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COMPLETE INTUITIONISTIC TEMPORAL LOGICS FOR TOPOLOGICAL DYNAMICS

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Abstract. The language of linear temporal logic can be interpreted on the class of dynamic topological systems, giving rise to the intuitionistic temporal logic $\mathsf{ITL}^{\mathsf{c}}_{\bigcirc\forall}$, recently shown to be decidable by Fernández-Duque. In this article we axiomatize this logic, some fragments, and prove completeness for several familiar spaces.

§1. Introduction. In 2004, Kremer [27] suggested that intuitionistic temporal logic (ITL) may be interpreted on dynamical systems, a proposal which only in recent years has gathered momentum due to recent developments showing that they provide an expressive yet decidable framework in which to reason about topological dynamics (see, e.g., [19]). However, techniques employed so far have mostly been semantic, with a proper deductive calculus missing. In this article we bridge this gap by showing that a fragment with 'everywhere' and 'eventually' also enjoys a natural axiomatization.

Dynamical systems may colloquially be described as mathematical models of change or movement. They are actively studied and are widely used in both pure and applied fields. Some of the most familiar examples arise from differential equations on $\mathbb{R}^n$. Recall that the solution to such equations yields a *flow*: a family of functions $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ indexed by $t \in \mathbb{R}$. To characterize the global structure of these systems, it is important to understand their *asymptotic* behavior, i.e., the behavior of $\phi_t(x)$ as $t \to \infty$. For this, it is often useful to employ a discrete-time approximation of $(\phi_t)_{t\in\mathbb{R}}$. By setting $f := \phi_\varepsilon$ for small ε, one may study the orbit $\{x, f(x), f^2(x), ...\}$ to obtain a picture of the behavior of $\phi_t(x)$.

The pair $(\mathbb{R}^n, f)$ thus obtained is a discrete-time dynamical system: more generally, we define *dynamic* (*topological*) *systems* as pairs (X, f), where X is a topological space and $f : X \to X$ is continuous [1]. This general definition allows for a wealth of examples where f is not obtained from a differential equation or X is not Euclidean. In this general setting, one may instead focus on other natural structural properties of dynamical systems: for example, *probability-preserving systems* are endowed with a probability measure P that is invariant under f (in the sense that $P = f^{-1}P$), and *minimal systems* are dynamical systems that contain no proper, closed, f-invariant systems (see Figure 1). In the setting of measure-preserving systems, the Poincaré recurrence theorem tells us that every set $A \subseteq X$ of positive

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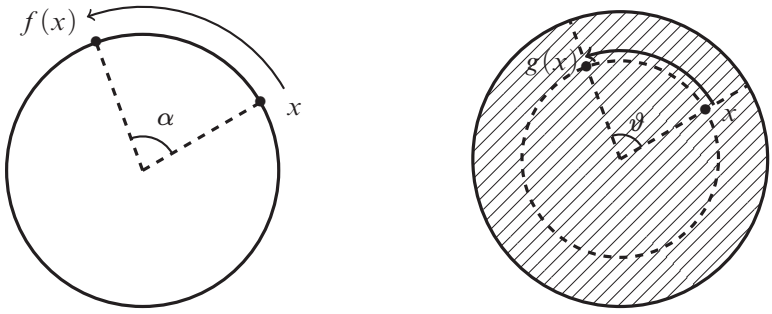


FIGURE 1. *Left*: If α is irrational multiple of π, a rotation f of a circle by α gives rise to a minimal system. *Right*: If ϑ is any angle (not necessarily an irrational multiple of π), a rotation g of a disk by an angle ϑ gives rise to a probability-preserving system, where the probability of a set is defined to be proportional to its area. Note that ϑ no longer needs to be an irrational multiple of π, unlike α. Moreover, such a system is not minimal, as (for example) the dashed circle is a closed, g-closed subsystem.

probability has a *recurrent point*, i.e., there are $x \in A$ and $n > 0$ such that $f^n(x) \in A$ [33]. In the setting of minimal systems, it is known that such systems may be characterized by the property that the orbit of every point is dense in the whole space [6].

The dynamic topological logic project aimed to provide a polymodal framework in which to reason about dynamical systems. It originated in the work of Artëmov et al. [3], who suggested that modal logic may be used to reason about dynamical systems (X, f) using the 'interior' ($\blacksquare$) modality based on the topology on X (in the sense of Tarski [36]) and the 'next' ($\circ$) modality to reason about the action of f. Kremer and Mints [28] later observed that 'henceforth' ($\Box$) could be used to model the asymptotic behavior of f, allowing one to represent a version of the Poincaré recurrence theorem. The resulting tri-modal system was called *dynamic topological logic* (DTL), and it was studied for some time with expectations that it may be applicable in, e.g., automated theorem proving. Fernández-Duque [13] suggested adding a universal modality ($\forall$) to further express minimality. However, Konev et al. [26] showed that the validity problem for DTL formulas is undecidable, hindering the prospect of automated reasoning in DTL.

In unpublished work, Kremer [27] also suggested an intuitionistic version of DTL, based on 'next' and 'henceforth'; no interior modality is required in this setting, as the topology is reflected in the semantics for implication. Years later, Fernández-Duque [19] showed that a mild variant based on 'next', 'eventually' ($\Diamond$), and the universal modality was decidable over the class of all dynamical systems, and Boudou et al. [7] showed that a logic with 'next', 'eventually', and 'henceforth' was decidable over the class of all dynamical systems based on a poset (see Section 3). These results have led us to recast the dynamic topological logic project in terms of intuitionistic temporal logics. Moreover, Poincaré recurrence may also be defined in terms of $\circ, \Diamond$ and minimality in terms of $\Diamond, \forall$, singling out the language with $\circ, \Diamond, \forall$ as suitable for reasoning about topological dynamics.

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These decidability results were proven using model-theoretic techniques, with little being known regarding axiomatic systems. Our goal in this paper is to develop deductive calculi for these logics.

1.1. State-of-the-art. There are several (poly)modal logics which may be used to model time, and some have already been studied in an intuitionistic setting, e.g., tense logics by Davoren [9] and propositional dynamic logic with iteration by Nishimura [32]. Here we are specifically concerned with intuitionistic analogues of discrete-time linear temporal logic. Versions of such a logic in finite time have been studied by Kojima and Igarashi [25] and Kamide and Wansing [24]. Nevertheless, logics over infinite time have proven to be rather difficult to understand, in no small part due to their similarity to intuitionistic modal logics such as IS4, whose decidability is still an open problem [34].

Balbiani and the authors have made some advances in this direction, showing that the intermediate logic of temporal here-and-there is decidable and enjoys a natural axiomatization [5], as well as identifying several conservative temporal extensions of intuitionistic logic, interpreted on dynamic topological systems [19], or what we call *dynamic posets*, also called *expanding posets* [7] due to their relation with expanding products of modal logics [20]. These logics are based on the temporal language with $\circ$ ('next'), $\Diamond$ ('eventually'), $\Box$ ('henceforth'), and the universal modality $\forall$. Note that unlike in the classical case, $\Diamond$ and $\Box$ are not inter-definable [4].

Fernández-Duque [19] showed that a formula in the $\Box$-free language $\mathcal{L}_{\Diamond\forall}$ is valid if and only if it is valid over a class of suitably defined *quasimodels*. With this, he showed that the validity problem for $\mathcal{L}_{\Diamond\forall}$ over the class of dynamical systems is decidable. Fernández-Duque also showed that $\mathcal{L}_{\Diamond\forall}$ is expressive enough to capture many of the recurrence phenomena that inspired interest in DTL, such as the aforementioned Poincaré recurrence and minimality. As we will show, quasimodels can also be used to prove the completeness of a natural deductive calculus, which we denote $\mathsf{ITL}_{\Diamond\forall}$, for this language. The calculus $\mathsf{ITL}_{\Diamond\forall}$ is obtained by adding $\Diamond$-versions of standard LTL axioms to the intuitionistic propositional calculus, as well as axioms for the universal modality.

On the other hand, we do not yet have a useful notion of quasimodel in the presence of $\Box$, and Kremer [27] showed that some key axioms of classical LTL (including $\Box\varphi \to \circ\Box\varphi$) are not valid for his topological semantics. At this point, it is unclear which weaker principles should replace them (although [8] suggests a calculus for this logic). For these reasons, in this manuscript we restrict our attention to $\Box$-free fragments of the temporal language.

1.2. Our main results. The main goal of this article is to prove that $\mathsf{LTL}_{\Diamond\forall}$ and some of its fragments are complete for the class of dynamical systems (Theorem 8.4). The completeness proof follows the general scheme of that for linear temporal logic [29]: a set of 'local states', which we will call *moments*, is defined, where a moment is a representation of a potential point in a model (or, in our case, a quasimodel). To each moment w one then assigns a characteristic formula $\chi(w)$ in such a way that $\chi(w)$ is consistent if and only if w can be included in a model, from which completeness can readily be deduced.

In the LTL setting, a moment is simply a maximal consistent subset of a suitable finite set Σ of formulas. For us, a moment is instead a finite labelled tree, and

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the formula $\chi(w)$ must characterize w up to *simulation*; for this reason we will henceforth write $\mathrm{Sim}(w)$ instead of $\chi(w)$. The required formulas $\mathrm{Sim}(w)$ can readily be constructed in our language (Proposition 6.4).

Note that it is *failure* of $\mathrm{Sim}(w)$ that characterizes the property of simulating w; hence the *possible* states will be those moments w such that $\mathrm{Sim}(w)$ is unprovable. The set of possible moments will form a quasimodel falsifying a given unprovable formula φ (Corollary 8.3), from which it follows that such a φ is falsified on some model as well (Theorem 4.9). Thus any unprovable formula is falsifiable, and Theorem 8.4 follows.

Our proof will be presented in such a way that completeness for the sub-logics $\mathsf{ITL}_\circ$ (the sublogic of $\mathsf{LTL}_{\Diamond\forall}$ whose only modality is $\circ$), $\mathsf{ITL}_{\circ\forall}$ (with modalities $\circ$ and $\forall$), and $\mathsf{ITL}_\Diamond$ (with $\circ$ and $\Diamond$) are obtained as partial results.[1] Once we have established our main completeness theorem, we will consider special classes of dynamical systems for which our logics are also complete. In summary, we obtain the following results.

1. $\mathsf{ITL}_\circ$ and $\mathsf{ITL}_\Diamond$ are sound and complete for the class of expanding posets, for dynamical systems based on $\mathbb{R}^n$ for any fixed $n \geq 2$, and for the Cantor space.
2. The logics $\mathsf{ITL}^+_\circ$ and $\mathsf{ITL}^+_{\circ\forall}$ obtained by adding the axiom $(\circ\varphi \to \circ\psi) \to \circ(\varphi \to \psi)$ are sound and complete for the class of *persistent* posets.
3. The logic $\mathsf{ITL}_{\Diamond\forall}$ is sound and complete for both the class of all dynamical systems and of all dynamical systems based on $\mathbb{Q}$.

In contrast, the logic $\mathsf{ITL}_\circ$ is incomplete for $\mathbb{R}$ [8].

1.3. Layout. Section 2 reviews some basic notions regarding topology and dynamical systems and Section 3 introduces the syntax and semantics of intuitionistic temporal logic. Section 4 then discusses labelled structures, which generalize both models and *quasimodels*, a central tool in proving completeness for languages with $\Diamond$. Section 5 discusses the canonical model, which properly speaking is only a *deterministic weak quasimodel*, but will be useful in establishing many of our main results, including completeness for logics over the language whose only modality is $\circ$. The rest of the paper is devoted to logics with $\Diamond$ and/or $\forall$, and readers only interested in the fragment with 'next' may stop reading at this point. Section 6 reviews simulations and dynamic simulations, including their definability in the intuitionistic language. Section 7 constructs the initial quasimodel and establishes its basic properties, but the fact that it is actually a quasimodel is proven only in Section 8, where we show that the quasimodel is *ω-sensible*, i.e., it satisfies the required condition to interpret $\Diamond$. The completeness of $\mathsf{ITL}_\Diamond$ follows from this. This is the central result of the paper, and may also be a natural stopping point for some readers. In Section 9 we extend our axiomatization to languages with the universal modality and prove that the logic $\mathsf{ITL}_{\Diamond\forall}$ is complete. We also prove completeness for $\mathsf{ITL}_{\circ\forall}$, and readers interested in this fragment may skip Sections 6 and 7. The remainder of the paper is devoted to special classes of dynamical systems. In Section 10, we prove via an unwinding construction that any formula falsifiable on a quasimodel is also falsifiable on an expanding poset, from which we conclude

[1] Sublogics or sublanguages are indicated by displaying the allowed modalities as a subindex, under the convention that logics with $\Diamond$ or $\Box$ always contain $\circ$, to avoid cluttering notation.

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that $\mathsf{ITL}_\diamond$ is complete for this class of systems. In Section 11, we show how to adapt results for DTL to our setting, obtaining completeness results for $\mathbb{Q}$, $\mathbb{R}^2$, and the Cantor space; the latter two results depend on Section 10, but completeness for $\mathbb{Q}$ is independent of that section. Finally, Section 12 provides some concluding remarks.

§2. Dynamical topological systems. In this section we briefly establish some conventions and basic definitions used throughout the text. First we settle some (fairly standard) notation for binary relations. Let A, B be sets and $R \subseteq A \times B$. As usual, R^{-1} denotes the inverse relation of R. If $X \subseteq A$ then $R[X] = \{b \in B : \exists a \in A(a\ R\ b)\}$. We then set $\mathrm{rng}(R) = R[A]$ and $\mathrm{dom}(R) = R^{-1}[B]$. If moreover $S \subseteq B \times C$, we may write RS instead of $R \circ S$, defined by $a\ RS\ c$ if there is $b \in B$ such that $a\ S\ b$ and $b\ R\ c$. We write $a\ R\ b\ S\ c$ if $a\ R\ b$ and $b\ S\ c$.

We assume familiarity with topological spaces and related concepts; the necessary background may be found in a text such as [31]. We will generally adopt the convention of denoting the domain of a structure $\mathfrak{S}$ by $|\mathfrak{S}|$, and in particular, topological spaces will typically be denoted $\mathfrak{X} = (|\mathfrak{X}|, \mathcal{T}_\mathfrak{X})$, i.e., $|\mathfrak{X}|$ is the set of points of $\mathfrak{X}$ and $\mathcal{T}_\mathfrak{X}$ the family of open sets; recall that this means that $\varnothing, |\mathfrak{X}| \in \mathcal{T}_\mathfrak{X}$ and that $\mathcal{T}_\mathfrak{X}$ is closed under finite intersections and arbitrary unions. We denote the interior of $A \subseteq |\mathfrak{X}|$ by A° and its closure by $\overline{A}$.

If $\mathfrak{X}, \mathfrak{Y}$ are topological spaces, a function $f : |\mathfrak{X}| \to |\mathfrak{Y}|$ is *continuous* if, whenever $U \subseteq |\mathfrak{Y}|$ is open, it follows that $f^{-1}[U]$ is open. A *dynamical* (*topological*) *system* is a triple $\mathfrak{Z} = (|\mathfrak{Z}|, \mathcal{T}_\mathfrak{Z}, f_\mathfrak{Z})$, where $(|\mathfrak{Z}|, \mathcal{T}_\mathfrak{Z})$ is a topological space and $f_\mathfrak{Z} : |\mathfrak{Z}| \to |\mathfrak{Z}|$ is a continuous function. The letters $\mathfrak{X}, \mathfrak{Y}, \ldots$ may denote either topological spaces or dynamical systems, but which one will always be specified in the text. Dynamic topological systems are the basis for our semantics and a central focus of this article.

