The Journal of Symbolic Logic
Volume 82, Number 4, December 2017

ON EQUIVALENCE RELATIONS GENERATED BY SCHAUDER BASES

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Abstract. In this article, a notion of Schauder equivalence relation $\mathbb{R}^{\mathbb{N}}/L$ is introduced, where L is a linear subspace of $\mathbb{R}^{\mathbb{N}}$ and the unit vectors form a Schauder basis of L. The main theorem is to show that the following conditions are equivalent:

(1) the unit vector basis is boundedly complete;
(2) L is F_σ in $\mathbb{R}^{\mathbb{N}}$;
(3) $\mathbb{R}^{\mathbb{N}}/L$ is Borel reducible to $\mathbb{R}^{\mathbb{N}}/\ell_\infty$.

We show that any Schauder equivalence relation generalized by a basis of ℓ_2 is Borel bireducible to $\mathbb{R}^{\mathbb{N}}/\ell_2$ itself, but it is not true for bases of c_0 or ℓ_1. Furthermore, among all Schauder equivalence relations generated by sequences in c_0, we find the minimum and the maximum elements with respect to Borel reducibility.

§1. Introduction. The notion of Borel reducibility becomes a tool to compare objects or problems from different branches of mathematics. In recent years, many equivalence relations concerning Banach space theory were investigated. One of the motivations of this article is the Borel reducibility among equivalence relations $\mathbb{R}^{\mathbb{N}}/\ell_p$ and $\mathbb{R}^{\mathbb{N}}/c_0$. It is proved by Dougherty and Hjorth that for $p, q \geq 1$,

$$\mathbb{R}^{\mathbb{N}}/\ell_p \leq_B \mathbb{R}^{\mathbb{N}}/\ell_q \iff p \leq q,$$

while $\mathbb{R}^{\mathbb{N}}/\ell_p$ and $\mathbb{R}^{\mathbb{N}}/c_0$ are Borel incomparable (see [4] and [11]). These results were generalized via different methods by several authors (see, e.g., [18] and [3]). In this article, we study equivalence relations of the form $\mathbb{R}^{\mathbb{N}}/L$, where L is a linear subspace of $\mathbb{R}^{\mathbb{N}}$. Moreover, the class of subspaces L we focus on can be specified by the following two equivalent conditions:

(i) there is a sequence (x_n) of non-zero elements in a Banach space X such that

$$L = \mathrm{coef}(X, (x_n)) \stackrel{\mathrm{Def}}{=} \{a \in \mathbb{R}^{\mathbb{N}} : \sum_n a(n)x_n \text{ converges in } X\}.$$

(ii) the unit vectors $e_n = (0, 0, \ldots, 0, \overset{n}{1}, 0, \ldots)$ form a Schauder basis of L.

If one, and thus all of the above conditions hold for L, we call $\mathbb{R}^{\mathbb{N}}/L$ a Schauder equivalence relation.

Schauder equivalence relations were already studied in different disguises. For example, it is obvious that all $\mathbb{R}^{\mathbb{N}}/\ell_p$ $(p \geq 1)$ and $\mathbb{R}^{\mathbb{N}}/c_0$ are Schauder equivalence

Received June 3, 2016.
2010 *Mathematics Subject Classification.* Primary 03E15, 46B15, 46B45.
Key words and phrases. equivalence relation, Borel reduction, Schauder basis.

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0022-4812/17/8204-0012
DOI:10.1017/jsl.2017.67

https://doi.org/10.1017/jsl.2017.67 Published online by Cambridge University Press

relations. Most recently, equivalence relations $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}/\ell_p(\ell_q)$ were considered by Gao and Yin [9]. We can easily see that, for any $p, q \geq 1$, $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}/\ell_p(\ell_q)$ is Borel bireducible to a Schauder equivalence relation. Mátrai [18] considered certain equivalence relation $\mathbf{E}_f$ (parameterized by continuous function f) on $[0,1]^{\mathbb{N}}$. Borel reducibility between equivalence relations of the form $\mathbf{E}_f$ was investigated in [18]. Yin [21] generalized Mátrai's results to show that the partial order structure $P(\omega)/\mathrm{Fin}$ can be embedded into the set of these $\mathbf{E}_f$'s equipped with the partial ordering of Borel reducibility. In fact, as noted in Yin [21], any such $\mathbf{E}_f$ appeared in [21] is Borel bireducible to a Schauder equivalence relation $\mathbb{R}^{\mathbb{N}}/L$, where L is an Orlicz sequence space.

The main theorem of this article is the following:

THEOREM 1.1. *If the unit vectors (e_n) form a Schauder basis of a Banach space L, then the following are equivalent*:

(1) *(e_n) is a boundedly complete basis*;
(2) *L is F_σ in $\mathbb{R}^{\mathbb{N}}$*;
(3) $\mathbb{R}^{\mathbb{N}}/L \leq_B \mathbb{R}^{\mathbb{N}}/\ell_\infty$.

Reducibility and nonreducibility between Schauder equivalence relations generated by different sequences of the same space are considered, especially in the case when the generating sequences are Schauder bases. In this article, we mainly focus on bases of three special Banach spaces: ℓ_2, c_0, and ℓ_1.

THEOREM 1.2. *For any basis (y_k) of ℓ_2, we have $E(\ell_2, (y_k)) \sim_B \mathbb{R}^{\mathbb{N}}/\ell_2$.*

In contrast, for c_0, we construct special bases (x_n^m) for each $m \geq 1$ and $m = \infty$, and denote $\mathrm{cs}^{(m)} = \mathrm{coef}(c_0, (x_n^m))$. For $m = 1$, we have

$$\mathrm{cs}^{(1)} = \{a \in \mathbb{R}^{\mathbb{N}} : \sum_n a(n) \text{ converges}\}.$$

We show that

THEOREM 1.3. (1) *For $m \geq 1$, we have*

$$\mathbb{R}^{\mathbb{N}}/c_0 <_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m)} <_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m+1)} <_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)}.$$

(2) *Let (x_n) be a non-zero sequence in c_0. Then*

$$\mathbb{R}^{\mathbb{N}}/c_0 \leq_B E(c_0, (x_n)) \leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)}.$$

While for ℓ_1, we construct a basis (y_n^1) with

$$\mathrm{coef}(\ell_1, (y_n^1)) = \mathrm{bv}_0 \stackrel{\mathrm{Def}}{=} c_0 \cap \{a \in \mathbb{R}^{\mathbb{N}} : \sum_n |a(n) - a(n+1)| < +\infty\},$$

and prove that $\mathbb{R}^{\mathbb{N}}/\ell_1 <_B \mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0 <_B \mathbb{R}^{\mathbb{N}}/\ell_1 \otimes \mathbb{R}^{\mathbb{N}}/c_0$, where $\otimes$ is the direct product operator between equivalence relations.

The article is organized as follows. In Section 2 we recall some notions in descriptive set theory and functional analysis, and introduce two kinds of equivalence relations. In Section 3 we prove Theorem 1.1. In Section 4 we prove an useful lemma for converting a Borel reduction to an additive reduction. In Section 5 we focus on Schauder equivalence relations generated by bases of ℓ_2, c_0, and ℓ_1. Finally Section 6 contains some further remarks and open problems.

https://doi.org/10.1017/jsl.2017.67 Published online by Cambridge University Press

§2. **Preliminaries.** A *Polish space* is a separable completely metrizable topological space. Let E and F be equivalence relations on Polish spaces X and Y respectively. A Borel function $\theta : X \to Y$ is called a *Borel reduction* from E to F if, for any $x, y \in X$,

$$(x, y) \in E \iff (\theta(x), \theta(y)) \in F.$$

We say E is *Borel reducible* to F, denoted $E \leq_B F$, if there exists a Borel reduction from E to F. If both $E \leq_B F$, $F \leq_B E$ hold, we say E and F are *Borel bireducible*, denoted $E \sim_B F$. We also denote $E \leq_B F$ and $F \not\leq_B E$ as $E <_B F$. We refer to [8] and [14] for background of Borel reducibility.

A sequence (x_n) in a Banach space X is called a *Schauder basis* (or basis, for the sake of brevity) of X if, for any $x \in X$, there is a unique sequence $a \in \mathbb{R}^{\mathbb{N}}$ such that $x = \sum_n a(n)x_n$. Let (x_n) be a Schauder basis of X. Define $P_n : X \to X$ as $P_n(\sum_i a(i)x_i) = \sum_{i \leq n} a(i)x_i$. Then all P_n are bounded and the *basis constant* $\sup_n \|P_n\| < +\infty$. It follows that, there is a sequence (x_n^*) of bounded linear functionals on X, such that $x = \sum_n x_n^*(x)x_n$. We call (x_n^*) the *biorthogonal functionals* associated to (x_n).

Let (x_n) be a basis of X. We say (x_n) is *unconditional* if, for any permutation $\pi : \mathbb{N} \to \mathbb{N}$, the sequence $(x_{\pi(n)})$ is also a basis. A basis (x_n) is said to be *boundedly complete* if, for every sequence $a \in \mathbb{R}^{\mathbb{N}}$ such that $\sup_n \|\sum_{i \leq n} a(i)x_i\| < \infty$, the series $\sum_n a(n)x_n$ converges.

Let (x_n) be a sequence of non-zero elements in a Banach space X. The closed linear span of $\{x_n : n \in \mathbb{N}\}$ is denoted by $[x_n]_{n \in \mathbb{N}}$. We denote

$$\mathrm{coef}(X, (x_n)) = \{a \in \mathbb{R}^{\mathbb{N}} : \sum_n a(n)x_n \text{ converges}\},$$

and for $a \in \mathrm{coef}(X, (x_n))$, we define

$$|\!|\!| a |\!|\!| = \sup_n \|\sum_{i \leq n} a(i)x_i\|.$$

By Cauchy criterion, it is routine to check that $(\mathrm{coef}(X, (x_n)), |\!|\!| \cdot |\!|\!|)$ is a Banach space, and the unit vectors $e_n = (0, 0, \ldots, 0, \overset{n}{1}, 0, \ldots)$ form a basis of it. From the definition of $\mathrm{coef}(X, (x_n))$, we can easily see that, if X is a closed subspace of Y, then $\mathrm{coef}(X, (x_n)) = \mathrm{coef}(Y, (x_n))$.

A sequence (x_n) is called *normalized* if $\|x_n\| = 1$ for each n, and is called *semi-normalized* if there are $A \geq B > 0$ such that $A \geq \|x_n\| \geq B$ for each n. It is easy to see that, for a semi-normalized sequence (x_n) of X, we always have $\ell_1 \subseteq \mathrm{coef}(X, (x_n)) \subseteq c_0$.

We say two sequences (x_n) and (y_n) of X are *equivalent* if $\mathrm{coef}(X, (x_n)) = \mathrm{coef}(X, (y_n))$.

Definition 2.1. Let (x_n) be a sequence in a Banach space X. We define an equivalence relation on $\mathbb{R}^{\mathbb{N}}$ as $E(X, (x_n)) = \mathbb{R}^{\mathbb{N}}/\mathrm{coef}(X, (x_n))$, i.e., for all $a, b \in \mathbb{R}^{\mathbb{N}}$,

$$(a, b) \in E(X, (x_n)) \iff a - b \in \mathrm{coef}(X, (x_n)).$$

We call this kind of equivalence relations *Schauder equivalence relations*.

