K \\ 1 - \sum_{j=1}^K p_{ij}a_{ij}(\alpha), & 1 \le i \le K, j = \sigma \\ 0, & i = \sigma, 1 \le j \le K \\ 1, & i = j = \sigma. \end{cases} \tag{4}$$

If we show that for the discrete Markov chain with transition matrix $\tilde{P} = (\tilde{p}_{ij})$ the states $1, \ldots, K$ are all transient, then if we denote the (i,j)th coordinate of $\tilde{P}^k$ by $\tilde{p}_{ij}^k$, then for every $(i,j) \ne (\sigma,\sigma)$ we have that $\tilde{p}_{ij}^k \to 0$ as $k \to \infty$. Since the states $1, \ldots, K$ cannot be reached from σ, it readily follows that for $1 \le i,j \le K$, we have that $(P \circ A(\alpha))^k = (\tilde{p}_{ij}^k)$ and thus $(P \circ A(\alpha))^k \to 0$ as $k \to \infty$. Therefore, $I - P \circ A(\alpha)$ is nonsingular, with

$$(I - P \circ A(\alpha))^{-1} = \sum_{k=0}^\infty (P \circ A(\alpha))^k \ge I, \tag{5}$$

and, thus, $-Q \circ A(\alpha)$ is also nonsingular, with

$$(-Q \circ A(\alpha))^{-1} = (I - P \circ A(\alpha))^{-1} D_q^{-1} \ge D_q^{-1}. \tag{6}$$

To argue that indeed the states $1, \ldots, K$ are transient in the chain with transition matrix $\tilde{P}$, we observe that since $\tilde{p}_{ij} \le p_{ij}$ for $1 \le i,j \le K$ and since the chain with transition

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matrix P is irreducible, the submatrix of $\tilde{P}$ associate with any strict subset of $1,\ldots,K$ cannot be stochastic, since this would imply that there are states that cannot be reached in the original chain which contradicts irreducibility. Since there exists a pair $1 \le i,j \le K$ for which $q_{ij} > 0$, hence $p_{ij} > 0$, for which $P[X_{ij} = 1] < 1$, it follows that for all $\alpha > 0$, $a_{ij}(\alpha) = \mathbb{E}X_{ij}^{\alpha} < 1$ and, thus, $\tilde{p}_{ij} < p_{ij}$. Thus, $P \circ A(\alpha)$, which is the submatrix of $\tilde{P}$ associate with the states $1,\ldots,K$, cannot be stochastic either. This means that, for $\tilde{P}$, every state other than σ must belong to an open class and is thus transient. ■

THEOREM 3: *If Condition* 1 *is fulfilled, then*

$$(\xi^n)^T = n!\pi^T \prod_{k=1}^{n} D_r(-Q \circ A(k))^{-1}, \qquad n \ge 1. \tag{7}$$

PROOF: If we denote by (W_*, J_*) a random pair having the stationary distribution, then it is well known that for every f in the domain of $\mathcal{A}$, we have that $\mathbb{E}[\mathcal{A}f(W^*, J^*)] = 0$. Since $\xi_i^0 = \pi_i$, it immediately follows from Theorem 2 and (2) that for $f(x,i) = c_i x^n$, where n is a nonnegative integer and c_i are arbitrary,

$$0 = \mathbb{E}[\mathcal{A}f(W_*, J_*)] = n(\xi^{n-1})^T D_r c + (\xi^n)^T Q \circ A(n)c. \tag{8}$$

Thus, if we take any square matrix C, then clearly the following system of equations is also valid:

$$n(\xi^{n-1})^T D_r C + (\xi^n)^T Q \circ A(n)C = 0. \tag{9}$$

Since, by Lemma 1, $Q \circ A(n)$ is nonsingular, it follows by taking $C = (-Q \circ A(n))^{-1}$ that the following recursion is valid:

$$(\xi^n)^T = n(\xi^{n-1})^T D_r(-Q \circ A(n))^{-1}, \tag{10}$$

from which (7) immediately follows. ■

Remark 1: We note that one could generalize the setup in a way that allows a geometric number of jumps down while in state i, but assuming that the jumps rate is some $\lambda_i < q_i$ with $p_{ii} = 1 - \lambda_i/q_i$. However, this effect can also be achieved in the above framework as well by simply allowing two different states, say i and k to have $q_{ij} = q_{kj}$ and $r_i = r_k$ and $q_{ik} = q_{ki} = q_i - \lambda_i$, where $\lambda_i = \lambda_k$.