CONVENTION 2.1. *Subindices will be used mainly for disambiguation and will be dropped when this does not lead to confusion. Thus if $\mathfrak{X}$ is a dynamic topological system, we may write, e.g., f instead of $f_\mathfrak{X}$. We may also on occasion introduce alternative notation for the components of a given structure: if we write, e.g., $\mathfrak{X} = (X, \mathcal{T}, f)$, this means that $|\mathfrak{X}| = X$, $\mathcal{T}_\mathfrak{X} = \mathcal{T}$, and $f_\mathfrak{X} = f$. Nevertheless, such usage will be preceded with due warning. A similar convention applies for topological spaces and other classes of structures.*

We will also work with posets, and it will be convenient to view them as a special case of topological spaces. As usual, a *poset* is a pair $\mathfrak{A} = (|\mathfrak{A}|, \preccurlyeq_\mathfrak{A})$, where $|\mathfrak{A}|$ is any set and $\preccurlyeq_\mathfrak{A} \subseteq |\mathfrak{A}| \times |\mathfrak{A}|$ is a reflexive, transitive, antisymmetric relation. We follow Convention 2.1 and write $\preccurlyeq$ instead of $\preccurlyeq_\mathfrak{A}$ when this does not lead to confusion, and write $a \prec b$ for $a \preccurlyeq b$ but $b \not\preccurlyeq a$. If $B \subseteq |\mathfrak{A}|$, $\mathfrak{A} \restriction B$ is the structure obtained by restricting each component of $\mathfrak{A}$ to B, so that $\mathfrak{A} \restriction B = (B, \preccurlyeq_\mathfrak{A} \cap (B \times B))$. Similar conventions apply to other classes of structures, so that for example, if $\mathfrak{X}$ is a topological space and $Y \subseteq |\mathfrak{X}|$ then $\mathfrak{X} \restriction Y$ is Y equipped with the subspace topology.

If $\mathfrak{A}$ is a poset, consider the topology $\mathcal{U}_\preccurlyeq$ on $|\mathfrak{A}|$ given by setting $U \subseteq |\mathfrak{A}|$ to be open if and only if $v \in U$ whenever $v \succcurlyeq w$ and $w \in U$. We call $\mathcal{U}_\preccurlyeq$ the *up-set topology of* $\preccurlyeq$. Topological spaces of this form are *Aleksandroff spaces* [2], which are fundamental for our completeness proof. If $\mathfrak{A}, \mathfrak{B}$ are posets, then it is not hard to check that $g : |\mathfrak{A}| \to |\mathfrak{B}|$ is continuous with respect to the up-set topologies on $\mathfrak{A}, \mathfrak{B}$ if and only if $v \preccurlyeq_\mathfrak{A} w$ implies that $g(v) \preccurlyeq_\mathfrak{B} g(w)$.

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§3. Syntax and semantics. Fix a countably infinite set $\mathbb{P}$ of propositional variables. The *full* (*intuitionistic temporal*) *language* $\mathcal{L}_*$ is defined by the grammar (in Backus–Naur form)

$$\varphi, \psi := \ \bot \mid p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \circ\varphi \mid \Diamond\varphi \mid \Box\varphi \mid \forall\varphi,$$

where $p \in \mathbb{P}$. Here, $\circ$ is read as 'next,' $\Diamond$ as 'eventually,' $\Box$ as 'henceforth,' and $\forall$ as 'everywhere'; note that this is a universal modality and not a quantifier. We also use $\neg\varphi$ as a shorthand for $\varphi \to \bot$, $\varphi \leftrightarrow \psi$ as a shorthand for $(\varphi \to \psi) \wedge (\psi \to \varphi)$, and define[2] $\exists\varphi := \neg\forall\neg\varphi$. We denote the set of subformulas of $\varphi \in \mathcal{L}_*$ by $\mathrm{sub}(\varphi)$, and we say that φ is a *superformula* of ψ if $\psi \in \mathrm{sub}(\varphi)$. We denote the set of superformulas of φ by $\mathrm{sup}(\varphi)$.

Let $M \subseteq \{\circ, \Diamond, \Box, \forall\}$. The sublanguage of $\mathcal{L}_*$ which only allows modalities in M is denoted $\mathcal{L}_M$, where we omit brackets and commas when writing M. For purposes of this article, a *temporal language* is any language of this form. Thus for example, $\mathcal{L}_{\circ\forall}$ is the sublanguage of $\mathcal{L}_*$ where $\Diamond$ and $\Box$ are not allowed. In order to avoid clutter, we adopt the convention that languages with $\Diamond$ or $\Box$ *always* include $\circ$, so that we write, e.g., $\mathcal{L}_\Diamond$ instead of $\mathcal{L}_{\circ\Diamond}$.

Next we define our semantics, based on dynamical systems.

Definition 3.1. Let $\mathfrak{X} = (X, \mathcal{T}, f)$ be a dynamic topological system. A *valuation on* $\mathfrak{X}$ is a function $[\![\cdot]\!] : \mathcal{L}_* \to \mathcal{T}_\mathfrak{X}$ such that

$$\begin{array}{rclcrcl}
[\![\bot]\!] & = & \varnothing, & & [\![\varphi \wedge \psi]\!] & = & [\![\varphi]\!] \cap [\![\psi]\!], \\
[\![\varphi \vee \psi]\!] & = & [\![\varphi]\!] \cup [\![\psi]\!], & & [\![\varphi \to \psi]\!] & = & \big((X \setminus [\![\varphi]\!]) \cup [\![\psi]\!]\big)^\circ, \\
[\![\circ\varphi]\!] & = & f^{-1}[\![\varphi]\!], & & [\![\Diamond\varphi]\!] & = & \bigcup_{n<\omega} f^{-n}[\![\varphi]\!], \\
[\![\Box\varphi]\!] & = & \big(\bigcap_{n<\omega} f^{-n}[\![\varphi]\!]\big)^\circ, & & [\![\forall\varphi]\!] & = & |\mathfrak{X}| \text{ if } [\![\varphi]\!] = |\mathfrak{X}|, \varnothing \text{ otherwise.}
\end{array}$$

A dynamical system $\mathfrak{X}$ equipped with a valuation $[\![\cdot]\!]_\mathfrak{X}$ is a (*dynamical topological*) *model*.

Validity is then defined in the usual way:

Definition 3.2. Given a model $\mathfrak{M}$ and a formula $\varphi \in \mathcal{L}_*$, we say that φ is *valid* on $\mathfrak{M}$, written $\mathfrak{M} \models \varphi$, if $[\![\varphi]\!]_\mathfrak{M} = |\mathfrak{M}|$. If $\mathfrak{M}$ is a dynamical system, we write $\mathfrak{M} \models \varphi$ if $(\mathfrak{M}, [\![\cdot]\!]) \models \varphi$ for every valuation $[\![\cdot]\!]$ on $\mathfrak{M}$. If Ω is a class of dynamical systems or models, we say that $\varphi \in \mathcal{L}_*$ is *valid in* Ω if, for every $\mathfrak{M} \in \Omega$, $\mathfrak{M} \models \varphi$. If φ is not valid in Ω, it is *falsifiable on* Ω. For a temporal language $\mathcal{L}_M$ and a class of dynamical systems Ω we define the logic ITL^Ω_M to be the set of formulas of $\mathcal{L}_M$ that are valid in Ω.

As before we assume that logics with $\Diamond$ or $\Box$ always include $\circ$, and for example write $\mathsf{ITL}^\Omega_\Diamond$ instead of $\mathsf{ITL}^\Omega_{\circ\Diamond}$. Some classes of interest are the class c of all dynamical systems (with a continuous function), the class o of all dynamical systems with a (continuous and) open map, the class e of all dynamical systems based on a poset (which we call *dynamic* or *expanding posets*), and the class $\mathsf{p} = \mathsf{e} \cap \mathsf{o}$ of *persistent*

[2] The existential and universal quantifiers are usually not inter-definable intuitionistically, but in [19] it is shown that this definition corresponds to the natural existential modality in the context of our semantics (as given in Definition 3.1).

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posets. If $\mathfrak{X}$ is a topological space, $\mathsf{ITL}^{\mathfrak{X}}_M$ denotes the set of $\mathcal{L}_M$-formulas valid in the class of dynamical systems of the form $(\mathfrak{X}, f)$.

EXAMPLE 3.3. Boudou et al. [8] showed that the formula $\varphi = (\neg\circ p \wedge \circ\neg\neg p) \to (\circ q \vee \neg\circ q)$ is valid in $\mathbb{R}$. Note that the formula φ tells us that $\mathsf{ITL}^{\mathbb{R}}_\circ$ does not enjoy Craig interpolation. On the other hand, consider the plane $\mathbb{R}^2$ with the projection function $\pi(x, y) = x$, and let $[\![p]\!]$ be the complement of the x axis and $[\![q]\!]$ the complement of the y axis. It is not hard to check that $0 \notin [\![\varphi]\!]$. This shows that $\mathsf{ITL}^{\mathbb{R}}_\circ \neq \mathsf{ITL}^{\mathbb{R}^2}_\circ$. In fact, as we will see, any $\mathcal{L}_\Diamond$-formula that is valid in $\mathbb{R}^2$ is valid in the class of all topological spaces (Theorem 11.2). Note that this is no longer the case for $\varphi \in \mathcal{L}_{\Diamond\forall}$ [19].

EXAMPLE 3.4. As we have discussed in the introduction, a dynamical system is *minimal* if the orbit of every point is dense. Fernández-Duque [19] proved (and is not hard to check) that $\mathfrak{X}$ is minimal iff $\mathfrak{X} \models \exists p \to \Diamond p$. We also say that $\mathfrak{X}$ is *Poincaré recurrent* if whenever $A \subseteq |\mathfrak{X}|$ is non-empty and open, there are $x \in A$ and $n > 0$ such that $f^n_{\mathfrak{X}}(x) \in A$. Then, $\mathfrak{X}$ is Poincaré recurrent in this sense iff $\mathfrak{X} \models \neg(p \wedge \neg\circ\Diamond p)$ [19].

3.1. Axiomatic systems. Our deductive calculi are obtained from propositional intuitionistic logic [30] by adding standard axioms and inference rules of LTL [29], although some modifications are needed to present them in terms of $\Diamond$ instead of $\Box$. For our purposes, a *logic* is a set of axioms and rules defining a subset of some temporal language $\mathcal{L}$. We say that Λ' *extends* Λ if the language of Λ' contains that of Λ and Λ' is closed under all substitution instances of the axioms and rules defining Λ.

Let us first give two axiomatizations for $\mathcal{L}_\circ$. The logic $\mathsf{ITL}^+_\circ$ is the least set of $\mathcal{L}_\circ$-formulas closed under the axioms of Intuitionistic Propositional Logic [30] plus the following axioms and inference rules:

(N1) $\neg\circ\bot$,
(N2) $\circ\varphi \wedge \circ\psi \to \circ(\varphi \wedge \psi)$,
(N3) $\circ(\varphi \vee \psi) \to \circ\varphi \vee \circ\psi$,
(N4) $\circ(\varphi \to \psi) \to (\circ\varphi \to \circ\psi)$,
(N5) $(\circ\varphi \to \circ\psi) \to \circ(\varphi \to \psi)$,
(NR1) $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$,
(NR2) $\dfrac{\varphi}{\circ\varphi}$.

All of the above axioms for $\circ$ are standard for a functional modality. We also define the logic $\mathsf{ITL}_\circ$ by omitting Axiom (N5), which is not valid in the class of dynamical systems, although it *is* valid in the class of open systems [8]. In contrast, we can derive the converse of (N3). Below, for a set of formulas Γ, we define $\circ\Gamma = \{\circ\varphi : \varphi \in \Gamma\}$, and empty conjunctions and disjunctions are defined by $\bigwedge \varnothing = \top$ and $\bigvee \varnothing = \bot$.

LEMMA 3.5. *Let $\mathcal{L}$ be a temporal language and Λ be a logic over $\mathcal{L}$ extending $\mathsf{ITL}^0_\circ$. Let $\Gamma \subseteq \mathcal{L}$ be finite. Then, the following are derivable in Λ*: (1) $\circ\bigwedge\Gamma \leftrightarrow \bigwedge\circ\Gamma$ *and* (2) $\circ\bigvee\Gamma \leftrightarrow \bigvee\circ\Gamma$.

PROOF. One direction is obtained from repeated use of Axioms (N2) or (N3) and the other is proven using (NR2) and (N4); note that the second claim requires (N1) to treat the case when $\Gamma = \varnothing$. Details are left to the reader. ⊣

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Next we define the logic $\mathsf{ITL}_\Diamond$ by extending $\mathsf{ITL}_\circ$ with the following axioms and rules. Recall that by convention, $\mathsf{ITL}_\Diamond$ includes $\circ$, given that any logic with $\Diamond$ that we consider contains $\circ$ as well.

$$(\mathrm{E1})\ \varphi \vee \circ\Diamond\varphi \to \Diamond\varphi, \qquad (\mathrm{ER1})\ \frac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}, \qquad (\mathrm{ER2})\ \frac{\circ\varphi \to \varphi}{\Diamond\varphi \to \varphi}.$$

Axiom (E1) is the dual of $\Box\varphi \to \varphi \wedge \circ\Box\varphi$. Rule (ER1) replaces the dual K-axiom $\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$ common in intuitionistic modal logic, while (ER2) is dual to the induction rule $\frac{\varphi \to \circ\varphi}{\varphi \to \Box\varphi}$. Of course we could also consider a logic $\mathsf{ITL}^+_\Diamond$ which includes Axiom (N5), but we do not have any completeness results for this logic.

LEMMA 3.6. *Let $\mathcal{L}$ be a temporal language and Λ be a logic over $\mathcal{L}$ extending $\mathsf{ITL}_\Diamond$. Then, for any $\varphi \in \mathcal{L}$, $\Lambda \vdash \Diamond\varphi \to \varphi \vee \circ\Diamond\varphi$.*

PROOF. Reasoning within $\mathsf{ITL}_\Diamond$, note that $\varphi \to \Diamond\varphi$ holds by (E1) and propositional reasoning; hence $\circ\varphi \to \circ\Diamond\varphi$ by (NR2), (N4), and (NR1). In a similar way, $\circ\Diamond\varphi \to \Diamond\varphi$ holds by (E1) and propositional reasoning, so $\circ\circ\Diamond\varphi \to \circ\Diamond\varphi$ does by (NR2), (N4), and (NR1). Hence, $\circ\varphi \vee \circ\circ\Diamond\varphi \to \circ\Diamond\varphi$ holds. Using (N3) and some propositional reasoning we obtain $\circ(\varphi \vee \circ\Diamond\varphi) \to \varphi \vee \circ\Diamond\varphi$. But then, by (ER2), $\Diamond(\varphi \vee \circ\Diamond\varphi) \to \varphi \vee \circ\Diamond\varphi$; since $\Diamond\varphi \to \Diamond(\varphi \vee \circ\Diamond\varphi)$ can be proven using (ER1), we obtain $\Diamond\varphi \to \varphi \vee \circ\Diamond\varphi$, as needed. ⊣

Finally, we define the logics $\mathsf{ITL}_{\circ\forall}$ and $\mathsf{ITL}_{\Diamond\forall}$ by extending $\mathsf{ITL}_\circ$ and $\mathsf{ITL}_\Diamond$, respectively, with the following axioms and rule.

$$\begin{array}{ll}
(\mathrm{UA1})\ \forall\varphi \vee \neg\forall\varphi, & (\mathrm{UA5})\ \forall\varphi \to \forall\forall\varphi, \\
(\mathrm{UA2})\ \forall(\varphi \to \psi) \to (\forall\varphi \to \forall\psi), & (\mathrm{UA6})\ \forall\varphi \leftrightarrow \circ\forall\varphi, \\
(\mathrm{UA3})\ \forall(\varphi \vee \forall\psi) \to \forall\varphi \vee \forall\psi, & (\mathrm{UR1})\ \dfrac{\varphi}{\forall\varphi}. \\
(\mathrm{UA4})\ \forall\varphi \to \varphi, &
\end{array}$$

The reader may observe that these axioms are designed to make the universal modality behave classically; indeed this is not surprising, as the only truth values that $\forall\varphi$ can take are the whole space or the empty set. With the exception of (N5), we will assume that all of the above rules and axioms are available when relevant.

DEFINITION 3.7. An *admissible intuitionistic temporal logic* is any logic Λ over a temporal language $\mathcal{L}_M$ such that Λ extends ITL_M.

As usual, a logic Λ is *sound* for a class of structures Ω if, whenever $\Lambda \vdash \varphi$, it follows that $\Omega \models \varphi$. The following is essentially proven in [8]:

THEOREM 3.8. *$\mathsf{ITL}_{\Diamond\forall}$ is sound for the class of dynamical systems.*

Note however that a few of the axioms and rules have been modified to fall within $\mathcal{L}_\Diamond$, and the axioms for the universal modality are new. However, these modifications are innocuous and their correctness may be readily checked by the reader. We will see that every admissible intuitionistic temporal logic is also complete for the class of dynamic topological systems.