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A non-zero sequence (x_n) of a Banach space X is said to be a *basic sequence* if it is a basis of $[x_n]_{n\in\mathbb{N}}$.

Let (x_n) be a basis of X, (r_n) a sequence of real numbers, and $0 = n_0 < n_1 < \cdots$ a strictly increasing sequence of natural numbers. If for every k, $u_k = \sum_{n=n_k}^{n_{k+1}-1} r_n x_n$ is not 0, we call sequence (u_k) a *block basis* of (x_n). A block basis is not necessarily a basis, but is always a basic sequence. A simple reduction θ witnesses that $E(X,(u_k)) \leq_B E(X,(x_n))$ defined as, for any $a \in \mathbb{R}^{\mathbb{N}}$ and $n \in \mathbb{N}$, $\theta(a)(n) = a(k)r_n$ for $n_k \leq n < n_{k+1}$. For any sequence (r_n) of non-zero numbers, we always have $E(X,(x_n)) \sim_B E(X,(r_n x_n))$. Therefore, we may assume any basis is normalized when we need.

Let (x_n) be a semi-normalized basis of X, (r_n) a sequence of positive real numbers with $\sum_n r_n < +\infty$. Since $\ell_1 \subseteq \mathrm{coef}(X,(x_n))$, we have

$$E(X,(x_n)) \sim_B (\prod_n r_n\mathbb{Z})/\mathrm{coef}(X,(x_n)).$$

A desired Borel reduction θ is defined as $\theta(a)(n) = r_n[a(n)/r_n]$ for $a \in \mathbb{R}^{\mathbb{N}}$ and $n \in \mathbb{N}$.

Let E and F be two equivalence relations on X and Y respectively. Denote $E \otimes F$ the equivalence relation on $X \times Y$ defined as

$$((x_1,y_1),(x_2,y_2)) \in E \otimes F \iff (x_1,x_2) \in E \ \&\ (y_1,y_2) \in F$$

for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Let (x_n) be a sequence in X and (y_n) a sequence in Y. We denote $z_{2k} = x_k$ and $z_{2k+1} = y_k$ for $k \in \mathbb{N}$. It is easy to see that

$$E(X \oplus Y,(z_n)) \sim_B E(X,(x_n)) \otimes E(Y,(y_n)).$$

So we may think $E(X,(x_n)) \otimes E(Y,(y_n))$ is still a Schauder equivalence relation. Furthermore, if (x_n) and (y_n) are Schauder bases on X and Y respectively, we can see that (z_n) is also a basis on $X \oplus Y$.

A sequence (X_n) of closed subspaces of a Banach space X is called a *Schauder decomposition* of X if every $x \in X$ has a unique representation of the form $x = \sum_n x_n$, with $x_n \in X_n$ for each n. Similar to Schauder bases, define $P_n : X \to X$ as $P_n(\sum_i x_i) = \sum_{i\leq n} x_i$ where $x_n \in X_n$ for each n. Then all P_n are bounded and the *decomposition constant* $\sup_n \|P_n\| < +\infty$.

Definition 2.2. Let (X_n) be a Schauder decomposition of a separable Banach space X. We define an equivalence relation $E(X,(X_n))$ on $\prod_n X_n$ as

$$(\alpha,\beta) \in E(X,(X_n)) \iff \sum_n (\alpha(n) - \beta(n)) \text{ converges in } X$$

for any $\alpha, \beta \in \prod_n X_n$. We call this kind of equivalence relations *decomposition equivalence relations*.

For more details for Schauder bases and Schauder decompositions, we refer to [16]. Note that in this article (x_n) means $(x_0, x_1, \ldots)$, not $(x_1, x_2, \ldots)$ as in [16].

§3. F_σ Schauder equivalence relations.

A theorem of Rosendal shows that any K_σ equivalence relation on a Polish space is Borel reducible to $\mathbb{R}^{\mathbb{N}}/\ell_\infty$ (see [20]).

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In the light of Rosendal's Theorem, we compare F_σ Schauder equivalence relations with $\mathbb{R}^{\mathbb{N}}/\ell_\infty$.

The following lemma will be used to convert a Borel reduction to a continuous reduction.

LEMMA 3.1. *Denote* $D = \{d \in \mathbb{R}^{\mathbb{N}} : \forall n(4^n d(n) \in \mathbb{Z})\}$. *Let* G *be a dense* G_δ *set in* D, $a \in \mathbb{R}^{\mathbb{N}}$ *with* $2^n a(n) \in \mathbb{Z}$ *for each* $n \in \mathbb{N}$, *and let* $-1 = n_0 < n_1 < \cdots < n_k < \cdots$. *Then there exist* $b^* \in D$ *and a strictly increasing sequence of natural numbers* (k_l) *with* $k_0 = 0$ *such that* $G \supseteq C$, *where*

$$C = \{d \in D : \forall l\ \exists i \le 2^l\ \forall n \in (n_{k_l}, n_{k_{l+1}}](d(n) = b^*(n) + ia(n)/2^l)\}.$$

PROOF. Assume that $O_0 \supseteq O_1 \supseteq \cdots \supseteq O_l \supseteq \cdots$ is a sequence of dense open sets with $G = \bigcap_l O_l$. We will construct b^* and (k_m) by induction on m such that $O_m \supseteq C_m$, where

$$C_m = \{d \in D : \forall l \le m\ \exists i \le 2^l\ \forall n \in (n_{k_l}, n_{k_{l+1}}](d(n) = b^*(n) + ia(n)/2^l)\}.$$

When we finish the construction, we shall have

$$C = \bigcap_m C_m \subseteq \bigcap_m O_m = G.$$

First, for $m = 0$, fix a $d_0^0 \in O_0$. Since O_0 is open, there is n_0^0 such that $O_0 \supseteq \{d \in D : \forall n \le n_0^0(d(n) = d_0^0(n))\}$. Set $b^*(n) = d_0^0(n)$ for $n \le n_0^0$. Denote

$$N_1^0 = \{d \in D : \forall n \le n_0^0(d(n) = d_0^0(n) + a(n))\}.$$

Since O_0 is dense, there is a $d_1^0 \in N_1^0 \cap O_0$. Then we can find an $n_1^0 \ge n_0^0$ such that $O_0 \supseteq \{d \in D : \forall n \le n_1^0(d(n) = d_1^0(n))\}$. Select k_1 such that $n_{k_1} \ge n_1^0$ and set $b^*(n) = d_1^0(n) - a(n)$ for $n_0^0 < n \le n_{k_1}$. Now we denote

$$C_0 = \{d \in D : \exists i \in \{0, 1\}\ \forall n \le n_{k_1}(d(n) = b^*(n) + ia(n))\}.$$

Then $C_0 \subseteq O_0$.

Secondly, assume that we have defined $k_1, \ldots, k_m$ and the value of $b^*(n)$ for $n \le n_{k_m}$. Let $s_0, s_1, \ldots, s_J$ be an enumeration of the following set

$$\{(i_0, \ldots, i_m) : \forall l \le m(0 \le i_l \le 2^l)\}.$$

We inductively find a sequence $n_0^m < n_1^m < \cdots < n_J^m$ as follows.

Denote

$$N_0^m = \{d \in D : \forall l < m\ \forall n \in (n_{k_l}, n_{k_{l+1}}](d(n) = b^*(n) + s_0(l)a(n)/2^l)\}.$$

Since O_m is dense, there is $d_0^m \in N_0^m \cap O_m$. Then since O_m is open, we can find an $n_0^m \ge n_{k_m}$ such that $O_m \supseteq \{d \in D : \forall n \le n_0^m(d(n) = d_0^m(n))$. Set $b^*(n) = d_0^m(n) - s_0(m)a(n)/2^m$ for $n_{k_m} < n \le n_0^m$.

Further assume that n_j^m and the value of $b^*(n)$ for $n \le n_j^m$ have been defined.

If $j < J$, denote by N_{j+1}^m the set of all $d \in D$ satisfying

$$d(n) = \begin{cases} b^*(n) + s_{j+1}(l)a(n)/2^l, & n \in (n_{k_l}, n_{k_{l+1}}] \text{ for } l < m, \\ b^*(n) + s_{j+1}(m)a(n)/2^m, & n_{k_m} < n \le n_j^m. \end{cases}$$

By the same reason, we can find $d_{j+1}^m \in N_{j+1}^m \cap O_m$ and $n_{j+1}^m \ge n_j^m$ with $O_m \supseteq \{d \in D : \forall n \le n_{j+1}^m(d(n) = d_{j+1}^m(n))\}$. Then set $b^*(n) = d_{j+1}^m - s_{j+1}(m)a(n)/2^m$ for $n_j^m < n \le n_{j+1}^m$.

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If $j = J$, select a k_{m+1} such that $n_{k_{m+1}} \geq n_J^m$. Set $b^*(n) = d_J^m(n) - s_J(m)a(n)/2^m$ for $n_J^m < n \leq n_{k_{m+1}}$. It easy to see $O_m \supseteq C_m$ as desired. ⊣

The next theorem is slightly more general than Theorem 1.1.

Recall that a Schauder decomposition (X_n) of a Banach space X is called *boundedly complete* if, for every sequence $\alpha \in \prod_n X_n$ such that $\sup_n \|\sum_{i\leq n} \alpha(i)\| < +\infty$, the series $\sum_n \alpha(n)$ converges (see [16]).

THEOREM 3.2. *Let (X_n) be a Schauder decomposition of a separable Banach space X. Then the following are equivalent*:

(1) *(X_n) is boundedly complete*;

(2) $\mathrm{cs}(X,(X_n)) \overset{\mathrm{Def}}{=} \{\alpha \in \prod_n X_n : \sum_n \alpha(n)$ *converges*$\}$ *is F_σ in $\prod_n X_n$*;

(3) $E(X,(X_n)) \leq_B \mathbb{R}^{\mathbb{N}}/\ell_\infty$.

PROOF. Define $S : X \to \prod_n X_n$ as $S(x) = (x_n)$ for $x = \sum_n x_n$ with each $x_n \in X_n$. Because all projections P_n of the decomposition (X_n) are bounded, we see S is a continuous injection whose range is $\mathrm{cs}(X,(X_n))$. Let M be the decomposition constant $\sup_n \|P_n\|$. For $m \geq 1$, we denote

$$\begin{aligned} B_m &= \{\alpha \in \textstyle\prod_i X_i : \sup_n \|\sum_{i\leq n} \alpha(i)\| \leq m\} \\ &= \textstyle\bigcap_n \{\alpha \in \prod_i X_i : \|\sum_{i\leq n} \alpha(i)\| \leq m\}. \end{aligned}$$

Then B_m is closed in $\prod_n X_n$.

(1)⇒(2). From the definition of boundedly complete decomposition, we have

$$\mathrm{cs}(X,(X_n)) = \bigcup_m B_m,$$

so $\mathrm{cs}(X,(X_n))$ is F_σ.