4. TRANSIENT MOMENTS

As in the stationary case, we define the vector $\xi^n(t) = (\xi_1^n(t),\ldots,\xi_K^n(t))$ of transient moments, with components $\xi_i^n(t) = \mathbb{E}[W_t^n 1_{\{J_t=i\}}]$. It follows as in (8) that

$$\mathbb{E}[\mathcal{A}f(W_t, J_t)] = n(\xi^{n-1}(t))^T D_r c + (\xi^n(t))^T Q \circ A(n)c. \tag{11}$$

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Dynkin's martingale is given by $f(W_t, J_t) - f(W_0, J_0) - \int_0^t \mathcal{A}f(X_s, J_s)\, ds$ for functions f in the domain of $\mathcal{A}$ (cf. [4]). In particular, if $f(x, i) = c_i x^n$, then

$$\mathbb{E}[c_{J_t} W_t^n] = \mathbb{E}[c_{J_0} W_0^n] + \int_0^t \mathbb{E}[\mathcal{A}f(W_s, J_s)]\, ds.$$

It follows that for any K-vector c

$$\xi^n(t)^T c = \xi^n(0)c + \int_0^t \left(n(\xi^{n-1}(s))^T D_r c + (\xi^n(s))^T Q \circ A(n)c\right) ds.$$

Letting C be an invertible $K \times K$ square matrix, we obtain, after differentiation w.r.t. t,

$$\left(\frac{d}{dt}\xi^n(t)\right)^T C = n(\xi^{n-1}(t))^T D_r C + (\xi^n(t))^T Q \circ A(n)C,$$

and after a multiplication by C^{-1} and transposition, we arrive at the following system of linear equations:

$$\frac{d}{dt}\xi^n(t) = nD_r\xi^{n-1}(t) + (Q \circ A(n))^T \xi^n(t).$$

A solution is given by

$$\xi^n(t) = e^{(Q\circ A(n))^T t}\xi^n(0) + n\int_0^t e^{(Q\circ A(n))^T (t-s)} D_r \xi^{n-1}(s)\, ds. \tag{12}$$

Letting $R_n(t) = e^{(Q\circ A(n))^T t} D_r$, we obtain

$$\xi^n(t) = R_n(t)D_r^{-1}\xi^n(0) + nR_n * \xi^{n-1}(t),$$

where $*$ denotes convolution, and then iteratively, with $R^{*n} = R_n * R^{*(n-1)}$, we have that

$$\xi^n(t) = n!\left(\sum_{k=1}^n \frac{D_r^{-1}}{(n-k+1)!}\xi^{n-k+1}(0)R^{*k}(t) + \int_0^t R^{*n}(s)\pi\, ds\right).$$

5. THE CASE OF PHASE-TYPE INTERJUMP TIMES

In this section we would like to consider a growth collapse process for which the time between jumps has a phase-type distribution and the jump is only of one type. To define a phase-type distribution, we need to define a continuous-time Markov chain with states, say, $0, \ldots, K$ such that 0 is accessible from any other state. Given some initial distribution β on $1, \ldots, K$, a phase-type distribution is the distribution of time

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until 0 is visited for the first time. Thus, we define a continuous-time Markov chain with the following transition matrix:

$$Q = \begin{pmatrix} -1 & \beta^T \\ -R\mathbf{1} & R \end{pmatrix}, \tag{13}$$

where the first row and column correspond to state 0 and the rest corresponds to states $1, \ldots, K$ and $\mathbf{1}$ denotes a K-vector of ones.

From the assumption, it follows that the distribution of time it takes to reach 0 given the initial distribution β on $1, \ldots, K$ is given by $\beta^T e^{Rt}\mathbf{1}$, where e^A is a matrix exponential. If the defined Markov chain is not irreducible, then, starting from state 0, some states will never be visited and thus can be removed. Thus, without loss of generality, irreducibility is assumed.

We now define the upward rates of our process as $r_0 = 0$ and $r_1 = \cdots = r_K = r$ and

$$X_{ij} = \begin{cases} X, & 1 \le i \le K, j = 0 \\ 1, & \text{otherwise}, \end{cases} \tag{14}$$

where $\mathbb{P}[X = 1] < 1$. Hence, the jumps when visiting state 0 are i.i.d. and independent of the originating state. We also observe that after 0 is visited, the growth collapse process remains constant until the modulating Markov chain makes a transition to a different state. This is not quite the structure of a growth collapse process with phase-type interarrival times. However, if we remove the time intervals where the modulating Markov chain is in zero, then the process does become one. Hence, with (W^*, J^*) denoting a random pair having the stationary distribution of our Markov-modulated growth collapse process, then the distribution of the growth collapse process with phase-type interarrival times is given by

$$F(t) = \frac{\sum_{i=1}^{K} \mathbb{P}[W^* \le t, J^* = i]}{1 - \pi_0}$$

and, in particular, the moments are given by

$$\mu^n = \int_{[0,t]} t^n \, dF(t) = \frac{\sum_{i=1}^{K} \xi_i^n}{1 - \pi_0}. \tag{15}$$