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§4. Labelled structures. The central ingredient of our completeness proof is given by non-deterministic *quasimodels*, introduced by Fernández-Duque [12] in the context of dynamic topological logic and later adapted to intuitionistic temporal logic [19].

4.1. Two-sided types. Our presentation will differ slightly from that of [19], since it will be convenient for us to use two-sided types, defined as follows.

DEFINITION 4.1. Let $\Sigma \subseteq \mathcal{L}_{\Diamond\forall}$ be closed under subformulas and $\Phi^+, \Phi^- \subseteq \Sigma$ be finite sets of formulas. We say that the pair $\Phi = (\Phi^+; \Phi^-)$ is a *two-sided* Σ-type if:

1. $\Phi^- \cap \Phi^+ = \varnothing$,
2. $\bot \notin \Phi^+$,
3. if $\varphi \wedge \psi \in \Phi^+$, then $\varphi, \psi \in \Phi^+$,
4. if $\varphi \wedge \psi \in \Phi^-$, then $\varphi \in \Phi^-$ or $\psi \in \Phi^-$,
5. if $\varphi \vee \psi \in \Phi^+$, then $\varphi \in \Phi^+$ or $\psi \in \Phi^+$,
6. if $\varphi \vee \psi \in \Phi^-$, then $\varphi, \psi \in \Phi^-$,
7. if $\varphi \to \psi \in \Phi^+$, then $\varphi \in \Phi^-$ or $\psi \in \Phi^+$,
8. if $\varphi \to \psi \in \Phi^-$, then $\psi \in \Phi^-$, and
9. if $\Diamond\varphi \in \Phi^-$ then $\varphi \in \Phi^-$.

If moreover $\Sigma = \Phi^- \cup \Phi^+$, we may say that Φ is *saturated*. The set of finite two-sided Σ-types will be denoted T_Σ.

Whenever Ξ is an expression denoting a two-sided type, we write Ξ^+ and Ξ^- to denote its components. We define $\Phi \preccurlyeq_T \Psi$ if $\Psi^- \subseteq \Phi^-$ and $\Phi^+ \subseteq \Psi^+$.

REMARK 4.2. Fernández-Duque [19] uses one-sided Σ-types, but it is readily checked that a one-sided type Φ as defined there can be regarded as a saturated two-sided type Ψ by setting $\Psi^+ = \Phi$ and $\Psi^- = \Sigma \setminus \Phi$. Henceforth we will write *type* instead of *two-sided type* and explicitly write *one-sided type* when discussing [19].

Many times we want Σ to be finite, and to indicate this, given $\Delta \subseteq \mathcal{L}_{\Diamond\forall}$ we write $\Sigma \Subset \Delta$ if $\Sigma \subseteq \Delta$ is finite and closed under subformulas. Note that T_Σ is partially ordered by $\preccurlyeq_T$, and we will endow it with the up-set topology $\mathcal{U}_{\preccurlyeq_T}$.

DEFINITION 4.3. Let $\Sigma \subseteq \mathcal{L}_{\Diamond\forall}$ be closed under subformulas. We say that a Σ-labelled space is a triple $\mathfrak{A} = (|\mathfrak{A}|, \mathcal{T}_\mathfrak{A}, \ell_\mathfrak{A})$, where $(|\mathfrak{A}|, \mathcal{T}_\mathfrak{A})$ is a topological space and $\ell_\mathfrak{A} \colon |\mathfrak{A}| \to T_\Sigma$ a continuous function such that for all $w \in |\mathfrak{A}|$, whenever $\varphi \to \psi \in \ell^-_\mathfrak{A}(w)$ and U is any neighborhood of w, there is $v \in U$ such that $\varphi \in \ell_\mathfrak{A}(v)$ and $\psi \notin \ell_\mathfrak{A}(v)$. If $\mathcal{T}_\mathfrak{A}$ is based on a poset, we call $\mathfrak{A}$ a Σ-labelled poset.

The $\mathcal{L}_M$-labelled space $\mathfrak{A}$ *satisfies* $\varphi \in \mathcal{L}$ if $\varphi \in \ell^+_\mathfrak{A}(w)$ for some $w \in |\mathfrak{A}|$, and *falsifies* $\varphi \in \mathcal{L}$ if $\varphi \in \ell^-_\mathfrak{A}(w)$ for some $w \in |\mathfrak{A}|$. We say that $\mathfrak{A}$ and $\ell_\mathfrak{A}$ are *saturated* if $\ell_\mathfrak{A}(w)$ is saturated for every $w \in |\mathfrak{A}|$.

If $\mathfrak{A}$ is a labelled space, elements of $|\mathfrak{A}|$ will sometimes be called *worlds*. As usual, we may write ℓ instead of $\ell_\mathfrak{A}$ when this does not lead to confusion. Since we have endowed T_Σ with the topology $\mathcal{U}_{\preccurlyeq_T}$, the continuity of ℓ means that for every $w \in |\mathfrak{A}|$, there is a neighborhood U of w such that, whenever $v \in U$, $\ell_\mathfrak{A}(w) \preccurlyeq_T \ell_\mathfrak{A}(v)$.

Labelled spaces respect the semantics of $\to$ by definition, but some additional conditions are required for them to respect the semantics of the modalities. The next definition gives the conditions needed to interpret $\forall$ correctly.

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DEFINITION 4.4. Let $\Sigma \subseteq \mathcal{L}_{\Diamond\forall}$ be closed under subformulas, $\mathfrak{A}$ be a Σ-labelled space, and $U \subseteq |\mathfrak{A}|$. We say that $\ell_{\mathfrak{A}}$ is *honest for* U if, for every $w \in U$ and every $\forall\varphi \in \Sigma$, we have that $\forall\varphi \in \ell^+_{\mathfrak{A}}(w)$ implies that $\varphi \in \ell^+_{\mathfrak{A}}(v)$ for every $v \in U$, and $\forall\varphi \in \ell^-_{\mathfrak{M}}(w)$ implies that $\varphi \in \ell^-_{\mathfrak{A}}(v)$ for some $v \in U$. We say that $\mathfrak{A}$ and $\ell_{\mathfrak{A}}$ are honest if $\ell_{\mathfrak{A}}$ is honest for $|\mathfrak{A}|$.

Note that not every subset U of $|\mathfrak{A}|$ gives rise to a substructure that is, also a labelled space; however, this is the case when U is open. The following is not hard to check.

LEMMA 4.5. *If $\Sigma \subseteq \mathcal{L}_{\Diamond\forall}$ is closed under subformulas, $\mathfrak{A}$ is a Σ-labelled space, and $U \subseteq |\mathfrak{A}|$ is open, then $\mathfrak{A} \upharpoonright U$ is a Σ-labelled space. If moreover $\ell_{\mathfrak{A}}$ is honest for U, it follows that $\mathfrak{A} \upharpoonright U$ is honest.*

For our purposes, a *continuous relation* on a topological space is a relation S under which the preimage of any open set is open.[3] In the context of a poset with the up-set topology, S is continuous if it satisfies the *forward confluence* property: if $w' \succcurlyeq w \; S \; v$, then there is v' such that $w' \; S \; v' \succcurlyeq v$. Similarly, an open relation S is one such that if $w \; S \; v \preccurlyeq v'$, then there is w' such that $w \preccurlyeq w' \; S \; v'$.

DEFINITION 4.6. Let $\Sigma \subseteq \mathcal{L}_{\Diamond\forall}$ be closed under subformulas and $\Phi, \Psi \in T_\Sigma$. The ordered pair (Φ, Ψ) is *sensible* if

(1) $\circ\varphi \in \Phi^+$ implies $\varphi \in \Psi^+$,
(2) $\circ\varphi \in \Phi^-$ implies $\varphi \in \Psi^-$,
(3) $\Diamond\varphi \in \Phi^+$ implies $\varphi \in \Phi^+$ or $\Diamond\varphi \in \Psi^+$,
(4) $\Diamond\varphi \in \Phi^-$ implies $\Diamond\varphi \in \Psi^-$,
(5) $\forall\varphi \in \Phi^+$ iff $\forall\varphi \in \Psi^+$, and
(6) $\forall\varphi \in \Phi^-$ iff $\forall\varphi \in \Psi^-$.

If (Φ, Ψ) is sensible, we write $\Phi \; S_T \; \Psi$. Likewise, a pair (w, v) of worlds in a labelled space $\mathfrak{A}$ is sensible if $(\ell(w), \ell(v))$ is sensible.

A continuous relation $S \subseteq |\mathfrak{A}| \times |\mathfrak{A}|$ is *sensible* if every pair in S is sensible. Further, S is ω-sensible if it is serial and, whenever $\Diamond\varphi \in \ell^+(w)$, there are $n \geq 0$ and v such that $w \; S^n \; v$ and $\varphi \in \ell^+(v)$.

A Σ-labelled system is a Σ-labelled space $\mathfrak{A}$ equipped with a sensible relation $S_{\mathfrak{A}} \subseteq |\mathfrak{A}| \times |\mathfrak{A}|$; if moreover $\ell_{\mathfrak{A}}$ is honest and $S_{\mathfrak{A}}$ is ω-sensible, we say that $\mathfrak{A}$ is a *well*-Σ-labelled system.

If $\mathfrak{A}$ is a Σ-labelled system and φ is any formula, we say that φ is *satisfied* on $\mathfrak{A}$ if there is $w \in |\mathfrak{A}|$ such that $\varphi \in \ell^+_{\mathfrak{A}}(w)$, and *falsified* on $\mathfrak{A}$ if there is $w \in |\mathfrak{A}|$ such that $\varphi \in \ell^-_{\mathfrak{A}}(w)$.

Given $\Sigma \subseteq \mathcal{L}_{\Diamond\forall}$ closed under subformulas, any dynamic topological model can be regarded as a well-Σ-labelled system. If $\mathfrak{M}$ is a model, we can assign a saturated Σ-type $\ell_{\mathfrak{M}}(x)$ to x given by $\ell^+_{\mathfrak{M}}(x) = \{\psi \in \Sigma : x \in [\![\psi]\!]_{\mathfrak{M}}\}$. We also set $S_{\mathfrak{M}} = f_{\mathfrak{M}}$; it is obvious that $\ell_{\mathfrak{M}}$ is honest and $S_{\mathfrak{M}}$ is ω-sensible. Henceforth we will tacitly identify $\mathfrak{M}$ by its associated well $\mathcal{L}_{\Diamond\forall}$-labelled system. However, not all labelled systems we

[3] Note that a more elaborate definition of *continuous relation* is often used in the literature; however, in the special case that S is a function, our definition does yield the standard definition of a continuous function.

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are interested in arise from models: another useful class of labelled systems is given by quasimodels.

DEFINITION 4.7. Given $\Sigma \subseteq \mathcal{L}_{\Diamond\forall}$ closed under subformulas, a *weak* Σ-quasimodel is a Σ-labelled system $\mathfrak{Q}$ such that there exists some partial order $\preccurlyeq_{\mathfrak{Q}}$ whose up-set topology is $\mathcal{T}_{\mathfrak{Q}}$. If moreover $\mathfrak{Q}$ is a well-Σ-labelled system, then we say that $\mathfrak{Q}$ is a Σ-quasimodel.

Note that quasimodels are very close to models, except that the relation S may be non-deterministic. Indeed, deterministic quasimodels are essentially models. The following can be checked by a standard structural induction on φ.

LEMMA 4.8. *Let $\Sigma \subseteq \mathcal{L}_{\Diamond\forall}$ be closed under subformulas and $\mathfrak{Q}$ be a deterministic Σ-quasimodel. Define a valuation $[\![\cdot]\!]_{\mathfrak{Q}}$ on $\mathfrak{Q}$ by setting $[\![p]\!]_{\mathfrak{Q}} = \{w \in W : p \in \ell(w)^+\}$ and extending to all of $\mathcal{L}$ recursively. Then, for all formulas $\varphi \in \mathcal{L}_{\Diamond}$ and for all $w \in W$, (1) if $\varphi \in \ell(w)^+$ then $w \in [\![\varphi]\!]_{\mathfrak{Q}}$, and (2) if $\varphi \in \ell(w)^-$ then $w \notin [\![\varphi]\!]_{\mathfrak{Q}}$.*

In the non-deterministic case, quasimodels are not models as they stand, but dynamical systems can nevertheless be extracted from them.

THEOREM 4.9 (Fernández-Duque [19]). *A formula $\varphi \in \mathcal{L}_{\Diamond\forall}$ is satisfiable (falsifiable) over the class of dynamic topological systems if and only if it is satisfiable (falsifiable) over the class of saturated, finite, sub(φ)-quasimodels.*

REMARK 4.10. To prove the 'if' direction of Theorem 4.9, given a quasimodel $\mathfrak{Q}$, we want to construct a dynamic topological model $\lim \mathfrak{Q}$ which validates the same formulas as $\mathfrak{Q}$. The construction is found in detail in [19], but we sketch some intuition. The issue when passing from $\mathfrak{Q}$ to $\lim \mathfrak{Q}$ is that $S_{\mathfrak{Q}}$ is a relation, not a function. To remedy this, the points of $\lim \mathfrak{Q}$ will be *paths* on $|\mathfrak{Q}|$, that is, infinite sequences $(w_i)_{i<\omega}$ of elements of $|\mathfrak{Q}|$ such that $w_i \, S_{\mathfrak{Q}} \, w_{i+1}$. We may then define $S_{\lim \mathfrak{Q}}(w_i)_{i<\omega} = (w_{i+1})_{i<\omega}$; in other words, $S_{\lim \mathfrak{Q}}$ (deterministically) removes the first element in the sequence. However, not all paths respect the semantics of $\Diamond$; thus we define $|\lim \mathfrak{Q}|$ to be the set of *realizing paths*, defined as those paths $(w_i)_{i<\omega}$ such that whenever $\Diamond\varphi \in \ell^+_{\mathfrak{Q}}(w_i)$ for some i, it follows that there is $j \geq i$ with $\varphi \in \ell^+_{\mathfrak{Q}}(w_j)$. We note that $|\lim \mathfrak{Q}| \subseteq |\mathfrak{Q}|^\omega$, and the latter is canonically equipped with a topology as an infinite product, so $|\lim \mathfrak{Q}|$ inherits this topology as a subspace. It follows from the continuity of $S_{\mathfrak{Q}}$ that the function $S_{\lim \mathfrak{Q}}$ is continuous for this topology. If we set $[\![p]\!]_{\lim \mathfrak{Q}} = \{(w_i)_{i<\omega} \in |\lim \mathfrak{Q}| : p \in \ell^+_{\mathfrak{Q}}(w_0)\}$, a standard induction on formulas shows that for $\varphi \in \Sigma$, $(w_i)_{i<\omega} \in [\![\varphi]\!]_{\lim \mathfrak{Q}}$ iff $\varphi \in \ell^+_{\mathfrak{Q}}(w_0)$.

Note that [19] uses quasimodels with one-sided types, but in view of Remark 4.2, Theorem 4.9 can easily be modified to obtain quasimodels with two-sided types. Two-sided types will be more convenient for us, especially in Section 10. Below, recall that for a structure $\mathfrak{A}$ and $U \subseteq |\mathfrak{A}|$, $\mathfrak{A} \upharpoonright U$ is the substructure of $\mathfrak{A}$ obtained by restricting all functions and relations of $\mathfrak{A}$ to U.

LEMMA 4.11. *Let $\mathfrak{Q}$ be a weak quasimodel and $U \subseteq |\mathfrak{Q}|$. Assume that U is open and $\ell_{\mathfrak{Q}}$ is honest for U. Suppose moreover that either (1) $S_{\mathfrak{Q}} \upharpoonright U$ is serial and ω-sensible, or (2) $S_{\mathfrak{Q}}$ is ω-sensible and U is $S_{\mathfrak{Q}}$-invariant (i.e., $S_{\mathfrak{Q}}[U] \subseteq U$). Then, $\mathfrak{Q} \upharpoonright U$ is a quasimodel.*

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PROOF. By Lemma 4.5, we know that $\mathfrak{Q} \restriction U$ is a labelled poset. Since U is open and $S_{\mathfrak{Q}}$ is continuous, $S_{\mathfrak{Q}\restriction U}$ is continuous as well, so $S_{\mathfrak{Q}\restriction U}$ is sensible. Thus it remains to show that $S_{\mathfrak{Q}\restriction U}$ is serial and ω-sensible, which in the first case holds by assumption and in the second follows easily from $S_{\mathfrak{Q}}$ already having these properties. ⊣

As usual, if φ is not derivable, we wish to produce a model where φ is falsified, but in view of Theorem 4.9, it suffices to falsify φ on a quasimodel. This is convenient, as quasimodels are much easier to construct than models.