(2)⇒(1). Assume that $\mathrm{cs}(X,(X_n)) = \bigcup_m F_m$ with each F_m closed in $\prod_n X_n$. Then each $S^{-1}(F_m)$ is closed in X. By Baire category theorem, there exists an m such that $S^{-1}(F_m)$ has an inner point. So there exist $y^\# \in X$ and $r > 0$ such that

$$S^{-1}(F_m) \supseteq B(y^\#, r) = \{x \in X : \|x - y^\#\| \leq r\}.$$

Now for any sequence $\alpha \in \prod_n X_n$ with $\sup_n \|\sum_{i\leq n} \alpha(i)\| < +\infty$, we will show $\alpha \in \mathrm{cs}(X,(X_n))$. Without loss of generality, we may assume that $\sup_n \|\sum_{i\leq n} \alpha(i)\| \leq r$. For each $j \in \mathbb{N}$, we define $\alpha_j \in \prod_n X_n$ as

$$\alpha_j(n) = \begin{cases} \alpha(n), & n \leq j, \\ 0, & n > j. \end{cases}$$

Then $\alpha_j \in \mathrm{cs}(X,(X_n))$ and $\|S^{-1}(\alpha_j)\| \leq r$. Therefore, for $j \in \mathbb{N}$, we have $y^\# + S^{-1}(\alpha_j) \in B(y^\#, r) \subseteq S^{-1}(F_m)$, i.e., $S(y^\#) + \alpha_j \in F_m$. Note that, in $\prod_n X_n$, $\lim_{j\to\infty} \alpha_j \to \alpha$. Since F_m is closed in $\prod_n X_n$, we have $S(y^\#) + \alpha \in F_m \subseteq \mathrm{cs}(X,(X_n))$. Hence $\alpha \in \mathrm{cs}(X,(X_n))$.

(1)⇒(3). Let $\{U_k : k \in \mathbb{N}\}$ be a basis for the topology of $\prod_n X_n$. For $\alpha \in \prod_n X_n$ and $k \in \mathbb{N}$, since $\bigcup_m (\alpha + B_m) = \alpha + \mathrm{cs}(X,(X_n))$ is dense in $\prod_n X_n$, there is some m such that $(\alpha + B_m) \cap U_k \neq \emptyset$. So we can define

$$\theta(\alpha)(k) = \min\{m : (\alpha + B_m) \cap U_k \neq \emptyset\}.$$

It is easy to see that $\theta : \prod_n X_n \to \mathbb{R}^{\mathbb{N}}$ is Borel.

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For $\alpha, \beta \in \prod_n X_n$, if $(\alpha, \beta) \in E(X, (X_n))$, let $\|S^{-1}(\alpha - \beta)\| \leq K/M$ with $K \in \mathbb{N}$. Then $\alpha - \beta \in B_K$, so for each m, $\alpha + B_m \subseteq \beta + B_{m+K}$, and $\beta + B_m \subseteq \alpha + B_{m+K}$. It follows that $|\theta(\alpha)(k) - \theta(\beta)(k)| \leq K$ for each k, and hence $\theta(\alpha) - \theta(\beta) \in \ell_\infty$. On the other hand, if $(\alpha, \beta) \notin E(X, (X_n))$, then

$$(\alpha + \mathrm{cs}(X, (X_n))) \cap (\beta + \mathrm{cs}(X, (X_n))) = \emptyset.$$

Thus for each $l \geq 1$, we have $(\alpha + B_1) \cap \bigcup_{m \leq l} (\beta + B_m) = \emptyset$. We can find a k such that $\alpha + B_1$ meets U_k, and $(\beta + B_m) \cap U_k = \emptyset$ for all $m \leq l$, so $|\theta(\alpha)(k) - \theta(\beta)(k)| \geq l$. It follows that $\theta(\alpha) - \theta(\beta) \notin \ell_\infty$.

Therefore, θ is a Borel reduction of $E(X, (X_n))$ to $\mathbb{R}^{\mathbb{N}}/\ell_\infty$.

(3)⇒(1). Assume for contradiction that there exists an $\alpha \in \prod_n X_n$ with $\sup_n \|\sum_{i \leq n} \alpha(i)\| < +\infty$, but $\alpha \notin \mathrm{cs}(X, (X_n))$. By Cauchy criterion, there exist an $\varepsilon_0 > 0$ and a sequence $-1 = n_0 < n_1 < \cdots < n_k < \cdots$ such that $\|\sum_{n=n_k+1}^{n_{k+1}} \alpha(n)\| \geq \varepsilon_0$. Denote $D = \{d \in \mathbb{R}^{\mathbb{N}} : \forall n (4^n d(n) \in \mathbb{Z})\}$. Without loss of generality, we may assume that $\alpha(n) \neq 0$ for each n, otherwise, we may replace α by $\alpha + \gamma$ for a suitable $\gamma \in \mathrm{cs}(X, (X_n))$. Define $T : D \to \prod_n X_n$ as $T(d)(n) = d(n)\alpha(n)$ for any $d \in D$ and $n \in \mathbb{N}$. It is clear that T is a homeomorphic embedding.

Suppose θ is a Borel reduction of $E(X, (X_n))$ to $\mathbb{R}^{\mathbb{N}}/\ell_\infty$. Then $\theta \circ T$ is also Borel on D. There exists a dense G_δ subset $G \subseteq D$ such that $\theta \circ T$ is continuous on G (cf. [15, (8.38)]). Then θ is also continuous on $T(G)$. Applying Lemma 3.1 with $a(n) = 1$ for each n, there exist $b^* \in D$ and a strictly increasing sequence (k_l) with $k_0 = 0$ such that $G \supseteq C$, where

$$C = \{d \in D : \forall l \, \exists i \leq 2^l \, \forall n \in (n_{k_l}, n_{k_{l+1}}](d(n) = b^*(n) + i/2^l)\}.$$

Let $C^* = T(C) \cap (T(b^*) + \mathrm{cs}(X, (X_n)))$. Note that $T(C)$ is closed in $\prod_n X_n$, so C^* is relatively closed in $T(b^*) + \mathrm{cs}(X, (X_n))$. By the definition of Borel reduction, $C^* = (\theta \restriction T(C))^{-1}(\theta(T(b^*)) + \ell_\infty)$. So C^* is F_σ in $T(C)$, since θ is continuous on $T(C)$ and ℓ_∞ is F_σ in $\mathbb{R}^{\mathbb{N}}$. Thus we can assume that $C^* = \bigcup_m F_m$ with each F_m closed in $T(C)$.

Now denote $Z = S^{-1}(C^* - T(b^*))$, $Z_m = S^{-1}(F_m - T(b^*))$. Then Z is closed in X, thus complete. Because each Z_m is closed in Z with $\bigcup_m Z_m = Z$, there exists an m such that Z_m has an inner point in Z. Thus there exist $y^\# \in Z$ and $r > 0$ such that

$$Z_m \supseteq W = \{x \in Z : \|x - y^\#\| \leq r\}.$$

Let $y^\# = \sum_n y_n$ with $y_n \in X_n$. By Cauchy criterion, we have

$$\lim_{k \to \infty} \| \sum_{n=n_k+1}^{n_{k+1}} y_n \| = 0.$$

Since $S(y^\#) \in (C^* - T(b^*)) \subseteq (T(C) - T(b^*))$, we have $T^{-1}(S(y^\#)) \in (C - b^*)$. So there is a sequence (i_l) such that, for $n_{k_l} < n \leq n_{k_{l+1}}$, $T^{-1}(S(y^\#))(n) = i_l/2^l$, i.e., $y_n = S(y^\#)(n) = (i_l/2^l)\alpha(n)$. Thus

$$\lim_{l \to \infty} \frac{i_l}{2^l} \| \sum_{n=n_{k_l}+1}^{n_{k_l+1}} \alpha(n) \| = 0.$$

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Taking into account that $\|\sum_{n=n_{k_l}+1}^{n_{k_l+1}}\alpha(n)\| \geq \varepsilon_0$, we get $\lim_{l\to\infty} i_l/2^l = 0$. Now fix a large enough natural number L such that $i_l/2^l < 1/2$ for $l \geq L$, and

$$\frac{1}{2^L}\sup_n \|\sum_{i\leq n}\alpha(i)\| \leq \frac{r}{2}.$$

We define

$$\alpha'(n) = \begin{cases} 0, & n \leq n_{k_L}, \\ \alpha(n), & n > n_{k_L}, \end{cases}$$

and for each $j > L$,

$$\alpha'_j(n) = \begin{cases} 0, & n \leq n_{k_L} \text{ or } n > n_{k_j}, \\ \alpha(n), & n_{k_L} < n \leq n_{k_j}. \end{cases}$$

It is clear that $T(b^*) + S(y^\#) + \alpha'/2^L \in T(C)$ and $T(b^*) + S(y^\#) + \alpha'_j/2^L \in C^*$ for each j. Note that $S^{-1}(\alpha'_j/2^L) = 1/2^L \sum_{n=n_{k_L}+1}^{n_{k_j}} \alpha(n)$ and

$$\frac{1}{2^L}\|\sum_{n=n_{k_L}+1}^{n_{k_j}}\alpha(n)\| \leq \frac{1}{2^L}\left(\|\sum_{i\leq n_{k_L}}\alpha(i)\| + \|\sum_{i\leq n_{k_j}}\alpha(i)\|\right) \leq r.$$

It follows that $y^\# + S^{-1}(\alpha'_j/2^L) \in W \subseteq Z_m = S^{-1}(F_m - T(b^*))$, i.e., $T(b^*) + S(y^\#) + \alpha'_j/2^L \in F_m$. Since F_m is closed in $T(C)$, we have

$$T(b^*) + S(y^\#) + \alpha'/2^L = \lim_{j\to\infty}(T(b^*) + S(y^\#) + \alpha'_j/2^L) \in F_m \subseteq C^*.$$

From the definition of C^*, we have $S(y^\#) + \alpha'/2^L \in \mathrm{cs}(X,(X_n))$, so $\alpha' \in \mathrm{cs}(X,(X_n))$. Hence $\alpha \in \mathrm{cs}(X,(X_n))$, a contradiction! ⊣

Indeed, the proof of (3)⇒(1) shows that, if (X_n) is not boundedly complete, then $E(X,(X_n))$ is not Borel reducible to any F_σ equivalence relation.

COROLLARY 3.3. *Let (x_n) be a Schauder basis of a Banach space X. Then the following are equivalent*:

(1) (x_n) *is boundedly complete*;
(2) $\mathrm{coef}(X,(x_n))$ *is F_σ in* $\mathbb{R}^\mathbb{N}$;
(3) $E(X,(x_n)) \leq_B \mathbb{R}^\mathbb{N}/\ell_\infty$.

§4. One lemma on additive reductions. A lemma for converting a Borel reduction to an additive reduction will be used again and again in the rest of this article, especially for proving nonreducibility. Before presenting this lemma, we introduce some relevent notions.

DEFINITION 4.1 (Farah [7]). (a) A map $\psi : \prod_n X_n \to \prod_n X'_n$ is *additive* if there are $0 = l_0 < l_1 < \cdots < l_j < \cdots$ and maps $H_j : X_j \to \prod_{n\in[l_i,l_{i+1})} X'_i$ such that

$$\psi(\alpha) = H_0(\alpha(0))^\frown H_1(\alpha(1))^\frown H_2(\alpha(2))^\frown \cdots.$$

(b) Let E and F be equivalence relations on $\prod_n X_n$ and $\prod_n X'_n$ respectively, we say E is *additively reducible* to F, denoted $E \leq_A F$, if there is an additive reduction of E to F.