Clearly, Condition 1 is satisfied, and with $a(\alpha) = \mathbb{E}X^\alpha$, we have that $a_{i0}(\alpha) = a(\alpha)$ for $i = 1, \ldots, K$ and $a_{ij}(\alpha) = 1$ for all other pairs. Hence,

$$Q \circ A(\alpha) = \begin{pmatrix} -1 & \beta^T \\ -a(\alpha)R\mathbf{1} & R \end{pmatrix}. \tag{16}$$

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Observe that since $\beta^T \mathbf{1} = 1$, it is straightforward to check that

$$(Q \circ A(\alpha))^{-1} = \begin{pmatrix} -\frac{1}{1-a(\alpha)} & \frac{1}{1-a(\alpha)}\beta^T R^{-1} \\ -\frac{a(\alpha)}{1-a(\alpha)}\mathbf{1} & \left(I + \frac{a(\alpha)}{1-a(\alpha)}\mathbf{1}\beta^T\right) R^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & R^{-1} \end{pmatrix} + \frac{1}{1-a(\alpha)} \begin{pmatrix} -1 & \beta^T R^{-1} \\ -a(\alpha)\mathbf{1} & a(\alpha)\mathbf{1}\beta^T R^{-1} \end{pmatrix}, \tag{17}$$

where $\mathbf{0}$ is a K-vector of zeros. Recall that we are assuming here that

$$D_r = \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & rI \end{pmatrix}, \tag{18}$$

so that

$$D_r(-Q \circ A(\alpha))^{-1} = r \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & -R^{-1} \end{pmatrix} + \frac{ra(\alpha)}{1-a(\alpha)} \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{1} & \mathbf{1}\beta^T(-R^{-1}) \end{pmatrix}. \tag{19}$$

Finally, noting that

$$\begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{a} & A \end{pmatrix} \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{b} & B \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{0} \\ A\mathbf{a} & AB \end{pmatrix}, \tag{20}$$

where $\mathbf{a}$ and $\mathbf{b}$ are some K-vectors and observing that from $\pi^T Q = 0$ we have that

$$p^T = (\pi_1, \ldots, \pi_K) = \pi_0 \beta^T(-R^{-1}),$$

we now have that

$$\mu^n = \frac{1}{1-\pi_0}(\xi^n)^T \begin{pmatrix} 0 \\ \mathbf{1} \end{pmatrix}$$
$$= \frac{n!}{1-\pi_0}\pi^T \prod_{k=1}^{n} D_r(-Q \circ A(k))^{-1} \begin{pmatrix} 0 \\ \mathbf{1} \end{pmatrix}$$
$$= \frac{n! r^n \pi_0}{1-\pi_0}\beta^T(-R^{-1}) \prod_{k=1}^{n} \left[\left(I + \frac{a(k)}{1-a(k)}\mathbf{1}\beta^T\right)(-R^{-1})\right]\mathbf{1}. \tag{21}$$

Finally, since $\pi^T = \pi_0(1, \beta^T(-R^{-1}))$ and since the coordinates of π sum up to 1, it follows that $\pi_0(1 + \beta^T(-R^{-1})\mathbf{1}) = 1$ so that

$$\frac{\pi_0}{1-\pi_0} = \frac{1}{\beta^T(-R^{-1})\mathbf{1}}. \tag{22}$$

This gives the following.

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THEOREM 4: *For a growth collapse model with linear increase with rate $r > 0$, the remaining proportion after a jump with distribution not concentrated at 1, with the nth moment $a(n)$ and with i.i.d. intercollapse times having the phase-type distribution $F(t) = \beta^T e^{Rt}\mathbf{1}$, a stationary distribution exists and has the following nth moment:*

$$\mu^n = n! r^n \frac{\beta^T(-R^{-1})}{\beta^T(-R^{-1})\mathbf{1}} \prod_{k=1}^{n} \left[\left(I + \frac{a(k)}{1-a(k)} \mathbf{1}\beta^T \right) (-R^{-1}) \right] \mathbf{1}. \tag{23}$$

COROLLARY 1: *If in Theorem 4, in addition, the remaining proportion after a jump is zero, then the growth collapse model becomes a clearing process and the corresponding moments are*

$$\mu^n = n! r^n \frac{\beta^T(-R^{-1})^{n+1}\mathbf{1}}{\beta^T(-R^{-1})\mathbf{1}}. \tag{24}$$

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