§5. The canonical model. In this section we construct a standard canonical model for any logic Λ extending $\mathsf{ITL}_\circ$. From this we will obtain some completeness results for logics over $\mathcal{L}_\circ$. In the presence of $\Diamond$, the standard canonical model is only a saturated, weak, deterministic quasimodel rather than a proper model. Nevertheless, it will be a useful ingredient in our completeness proofs for $\mathsf{ITL}_\Diamond$ and $\mathsf{ITL}_{\Diamond\forall}$. Since we are working over an intuitionistic logic, the role of maximal consistent sets will be played by prime types, as defined below.

DEFINITION 5.1. Let $\mathcal{L}$ be a temporal language and Λ a logic over $\mathcal{L}$. Given two sets of formulas Γ and Δ, we say that Δ is a consequence of Γ(with respect to Λ), denoted by $\Gamma \vdash \Delta$, if there exist finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\Lambda \vdash \bigwedge \Gamma' \to \bigvee \Delta'$.

We say that a pair of sets $\Phi = (\Phi^+, \Phi^-)$ is *consistent* if $\Phi^+ \nvdash \Phi^-$. A saturated, consistent $\mathcal{L}$-type is a *prime type*. The set of prime $\mathcal{L}$-types will be denoted $T^\infty_{\mathcal{L}}$.

Note that we are using the standard interpretation of $\Gamma \vdash \Delta$ in Gentzen-style calculi. The logic Λ will always be clear from context, which is why we do not reflect it in the notation. When working within a turnstyle, we will follow the usual proof-theoretic conventions of writing Γ, Δ instead of $\Gamma \cup \Delta$ and φ instead of $\{\varphi\}$. Observe that there is no clash in terminology regarding the use of the word *type*:

LEMMA 5.2. *If Λ is an admissible temporal logic over a language $\mathcal{L}$ and Φ is a prime $\mathcal{L}$-type then Φ is an $\mathcal{L}$-type.*

PROOF. Let Φ be a prime $\mathcal{L}$-type. Observe that Φ is already saturated by definition, so it remains to check that it satisfies all conditions of Definition 4.1. Conditions 1 and 2 follow from the consistency of Φ. The proofs of the other conditions are all similar to each other. For example, for 7, suppose that $\varphi \to \psi \in \Phi^+$ and $\varphi \notin \Phi^-$. Since Φ is saturated, it follows that $\varphi \in \Phi^+$. But $\big(\varphi \wedge (\varphi \to \psi)\big) \to \psi$ is an intuitionistic tautology, so using the fact that Φ is consistent we see that $\psi \notin \Psi^-$, which using the assumption that Φ is saturated gives us $\psi \in \Phi^+$. For condition 9 we use (E1): if $\Diamond\varphi \in \Phi^-$ and $\varphi \in \Phi^+$ we would have that Φ is inconsistent; hence $\varphi \in \Phi^-$. The rest of the conditions are left to the reader. ⊣

As with maximal consistent sets, prime types satisfy a Lindenbaum property.

LEMMA 5.3 (Lindenbaum Lemma). *Fix an admissible temporal logic Λ over $\mathcal{L}$. Let $\Gamma, \Delta \subseteq \mathcal{L}$. If $\Gamma \nvdash \Delta$, then there exists a prime type Φ such that $\Gamma \subseteq \Phi^+$ and $\Delta \subseteq \Phi^-$.*

PROOF. The proof is standard, but we provide a sketch. Let $\varphi \in \mathcal{L}$. Note that either $\Gamma, \varphi \nvdash \Delta$ or $\Gamma \nvdash \Delta, \varphi$, for otherwise by a cut rule (which is intuitionistically derivable) we would have $\Gamma \vdash \Delta$. Thus we can add φ to $\Gamma \cup \Delta$, and by repeating this process for each element of $\mathcal{L}$ (or using Zorn's lemma) we can find suitable Φ. ⊣

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Given a set A, let $\mathbb{I}_A$ denote the identity function on A. Let Λ be an admissible temporal logic over $\mathcal{L}$. The canonical model $\mathfrak{M}_\Lambda$ for Λ is defined as the labelled structure

$$\mathfrak{M}_\Lambda = (|\mathfrak{M}_\Lambda|, \preccurlyeq_\Lambda, S_\Lambda, \ell_\Lambda) \stackrel{\text{def}}{=} (T_\mathcal{L}, \preccurlyeq_T, S_T, \mathbb{I}_{T^\infty_\mathcal{L}}) \restriction T_\infty;$$

in other words, $\mathfrak{M}_\Lambda$ is the set of prime types with $\Phi \preccurlyeq_\Lambda \Psi$ if $\Psi^- \subseteq \Phi^-$ and $\Phi^+ \subseteq \Psi^+$, and $\Phi\ S_\Lambda\ \Psi$ if the pair (Φ, Ψ) is sensible in the sense of Definition 4.6. Note that ℓ_Λ is just the identity (i.e., $\ell_\Lambda(\Phi) = \Phi$). We will usually omit writing ℓ_Λ, as it has no effect on its argument.

Next we show that $\mathfrak{M}_\Lambda$ is a saturated, weak, deterministic quasimodel. For this, we must prove that it has all the required properties.

LEMMA 5.4. *Let Λ be an admissible temporal logic over $\mathcal{L}$. Then, $\mathfrak{M}_\Lambda$ is a labelled poset.*

PROOF. We know that $\preccurlyeq_T$ is a partial order and restrictions of partial orders are partial orders, so $\preccurlyeq_\Lambda$ is a partial order. Moreover, ℓ_Λ is the identity, so $\Phi \preccurlyeq_\Lambda \Psi$ implies that $\ell_\Lambda(\Phi) \preccurlyeq_T \ell_\Lambda(\Psi)$.

Now let $\Phi \in |\mathfrak{M}_\Lambda|$ and assume that $\varphi \to \psi \in \Phi^-$. Note that $\Phi^+, \varphi \nvdash \psi$, for otherwise by intuitionistic reasoning we would have $\Phi^+ \vdash \varphi \to \psi$; hence $\Phi^+, \varphi \nvdash \psi$, contrary to our assumption. By Lemma 5.3, there is a prime type Ψ with $\Phi^+ \cup \{\varphi\} \subseteq \Psi^+$ and $\psi \in \Psi^-$. It follows that $\Phi \preccurlyeq_\Lambda \Psi$, $\varphi \in \Psi^+$, and $\psi \in \Psi^-$, as needed. ⊣

LEMMA 5.5. *Let Λ be an admissible temporal logic over $\mathcal{L}$. Then, S_Λ is a continuous function. If moreover $(N5)$ is an axiom of Λ, then S_Λ is also open.*

PROOF. For a set $\Gamma \subseteq \mathcal{L}$, recall that we had defined $\circ\Gamma = \{\circ\varphi : \varphi \in \Gamma\}$. Define also $\ominus\Gamma = \{\varphi : \circ\varphi \in \Gamma\}$. With this, we show that S_Λ is functional and forward-confluent.

Functionality. Let $\Phi \in |\mathfrak{M}_\Lambda|$ and define $\Psi = (\ominus\Phi^+, \ominus\Phi^-)$. We claim that $\Phi\ S_T\ \Psi$ and that if $\Phi\ S_T\ \Theta$, it follows that $\Theta = \Psi$. Recall that $\Phi\ S_T\ \Theta$ means that the pair (Φ, Θ) is sensible (in the sense of Definition 4.6).

First we must check that $\Psi \in |\mathfrak{M}_\Lambda|$. To see that Ψ is saturated, let $\varphi \in \mathcal{L}_\Diamond$ be so that $\varphi \notin \Psi^-$. It follows that $\circ\varphi \notin \Phi^-$, but Φ is saturated, so $\circ\varphi \in \Phi^+$ and thus $\varphi \in \Psi^+$. Since φ was arbitrary, $\Psi^- \cup \Psi^+ = \mathcal{L}_\Diamond$. Next we check that Ψ is consistent. If not, let $\Gamma \subseteq \Psi^+$ and $\Delta \subseteq \Psi^-$ be finite such that $\bigwedge\Gamma \to \bigvee\Delta$ is derivable. Using (NR2) and (N4) we see that $\circ\bigwedge\Gamma \to \circ\bigvee\Delta$ is derivable, which in view of Lemma 3.5 implies that $\bigwedge\circ\Gamma \to \bigvee\circ\Delta$ is derivable as well. But $\circ\Gamma \subseteq \Phi^+$ and $\circ\Delta \subseteq \Phi^-$, contradicting the fact that Φ is consistent. We conclude that $\Psi \in |\mathfrak{M}_\Lambda|$.

We must now check that $\Phi\ S_T\ \Psi$. It is clear that clauses (1) and (2) of Definition 4.6 hold. If $\Diamond\varphi \in \Phi^+$ (so that $\Diamond$ is allowed in $\mathcal{L}$) and $\varphi \notin \Phi^+$, it follows that $\varphi \in \Phi^-$. By Lemma 3.6 $\Diamond\varphi \to \varphi \vee \circ\Diamond\varphi$ is derivable, so we cannot have that $\circ\Diamond\varphi \in \Phi^-$ and hence $\circ\Diamond\varphi \in \Phi^+$, so that $\Diamond\varphi \in \Psi^+$. Similarly, if $\Diamond\varphi \in \Phi^-$ we have that $\circ\Diamond\varphi \in \Phi^-$, for otherwise we obtain a contradiction from (E1). Therefore, $\Diamond\varphi \in \Psi^-$ as well. The clauses for $\forall\varphi$ follow a similar line of reasoning using (UA6).

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To check that Ψ is unique, suppose that $\Theta \in |\mathfrak{M}_\Lambda|$ is such that $\Phi\ S_\Lambda\ \Theta$. Then, if $\varphi \in \Psi^+$ it follows from the definition of S_T that $\circ\varphi \in \Phi^+$ and hence $\varphi \in \Theta^+$; by the same argument, if $\varphi \in \Psi^-$ it follows that $\varphi \in \Theta^-$, and hence $\Theta = \Psi$.

Continuity: Now that we have shown that S_Λ is a function, we may treat it as such. Suppose that $\Phi \preccurlyeq_\Lambda \Psi$; we must check that $S_\Lambda(\Phi) \preccurlyeq_\Lambda S_\Lambda(\Psi)$. Let $\varphi \in S^+_\Lambda(\Phi)$. Using the definition of S_T, we have that $\circ\varphi \in \Phi^+$; hence $\circ\varphi \in \Psi^+$ and thus $\varphi \in S^+_\Lambda(\Psi)$. By a similar argument, if $\varphi \in S^-_\Lambda(\Psi)$, it follows that $\varphi \in S^-_\Lambda(\Phi)$. Since $\varphi \in S_\Lambda(\Phi)$ was arbitrary we obtain $S^+_\Lambda(\Phi) \preccurlyeq_\Lambda S^+_\Lambda(\Psi)$, as needed.

Openness: Suppose that (N5) is an axiom of Λ and $\Psi \succcurlyeq_\Lambda S_\Lambda(\Phi)$. We claim that $\Theta' := (\circ\Psi^+ \cup \Phi^+, \circ\Psi^-)$ is consistent. If not, there are finite $\Xi \subseteq \Psi^+$, $\Gamma \subseteq \Phi^+$, and $\Delta \subseteq \Psi^-$ such that $\vdash \bigwedge \circ\Xi \wedge \bigwedge \Gamma \to \bigvee \circ\Delta$. Since $\Gamma \subseteq \Phi^+$, this yields $\bigwedge \circ\Xi \to \bigvee \circ\Delta \in \Phi^+$. Using Lemma 3.5, (N5), and propositional reasoning, this gives us $\circ(\bigwedge \Xi \to \bigvee \Delta) \in \Phi^+$; hence $\bigwedge \Xi \to \bigvee \Delta \in \Psi^+$. But then $\bigwedge \Xi \wedge (\bigwedge \Xi \to \bigvee \Delta) \to \bigvee \Delta$ would be an intuitionistic tautology witnessing that Ψ is inconsistent, contrary to our assumption. We conclude that Θ' is consistent; hence it can be extended to a prime type Θ using the Lindenbaum lemma, and clearly $\Phi \preccurlyeq_\Lambda \Theta\ S_\Lambda\ \Psi$, as required. ⊣

Proposition 5.6. *Let Λ be an admissible temporal logic over $\mathcal{L}$. Then the canonical model for Λ is a deterministic weak $\mathcal{L}$-quasimodel.*

Proof. We need for the following three properties to hold: 1. $(|\mathfrak{M}_\Lambda|, \preccurlyeq_\Lambda, \ell_\Lambda)$ is a labelled poset; 2. S_Λ is a sensible forward-confluent function; and3.ℓ_Λ has $T_\mathcal{L}$ as its codomain. The first item is Lemma 5.4. That S_Λ is a forward-confluent function is Lemma 5.5, and it is sensible since $\Phi\ S_\Lambda\ \Psi$ precisely when $\Phi\ S_T\ \Psi$. Finally, if $\Phi \in |\mathfrak{M}_\Lambda|$ then $\ell_\Lambda(\Phi) = \Phi$, which is an element of $T_\mathcal{L}$ by Lemma 5.2. ⊣

From this we may already obtain our first completeness results.

Theorem 5.7. $\mathsf{ITL}_\circ$ *is complete for the class of expanding posets, and* $\mathsf{ITL}^+_\circ$ *is complete for the class of persistent posets.*

Proof. Let Λ be either $\mathsf{ITL}_\circ$ or $\mathsf{ITL}^+_\circ$. By Lemma 5.3, if $\Lambda \nvdash \varphi$ then there is $\Phi \in |\mathfrak{M}_\Lambda|$ such that $\varphi \in \Phi^-$. By Proposition 5.6 $\mathfrak{M}_\Lambda$ is a deterministic, weak quasimodel falsifying φ, and moreover it is trivially ω-sensible as $\Diamond$ is not in our language. By Lemma 4.8, φ is not valid in the class of expanding posets, as required. In the case that $\Lambda = \mathsf{ITL}^+_\circ$, we additionally use the fact that S_Λ is open, so that $\mathfrak{M}_\Lambda$ is persistent. ⊣

§6. Simulation formulas. Simulations are relations between labelled spaces, and give rise to the appropriate notion of 'substructure' for modal and intuitionistic logics. We have used them to prove that $\mathsf{ITL}^{\mathsf{c}}_{\Diamond\forall}$ has the finite quasimodel property [19], and they will also be useful for our completeness proof. Recall that a relation is continuous if preimages of open sets are open. If Ψ and Φ are types, we write $\Phi \subseteq_T \Psi$ if $\Phi^- \subseteq \Psi^-$ and $\Phi^+ \subseteq \Psi^+$.[4]

Definition 6.1. Let $\Sigma \subseteq \Delta \subseteq \mathcal{L}_{\Diamond\forall}$ be closed under subformulas, $\mathfrak{X}$ be a Σ-labelled space, and $\mathfrak{Y}$ be Δ-labelled. A continuous relation $E \subseteq |\mathfrak{X}| \times |\mathfrak{Y}|$ is a *simulation* if $\ell_\mathfrak{X}(x) \subseteq_T \ell_\mathfrak{Y}(y)$ whenever $x\ E\ y$. If there exists a simulation E such that $x\ E\ y$, we

[4]Note that this is *not* the same as $\Phi \preccurlyeq_T \Psi$, in which case the first expression is reversed.