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Let E be an equivalence relation on $\prod_n X_n$, and let $I \subseteq \mathbb{N}$ be infinite. Fix an element $\mu \in \prod_{n\notin I} X_n$. For $\alpha \in \prod_{n\in I} X_n$, put $\alpha \oplus \mu = \begin{cases}\alpha(n), & n \in I,\\ \mu(n), & n \notin I.\end{cases}$ We define $E|_I^\mu$ on $\prod_{n\in I} X_n$ as

$$(\alpha,\beta) \in E|_I^\mu \iff (\alpha\oplus\mu, \beta\oplus\mu) \in E.$$

If for any $\alpha_1, \alpha_2 \in \prod_n X_n$, $(\alpha_1,\alpha_2) \in E$ is equivalent to saying $\alpha_1 - \alpha_2$ is in a specified set, then the exact value of $\mu(n)$ is meaningless, thus we may assume, say, $\mu(n) = 0$ for all $n \notin I$, if we need.

Let (F_n) be a sequence of finite sets. A special equivalence relation $E_0(\prod_n F_n)$ defined as

$$(\alpha,\beta) \in E_0(\prod_n F_n) \iff \exists m\, \forall n > m(\alpha(n) = \beta(n))$$

for all $\alpha, \beta \in \prod_n F_n$.

A weaker version of the following lemma is due to Dougherty and Hjorth [4].

LEMMA 4.2. *Let E be an equivalence relation on $\prod_n F_n$ with $E_0(\prod_n F_n) \subseteq E$, where all F_n are finite sets. Let (X_n) be a Schauder decomposition of a separable Banach space X. If $E \leq_B E(X,(X_n))$, then there exist an infinite $I \subseteq \mathbb{N}$ and a $\mu \in \prod_{n\notin I} X_n$ such that $E|_I^\mu \leq_A E(X,(X_n))$.*

PROOF. The following proof is a modification of the proof of [4], Theorem 2.2, claims (i)–(iii). We omit some similar arguments.

Assume that θ is a Borel reduction of E to $E(X,(X_n))$. Following claims (i) and (ii), and the arguments after Claim (ii) of [4], Theorem 2.2, we construct two sequences of natural numbers $n_0 < n_1 < n_2 < \cdots$ and $l_0 < l_1 < l_2 < \cdots$, a sequence (s_j) with $\mathrm{lh}(s_0) = n_0$ and $\mathrm{lh}(s_j) = n_j - n_{j-1} - 1$ for each $j > 0$, and dense open sets $D_i^j \subseteq \prod_n F_n$ $(i, j \in \mathbb{N})$.

Denote $I = \{n_j : j \in \mathbb{N}\}$. For $n \notin I$, define

$$\mu(n) = \begin{cases} s_0(n), & n < n_0,\\ s_j(n - n_{j-1} - 1), & n_{j-1} < n < n_j.\end{cases}$$

The construction confirms that, for any $\alpha, \hat\alpha \in \prod_{n\in I} F_n$, we have

(a) if $\alpha(n) = \hat\alpha(n)$ for $n > n_j$, then

$$\|\sum_{n\geq l_{j+1}} (\theta(\alpha\oplus\mu)(n) - \theta(\hat\alpha\oplus\mu)(n))\| < 2^{-j};$$

(b) if $\alpha(n) = \hat\alpha(n)$ for $n \leq n_j$, then

$$\|\sum_{n< l_{j+1}} (\theta(\alpha\oplus\mu)(n) - \theta(\hat\alpha\oplus\mu)(n))\| < 2^{-j}.$$

Now we are ready to define a sequence of mappings $(H_n)_{n\in I}$ to assemble the desired additive reduction ψ. For each $n \in I$, fix an element $x_n^\# \in F_n$. We define $p_j : F_{n_j} \to \prod_{n\in I} F_n$ for each $j \in \mathbb{N}$ as $p_j(x)(n) = \begin{cases} x, & n = n_j,\\ x_n^\#, & n \neq n_j\end{cases}$ for $x \in F_n$ and $n \in \mathbb{N}$. Then for $n \in I$ with $n = n_j$, we define $H_n : F_n \to \prod_{n\in[l_j,l_{j+1})} X_n$ as, for $x \in F_n$,

$$H_n(x) = \theta(p_j(x)\oplus\mu) \restriction [l_j, l_{j+1}).$$

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The additive mapping $\psi : \prod_{n\in I} F_n \to \prod_n X_n$ is defined as

$$\psi(\alpha) = H_{n_0}(\alpha(n_0))^\frown H_{n_1}(\alpha(n_1))^\frown H_{n_2}(\alpha(n_2))^\frown \cdots,$$

for $\alpha \in \prod_{n\in I} F_n$.

Now, we will show that ψ is a reduction of $E|_I^\mu$ to $E(X,(X_n))$.

For any $\alpha \in \prod_{n\in I} F_n$ and $j \in \mathbb{N}$, define $e_j(\alpha), e'_j(\alpha) \in \prod_{n\in I} F_n$ as

$$e_j(\alpha)(n) = \begin{cases} \alpha(n), & n = n_j, \\ x_n^\#, & n \neq n_j, \end{cases} \quad e'_j(\alpha)(n) = \begin{cases} \alpha(n), & n \leq n_j, \\ x_n^\#, & n > n_j. \end{cases}$$

Applying (a) for $j-1$ and (b) for j, we have

$$\|\sum_{n\geq l_j}(\theta(e_j(\alpha)\oplus\mu)(n) - \theta(e'_j(\alpha)\oplus\mu)(n))\| < 2^{-(j-1)},$$

$$\|\sum_{n<l_{j+1}}(\theta(\alpha\oplus\mu)(n) - \theta(e'_j(\alpha)\oplus\mu)(n))\| < 2^{-j}.$$

Let M be the decompositon constant of (X_n). We have

$$\begin{array}{rl} & \|\sum_{n\in[l_j,l_{j+1})}(\theta(\alpha\oplus\mu)(n) - \theta(e_j(\alpha)\oplus\mu)(n))\| \\ \leq & \|\sum_{n\in[l_j,l_{j+1})}(\theta(\alpha\oplus\mu)(n) - \theta(e'_j(\alpha)\oplus\mu)(n))\| \\ & +\|\sum_{n\in[l_j,l_{j+1})}(\theta(e'_j(\alpha)\oplus\mu)(n)) - \theta(e_j(\alpha)\oplus\mu)(n))\| \\ \leq & (1+M)\|\sum_{n<l_{j+1}}(\theta(\alpha\oplus\mu)(n) - \theta(e'_j(\alpha)\oplus\mu)(n))\| \\ & +M\|\sum_{n\geq l_j}(\theta(e'_j(\alpha)\oplus\mu)(n)) - \theta(e_j(\alpha)\oplus\mu)(n))\| \\ < & (1+3M)2^{-j}. \end{array}$$

Note that $p_j(\alpha(n_j)) = e_j(\alpha)$. We have

$$\psi(\alpha)\restriction[l_j,l_{j+1}) = H_{n_j}(\alpha(n_j)) = \theta(e_j(\alpha)\oplus\mu)\restriction[l_j,l_{j+1})$$

for each $j \in \mathbb{N}$. For each $m \in \mathbb{N}$, let $n_i \leq m < n_{i+1}$. We have

$$\begin{array}{rl} & \|\sum_{n\geq m}(\theta(\alpha\oplus\mu)(n) - \psi(\alpha)(n))\| \\ \leq & \|\sum_{n\in[m,l_{i+1})}(\theta(\alpha\oplus\mu)(n) - \theta(e_i(\alpha)\oplus\mu)(n))\| \\ & +\sum_{j\geq i+1}\|\sum_{n\in[l_j,l_{j+1})}(\theta(\alpha\oplus\mu)(n) - \theta(e_j(\alpha)\oplus\mu)(n))\| \\ < & (1+M)(1+3M)2^{-i} + \sum_{j\geq i+1}(1+3M)2^{-j} \\ = & (2+M)(1+3M)2^{-i}. \end{array}$$

It follows that

$$\lim_{m\to\infty}\sum_{n\geq m}(\theta(\alpha\oplus\mu)(n) - \psi(\alpha)(n)) = 0,$$

i.e., $(\theta(\alpha\oplus\mu), \psi(\alpha)) \in E(X,(X_n))$.

In the end, for $\alpha, \hat{\alpha} \in \prod_{n\in I} F_n$, we have

$$\begin{array}{lll} (\psi(\alpha),\psi(\hat{\alpha})) \in E(X,(X_n)) & \iff & (\theta(\alpha\oplus\mu),\theta(\hat{\alpha}\oplus\mu)) \in E(X,(X_n)) \\ & \iff & (\alpha\oplus\mu,\hat{\alpha}\oplus\mu) \in E \\ & \iff & (\alpha,\hat{\alpha}) \in E|_I^\mu. \end{array}$$

This completes the proof. ⊣

REMARK 4.3. It is worth noting that the preceding lemma can be applied with some variations. If there is a subset $M_n \subseteq X_n$ for each n such that

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$E \leq_B E(X,(X_n)) \restriction \prod_n M_n$, i.e., there is a Borel reduction θ from $\prod_n F_n$ to $\prod_n M_n$, then the resulted additive reduction ψ is also mapping into $\prod_n M_n$. Thus $E|_I^\mu \leq_A E(X,(X_n)) \restriction \prod_n M_n$.

As an application, we prove the following theorem.

THEOREM 4.4. *Let (x_n) and (y_n) be bases of Banach spaces X and Y respectively. If (x_n) is unconditional, and every subsequence of (y_n) is conditional, then $E(Y,(y_n)) \not\leq_B E(X,(x_n))$.*

PROOF. Assume for contradiction that $\theta : \mathbb{R}^\mathbb{N} \to \mathbb{R}^\mathbb{N}$ is a Borel reduction of $E(Y,(y_n))$ to $E(X,(x_n))$. Denote $F_n = \{i/2^n : i = 0,1,\dots,2^n\}$, and denote by E the restriction of $E(Y,(y_n))$ on $\prod_n F_n$. Then $\theta \restriction \prod_n F_n$ is also a Borel reduction of E to $E(X,(x_n))$. From Lemma 4.2, we can find an infinite set $I \subseteq \mathbb{N}$ and an element $\mu \in \prod_{n\notin I} F_n$ such that $E|_I^\mu$ is additively reducible to $E(X,(x_n))$. Without loss of generality, we may assume that $\mu(n) = 0$ for $n \notin I$. Therefore, we can find, for each $n \in I$, a natural number $l_n \geq 1$ and a map $H_n : F_n \to \mathbb{R}^{l_n}$ such that the following ψ is a reduction of $E|_I^\mu$ to $E(X,(x_n))$. Let (n_k) be the strictly increasing enumeration of I, then ψ is defined as

$$\psi(a) = H_{n_0}(a(n_0))^\frown H_{n_1}(a(n_1))^\frown H_{n_2}(a(n_2))^\frown \cdots,$$

for any $a \in \prod_{n\in I} F_n$.