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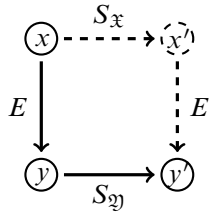


FIGURE 2. If $E \subseteq |\mathfrak{X}| \times |\mathfrak{Y}|$ is a dynamical simulation, this diagram can always be completed.

write $(\mathfrak{X}, x) \rightharpoonup (\mathfrak{Y}, y)$. The relation E is a *dynamic simulation* between $\mathfrak{X}$ and $\mathfrak{Y}$ if $S_{\mathfrak{Y}} E \subseteq E S_{\mathfrak{X}}$ (Figure 2).

When the structures $\mathfrak{X}$ and $\mathfrak{Y}$ are clear from context, we may write $x \rightharpoonup y$ instead of $(\mathfrak{X}, x) \rightharpoonup (\mathfrak{Y}, y)$. It is well known (and easily verified) that the union of simulations is a simulation. Hence $\rightharpoonup$ is in fact a simulation and is the maximal simulation between $\mathfrak{X}$ and $\mathfrak{Y}$.

Next we show that there exist formulas defining points in finite models up to simulability, i.e., that if $\mathfrak{A}$ is a finite model and $w \in |\mathfrak{A}|$, there exists a formula $\mathrm{Sim}(w)$ such that for all labelled posets $\mathfrak{M}$ and all $x \in |\mathfrak{M}|$, we have that $x \notin [\![\mathrm{Sim}(w)]\!]_{\mathfrak{M}}$ if and only if $(\mathfrak{A}, w) \rightharpoonup (\mathfrak{M}, x)$. Compare this to the classical setting, where simulability formulas for finite S4 models are not definable in the modal language [14], but they *can* be constructed using a polyadic extension of the modal language representing the *tangled closure* of a family of sets [15, 21, 22].[5] Fernández-Duque [17] uses simulation formulas to axiomatize the resulting polyadic extension of DTL; in contrast, the natural axiomatization suggested by Kremer and Mints [28] of dynamic topological logic is incomplete [18]. In the intuitionistic setting the situation is simplified somewhat, as finite posets [23] (and hence models) are already definable up to simulation in the intuitionistic language. This may be surprising, as the intuitionistic language is less expressive than the modal language; however, intuitionistic models are posets rather than arbitrary preorders, and this allows us to define simulability formulas by recursion on $\prec$.

DEFINITION 6.2. Fix $\Sigma \Subset \mathcal{L}_{\Diamond\forall}$ and let $\mathfrak{A}$ be a finite Σ-labelled poset. Given $w \in |\mathfrak{A}|$, we define a formula $\mathrm{Sim}(w)$ by backwards induction on $\preccurlyeq = \preccurlyeq_{\mathfrak{A}}$ by

$$\mathrm{Sim}(w) = \bigwedge \ell^+(w) \to \bigvee \ell^-(w) \vee \bigvee_{v \succ w} \mathrm{Sim}(v).$$

REMARK 6.3. Observe that if $\mathcal{L}$ is a temporal language and $\mathfrak{A}$ a finite Σ-labeled poset with $\Sigma \Subset \mathcal{L}$, then $\mathrm{Sim}(w) \in \mathcal{L}$ for all $w \in |\mathfrak{A}|$.

PROPOSITION 6.4. *Given $\Sigma \Subset \Delta \subseteq \mathcal{L}_{\Diamond\forall}$, a finite Σ-labelled poset $\mathfrak{A}$, a Δ-labelled poset $\mathfrak{X}$, and $w \in |\mathfrak{A}|$, $x \in |\mathfrak{X}|$:*

1. *if $\mathrm{Sim}(w) \in \ell^-_{\mathfrak{X}}(x)$ then there is $y \succcurlyeq x$ such that $(\mathfrak{A}, w) \rightharpoonup (\mathfrak{X}, y)$ and*
2. *if there is $y \succcurlyeq x$ such that $(\mathfrak{A}, w) \rightharpoonup (\mathfrak{X}, y)$ then $\mathrm{Sim}(w) \notin \ell^+_{\mathfrak{X}}(x)$.*

[5] The tangled closure is a topological operation definable in the μ-calculus, and is in fact expressively equivalent to the full μ-calculus over S4 models [10].

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PROOF. Each claim is proved by backward induction on $\preccurlyeq$.

(1) Let us first consider the base case, when there is no $v \succ w$. Assume that $\mathrm{Sim}(w) \in \ell^-(x)$. From the definition of labelled poset $\bigwedge \ell^+_{\mathfrak{A}}(w) \in \ell^+_{\mathfrak{X}}(y)$ and $\bigvee \ell^-_{\mathfrak{A}}(w) \in \ell^-_{\mathfrak{X}}(y)$ for some $y \succcurlyeq x$. From the definition of type it follows that $\ell^+_{\mathfrak{A}}(w) \subseteq \ell^+_{\mathfrak{X}}(y)$ and $\ell^-_{\mathfrak{A}}(w) \subseteq \ell^-_{\mathfrak{X}}(y)$, so that $\ell_{\mathfrak{A}}(w) \subseteq_T \ell_{\mathfrak{X}}(y)$. It follows that $E \stackrel{\text{def}}{=} \{(w, y)\}$ is a simulation, so $(\mathfrak{A}, w) \rightharpoonup (\mathfrak{X}, y)$.

For the inductive step, let us assume that the lemma holds for all $v \succ w$. Assume that $\mathrm{Sim}(w) \in \ell^-_{\mathfrak{X}}(x)$. From the definition of labelled poset, it follows that $\bigwedge \ell^+_{\mathfrak{A}}(w) \in \ell^+_{\mathfrak{X}}(y)$, $\bigvee \ell^-_{\mathfrak{A}}(w) \in \ell^-_{\mathfrak{X}}(y)$, and $\bigvee_{v \prec w} \mathrm{Sim}(v) \in \ell^-_{\mathfrak{X}}(y)$ for some $y \succcurlyeq x$. Following similar reasoning as in the base case we can conclude that $\ell_{\mathfrak{A}}(w) \subseteq \ell_{\mathfrak{X}}(y)$, and moreover, that $\mathrm{Sim}(v) \in \ell^-_{\mathfrak{X}}(y)$ for all $v \succ w$. By induction hypothesis we conclude that for all $v \succ w$, there exists a simulation E_v such that $v \; E_v \; z_v$ for some $z_v \succcurlyeq y$. Let $E \stackrel{\text{def}}{=} \{(w, y)\} \cup \bigcup_{v \succ w} E_v$. The reader may check that E is a simulation and that $w \; E \; y \succcurlyeq x$, so that $(\mathfrak{A}, w) \rightharpoonup (\mathfrak{X}, y)$, as needed.

(2) For the base case, assume that $(\mathfrak{A}, w) \rightharpoonup (\mathfrak{X}, y)$ for some $y \succcurlyeq x$, so there exists a simulation E such that $w \; E \; y$. It follows that $\ell^+_{\mathfrak{A}}(w) \subseteq \ell^+_{\mathfrak{X}}(y)$ and $\ell^-_{\mathfrak{A}}(w) \subseteq \ell^-_{\mathfrak{X}}(y)$. From conditions 3 and 5 of the definition of type (Definition 4.1), it follows that $\bigwedge \ell^+_{\mathfrak{A}}(w) \notin \ell^-_{\mathfrak{X}}(y)$ and $\bigvee \ell^-_{\mathfrak{A}}(w) \notin \ell^+_{\mathfrak{X}}(y)$. But then, condition 7 gives us $\mathrm{Sim}(w) \notin \ell^+_{\mathfrak{X}}(y)$, so $\mathrm{Sim}(w) \notin \ell^+_{\mathfrak{X}}(x)$.

For the inductive step, by the same reasoning as in the base case it follows that $\bigwedge \ell^+_{\mathfrak{A}}(w) \notin \ell^-_{\mathfrak{X}}(y)$ and $\bigvee \ell^-_{\mathfrak{A}}(w) \notin \ell^+_{\mathfrak{X}}(y)$. Now, let v be such that $v \succ w$. Since E is forward confluent then $v \; E \; z_v$ for some $z_v \succcurlyeq y$. By induction hypothesis, $\mathrm{Sim}(v) \notin \ell^+(z_v)$, so $\mathrm{Sim}(v) \notin \ell^+(y)$. Since v was arbitrary we conclude that $\bigvee_{v \succ w} \mathrm{Sim}(v) \notin \ell^+(y)$. Finally, from condition 7 of Definition 4.1 and the fact that $y \preccurlyeq x$ we get that $\mathrm{Sim}(w) \notin \ell^+(x)$. ⊣

REMARK 6.5. Proposition 6.4 more generally holds when $\mathfrak{X}$ is any labelled space (not necessarily Aleksandroff), but this restricted version will suffice for our purposes.

§7. The initial quasimodel. In this section we review the initial weak quasimodel $\mathfrak{I}_\Sigma$ [19] and use it to define an initial (non-weak) quasimodel $\mathfrak{I}^\blacklozenge_\Sigma$. The structure $\mathfrak{I}_\Sigma$ is 'initial' in the sense that if $\mathfrak{A}$ is any labelled system, there exists a surjective simulation from $\mathfrak{I}_\Sigma$ to $\mathfrak{A}$, i.e., it is initial in a category-theoretic sense. The issue is that $\mathfrak{I}_\Sigma$ is not ω-sensible; hence it is not a proper quasimodel, which is what we need to have access to Theorem 4.9. Thus we identify a substructure $\mathfrak{I}^\blacklozenge_\Sigma$ of $\mathfrak{I}_\Sigma$ which *is* a quasimodel. In particular, $\mathfrak{I}_\Sigma$ may falsify formulas which are valid in the class of dynamical systems, but such formulas will not be falsified in $\mathfrak{I}^\blacklozenge_\Sigma$.

If we write $\rightharpoonup$ instead of $(\mathfrak{I}_\Sigma, \cdot) \rightharpoonup (\mathfrak{A}, \cdot)$, then $\rightharpoonup \subseteq |\mathfrak{I}_\Sigma| \times |\mathfrak{A}|$ is the maximal simulation between the two structures. With this in mind, we obtain the following presentation of one of the main results of [19].

THEOREM 7.1. *Given $\Sigma \Subset \mathcal{L}_{\Diamond\forall}$, there exists a finite, saturated weak quasimodel $\mathfrak{I}_\Sigma$ such that if $\mathfrak{A}$ is any deterministic weak quasimodel then $\rightharpoonup \subseteq |\mathfrak{I}_\Sigma| \times |\mathfrak{A}|$ is a surjective dynamic simulation.*

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REMARK 7.2. We will not elaborate on the construction of $\mathfrak{I}_\Sigma$ here, but we provide some intuition. Points of $\mathfrak{I}_\Sigma$ are called *moments*. The key feature is that moments may be glued together to form more complex moments. Thus if $E \subseteq |\mathfrak{I}_\Sigma| \times |\mathfrak{A}|$ is any simulation that is not surjective, we may find suitable $x \notin \mathrm{rng}(E)$ and construct suitable $w \in |\mathfrak{I}_\Sigma|$ by gluing together simpler moments such that $E \cup \{(x, w)\}$ is also a simulation. It follows that the maximal simulation is surjective. A similar idea is used to show that the maximal simulation is an ω-simulation. The full proof is rather elaborate and goes beyond the scope of this paper, but full details are provided in [19].

One can thus think of $\mathfrak{I}_\Sigma$ as a finite initial structure over the category of labelled weak quasimodels. Next, we will internalize the notion of simulating elements of $\mathfrak{I}_\Sigma$ into the temporal language. This is achieved by the formulas $\mathrm{Sim}(w)$.

PROPOSITION 7.3. *Let Λ be a logic extending $\mathsf{ITL}_\circ$ over a temporal language $\mathcal{L}$. Fix $\Sigma \Subset \mathcal{L}$ and let $\mathfrak{I} = \mathfrak{I}_\Sigma$, $w \in |\mathfrak{I}|$, and $\psi \in \Sigma$.*

(1) *If $\psi \in \ell^-(w)$, then $\vdash \psi \to Sim(w)$.*
(2) *If $\psi \in \ell^+(w)$, then $\vdash \big(\psi \to Sim(w)\big) \to Sim(w)$.*
(3) *If $w \preccurlyeq v$, then $\vdash Sim(v) \to Sim(w)$.*
(4) $\vdash \bigwedge_{\psi \in \ell^-_{\mathfrak{I}}(w)} Sim(w) \to \psi$.
(5) $\vdash \circ \bigwedge_{wS_{\mathfrak{I}}v} Sim(v) \to Sim(w)$.

PROOF. (1) First assume that $\psi \in \ell^-(w)$, and toward a contradiction that $\nvdash \psi \to \mathrm{Sim}(w)$. Recall that $\mathfrak{M}_\Lambda$ denotes the canonical model for Λ. By the Lindenbaum lemma there is $\Gamma \in |\mathfrak{M}_\Lambda|$ such that $\psi \to \mathrm{Sim}(w) \in \Gamma^-$. Thus for some $\Theta \succcurlyeq_\Lambda \Gamma$ we have that $\psi \in \Theta^+$ and $\mathrm{Sim}(w) \in \Theta^-$. But then by Proposition 6.4 we have that $(\mathfrak{I}, w) \rightharpoonup (\mathfrak{M}_\Lambda, \Delta)$ for some $\Delta \succcurlyeq_\Lambda \Theta$, so that $\psi \in \Delta^-$, and by monotonicity $\psi \in \Theta^-$, contradicting the consistency of Θ.

(2) If $\psi \in \ell^+(w)$, we proceed similarly. Assume toward a contradiction that $\nvdash \big(\psi \to \mathrm{Sim}(w)\big) \to \mathrm{Sim}(w)$. Then, reasoning as above there is $\Theta \in |\mathfrak{M}_\Lambda|$ such that $\psi \to \mathrm{Sim}(w) \in \Theta^+$ and $\mathrm{Sim}(w) \in \Theta^-$. From Proposition 6.4 we see that there is $\Delta \succcurlyeq_c \Theta$ such that $(\mathfrak{I}, w) \rightharpoonup (\mathfrak{M}_\Lambda, \Delta)$, so that $\psi \in \Delta^+$ and, once again by Proposition 6.4, $\mathrm{Sim}(w) \in \Delta^-$. It follows that $\psi \to \mathrm{Sim}(w) \notin \Delta^+$; but in view of upward persistence, this contradicts that $\psi \to \mathrm{Sim}(w) \in \Theta^+$.

(3) Suppose that $v \succcurlyeq w$. Reasoning as above, it suffices to show that if $\Gamma \in |\mathfrak{M}_\Lambda|$ is such that $\mathrm{Sim}(w) \in \Gamma^-$, then also $\mathrm{Sim}(v) \in \Gamma^-$. But if $\mathrm{Sim}(w) \in \Gamma^-$, there is $\Theta \succcurlyeq_\Lambda \Gamma$ such that $(\mathfrak{I}, w) \rightharpoonup (\mathfrak{M}_\Lambda, \Theta)$. By forward confluence $(\mathfrak{I}, v) \rightharpoonup (\mathfrak{M}_\Lambda, \Delta)$ for some $\Delta \succcurlyeq_\Lambda \Theta$. Thus by Proposition 6.4, $\mathrm{Sim}(v) \in \Delta^-$ and by upwards persistence $\mathrm{Sim}(v) \in \Gamma^-$. Since $\Gamma \in |\mathfrak{M}_\Lambda|$ was arbitrary, the claim follows.

(4) We prove that if $\Gamma \in |\mathfrak{M}_\Lambda|$ is such that

$$\bigwedge_{\psi \in \ell^-(w)} \mathrm{Sim}(w) \in \Gamma^+, \tag{1}$$

then $\psi \in \Gamma^+$. If (1) holds then by Theorem 7.1, there is $w \in |\mathfrak{I}|$ with $(\mathfrak{I}, w) \rightharpoonup (\mathfrak{M}_\Lambda, \Gamma)$. By Proposition 6.4, $\mathrm{Sim}(w) \in \Gamma^-$, and hence it follows from (1) that $\psi \notin \ell^-(w)$, but $\mathfrak{I}$ is saturated and $\psi \in \Sigma$, so $\psi \in \ell^+(w)$ and thus $\psi \in \Gamma^+$, as required.