From the assumption on (y_n), the subsequence (y_{n_k}) is conditional. By Definition 1.c.5 and Proposition 1.c.6 of [16], there exists an $a_0 \in \mathbb{R}^\mathbb{N}$ such that $\sum_k a_0(k)y_{n_k}$ converges conditionally. Then by Proposition 1.c.1 of [16], there is an $\epsilon \in \{-1,1\}^\mathbb{N}$ such that $\sum_k a_0(k)y_{n_k}$ converges while $\sum_k \epsilon(k)a_0(k)y_{n_k}$ diverges.

Without loss of generality, we may assume that (y_n) is normalized. Then we have $\ell_1 \subseteq \mathrm{coef}(Y,(y_{n_k})) \subseteq c_0$, so we may further assume that $a_0(k) \in F_{n_k}$. For each $n_k \in I$, denote $a^I(n_k) = a_0(k)$, $a^\emptyset(n) = 0$, and define

$$a^+(n_k) = \begin{cases} a_0(k), & \epsilon(k) = 1, \\ 0, & \epsilon(k) = -1, \end{cases} \qquad a^-(n_k) = \begin{cases} 0, & \epsilon(k) = 1, \\ a_0(k), & \epsilon(k) = -1. \end{cases}$$

Since $\sum_{n\in I}(a^I(n) - a^\emptyset(n))y_n$ converges and $\sum_{n\in I}(a^+(n) - a^-(n))y_n$ diverges, we have $\psi(a^I) - \psi(a^\emptyset) \in \mathrm{coef}(X,(x_n))$ and $\psi(a^+) - \psi(a^-) \notin \mathrm{coef}(X,(x_n))$. Denote $t_k = H_{n_k}(a_0(k)) - H_{n_k}(0)$. Then we have

$$t_0^\frown t_1^\frown t_2^\frown \cdots \in \mathrm{coef}(X,(x_n)),$$

$$\epsilon(0)t_0^\frown \epsilon(1)t_1^\frown \epsilon(2)t_2^\frown \cdots \notin \mathrm{coef}(X,(x_n)).$$

This contradicts the unconditionality of (x_n) (cf. Proposition 1.c.6 of [16]). ⊣

§5. Different Schauder bases of a Banach space.

A question arises naturally:

If (x_n) and (y_n) are two Schauder bases of a Banach space X, does $E(X,(x_n)) \sim_B E(X,(y_n))$?

Recall that two sequences (x_n) and (y_n) of X are equivalent if $\mathrm{coef}(X,(x_n)) = \mathrm{coef}(X,(y_n))$. It is well known that, in every infinite dimensional Banach space with a basis, there exist continuum many mutually non-equivalent normalized bases (see [19], 4.1). If we restrict to unconditional bases, only c_0, ℓ_1 and ℓ_2 have the property that all unconditional bases are equivalent (see [16], Theorem 2.b.10).

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When we return to Borel bireducibility between $E(X,(x_n))$ and $E(X,(y_n))$, the question becomes very complicated. From Corollary 3.3, if (x_n) is boundedly complete and (y_n) is not, then $E(X,(x_n)) \not\sim_B E(X,(y_n))$. In [22], Zippin proved that, for a Banach space X with a basis, X is reflexive iff all bases of X are boundedly complete. So if a non-reflexive space X has a boundedly complete basis, the question turns out to fail for X. Which kind of spaces do the question hold for? Perhaps, the most likely candidates might be the reflexive spaces, but so far the only known example of such a space is the Hilbert space ℓ_2.

LEMMA 5.1. *Let (x_n) and (y_k) be two bases of a Banach space X. If $y_k = \sum_n \alpha_{nk} x_n$ with $\alpha_{nk} = 0$ for any $n < k$, then $E(X,(x_n)) \sim_B E(X,(y_k))$.*

PROOF. For any $a \in \mathbb{R}^{\mathbb{N}}$ and $n \in \mathbb{N}$, define $\theta(a)(n) = \sum_{k\le n} \alpha_{nk} a(k)$. Since (y_k) is a basis of X, we can assume that $x_n = \sum_k \beta_{kn} y_k$ for each n. Then

$$\delta_{mn} = \begin{cases} 1, & m = n, \\ 0, & m \neq n \end{cases} = x_m^*(x_n) = \sum_k \beta_{kn} x_m^*(y_k) = \sum_{k\le m} \beta_{kn}\alpha_{mk}.$$

By induction on m, we can prove that $\beta_{kn} = 0$ for any $k < n$. Furthermore, we also have $\sum_{n\le m} \alpha_{nk}\beta_{mn} = y_m^*(y_k) = \delta_{mk}$. Therefore, $\theta : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ is invertible and $\theta^{-1}(d)(k) = \sum_{n\le k} \beta_{kn} d(n)$ for any $d \in \mathbb{R}^{\mathbb{N}}$ and $k \in \mathbb{N}$.

Let $a, b \in \mathbb{R}^{\mathbb{N}}$. If $a - b \in \operatorname{coef}(X,(y_k))$, then there is $x \in X$ such that $x = \sum_k (a(k) - b(k)) y_k$. Note that

$$x_n^*(x) = \sum_k (a(k) - b(k)) x_n^*(y_k) = \sum_{k\le n} \alpha_{nk}(a(k) - b(k)).$$

We have $x = \sum_n x_n^*(x) x_n = \sum_n (\theta(a)(n) - \theta(b)(n)) x_n$. Thus $\theta(a) - \theta(b) \in \operatorname{coef}(X,(x_n))$. On the other hand, using θ^{-1}, we can prove that, if $\theta(a) - \theta(b) \in \operatorname{coef}(X,(x_n))$, then $a - b \in \operatorname{coef}(X,(y_k))$.

Therefore, θ and θ^{-1} witness $E(X,(x_n)) \sim_B E(X,(y_k))$. ⊣

THEOREM 5.2. *For any basis (y_k) of ℓ_2, we have $E(\ell_2,(y_k)) \sim_B \mathbb{R}^{\mathbb{N}}/\ell_2$.*

PROOF. For each $n \in \mathbb{N}$, denote $X_n = [y_k]_{k\ge n}$. Find a normalized $x_n \in X_n$ such that $x_n \perp X_{n+1}$. Then we have $\{x_0, \dots, x_n\}^{\perp} = X_{n+1}$. We claim that (x_n) is an orthonormal basis of ℓ_2. It is easy to see that $x_m \perp x_n$ for $m \neq n$. Thus (x_n) is an orthonormal basis of $[x_n]_{n\in\mathbb{N}}$. Let $x \perp [x_n]_{n\in\mathbb{N}}$. Then $x \in \bigcap_n X_n$. Since (y_k) is a basis, let $x = \sum_k r_k y_k$. We can see that $r_k = 0$ for any k, so $x = 0$. It follows that $[x_n]_{n\in\mathbb{N}} = \ell_2$.

Let $\langle\cdot,\cdot\rangle$ be the inner product of ℓ_2. For any $n, k \in \mathbb{N}$, denote $\alpha_{nk} = \langle y_k, x_n\rangle$. Since $y_k \in X_k$, we have $\alpha_{nk} = 0$ for $n < k$. By Lemma 5.1, we have $E(\ell_2,(y_k)) \sim_B E(\ell_2,(x_n)) = \mathbb{R}^{\mathbb{N}}/\ell_2$. ⊣

Besides Hilbert space ℓ_2, we would like to investigate the Borel reducibility between $E(X,(x_n))$'s with (x_n) being a basis of X. In this section, we focus on two special spaces: c_0 and ℓ_1. Both of them are non-reflexive. ℓ_1 has boundedly complete bases, while c_0 has none.

THEOREM 5.3. *Let (x_n) be a basis of a Banach space X. Let $y_k = \sum_n \alpha_{nk} x_n$ satisfy that, for any n, there are only finitely many k such that $\alpha_{nk} \neq 0$. Then*

$$E(X,(y_k)) \le_B E(X,(x_n)) \otimes \mathbb{R}^{\mathbb{N}}/c_0.$$

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PROOF. Fix a k_0 such that $\alpha_{0k} = 0$ for $k > k_0$. For $m \in \mathbb{N}$, denote

$$N_m = \max\{N : \forall n \leq N\ \forall k > k_0 + m\ (\alpha_{nk} = 0)\}.$$

From the assumption on α_{nk}, we can see $\lim_{m\to\infty} N_m = \infty$. Define $\theta_1 : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by $\theta_1(a)(n) = \sum_k a(k)\alpha_{nk}$ for $a \in \mathbb{R}^{\mathbb{N}}$ and $n \in \mathbb{N}$, and define $\theta_2 : \mathbb{R}^{\mathbb{N}} \to X^{\mathbb{N}}$ by

$$\theta_2(a)(m) = \sum_{k\leq k_0+m} a(k) \sum_{n>N_m} \alpha_{nk} x_n$$

for $a \in \mathbb{R}^{\mathbb{N}}$ and $m \in \mathbb{N}$. The definition of N_m implies that, for $n \leq N_m$, $\theta_1(a)(n) = \sum_{k\leq k_0+m} a(k)\alpha_{nk}$.

Let $a, b \in \mathbb{R}^{\mathbb{N}}$. We have

$$\begin{aligned}
&\textstyle\sum_{k\leq k_0+m}(a(k) - b(k))y_k \\
=\ &\textstyle\sum_{k\leq k_0+m}(a(k) - b(k)) \left(\sum_{n\leq N_m} \alpha_{nk}x_n + \sum_{n>N_m} \alpha_{nk}x_n\right) \\
=\ &\textstyle\sum_{n\leq N_m} \left(\sum_{k\leq k_0+m}(a(k) - b(k))\alpha_{nk}\right) x_n + (\theta_2(a)(m) - \theta_2(b)(m)) \\
=\ &\textstyle\sum_{n\leq N_m} \left(\theta_1(a)(n) - \theta_1(b)(n)\right) x_n + (\theta_2(a)(m) - \theta_2(b)(m)).
\end{aligned}$$

If $a - b \in \mathrm{coef}(X, (y_k))$, we denote $x = \sum_k (a(k) - b(k))y_k$. Since (x_n) is a basis, we have

$$\begin{aligned}
x &= \textstyle\sum_n x_n^*(x)x_n \\
&= \textstyle\sum_n \left(\sum_k (a(k) - b(k))x_n^*(y_k)\right) x_n \\
&= \textstyle\sum_n \left(\sum_k (a(k) - b(k))\alpha_{nk}\right) x_n \\
&= \textstyle\sum_n (\theta_1(a)(n) - \theta_1(b)(n))x_n.
\end{aligned}$$

Thus $\theta_1(a) - \theta_1(b) \in \mathrm{coef}(X, (x_n))$. Furthermore, by $\lim_{m\to\infty} N_m = \infty$, we have

$$\lim_{m\to\infty} \sum_{k\leq k_0+m} (a(k) - b(k))y_k = x = \lim_{m\to\infty} \sum_{n\leq N_m} (\theta_1(a)(n) - \theta_1(b)(n))x_n.$$

Therefore $\lim_{m\to\infty} \|\theta_2(a)(m) - \theta_2(b)(m)\| = 0$.

On the other hand, assume that

$$\theta_1(a) - \theta_1(b) \in \mathrm{coef}(X, (x_n)) \quad \& \quad \lim_{m\to\infty} \|\theta_2(a)(m) - \theta_2(b)(m)\| = 0.$$

Then $\sum_n \left(\theta_1(a)(n) - \theta_1(b)(n)\right) x_n$ is convergent. Furthermore, we have

$$\begin{aligned}
\textstyle\sum_k (a(k) - b(k))y_k &= \lim_{m\to\infty} \textstyle\sum_{k\leq k_0+m}(a(k) - b(k))y_k \\
&= \lim_{m\to\infty} \textstyle\sum_{n\leq N_m}(\theta_1(a)(n) - \theta_1(b)(n))x_n \\
&= \textstyle\sum_n (\theta_1(a)(n) - \theta_1(b)(n))x_n.
\end{aligned}$$

It follows that $a - b \in \mathrm{coef}(X, (y_k))$.

By Theorem 3.4 of [3], there is a Borel function $\theta' : X^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ such that, for $x, y \in X^{\mathbb{N}}$,

$$\lim_{m\to\infty} \|x(m) - y(m)\| = 0 \iff (\theta'(x) - \theta'(y)) \in c_0.$$

Now we can define the desired Borel reduction of $E(X, (y_k))$ to $E(X, (x_n)) \otimes \mathbb{R}^{\mathbb{N}}/c_0$ as $\theta(a) = (\theta_1(a), \theta'(\theta_2(a)))$ for all $a \in \mathbb{R}^{\mathbb{N}}$. ⊣

Recall that a basis (x_n) of a Banach space X is called *subsymmetric* if it is unconditional and any subsequence (x_{n_k}) of (x_n) is equivalent to (x_n) itself. The

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unit vector bases of c_0 and ℓ_p are subsymmetric. The following theorem is due to Xin Ma.

THEOREM 5.4 ([17], Theorem 1.1). *Let (y_n) be a basis of a Banach space Y, and (x_n) a subsymmetric basis of a closed subspace X of Y. Then $E(X,(x_n)) \leq_B E(Y,(y_n))$.*

5.1. Bases in c_0. By Theorem 5.4, among all $E(c_0,(x_n))$ with (x_n) a basis of c_0, $\mathbb{R}^{\mathbb{N}}/c_0$ is the minimum element with respect to Borel reducibility. We are going to find a maximum among them.

We denote

$$\mathrm{cs} = \{a \in \mathbb{R}^{\mathbb{N}} : \sum_n a(n) \text{ converges}\}.$$

Let $x^1_n = \sum_{k\leq n} e_k$, where (e_k) is the unit vector basis of c_0. Then (x^1_n) is a basis of c_0 too. Since

$$\sum_n a(n)x^1_n = (\sum_{n\geq 0} a(n), \sum_{n\geq 1} a(n), \ldots, \sum_{n\geq k} a(n), \ldots),$$

we can see $\mathrm{cs} = \mathrm{coef}(c_0,(x^1_n))$.

A simple reduction $\theta(a) = (a(0), -a(0), a(1), -a(1), \ldots)$ witnesses that $\mathbb{R}^{\mathbb{N}}/c_0 \leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}$. We can easily prove $\mathbb{R}^{\mathbb{N}}/\mathrm{cs} \sim_B \mathbb{R}^{\mathbb{N}}/c$, where c is the set of all convergent sequences. It is worth noting that the unit vectors cannot form a basis of c.

For $m \geq 1$, note that $\bigoplus_{i=1}^m c_0 \cong c_0$. We choose a suitable basis (x^m_n) of c_0 and define $\mathrm{cs}^{(m)} = \mathrm{coef}(c_0,(x^m_n))$. To do so, let (e^i_k) be the unit vector basis of the i-th component space c_0, then we set $x^m_{mj+i-1} = \sum_{k\leq j} e^i_k$ for $i \leq m$ and $j \in \mathbb{N}$.

Furthermore, recall that

$$\left(\bigoplus_{i\in\mathbb{N}} c_0\right)_0 = \{(a_n) \in (c_0)^{\mathbb{N}} : \lim_{n\to\infty} \|a_n\| = 0\}.$$

We still have $\left(\bigoplus_{i\in\mathbb{N}} c_0\right)_0 \cong c_0$. Now fix a bijection $\langle\cdot,\cdot\rangle : \mathbb{N}^2 \to \mathbb{N}$ such that, for any i, $\langle i,j\rangle$ is strictly increasing with respect to variable j. Let (e^i_k) be the unit vector basis of the i-th component space c_0 in $\left(\bigoplus_{i\in\mathbb{N}} c_0\right)_0$. We set $x^\infty_n = \sum_{k\leq j} e^i_k$ for $n = \langle i,j\rangle$. We can see that (x^∞_n) is also a basis of c_0. Now we denote

$$\mathrm{cs}^{(\infty)} = \mathrm{coef}(c_0,(x^\infty_n)).$$

It is straightforward to check that

$$\mathrm{cs}^{(m)} = \{a \in \mathbb{R}^{\mathbb{N}} : \forall i \leq m(\sum_j a(mj+i-1) \text{ converges})\},$$

$$\mathrm{cs}^{(\infty)} = \{a \in \mathbb{R}^{\mathbb{N}} : \forall j(a(\langle\cdot,j\rangle) \in c_0) \;\&\; \sum_j a(\langle\cdot,j\rangle) \text{ converges in } c_0\}.$$

For any Banach space X, we define

$$\mathrm{cs}(X) = \{\alpha \in X^{\mathbb{N}} : \sum_n \alpha(n) \text{ converges in } X\},$$

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and for any $\alpha \in \mathrm{cs}(X)$, define

$$\|\alpha\|_X = \sup_n \|\sum_{i\le n} \alpha(i)\|.$$

By Cauchy criterion, it is straightforward to check that $(\mathrm{cs}(X), \|\cdot\|_X)$ is a Banach space. Furthermore, letting $X_n = \{\alpha \in \mathrm{cs}(X) : \forall i \neq n(\alpha(i) = 0)\}$, we can see that (X_n) forms a Schauder decomposition of $\mathrm{cs}(X)$. For any sequence (x_n) in X, we claim that

$$E(X, (x_n)) \le_B X^{\mathbb{N}}/\mathrm{cs}(X).$$

Indeed, for any $a \in \mathbb{R}^{\mathbb{N}}$ and $k \in \mathbb{N}$, we define $\theta(a)(k) = a(k)x_k$. Then θ is a Borel reduction of $E(X, (x_n))$ to $X^{\mathbb{N}}/\mathrm{cs}(X)$.

An easy observation shows that

$$(\mathbb{R}^m)^{\mathbb{N}}/\mathrm{cs}(\mathbb{R}^m) \sim_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m)},$$

$$(c_0)^{\mathbb{N}}/\mathrm{cs}(c_0) \le_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)} = E(c_0, (x_n^\infty)) \le_B (c_0)^{\mathbb{N}}/\mathrm{cs}(c_0).$$

Therefore, $\mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)}$ is the desired maximum element. Furthermore, we have the following theorem.

THEOREM 5.5. *Let (x_n) be a non-zero sequence in c_0. Then*

$$\mathbb{R}^{\mathbb{N}}/c_0 \le_B E(c_0, (x_n)) \le_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)}.$$

PROOF. $E(c_0, (x_n)) \le_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)}$ follows from $(c_0)^{\mathbb{N}}/\mathrm{cs}(c_0) \sim_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)}$. We only need to prove $\mathbb{R}^{\mathbb{N}}/c_0 \le_B E(c_0, (x_n))$.

We may assume that (x_n) is normalized. Note that $c_0^{**} = \ell_\infty$ and the unit ball of ℓ_∞ is weak* compact. There is a subsequence of (x_n) which is weak* convergent in ℓ_∞. Without loss of generality, assume that (x_n) itself is weak* convergent.

CASE 1. If (x_n) is not weakly convergent in c_0, by the Eberlein-Šmulian theorem (cf. [2, p. 41]), there is a subsequence (x_{n_k}) of (x_n) which is a basic sequence. By Proposition 1.a.11 of [16], there is a basic sequence (y_n) in $[x_{n_k}]_{k\in\mathbb{N}}$ which is equivalent to a block basis (u_j) of the unit vector basis (e_n). We may also assume that (u_j) is normalized. From Proposition 2.a.1 of [16], (u_j) is equivalent to (e_n). Let $(x_{n_k}^*)$ be the biorthogonal functionals in $[x_{n_k}]_{k\in\mathbb{N}}^*$. Then $x_{n_k}^* \restriction [y_n]_{n\in\mathbb{N}} \in [y_n]_{n\in\mathbb{N}}^*$. Since (y_n) is equivalent to (e_n), we have that (y_n^*) is equivalent to the unit vector basis of ℓ_1. It follows that $\lim_{n\to\infty} x_{n_k}^*(y_n) = 0$ for each $k \in \mathbb{N}$. Then by Proposition 1.a.12 of [16], there is a subsequence (y_{n_j}) of (y_n) which is equivalent to a block basis of (x_{n_k}). Note that (y_{n_j}) is still equivalent to the unit vector basis of c_0, so we have

$$\mathbb{R}^{\mathbb{N}}/c_0 = E(c_0, (y_{n_j})) \le_B E(c_0, (x_{n_k})) \le_B E(c_0, (x_n)).$$

CASE 2. If (x_n) converges to $x \in c_0$. Then $\|x\| = 1$, since (x_n) is normalized. There is a subsequence (x_{n_k}) such that $\|x_{n_k} - x\| \le 2^{-k}$. Then for any $a \in \mathbb{R}^{\mathbb{N}}$, we have

$$\sum_k a(k)x_{n_k} \text{ converges} \iff \sum_k a(k)x \text{ converges} \iff a \in \mathrm{cs}.$$

Thus $\mathbb{R}^{\mathbb{N}}/\mathrm{cs} \le_B E(c_0, (x_n))$, and hence $\mathbb{R}^{\mathbb{N}}/c_0 \le_B E(c_0, (x_n))$.

CASE 3. If (x_n) weakly converges to $x \in c_0$ but does not converge to x. We can find a subsequence (x_{n_k}) such that $\|x_{n_k} - x\| \ge C$ for some $C > 0$. By Proposition

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1.a.12 of [16], there is a subsequence of $(x_{n_k} - x)$ which is equivalent to a block basis (u_j) of (e_n). Without loss of generality, assume that $(x_{n_k} - x)$ itself is equivalent to (u_j). Since $C \leq \|x_{n_k} - x\| \leq 1 + \|x\|$, we have that $(x_{n_k} - x)$ is semi-normalized. It is routine to check that (u_j) is also semi-normalized, so (u_j) is equivalent to (e_n). If $x = 0$, then $\mathrm{coef}(c_0, (x_{n_k})) = c_0$; else if $x \neq 0$, for any $a \in \mathbb{R}^{\mathbb{N}}$, we have

$$\sum_k a(k)x_{n_k} \text{ converges } \iff a \in c_0 \cap \mathrm{cs} = \mathrm{cs}.$$

We always have $\mathbb{R}^{\mathbb{N}}/c_0 \leq_B E(c_0, (x_n))$. ⊣

Remark 5.6. Following the proof of the last theorem, we can also get: for any non-zero sequence (x_n) in ℓ_p with $p > 1$, we have

$$\mathbb{R}^{\mathbb{N}}/\ell_p \leq_B E(\ell_p, (x_n)) \quad \text{or} \quad \mathbb{R}^{\mathbb{N}}/\mathrm{cs} \leq_B E(\ell_p, (x_n)).$$

For Case 3 with $x \neq 0$, we can prove that $\mathrm{coef}(\ell_p, (x_{n_k})) = \ell_p \cap \mathrm{cs}$. It is trivial that $\mathbb{R}^{\mathbb{N}}/\ell_p \leq_B \mathbb{R}^{\mathbb{N}}/\ell_p \cap \mathrm{cs}$.