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(5) Suppose that $\Gamma \in |\mathfrak{M}_\Lambda|$ is such that

$$\circ \bigwedge_{wS_\mathfrak{I} v} \mathrm{Sim}(v) \in \Gamma^+, \tag{2}$$

and assume toward a contradiction that $\mathrm{Sim}(w) \in \Gamma^-$. By Proposition 6.4 $(\mathfrak{I}, w) \rightharpoonup (\mathfrak{M}_\Lambda, \Delta)$ for some $\Delta \succcurlyeq_\Lambda \Gamma$. Since $\rightharpoonup$ is a dynamic simulation, it follows that there is $v \in |\mathfrak{I}|$ with $w\, S_\mathfrak{I}\, v$ and $(\mathfrak{I}, v) \rightharpoonup \big(\mathfrak{M}_\Lambda, S_\Lambda(\Delta)\big)$, so that $\mathrm{Sim}(v) \in \big(S_\Lambda(\Delta)\big)^-$. It follows that $\circ\mathrm{Sim}(v) \in \Gamma^-$, since S_Λ is sensible and Γ is saturated. But $\Delta \succcurlyeq_\Lambda \Gamma$, so that $\circ\mathrm{Sim}(v) \in S_\Lambda^-(\Gamma)$ as well, contradicting (2). ⊣

Given a finite set Σ of formulas, we will define a quasimodel $\mathfrak{I}_\Sigma^\blacklozenge$ falsifying all consistent Σ-types. This quasimodel is a substructure of $\mathfrak{I}_\Sigma$, containing only moments which are *possible* in the following sense.

Definition 7.4. Let Λ be an admissible temporal logic over $\mathcal{L} \subseteq \mathcal{L}_{\Diamond\forall}$. Fix $\Sigma \Subset \mathcal{L}$. We say that a moment $w \in |\mathfrak{I}_\Sigma|$ is *possible* if $\not\vdash \mathrm{Sim}(w)$, and denote the set of possible Σ-moments by Poss_Σ.

The following gives an alternative characterization of Definition 7.4 and can be checked using Proposition 6.4 and the Lindenbaum lemma.

Lemma 7.5. *Let Λ be an admissible temporal logic over $\mathcal{L} \subseteq \mathcal{L}_{\Diamond\forall}$ and $\Sigma \Subset \mathcal{L}$. Then, $w \in |\mathfrak{I}_\Sigma|$ is possible if and only if there is $\Gamma \in |\mathfrak{M}_\Lambda|$ such that $(\mathfrak{I}_\Sigma, w) \rightharpoonup (\mathfrak{M}_\Lambda, \Gamma)$.*

With this we are ready to define our initial structure, which as we will see later is indeed a quasimodel.

Definition 7.6. Let Λ be an admissible temporal logic over $\mathcal{L} \subseteq \mathcal{L}_{\Diamond\forall}$. Given $\Sigma \Subset \mathcal{L}$, we define the *initial structure* for Σ by $\mathfrak{I}_\Sigma^\blacklozenge = \mathfrak{I}_\Sigma \restriction \mathrm{Poss}_\Sigma$.

Remark 7.7. In principle $\mathfrak{I}_\Sigma^\blacklozenge$ depends on Λ, but we do not reflect this in the notation since Λ will always be either $\mathsf{ITL}_\Diamond$ or $\mathsf{ITL}_{\Diamond\forall}$, depending on whether $\forall$ appears in Σ.

Our strategy from here on will be to show that canonical structures are indeed quasimodels; once we establish this, completeness of $\mathsf{ITL}_\Diamond$ is an easy consequence. The most involved step will be showing that the successor relation on $\mathfrak{I}_\Sigma^\blacklozenge$ is ω-sensible, but we begin with some simpler properties.

Lemma 7.8. *Let Λ be a logic extending $\mathsf{ITL}_\circ$ over a temporal language $\mathcal{L} \subseteq \mathcal{L}_{\Diamond\forall}$. Let $\Sigma \Subset \mathcal{L}$, $\mathfrak{I} = \mathfrak{I}_\Sigma$, and $\mathfrak{I}^\blacklozenge = \mathfrak{I}_\Sigma^\blacklozenge$. Then, $|\mathfrak{I}^\blacklozenge|$ is an open subset of $|\mathfrak{I}|$ and $S_{\mathfrak{I}^\blacklozenge}$ is serial.*

Proof. To check that $|\mathfrak{I}^\blacklozenge|$ is upward closed, let $w \in |\mathfrak{I}^\blacklozenge|$ and suppose $v \succcurlyeq w$. Now, by Proposition 7.3(3), we have that $\vdash \mathrm{Sim}(v) \to \mathrm{Sim}(w)$; hence if w is possible, so is v. To see that $S_{\mathfrak{I}^\blacklozenge}$ is serial, observe that by Proposition 7.3(5), if $w \in |\mathfrak{I}^\blacklozenge| \subseteq |\mathfrak{I}|$, $\vdash \circ \bigwedge_{wS_\mathfrak{I} v} \mathrm{Sim}(v) \to \mathrm{Sim}(w)$. Since w is possible, it follows that for some v with $w\, S_\mathfrak{I}\, v$, v is possible as well, for otherwise $\circ \bigwedge_{wS_\mathfrak{I} v} \mathrm{Sim}(v)$ would be equivalent to $\circ\top$, allowing us to deduce $\mathrm{Sim}(w)$. But then $v \in |\mathfrak{I}^\blacklozenge|$, as needed. ⊣

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§8. ω-sensibility. In this section we will show that $S_{\mathfrak{J}^\blacklozenge}$ is ω-sensible, the most difficult step in proving that $\mathfrak{I}^\blacklozenge = \mathfrak{I}^\blacklozenge_\Sigma$ is a quasimodel. Fix an admissible temporal logic Λ over $\mathcal{L} \subseteq \mathcal{L}_{\Diamond\forall}$, and let R denote the transitive, reflexive closure of $S_{\mathfrak{J}^\blacklozenge}$. If $w\ R\ v$, we say that v is *reachable* from w.

LEMMA 8.1. *Let Λ be an admissible temporal logic over $\mathcal{L} \subseteq \mathcal{L}_{\Diamond\forall}$. If $\Sigma \Subset \mathcal{L}$ and $w \in |\mathfrak{I}^\blacklozenge_\Sigma|$, then* $\vdash \circ \bigwedge_{wRv} \mathrm{Sim}(v) \to \bigwedge_{wRv} \mathrm{Sim}(v)$.

PROOF. Let $\mathfrak{I} = \mathfrak{I}_\Sigma$. By Proposition 7.3(5) we have that, for all $v \in R(w)$,

$$\vdash \circ \bigwedge_{vS_{\mathfrak{J}}u} \mathrm{Sim}(u) \to \mathrm{Sim}(v).$$

Now, if $u \notin |\mathfrak{I}^\blacklozenge_\Sigma|$, then $\vdash \mathrm{Sim}(u)$; hence by (NR2) we have $\vdash \circ\mathrm{Sim}(u)$, and we can remove $\mathrm{Sim}(u)$ from the conjunction using Lemma 3.5 and propositional reasoning. Since $v \in R(w)$ was arbitrary, this shows that

$$\vdash \circ \bigwedge_{wRv} \mathrm{Sim}(v) \to \bigwedge_{wRv} \mathrm{Sim}(v). \qquad \dashv$$

From this we obtain the following, which evidently implies ω-sensibility:

PROPOSITION 8.2. *Let Λ be a logic extending* $\mathsf{ITL}_\Diamond$ *over a temporal language $\mathcal{L} \subseteq \mathcal{L}_{\Diamond\forall}$ and $\Sigma \Subset \mathcal{L}$. If $w \in |\mathfrak{I}^\blacklozenge_\Sigma|$ and $\Diamond\psi \in \ell^+(w)$, then there is $v \in R(w)$ such that $\psi \in \ell^+(v)$.*

PROOF. Towards a contradiction, assume that $w \in \mathrm{Poss}_\Sigma$ and $\Diamond\psi \in \ell^+(w)$ but, for all $v \in R(w)$, $\psi \in \ell^-(w)$. By Lemma 8.1, $\vdash \circ \bigwedge_{wRv} \mathrm{Sim}(v) \to \bigwedge_{wRv} \mathrm{Sim}(v)$. By the $\Diamond$-induction Rule (ER2), $\vdash \Diamond \bigwedge_{wRv} \mathrm{Sim}(v) \to \bigwedge_{wRv} \mathrm{Sim}(v)$; in particular,

$$\vdash \Diamond \bigwedge_{wRv} \mathrm{Sim}(v) \to \mathrm{Sim}(w). \tag{3}$$

Now let $v \in R(w)$. By Proposition 7.3(1) and the assumption that $\psi \in \ell^-(v)$ we have that $\vdash \psi \to \mathrm{Sim}(v)$, and since v was arbitrary, $\vdash \psi \to \bigwedge_{wRv} \mathrm{Sim}(v)$. Using distributivity (ER1) we further have that $\vdash \Diamond\psi \to \Diamond \bigwedge_{wRv} \mathrm{Sim}(v)$. This, along with (3), shows that $\vdash \Diamond\psi \to \mathrm{Sim}(w)$; however, by Proposition 7.3(2) and our assumption that $\Diamond\psi \in \ell^+(w)$ we have that $\vdash \big(\Diamond\psi \to \mathrm{Sim}(w)\big) \to \mathrm{Sim}(w)$; hence by modus ponens we obtain $\vdash \mathrm{Sim}(w)$, which contradicts the assumption that $w \in \mathrm{Poss}_\Sigma$. We conclude that there can be no such w. $\dashv$

COROLLARY 8.3. *Let Λ be an admissible temporal logic over $\mathcal{L} \subseteq \mathcal{L}_{\Diamond\forall}$. Then, if $\Sigma \Subset \mathcal{L}$, $\mathfrak{I}^\blacklozenge_\Sigma$ is a quasimodel.*

PROOF. Let $\mathfrak{I}^\blacklozenge = \mathfrak{I}^\blacklozenge_\Sigma$. By Lemma 7.8, $|\mathfrak{I}^\blacklozenge|$ is upwards closed in $|\mathfrak{I}_\Sigma|$ and $S_{\mathfrak{J}^\blacklozenge}$ is serial, while by Proposition 8.2, $S_{\mathfrak{J}^\blacklozenge}$ is ω-sensible. It follows from Lemma 4.11 that $\mathfrak{I}^\blacklozenge$ is a quasimodel. $\dashv$

We are now ready to prove that $\mathsf{ITL}_\Diamond$ is complete.

THEOREM 8.4. *If $\varphi \in \mathcal{L}_\Diamond$ is valid in the class of dynamical systems, then* $\mathsf{ITL}_\Diamond \vdash \varphi$.

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PROOF. We prove the contrapositive. Suppose φ is an unprovable formula and let

$$W = \left\{w \in \mathfrak{I}_{\mathrm{sub}(\varphi)} : \varphi \in \ell^-(w)\right\}.$$

Then, by Proposition 7.3(4) we have that $\vdash \bigwedge_{w\in W} \mathrm{Sim}(w) \to \varphi$; since φ is unprovable, it follows that some $w^* \in W$ is possible and hence $w^* \in \mathrm{Poss}_{\mathrm{sub}(\varphi)}$. By Corollary 8.3, $\mathfrak{I}^{\blacklozenge}_{\mathrm{sub}(\varphi)}$ is a quasimodel, so that by Theorem 4.9, φ is falsifiable in some dynamical system. ⊣

§9. The universal modality. Now let us show that $\mathsf{ITL}_{\diamond\forall}$ is complete for the class of dynamical systems. As before, our completeness proof relies on the canonical model $\mathcal{M}_{\mathsf{ITL}_{\diamond\forall}}$ and the initial quasimodel $\mathfrak{I}^{\blacklozenge}_{\Sigma}$ (for suitable Σ), but now we cannot use these structures as they are, given that they are not honest (see Definition 4.3). We will first exhibit an honest substructure of $\mathcal{M}_{\mathsf{ITL}_{\diamond\forall}}$.

DEFINITION 9.1. Let $\Sigma \subseteq \mathcal{L}_{\diamond\forall}$. We define $\Sigma_\forall$ to be the set of formulas of Σ of the form $\forall\varphi$. A *universal* Σ-profile is a partition $\Pi = (\Pi^+, \Pi^-)$ of $\Sigma_\forall$. If $\Phi = (\Phi^+, \Phi^-)$ is a pair of sets of formulas, we define $\Phi_\forall = \big((\Phi^+)_\forall, (\Phi^-)_\forall\big)$, which we will henceforth write as $(\Phi^+_\forall, \Phi^-_\forall)$.

Recall that we defined $\Pi \subseteq_T \Phi$ if $\Pi^- \subseteq \Phi^-$ and $\Pi^+ \subseteq \Phi^+$

DEFINITION 9.2. Given $\Sigma \subseteq \mathcal{L}_{\diamond\forall}$, a Σ-labelled structure $\mathfrak{A}$ and a universal Σ-profile Π, we define $\mathfrak{A}[\Pi] = \mathfrak{A} \upharpoonright \{w \in |\mathfrak{A}| : \Pi \subseteq_T \ell_\forall(w)\}$.

LEMMA 9.3. *Let $\Sigma \Subset \mathcal{L}_{\diamond\forall}$, $\Lambda = \mathsf{ITL}_{\diamond\forall}$, $\mathfrak{M}_\Lambda$ be the canonical Λ-model, and Π a universal Σ-profile. Then, $|\mathfrak{M}_\Lambda[\Pi]|$ is open and honest as a Σ-labelled structure.*

PROOF. It can be checked using universal excluded middle (UA1) that $|\mathfrak{M}_\Lambda[\Pi]|$ is open and using (UA6) that it is S_Λ-invariant. In view of Lemma 10.9, we may moreover conclude that $\mathfrak{M}_\Lambda[\Pi]$ is a weak quasimodel.

It remains to show that $\mathfrak{M}_\Lambda[\Pi]$ is honest. That $\forall\varphi \in \Phi^+ \cap \Sigma$ implies that $\varphi \in \Psi^+$ for all $\Phi, \Psi \in |\mathfrak{M}_\Lambda[\Pi]|$ follows readily from the truth Axiom (UA4). For the remaining condition, let us take $\Phi \in |\mathfrak{M}_\Lambda[\Pi]|$ and let $\forall\varphi \in \Sigma$ be such that $\forall\varphi \in \Phi^-$. Let $\Psi' = (\Pi^+, \{\varphi\} \cup \Pi^-)$. Assume towards a contradiction that Ψ' is not consistent, so that $\vdash \bigwedge \Pi^+ \to \varphi \vee \bigvee \Pi^-$. By Rule (UR1), $\vdash \forall\big(\bigwedge \Pi^+ \to \varphi \vee \bigvee \Pi^-\big)$. By Axiom (UA2), $\vdash \forall \bigwedge \Pi^+ \to \forall(\varphi \vee \bigvee \Pi^-)$. From $\Phi_\forall = \Pi$, Axioms (UA5) and (UA2), and propositional reasoning we conclude that $\vdash \bigwedge \Pi^+ \to \forall \bigwedge \Pi^+$; hence $\vdash \bigwedge \Pi^+ \to \forall(\varphi \vee \bigvee \Pi^-)$. By several applications of Axiom (UA3), $\vdash \forall(\varphi \vee \bigvee \Pi^-) \to \forall\varphi \vee \bigvee \Pi^-$; hence $\vdash \bigwedge \Pi^+ \to \forall\varphi \vee \bigvee \Pi^-$, which since $\forall\varphi \in \Pi^- \subseteq \Phi^-$ implies that Φ is inconsistent, a contradiction.

Hence Ψ' is consistent. By Lemma 5.3, Ψ' can be extended to a prime type Ψ. Since, by construction, $\Pi \subseteq_T \Psi$, we have that $\Psi \in |\mathfrak{M}_\Lambda[\Pi]|$ and, moreover, $\varphi \in \Psi^-$, as required. ⊣

This already is sufficient to prove that our logics over $\mathcal{L}_{\circ\forall}$ are complete. The proof of the following is analogous to that of Theorem 5.7, but using the structures $\mathfrak{M}_\Lambda[\Pi]$ instead of $\mathfrak{M}_\Lambda$.

THEOREM 9.4. *$\mathsf{ITL}_{\circ\forall}$ is complete for the class of expanding posets and $\mathsf{ITL}^+_{\circ\forall}$ for the class of persistent posets.*

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For the language with $\Diamond$ we will use an honest substructure of $\mathfrak{I}_\Sigma$, for which we use the following result of [19].

LEMMA 9.5. *Suppose that $\Sigma \subseteq \Delta \subseteq \mathcal{L}_{\Diamond\forall}$ are both closed under subformulas, $\mathfrak{X}$ is a Σ-labelled space, $\mathfrak{Y}$ is a Δ-labelled space, and $E \subseteq |\mathfrak{X}| \times |\mathfrak{Y}|$ is a total, surjective simulation. Then, if $\ell_\mathfrak{Y}$ is honest, it follows that $\ell_\mathfrak{X}$ is honest as well.*

Finally, we observe that $|\mathfrak{I}^\blacklozenge_\Sigma[\Pi]|$ is a quasimodel. Below, we write $w \rightharpoonup \Gamma$ instead of $(\mathfrak{I}_\Sigma, w) \rightharpoonup (\mathfrak{M}_\Lambda, \Gamma)$.

LEMMA 9.6. *If $\Sigma \Subset \mathcal{L}_{\Diamond\forall}$ and Π is a Σ-universal profile then the restriction of $\rightharpoonup$ to $|\mathfrak{I}^\blacklozenge_\Sigma[\Pi]| \times |\mathfrak{M}_\Lambda[\Pi]|$ is total and surjective. Moreover, $\mathfrak{I}^\blacklozenge_\Sigma[\Pi]$ is a quasimodel.*

PROOF. Let $\mathfrak{I}^\blacklozenge = \mathfrak{I}^\blacklozenge_\Sigma$. If $w \in |\mathfrak{I}^\blacklozenge[\Pi]|$, by Lemma 7.5 there is $\Gamma \in |\mathfrak{M}_\Lambda|$ with $w \rightharpoonup \Gamma$, and by label-preservation $\Pi \subseteq_T \Gamma$, so that $\Gamma \in |\mathfrak{M}_\Lambda[\Pi]|$. Hence $\rightharpoonup$ is total. Conversely, if $\Gamma \in |\mathfrak{M}_\Lambda[\Pi]|$, then by Theorem 7.1 there is $w \in |\mathfrak{I}_\Sigma|$ such that $w \rightharpoonup \Gamma$, and once again by label-preservation $w \in |\mathfrak{I}^\blacklozenge[\Pi]|$.

To see that $\mathfrak{I}^\blacklozenge[\Pi]$ is a quasimodel, in view of Lemma 4.11, it suffices to show that $|\mathfrak{I}^\blacklozenge[\Pi]|$ is open and $S_{\mathfrak{I}^\blacklozenge}$-invariant. However, if $w \in |\mathfrak{I}^\blacklozenge[\Pi]|$, then since $\rightharpoonup$ is total we have that there is $\Gamma \in |\mathfrak{M}_\Lambda[\Pi]|$ with $w \rightharpoonup \Gamma$. By Lemma 9.3 $|\mathfrak{M}_\Lambda[\Pi]|$ is open; hence by continuity there is $\Delta \in |\mathfrak{M}_\Lambda[\Pi]|$ such that $v \rightharpoonup \Delta$, and once again by label-preservation this shows that $\Pi \subseteq_T \ell(v)$, so that $v \in |\mathfrak{I}^\blacklozenge[\Pi]|$. That $|\mathfrak{I}^\blacklozenge[\Pi]|$ is $S_{\mathfrak{I}^\blacklozenge}$-invariant is immediate from conditions (5) and (6) of Definition 4.6 and the fact that $S_{\mathfrak{I}^\blacklozenge}$ is sensible. ⊣

THEOREM 9.7. *The logic $\mathsf{ITL}_{\Diamond\forall}$ is complete for the class of dynamical systems.*

PROOF. If φ is unprovable, then $\varphi \in \Phi^-$ for some $\Phi \in |\mathfrak{M}_\Lambda|$, and by Theorem 7.1, there is some $w \in |\mathfrak{I}^\blacklozenge_\Sigma|$ such that $(\mathfrak{I}^\blacklozenge_\Sigma, w) \rightharpoonup (\mathfrak{M}_\Lambda, \Phi)$. Let $\Pi = (\ell(w))_\forall$. Then, $\mathfrak{M}_\Lambda[\Pi]$ is an honest weak quasimodel, so that by Lemmas 9.5 and 9.6, so is $\mathfrak{I}^\blacklozenge_\Sigma[\Pi]$. It follows by Theorem 4.9 that φ is falsifiable on some dynamic topological model. ⊣

§10. Completeness for expanding posets. Our goal for this section is to show that the temporal logics of dynamic posets and of dynamical systems coincide over $\mathcal{L}_\Diamond$. We will show this by 'unwinding' a quasimodel to produce a dynamical poset. First we discuss some operations on types that will be used in the unwinding. If Σ is a set of formulas and Ψ is a type, define $\Psi/\Sigma = (\Psi^+ \cap \Sigma, \Psi^-)$, and $\mathrm{sub}(\Sigma) = \bigcup_{\varphi\in\Sigma} \mathrm{sub}(\varphi)$. If Φ is also a type, we further set $\Phi \sqsubseteq_T \Psi$ if $\Phi^- = \Psi^-$ and $\Phi^+ = \Psi^+ \cap \Delta$, where Δ is some set closed under subformulas; note that $\sqsubseteq_T$ is a refinement of $\preccurlyeq_T$. With this, we have the following:

LEMMA 10.1. *Let $\Phi, \Psi, \Gamma, \Theta$ be $\mathcal{L}_\Diamond$-types and $\Sigma \subseteq \mathcal{L}_\Diamond$ be closed under subformulas. Then*:

1. *Φ/Σ is also a type.*
2. *If $\Gamma \sqsubseteq_T \Phi \preccurlyeq_T \Psi$ or $\Gamma \preccurlyeq_T \Phi \sqsubseteq_T \Psi$ then $\Gamma \preccurlyeq_T \Psi$.*
3. *If $\Gamma \sqsubseteq_T \Phi\ S_T\ \Psi$ and $\mathrm{sub}(\Gamma^+) \subseteq \Sigma$, then $\Gamma\ S_T\ \Psi/\Sigma$.*

PROOF. To prove item 1, it is sufficient to check that the conditions of Definition 4.1 hold. Conditions 1 and 2 of Definition 4.1 are straightforward. Since $\Phi^- = (\Phi/\Sigma)^-$, conditions 4 and 6 clearly hold. For condition 7, suppose that $\varphi \to \psi \in (\Psi/\Sigma)^+$. Since

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Σ is closed under subformulas, $\varphi, \psi \in \Sigma$ and, since Ψ is a type it follows that either $\varphi \in \Psi^-$ or $\psi \in \Psi^+$. By definition either $\varphi \in \Psi^-$ or $\psi \in \Psi^+ \cap \Sigma$. The proofs for conditions 3 and 5 of Definition 4.1 are similar and left to the reader.

For item 2 of the lemma, it suffices to observe that $\sqsubseteq_T$ is a refinement of $\preccurlyeq_T$, and that $\preccurlyeq_T$ is already transitive.

For item 3, note that $\Gamma^- = \Phi^-$ and $\Gamma^+ = \Phi^+ \cap \Delta$ for some Δ closed under subformulas. We must check that each condition of Definition 4.6 holds. As an example, we work out (3). If $\Diamond\psi \in \Gamma^+$, since $\mathrm{sub}(\Gamma^+) \subseteq \Sigma$ then $\Diamond\psi, \psi \in \Sigma$. From $\Gamma \sqsubseteq_T \Phi\ S_T\ \Psi$ we conclude that $\Diamond\psi \in \Phi^+$, and either $\psi \in \Phi^+$ or $\Diamond\psi \in \Psi^+$. In the first case, we note that $\Diamond\psi \in \Gamma^+$ yields $\psi \in \Delta$; hence $\psi \in \Phi^+ \cap \Delta = \Gamma^+$. In the second, $\Diamond\psi \in \Psi^+ \cap \Sigma$, which means that $\Diamond\psi \in (\Psi/\Sigma)^+$. Other conditions follow similar reasoning and are left to the reader.

We may also wish to 'forget' temporal formulas that have been realized. Recall that $\sup(\varphi)$ denotes the set of superformulas of φ. Say that a formula φ is a *temporal formula* if it is of the forms $\circ\psi$ or $\Diamond\psi$, and if Φ is a set of formulas, say that $\varphi \in \Phi$ is *maximal in* Φ if it does not have any temporal superformulas in Φ (except φ). Then, define $\Phi \setminus \varphi = (\Phi^+ \setminus \sup(\varphi), \Phi^-)$.

LEMMA 10.2. *Suppose that* $\Phi\ S_T\ \Psi$. (1) *If* $\circ\varphi$ *is maximal in* Φ^+, *then* $\Phi\ S_T\ (\Psi \setminus \circ\varphi)$.(2) *If* $\Diamond\varphi$ *is maximal in* Φ^+ *and* $\varphi \in \Phi^+$, *then* $\Phi\ S_T\ (\Psi \setminus \Diamond\varphi)$.

PROOF SKETCH. We consider the first item; the second is analogous. Assuming that $\circ\varphi$ is maximal in Φ^+, it must be checked that the four conditions of Definition 4.6 hold. For conditions (1) and (3), note that if $\circ\theta$ or $\Diamond\theta$ belong to Φ^+, then neither θ nor $\Diamond\theta$ belong to $\sup(\circ\varphi)$. For conditions (2) and (4), it suffices to observe that $(\Psi \setminus \circ\varphi)^- = \Psi^-$. ⊣

The unwinding procedure defined below is similar to the construction sketched in Remark 4.10. There, the points of the limit model obtained from a quasimodel are the infinite paths satisfying all $\Diamond$-formulas in their labels. To obtain a poset rather than a topological space, we will instead work with finite paths. We can do this because in absence of $\Box$ or $\forall$, once a path has realized all formulas $\Diamond\psi$ in its first world, no new formulas need to be generated. The following construction makes this precise.

DEFINITION 10.3. If $\mathfrak{Q} = (W, \preccurlyeq, S, \ell)$ is an $\mathcal{L}_\Diamond$-quasimodel, a *path* (*on* $\mathfrak{Q}$) is a sequence $(w_i)_{i<n} \subseteq W$ such that $w_i\ S\ w_{i+1}$ for all $i < n-1$. We define a *typed path* (*on* $\mathfrak{Q}$) to be a sequence $\big((w_i, \Phi_i)\big)_{i<n}$ such that $(w_i)_{i<n}$ is a path, for all $i < n$, $\Phi_i \sqsubseteq_T \ell(w_i)$, and for all $i < n-1$, $\Phi_i\ S_T\ \Phi_{i+1}$. We will henceforth write $(w_i, \Phi_i)_{i<n}$ instead of $\big((w_i, \Phi_i)\big)_{i<n}$.

We say that $(w_i, \Phi_i)_{i<n}$ is *properly typed* if $\mathrm{sub}(\Phi^+_{i+1}) \subseteq \mathrm{sub}(\Phi^+_i)$ for all $i < n-1$, and *terminal* if $\Phi^+_{n-1} = \varnothing$.

Note that we allow $\Phi_i \sqsubseteq_T \ell(w_i)$ and not only $\Phi_i = \ell(w_i)$. This will allow us to use finite paths, as temporal formulas can be 'forgotten' once they have been realized.

DEFINITION 10.4. Let $\mathfrak{Q} = (W, \preccurlyeq, S, \ell)$ be an $\mathcal{L}_\Diamond$-quasimodel. We define the *weak limit model* $\vec{\mathfrak{Q}} = (\vec{W}, \vec{\preccurlyeq}, \vec{S}, \vec{\ell})$ of $\mathfrak{Q}$ as follows:

- $\vec{W}$ is the set of terminal typed paths on $\mathfrak{Q}$ together with the empty path, which we denote ε.

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- For $\alpha = (w_i, \Phi_i)_{i<n}$, $\beta = (v_i, \Psi_i)_{i<m} \in \vec{W}$, define $\alpha \vec{\preccurlyeq} \beta$ if $n \leq m$ and for all $i < n$, $w_i \preccurlyeq v_i$ and $\Phi_i \preccurlyeq_T \Psi_i$.
- Define $\vec{S}(w_i, \Phi_i)_{i<n} = (w_{i+1}, \Phi_{i+1})_{i<n-1}$; note that $\vec{S}(\varepsilon) = \varepsilon$.
- If $n > 0$, define $\vec{\ell}(w_i, \Phi_i)_{i<n} = \Phi_0$. Set $\vec{\ell}^-(\varepsilon) = \bigcup_{w \in W} \ell^-(w)$ and $\vec{\ell}^+(\varepsilon) = \varnothing$.

The structure $\vec{\mathfrak{Q}}$ we have just defined is always a deterministic quasimodel, as we show in the following lemmas.

LEMMA 10.5. *If $\mathfrak{Q}$ is an $\mathcal{L}_\diamond$-quasimodel then $\vec{\mathfrak{Q}}$ is a dynamic poset.*

PROOF. Write $\mathfrak{Q} = (W, \preccurlyeq, S, \ell)$ (so that $\vec{\mathfrak{Q}} = (\vec{Q}, \vec{\preccurlyeq}, \vec{S}, \vec{\ell})$). We have to prove that $\vec{\preccurlyeq}$ is a partial order on $\vec{W}$, $\vec{S}$ is a function, and that it is continuous. We prove only continuity and leave the other properties to the reader. Let $\alpha = (w_i, \Phi_i)_{i<n}$ and $\beta = (v_i, \Psi_i)_{i<m}$. If $\alpha \vec{\preccurlyeq} \beta$ then $n \leq m$ and for all $i < n$, $w_i \preccurlyeq v_i$ and $\Phi_i \preccurlyeq_T \Psi_i$. If $n > 0$, then we also have $n - 1 \leq m - 1$ and for all $i < n - 1$, $w_{i+1} \preccurlyeq v_{i+1}$ and $\Phi_{i+1} \preccurlyeq_T \Psi_{i+1}$, i.e., $\vec{S}(\alpha) = (w_{i+1}, \Phi_{i+1})_{i<n-1} \vec{\preccurlyeq} (v_{i+1}, \Psi_{i+1})_{i<m-1} = \vec{S}(\beta)$, as needed. If $n = 0$ then $\alpha = \varepsilon$, so that $\vec{S}(\alpha) = \varepsilon$ and, vacuously, $\varepsilon \vec{\preccurlyeq} S(\beta)$. ⊣

Next, we must show that $\vec{\mathfrak{Q}}$ has 'enough' paths to form a model. First we show that we can iterate the forward-confluence property.

LEMMA 10.6. *If $\mathfrak{Q}$ is an $\mathcal{L}_\diamond$-quasimodel, $(w_i, \Phi_i)_{i<n}$ is a typed path in $\mathfrak{Q}$, and $w_0 \preccurlyeq v_0$, then there is a typed path $(v_i, \Psi_i)_{i<n}$ such that $w_i \preccurlyeq v_i$ and $\Phi_i \preccurlyeq_T \Psi_i$ for all $i < n$.*

PROOF. Let $\mathfrak{Q} = (W, \preccurlyeq, S, \ell)$. First we find v_i by induction on i; v_0 is already given, and once we have found v_i, we use forward confluence to choose v_{i+1} so that $v_i \; S \; v_{i+1}$ and $w_{i+1} \preccurlyeq v_{i+1}$. Then we set $\Psi_i = \ell(v_i)$; since S is sensible, $\Psi_i \; S_T \; \Psi_{i+1}$, and by Lemma 10.1.2, $\Phi_i \preccurlyeq_T \Psi_i$. ⊣

Now we want to prove that any point can be included in a terminal typed path. For this we will first show that we can work mostly with properly typed paths (where no new formulas are introduced along the path), thanks to the following.