Corollary 5.7. *Let (x_k) be a non-zero sequence in c_0. If for any n there are only finitely many k such that $x_k(n) \neq 0$, then $E(c_0, (x_k)) \sim_B \mathbb{R}^{\mathbb{N}}/c_0$.*

Proof. The last theorem implies $\mathbb{R}^{\mathbb{N}}/c_0 \leq_B E(c_0, (x_k))$. From Theorem 5.3, we get $E(c_0, (x_k)) \leq_B \mathbb{R}^{\mathbb{N}}/c_0 \otimes \mathbb{R}^{\mathbb{N}}/c_0$. So we have $E(c_0, (x_k)) \leq_B \mathbb{R}^{\mathbb{N}}/c_0$, since $\mathbb{R}^{\mathbb{N}}/c_0 \otimes \mathbb{R}^{\mathbb{N}}/c_0 \leq_B \mathbb{R}^{\mathbb{N}}/c_0$ is trivial. ⊣

Now we are going to compare $\mathbb{R}^{\mathbb{N}}/c_0$, $\mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m)}$, and $\mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)}$.

Lemma 5.8. *If X, Y are two finite dimensional spaces with $\dim(X) > \dim(Y)$, then $X^{\mathbb{N}}/\mathrm{cs}(X) \not\leq_B Y^{\mathbb{N}}/\mathrm{cs}(Y)$.*

Proof. Assume for contradiction that there is a Borel reduction of $X^{\mathbb{N}}/\mathrm{cs}(X)$ to $Y^{\mathbb{N}}/\mathrm{cs}(Y)$. Let $m = \dim(X)$,

$$F_n = \{i/2^n : i = 0, 1, \ldots, 2^n\}^m,$$

and E the restriction of $X^{\mathbb{N}}/\mathrm{cs}(X)$ on $\prod_n F_n$. We can find an infinite set $I \subseteq \mathbb{N}$ such that $E|_I^{\mu}$ is additively reducible to $Y^{\mathbb{N}}/\mathrm{cs}(Y)$ with $\mu(n) = 0$ for $n \notin I$. Therefore, we can find, for each $n \in I$, a natural number $l_n \geq 1$ and a map $H_n : F_n \to Y^{l_n}$ such that the following ψ is a reduction of $E|_I^{\mu}$ to $Y^{\mathbb{N}}/\mathrm{cs}(Y)$. Let (n_k) be the strictly increasing enumeration of I, then ψ is defined as

$$\psi(\alpha) = H_{n_0}(\alpha(n_0))^\frown H_{n_1}(\alpha(n_1))^\frown H_{n_2}(\alpha(n_2))^\frown \cdots,$$

for any $\alpha \in \prod_{n \in I} F_n$.

For $(y_0, \ldots, y_{l-1}) \in Y^l$, denote

$$\|(y_0, \ldots, y_{l-1})\|_Y = \max_{i<l} \|y_0 + \cdots + y_i\|.$$

Claim 1. $\forall q\ \exists \delta_q > 0\ \forall^\infty n \in I\ \forall x, y \in F_n (\|x - y\| < \delta_q \Rightarrow \|H_n(x) - H_n(y)\|_Y < 2^{-q})$.

Proof of Claim 1. If not, there exist an $\varepsilon_0 > 0$, a strictly increasing sequence (k_p), and $x_{k_p}, y_{k_p} \in F_{n_{k_p}}$ for each p, such that $\|x_{k_p} - y_{k_p}\| < 2^{-p}$ but $\|H_{n_{k_p}}(x_{k_p}) - H_{n_{k_p}}(y_{k_p})\|_Y \geq \varepsilon_0$. For $n \in I$, put

$$\alpha(n) = \begin{cases} x_{k_p}, & n = n_{k_p}, \\ 0, & \text{otherwise}, \end{cases} \qquad \beta(n) = \begin{cases} y_{k_p}, & n = n_{k_p}, \\ 0, & \text{otherwise}. \end{cases}$$

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Then $(\alpha,\beta)\in E|_I^\mu$ but $\psi(\alpha)-\psi(\beta)\notin \mathrm{cs}(Y)$. $\dashv_{\text{Claim 1}}$

Without loss of generality, we can assume $H_n(0)=0$ for each $n\in I$. Denote $S_n(x)=\sum_{i<l_n}H_n(x)(i)$ for $x\in F_n$. By Claim 1, we can see that $(S_n(x))$ is bounded for any given $x\in\bigcup_n F_n$. Fix an ultrafilter $\mathfrak{A}$ on I such that $\mathfrak{A}$ does not contain any finite set. We now define $S(x)=\lim_{\mathfrak{A}} S_n(x)$. From the preceding claim, we see that

$$\forall\varepsilon>0\ \exists\delta>0\ \forall x,y\in\bigcup_n F_n(\|x-y\|<\delta\Rightarrow\|S(x)-S(y)\|\le\varepsilon),$$

i.e., S is uniformly continuous on $\bigcup_n F_n$. Thus S can be continuously extended onto $[0,1]^m$.

Claim 2. $\exists r>0\ \forall x,y\in[0,1]^m(0<\|x-y\|\le r\Rightarrow S(x)\neq S(y))$.

Proof of Claim 2. If not, there exist two sequences $(x_q),(y_q)\in[0,1]^m$ such that $0<\|x_q-y_q\|<\delta_q$, and $S(x_q)=S(y_q)$. We can find two sequences $(x_{q,j}),(y_{q,j})\in\bigcup_n F_n$ for each q such that

$$\|x_{q,j}-x_q\|,\|y_{q,j}-y_q\|<2^{-(j+2)},$$
$$\|x_{q,j}-y_{q,j}\|<\delta_q,$$
$$\|S(x_{q,j})-S(y_{q,j})\|<2^{-(q+j+2)}.$$

Denote $J_q=[2/\|x_q-y_q\|]$. For each q and $j=0,1,\dots,J_q$, choose an $n(q,j)\in I$ such that

(0) $x_{q,j},y_{q,j}\in F_{n(q,j)}$;
(1) $n(q,0)<n(q,1)<\cdots<n(q,J_q)<n(q+1,0)$;
(2) $\|S_{n(q,j)}(x_{q,j})-S(x_{q,j})\|<2^{-(q+j+3)}$ and $\|S_{n(q,j)}(y_{q,j})-S(y_{q,j})\|<2^{-(q+j+3)}$;
(3) $\|\!|H_{n(q,j)}(x_{q,j})-H_{n(q,j)}(y_{q,j})|\!\|_Y<2^{-q}$.

For $n\in I$, set

$$\alpha(n)=\begin{cases}x_{q,j}, & n=n(q,j),\\ 0, & \text{otherwise},\end{cases}\qquad \beta(n)=\begin{cases}y_{q,j}, & n=n(q,j),\\ 0, & \text{otherwise}.\end{cases}$$

Then

$$\begin{aligned}&\|\textstyle\sum_{j=0}^{J_q}(\alpha(n(q,j))-\beta(n(q,j)))\|\\ =\ &\|\textstyle\sum_{j=0}^{J_q}((x_q-y_q)+(x_{q,j}-x_q)-(y_{q,j}-y_q))\|\\ \ge\ &(1+J_q)\|x_q-y_q\|-\textstyle\sum_{j=0}^{J_q}2^{-(j+1)}\ge 1.\end{aligned}$$

Thus $(\alpha,\beta)\notin E|_I^\mu$. On the other hand,

$$\sum_q\sum_{j=0}^{J_q}\|S_{n(q,j)}(x_{q,j})-S_{n(q,j)}(y_{q,j})\|\le\sum_q\sum_{j=0}^{J_q}2^{-(q+j+1)}\le 2.$$

Comparing with clause (3), we have $\psi(\alpha)-\psi(\beta)\in\mathrm{cs}(Y)$. This is a contradiction! $\dashv_{\text{Claim 2}}$

Therefore, the restriction of S on $[0,1]^m\cap\{x\in X:\|x\|\le r\}$ is a continuous embedding into Y. This contradicts with $m>\dim(Y)$. $\dashv$

Theorem 5.9. *For $m\ge 1$, we have*

(i) $\mathbb{R}^{\mathbb{N}}/c_0<_B\mathbb{R}^{\mathbb{N}}/\mathrm{cs}$,
(ii) $\mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m)}<_B\mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m+1)}<_B\mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)}$.

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PROOF. We have already known that $\mathbb{R}^{\mathbb{N}}/c_0 \leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}$, Theorem 4.4 gives $\mathbb{R}^{\mathbb{N}}/c_0 <_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}$. For proving clause (ii), we only need to prove

$$(\mathbb{R}^m)^{\mathbb{N}}/\mathrm{cs}(\mathbb{R}^m) <_B (\mathbb{R}^{m+1})^{\mathbb{N}}/\mathrm{cs}(\mathbb{R}^{m+1}).$$

Since $(\mathbb{R}^m)^{\mathbb{N}}/\mathrm{cs}(\mathbb{R}^m) \leq_B (\mathbb{R}^{m+1})^{\mathbb{N}}/\mathrm{cs}(\mathbb{R}^{m+1})$ is trivial, the assertion follows from Lemma 5.8. ⊣

5.2. Bases of ℓ_1. As in the last subsection, we know $\mathbb{R}^{\mathbb{N}}/\ell_1$ is the minimum element among $E(\ell_1, (x_n))$ with (x_n) a basis of ℓ_1. Unfortunately, we did not find a maximum for them so far. We managed to find some bases such that the equivalence relations generated by them are not Borel reducible to $\mathbb{R}^{\mathbb{N}}/\ell_1$.

We denote $\mathrm{bv}_0 = \mathrm{bv} \cap c_0$ where bv stands for bounded variation, i.e.,

$$\mathrm{bv} = \{a \in \mathbb{R}^{\mathbb{N}} : \sum_n |a(n) - a(n+1)| < +\infty\}.$$

It is trivial that $\ell_1 \subseteq \mathrm{bv}_0$. Let $y^1_0 = e_0$ and $y^1_n = e_n - e_{n-1}$ for $n > 0$, where (e_n) is the unit vector basis of ℓ_1. Then (y^1_n) is a basis of ℓ_1 too. Since

$$\sum_n a(n) y^1_n = (a(0) - a(1), a(1) - a(2), \ldots, a(n) - a(n+1), \ldots),$$

we can see $\mathrm{bv}_0 = \mathrm{coef}(\ell_1, (y^1_n))$.

THEOREM 5.10. $\mathbb{R}^{\mathbb{N}}/\ell_1 <_B \mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0$.

PROOF. For $a \in \mathbb{R}^{\mathbb{N}}$ and $k \in \mathbb{N}$, define $\theta(a)(2k) = a(k)$ and $\theta(a)(2k+1) = 0$. It is clear that θ is a Borel reduction of $\mathbb{R}^{\mathbb{N}}/\ell_1$ to $\mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0$.