LEMMA 10.7. *Let $\mathfrak{Q}$ be an $\mathcal{L}_\diamond$-quasimodel, $(w_i)_{i<n}$ be a path on $\mathfrak{Q}$, and $\Phi_0 \sqsubseteq_T \ell(w_0)$. Then, there exist $(\Phi_i)_{i<n}$ such that $(w_i, \Phi_i)_{i<n}$ is a properly typed path.*

PROOF. For $i < n - 1$ define recursively $\Phi_{i+1} = {}^{\ell(w_{i+1})}\!/_{\mathrm{sub}(\Phi_i^+)}$; by the assumption that $S_{\mathfrak{Q}}$ is sensible and Lemma 10.1, $\Phi_i \; S_T \; \Phi_{i+1}$ for each $i < n - 1$. It is easy to see that $(w_i, \Phi_i)_{i<n}$ thus defined is properly typed. ⊣

However, the properly typed paths we have constructed need not be terminal. This will typically require extending them to a long-enough path. The extension procedure is precisely the crux of our unwinding procedure.

LEMMA 10.8. *If $\mathfrak{Q}$ is an $\mathcal{L}_\diamond$-quasimodel, then any non-empty typed path on $\mathfrak{Q}$ can be extended to a terminal path.*

PROOF. Let $\alpha = (w_i, \Phi_i)_{i<m}$ be any typed path on $\mathfrak{Q}$ and write $\mathfrak{Q} = (W, \preccurlyeq, S, \ell)$. For a type Φ, define $\|\Phi\| = |\mathrm{sub}(\Phi^+)|$. We proceed to prove the claim by induction on $\|\Phi_{m-1}\|$. Consider first the case where Φ_{m-1}^+ contains no temporal formulas; that is, formulas of the form $\circ\psi$ or $\diamond\psi$ for some ψ. In this case, using the seriality of S choose w_m such that $w_{m-1} \; S \; w_m$, and define $\Phi_{m+1} = (\ell(w_m)^-; \varnothing)$; it is easy to see

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that $(w_i, \Phi_i)_{i \leq m}$ is a terminal path. Otherwise, let φ be a maximal temporal formula of Φ^+_{m-1}, i.e., it does not appear as a proper subformula of any other temporal formula in Φ^+_{m-1}. We consider two sub-cases.

Assume first that $\varphi = \circ\psi$. Then, by the seriality of S, we may choose w_m so that $w_{m-1}\ S\ w_m$. Applying Lemma 10.7, let $\widetilde{\Phi}_m$ be such that $((w_{m-1}, \Phi_{m-1}), (w_m, \widetilde{\Phi}_m))$ is a properly typed path. Setting $\Phi_m = \widetilde{\Phi}_m \setminus \circ\psi$, we see by Lemma 10.2(1) that $((w_{m-1}, \Phi_{m-1}), (w_m, \Phi_m))$ is a properly typed path, and $\|\Phi_m\| < \|\Phi_{m-1}\|$, since the left-hand side does not count $\circ\psi$. Thus we may apply the induction hypothesis to obtain a terminal typed path $(w_i, \Phi_i)_{i<n}$ extending α.

Now consider the case where $\varphi = \Diamond\psi$. Since S is ω-sensible, there is a path $w_{m-1}\ S\ w_m\ S \cdots S\ w_k$ so that $\varphi \in \ell(w_k)$. Using the seriality of S, choose w_{k+1} so that $w_k\ S\ w_{k+1}$. By Lemma 10.7, there are types Φ_i for $m \leq i \leq k$ and a type $\widetilde{\Phi}_{k+1}$ such that $((w_{m-1}, \Phi_{m-1}), \ldots, (w_k, \Phi_k), (w_{k+1}, \widetilde{\Phi}_{k+1}))$ is a properly typed path. Then, define $\Phi_{k+1} = \widetilde{\Phi}_{k+1} \setminus \Diamond\psi$. Using Lemma 10.2 we see that (Φ_k, Φ_{k+1}) is sensible; moreover, $\|\Phi_{k+1}\| < \|\Phi_{m-1}\|$. Hence we can apply the induction hypothesis to obtain a terminal typed path $(w_i, \Phi_i)_{i<n}$ extending α. ⊣

With this, we are ready to show that our unwinding is indeed a deterministic quasimodel.

LEMMA 10.9. *If $\mathfrak{Q}$ is an $\mathcal{L}_\Diamond$-quasimodel, then $\vec{\mathfrak{Q}}$ is a deterministic $\mathcal{L}_\Diamond$-quasimodel.*

PROOF. Let $\mathfrak{Q} = (W, \preccurlyeq, S, \ell)$. We have already seen in Lemma 10.5 that $(\vec{W}, \vec{\preccurlyeq}, \vec{S})$ is a dynamic poset, so it remains to check that $(\vec{W}, \vec{\preccurlyeq}, \vec{\ell})$ is a labelled poset and $\vec{S}$ is sensible and ω-sensible. Let $\alpha = (w_i, \Phi_i)_{i<n} \in \vec{W}$.

First we must check that if $\alpha \vec{\preccurlyeq} \beta$, then $\vec{\ell}(\alpha) \preccurlyeq_T \vec{\ell}(\beta)$. Consider two cases; if $n > 0$, then β is also of the form $(v_i, \Psi_i)_{i<m}$ with $m > 0$ and by definition, $\vec{\ell}(\alpha) = \Phi_0 \preccurlyeq_T \Psi_0 = \vec{\ell}(\beta)$. Otherwise, $\alpha = \varepsilon$, and it is clear from the definition of $\vec{\ell}(\varepsilon)$ that $\vec{\ell}(\varepsilon) \preccurlyeq_T \vec{\ell}(\beta)$ regardless of β.

Now assume that $\varphi \to \psi \in \vec{\ell}^-(\alpha)$. If $n > 0$, then since $\mathfrak{Q}$ is a labelled poset, we can pick $v_0 \succcurlyeq w_0$ with $\varphi \in \ell^+(v_0)$ and $\psi \in \ell^-(v_0)$. Since $\Phi_0 \sqsubseteq_T \ell(w_0) \preccurlyeq_T \ell(v_0)$, by Lemma 10.1.2, $\Phi_0 \preccurlyeq_T \ell(v_0)$, so that by Lemma 10.6, there is a typed path $\beta' = (v_i, \Psi_i)_{i<n}$ with $\Psi_0 = \ell(v_0)$ such that $w_i \preccurlyeq v_i$ and $\Phi_i \preccurlyeq_T \Psi_i$ for all $i < n$. By Lemma 10.8, we can extend β' to a terminal path β. Then, it is easy to see that $\alpha \vec{\preccurlyeq} \beta$, $\varphi \in \vec{\ell}^+(\beta)$, and $\psi \in \vec{\ell}^-(\beta)$, as required.

To check that $\vec{S}$ is sensible, consider two cases. If $\vec{S}(\alpha) \neq \varepsilon$, then α has length at least two, but since α is a typed path, $\vec{\ell}(\alpha) = \Phi_0\ S_T\ \Phi_1 = \vec{\ell}\big(\vec{S}(\alpha)\big)$. Otherwise, $\vec{S}(\alpha) = \varepsilon$; this means that either $\alpha = \varepsilon$ and thus $\vec{\ell}^+(\alpha) = \varnothing$, or α has length 1, in which case α is terminal, so we also have that $\vec{\ell}^+(\alpha) = \varnothing$. In either case, there can be no temporal formula in $\vec{\ell}^+(\alpha)$. Now, if $\circ\varphi \in \vec{\ell}^-(\alpha)$, then $\circ\varphi \in \ell^-(w)$ for some $w \in W$, and hence $\varphi \in \ell^- S(v)$ for any v with $w\ S\ v$. But at least one such v exists since S is serial, and thus $\varphi \in \vec{\ell}^-(\varepsilon)$, as needed. Similarly, if $\Diamond\varphi \in \vec{\ell}^-(\alpha)$, then by Definition 4.1.9 $\varphi \in \ell^-(\alpha)$, so that $\varphi \in \vec{\ell}^-(\varepsilon)$.

Finally, we check that $\vec{S}$ is ω-sensible. Suppose that $\Diamond\varphi \in \ell(\alpha)^+$. This means that $\alpha \neq \varepsilon$, so $n > 0$, and since α is terminal, $\Diamond\alpha \notin \Phi_{n-1}$. But this is only possible if $\varphi \in \Phi_i$ for some $i < n - 1$, in which case $\varphi \in \vec{\ell}\big(\vec{S}^i(\alpha)\big)$. ⊣

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THEOREM 10.10. *A formula $\varphi \in \mathcal{L}_\Diamond$ is satisfiable (falsifiable) over the class of dynamic posets if and only if it is satisfiable (falsifiable) over the class of dynamical systems.*

PROOF. Suppose that $\varphi \in \mathcal{L}_\Diamond$ is satisfied (falsified) on a dynamical topological model. Then, by Theorem 4.9, φ is satisfied (falsified) on some point w_* of a $\mathrm{sub}(\varphi)$-quasimodel $\mathfrak{Q} = (W, \preccurlyeq, S, \ell)$. By Lemma 10.9, $\vec{\mathfrak{Q}}$ is a deterministic quasimodel, and by Lemma 10.8, $(w_*, \ell(w_*))$ can be extended to a terminal path $\alpha_* \in \vec{W}$. By Lemma 4.8, α_* satisfies (falsifies) φ on the dynamic poset model $(\vec{W}, \vec{\preccurlyeq}, \vec{S}, [\![\cdot]\!]_{\vec{\mathfrak{Q}}})$. ⊣

As an immediate consequence of Theorems 8.4 and 10.10, we conclude that $\mathsf{ITL}_\Diamond$ is complete for the class of dynamic posets, also called expanding posets.

COROLLARY 10.11. $\mathsf{ITL}_\Diamond = \mathsf{ITL}^{\mathsf{c}}_\Diamond = \mathsf{ITL}^{\mathsf{e}}_\Diamond$.

§11. Completeness for specific spaces. In this section we will show that the above completeness theorems already hold for some familiar spaces, which follows from known results regarding dynamic topological logic. Thus it will be convenient to briefly review DTL and how ITL embeds into it. Since the base logic of DTL is classical, we may use a simpler syntax, using the language $\mathcal{L}^C_*$ given by the grammar

$$\bot \mid p \mid \varphi \to \psi \mid \blacksquare\varphi \mid \circ\varphi \mid \Box\varphi \mid \forall\varphi.$$

We can then define $\neg, \wedge, \vee, \blacklozenge, \Diamond$ using standard classical validities. We follow the same convention for denoting sublanguages of $\mathcal{L}^C_*$ as for $\mathcal{L}_*$, where the allowed modalities are indicated as a subindex and languages with $\Box$ always allow $\circ$, so that for example, $\mathcal{L}^C_{\blacksquare\Box}$ denotes the sublanguage with modalities $\circ$, $\blacksquare$, and $\Box$.

Given a dynamical system $\mathfrak{X}$, a *classical valuation on* $\mathfrak{X}$ is a function $[\![\cdot]\!]^C : \mathcal{L}^C_* \to \mathcal{P}(|\mathfrak{X}|)$ such that:

$$\begin{aligned}
[\![\bot]\!]^C &= \varnothing, & [\![\varphi \to \psi]\!]^C &= (|\mathfrak{X}| \setminus [\![\varphi]\!]^C) \cup [\![\psi]\!]^C,\\
[\![\blacksquare\varphi]\!]^C &= ([\![\varphi]\!]^C)^\circ, & [\![\circ\varphi]\!]^C &= f^{-1}_{\mathfrak{X}}\, [\![\varphi]\!]^C,\\
[\![\Box\varphi]\!]^C &= \bigcap_{n<\omega} f^{-n}_{\mathfrak{X}}\, [\![\varphi]\!]^C, & [\![\forall\varphi]\!]^C &= |\mathfrak{X}| \text{ if } [\![\varphi]\!]^C = |\mathfrak{X}|, \text{ else } \varnothing.
\end{aligned}$$

Note that valuations of formulas are no longer restricted to open sets. We then have the following results regarding satisfiability on some standard metric spaces.

THEOREM 11.1 [16]. *If $\varphi \in \mathcal{L}^C_*$ is classically satisfiable on any topological space, then it is classically satisfiable on $\mathbb{Q}$.*

THEOREM 11.2 [11]. *For any $n \geq 2$, if $\varphi \in \mathcal{L}^C_{\blacksquare\Box}$ is classically satisfiable on an expanding poset, then it is classically satisfiable on $\mathbb{R}^n$.*

THEOREM 11.3 [16]. *If $\varphi \in \mathcal{L}^C_{\blacksquare\Box}$ is classically satisfiable on any complete metric space, then it is classically satisfiable on the Cantor space.*

REMARK 11.4. Fernández-Duque [16] states Theorem 11.1 for $\mathcal{L}^C_{\blacksquare\Box}$, but the proof provided yields the result for all of $\mathcal{L}^C_*$. Roughly speaking, this is due to the fact that the simulations constructed in the proof are total and surjective.

Theorem 11.2 is a strengthening of a result of Slavnov [35].

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Our intuitionistic temporal logic may then be interpreted in DTL via the Gödel–Tarski translation $\cdot^{\blacksquare}$, defined as follows:

DEFINITION 11.5. Given $\varphi \in \mathcal{L}_*$, we define $\varphi^{\blacksquare} \in \mathcal{L}_*^C$ recursively by setting

$$\begin{array}{llllll} \bot^{\blacksquare} & = & \bot, & (\Box\varphi)^{\blacksquare} & = & \blacksquare\Box\varphi^{\blacksquare}, \\ p^{\blacksquare} & = & \blacksquare p, & (\boxdot\varphi)^{\blacksquare} & = & \boxdot\varphi^{\blacksquare}, \end{array} \qquad \begin{array}{lll} (\varphi \odot \psi)^{\blacksquare} & = & \varphi^{\blacksquare} \odot \psi^{\blacksquare}, \\ (\varphi \to \psi)^{\blacksquare} & = & \blacksquare(\varphi^{\blacksquare} \to \psi^{\blacksquare}), \end{array}$$

where $\odot \in \{\wedge, \vee\}$ and $\boxdot \in \{\circ, \Diamond, \forall\}$.

In words, we put $\blacksquare$ in front of variables, implication, and $\Box$. The following can then be verified by a simple induction on φ:

LEMMA 11.6. *Let $\varphi \in \mathcal{L}_*$, and $\mathfrak{X}$ be any dynamic topological system. Suppose that $[\![\cdot]\!]$ is an intuitionistic valuation and $[\![\cdot]\!]^C$ a classical valuation such that, for every atom p, $([\![p]\!]^C)^\circ = [\![p]\!]$. Then, for every formula φ, $[\![\varphi]\!] = [\![\varphi^{\blacksquare}]\!]^C$.*

Then we obtain the following.

THEOREM 11.7. *Given $n \geq 2$, $\mathsf{ITL}_\Diamond$ is complete for $\mathbb{R}^n$, as well as for the Cantor space.*

PROOF. If φ is valid in $\mathbb{R}^n$ then by Lemma 11.6, $\varphi^{\blacksquare}$ is valid in $\mathbb{R}^n$ and hence, by Theorem 11.2, $\varphi^{\blacksquare}$ (and thus φ) is valid in the class of expanding posets. By Corollary 10.11, φ is derivable in $\mathsf{ITL}_\Diamond$. Completeness for the Cantor space follows by similar reasoning using Theorem 11.3. ⊣

By similar reasoning, but using Theorems 9.7 and 11.1, we obtain the following.

THEOREM 11.8. *$\mathsf{ITL}_{\Diamond\forall}$ is complete for $\mathbb{Q}$.*

In contrast Example 3.3 shows that $\mathsf{ITL}_\circ$ is already incomplete for $\mathbb{R}$. We leave the problem of axiomatizing $\mathsf{ITL}_\circ^{\mathbb{R}}$ open, along with the long-standing problem of axiomatizing $\mathsf{DTL}_\circ^{\mathbb{R}}$.

§12. Concluding remarks. We have provided a sound and complete axiomatization for $\Box$-free fragments of intuitionistic temporal logics interpreted on various classes of dynamical systems. Many questions remain open in this direction, perhaps most notably an extension to the full language with $\Box$. Boudou et al. [8] sketch a possible axiomatization. They also define an alternative interpretation for 'henceforth' which satisfies all of the standard axioms for $\Box$ of classical LTL. Whether either of the two interpretations for 'henceforth' yields a decidable logic, or whether the proposed axiomatizations are complete, remain open. One may also consider the corresponding logics with 'henceforth' for the class of dynamical posets, for spaces with an open map, or for persistent posets; none of these logics have been axiomatized, but we know that they are all distinct [8].

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