Note that $\|y^1_0 + \cdots + y^1_n\| = 1$ for any $n \in \mathbb{N}$, but $\sum_n y^1_n$ does not converge. It follows that (y^1_n) is not boundedly complete. By Corollary 3.3, we have $\mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0 \not\leq_B \mathbb{R}^{\mathbb{N}}/\ell_\infty$. And hence $\mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0 \not\leq_B \mathbb{R}^{\mathbb{N}}/\ell_1$. ⊣

For $m \geq 1$, similarly to $\mathrm{cs}^{(m)}$, note that $\bigoplus_{i=1}^m \ell_1 \cong \ell_1$, we choose a suitable basis (y^m_n) of ℓ_1 such that $\mathrm{coef}(\ell_1, (y^m_n)) = \mathrm{bv}_0^{(m)} = \mathrm{bv}^{(m)} \cap c_0$, where

$$\mathrm{bv}^{(m)} = \{a \in \mathbb{R}^{\mathbb{N}} : \forall i \leq m (\sum_j |a(mj + i - 1) - a(m(j+1) + i - 1)| < +\infty)\}.$$

Also similarly to $\mathrm{cs}^{(\infty)}$, we define $\mathrm{bv}_0^{(\infty)}$ as follows. Recall that

$$\left(\bigoplus_{i \in \mathbb{N}} \ell_1\right)_1 = \{(a_n) \in (\ell_1)^{\mathbb{N}} : \sum_n \|a_n\| < +\infty\}.$$

We still have $\left(\bigoplus_{i \in \mathbb{N}} \ell_1\right)_1 \cong \ell_1$. Fix a bijection $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$ such that, for any i, $\langle i, j \rangle$ is strictly increasing with respect to variable j. We can find a basis (y^∞_n) of ℓ_1 such that $\mathrm{coef}(\ell_1, (y^\infty_n)) = \mathrm{bv}_0^{(\infty)} = \mathrm{bv}^{(\infty)} \cap c_0$, where

$$\mathrm{bv}^{(\infty)} = \{a \in \mathbb{R}^{\mathbb{N}} : \sum_i \sum_j |a(\langle i, j \rangle) - a(\langle i, j+1 \rangle)| < +\infty\}.$$

It is trivial that

$$\mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0^{(m)} \leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0^{(m+1)} \leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0^{(\infty)}.$$

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We do not know whether they are Borel bireducible to each other. We also compare them with the equivalence relations that appeared in the last subsection.

THEOREM 5.11. *For any $m \in \mathbb{N}$, we have*

(i) $\mathbb{R}^{\mathbb{N}}/\ell_1 \not\leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m)}$,
(ii) $\mathbb{R}^{\mathbb{N}}/c_0 \not\leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0^{(\infty)}$,
(iii) $\mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0^{(\infty)} <_B \mathbb{R}^{\mathbb{N}}/\ell_1 \otimes \mathbb{R}^{\mathbb{N}}/c_0$.

PROOF. (i) The proof is a combination of proofs of Theorem 4.4 and Lemma 5.8, So we omit some similar arguments.

Since $(\mathbb{R}^m)^{\mathbb{N}}/\mathrm{cs}(\mathbb{R}^m) \sim_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(m)}$, we assume for contradiction that $\mathbb{R}^{\mathbb{N}}/\ell_1 \leq_B (\mathbb{R}^m)^{\mathbb{N}}/\mathrm{cs}(\mathbb{R}^m)$. Denote $F_n = \{i/2^n : i = 0, 1, \ldots, 2^n\}$. From Lemma 4.2, we can find an infinite set $I \subseteq \mathbb{N}$, a natural number $l_n \geq 1$ and a map $H_n : F_n \to (\mathbb{R}^m)^{l_n}$ for each $n \in I$, satisfying the following requirements. Letting (n_k) be the strictly increasing enumeration of I, we define ψ as

$$\psi(a) = H_{n_0}(a(n_0))^\frown H_{n_1}(a(n_1))^\frown H_{n_2}(a(n_2))^\frown \cdots,$$

for any $a \in \prod_{n\in I} F_n$, then we have, for $a, b \in \prod_{n\in I} F_n$,

$$\sum_{n\in I} |a(n) - b(n)| < +\infty \iff (\psi(a) - \psi(b)) \in \mathrm{cs}(\mathbb{R}^m).$$

Choose an $i_n \in \{0, 1, \ldots, 2^n\}$ for each $n \in I$ such that

$$\lim_{k\to\infty} \frac{i_{n_k}}{2^{n_k}} = 0, \quad \sum_{n\in I} \frac{i_n}{2^n} = +\infty.$$

Now denote $x_k = i_{n_k}/2^{n_k}$ for each $k \in \mathbb{N}$. We claim that $\lim_{k\to\infty} \|H_{n_k}(x_k) - H_{n_k}(0)\|_{\mathbb{R}^m} = 0$. If not, there is an infinite $K \subseteq \mathbb{N}$ and $\varepsilon > 0$ such that $\|H_{n_k}(x_k) - H_{n_k}(0)\|_{\mathbb{R}^m} \geq \varepsilon$ for each $k \in K$. Since $\lim_{k\to\infty} x_k = 0$, we can find an infinite $J \subseteq K$ with $\sum_{k\in J} x_k$ converging. Now define $a^\emptyset(n) = 0$ for each $n \in I$ and $a^J \in \prod_{n\in I} F_n$ as

$$a^J(n) = \begin{cases} x_k, & n = n_k, k \in J, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\sum_{n\in I}(a^J(n) - a^\emptyset(n)) = \sum_{k\in J} x_k$ converges, so $(a^J, a^\emptyset) \in E|_I^\mu$. Therefore $\psi(a^J) - \psi(a^\emptyset) \in \mathrm{cs}(\mathbb{R}^m)$. By Cauchy criterion, we have

$$\lim_{k\to\infty} \|H_{n_k}(a^J(n_k)) - H_{n_k}(a^\emptyset(n_k))\|_{\mathbb{R}^m} = 0.$$

This contradicts with $J \subseteq K$.

By Dvoretzky-Hanani's Theorem (see, e.g., [13], Theorem 2.2.1), there is a sequence $\epsilon \in \{-1, 1\}^{\mathbb{N}}$ such that

$$\sum_k \epsilon(k)(H_{n_k}(x_k) - H_{n_k}(0)) \text{ converges.}$$

Now define $a^+, a^- \in \prod_{n\in I} F_n$ as

$$a^+(n_k) = \begin{cases} x_k, & \epsilon(k) = 1, \\ 0, & \epsilon(k) = -1, \end{cases} \qquad a^-(n_k) = \begin{cases} 0, & \epsilon(k) = 1, \\ x_k, & \epsilon(k) = -1. \end{cases}$$

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We have $\sum_{n\in I}|a^+(n)-a^-(n)|=\sum_k x_k=+\infty$. It follows that $(\psi(a^+)-\psi(a^-))\notin \mathrm{cs}(Y)$. On the other hand, we can see that

$$\sum_k (H_{n_k}(a^+(n_k))-H_{n_k}(a^-(n_k)))=\sum_k \epsilon(k)(H_{n_k}(x_k)-H_{n_k}(0)) \text{ converges.}$$

Comparing with $\lim_{k\to\infty}\|H_{n_k}(x_k)-H_{n_k}(0)\|_{\mathbb{R}^m}=0$, we have $(\psi(a^+)-\psi(a^-))\in \mathrm{cs}(Y)$. A contradiction!

(ii) By Theorem 8.5.2 and Lemma 8.5.3 of [8], we only need to prove that $\mathrm{bv}_0^{(\infty)}$ is $\mathbf{\Sigma}^0_3$. Define a subset $A\subseteq\mathbb{R}^{\mathbb{N}}$ satisfying that, for any $a\in\mathbb{R}^{\mathbb{N}}$,

$$\begin{aligned} a\in A &\iff \forall i\in\mathbb{N}(\text{there is a subsequence of } a(\langle i,\cdot\rangle) \text{ converging to } 0)\\ &\iff \forall i,p,N\in\mathbb{N}\exists j>N(|a(\langle i,j\rangle)|<1/p).\end{aligned}$$

Note that, for any $a\in \mathrm{bv}^{(\infty)}$, we have $\lim_{j\to\infty}a(\langle i,j\rangle)$ converges for each $i\in\mathbb{N}$, and $a(\langle i,\cdot\rangle)$ is uniformly convergent to 0 as $i\to\infty$. It follows that $\mathrm{bv}_0^{(\infty)}=\mathrm{bv}^{(\infty)}\cap c_0=\mathrm{bv}^{(\infty)}\cap A$. Since $\mathrm{bv}^{(\infty)}$ is F_σ and A is G_δ, we see that $\mathrm{bv}_0^{(\infty)}$ is $\mathbf{\Delta}^0_3$. Indeed, $\mathrm{bv}_0^{(\infty)}$ is a $D_2(\mathbf{\Sigma}^0_2)$ set (for definition of $D_2(\mathbf{\Sigma}^0_2)$, see, e.g., [15], 22.E).

(iii) $\mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0^{(\infty)}<_B \mathbb{R}^{\mathbb{N}}/\ell_1\otimes\mathbb{R}^{\mathbb{N}}/c_0$ follows from Theorem 5.3 and (ii). ⊣

However, we do not know whether $\mathbb{R}^{\mathbb{N}}/\ell_1\leq_B \mathbb{R}^{\mathbb{N}}/\mathrm{cs}^{(\infty)}$.

For $p>1$, (y^1_n) is also a sequence in ℓ_p, but not a basis of ℓ_p. We have

$$\mathrm{coef}(\ell_p,(y^1_n))=c_0\cap\{a\in\mathbb{R}^{\mathbb{N}}:\sum_n|a(n)-a(n+1)|^p<+\infty\}.$$

For the Borel reducibility, we claim $E(\ell_p,(y^1_n))\sim_B \mathbb{R}^{\mathbb{N}}/\ell_p\otimes\mathbb{R}^{\mathbb{N}}/c_0$. This is because of Theorem 5.3 and the following θ which witnesses $\mathbb{R}^{\mathbb{N}}/\ell_p\otimes[0,1]^{\mathbb{N}}/c_0\leq_B E(\ell_p,(y^1_n))$. For $a\in\mathbb{R}^{\mathbb{N}}, b\in[0,1]^{\mathbb{N}}$ and $n\in\mathbb{N}$, we define

$$\theta(a,b)(n)=\begin{cases} b(0), & 0\leq n\leq 3,\\ a(k)+b(k), & n=2^{k+2},\\ b(k)+\frac{(i-1)(b(k+1)-b(k))}{2^{k+2}-2}, & n=2^{k+2}+i, 1\leq i<2^{k+2}.\end{cases}$$

In the light of Theorem 4.4, we are interested in these examples because the unit vector basis of $\mathrm{coef}(\ell_p,(y^1_n))$ is conditional while $\mathbb{R}^{\mathbb{N}}/\ell_p\otimes\mathbb{R}^{\mathbb{N}}/c_0$ is generated by an unconditional basis.

§6. Further remarks. Perhaps the most interesting question is:

QUESTION 6.1. *If (x_n) and (y_n) are two bases of a reflexive Banach space X, does $E(X,(x_n))\sim_B E(Y,(y_n))$?*

Let us denote by SER the set of all Schauder equivalence relations. Ma [17] showed that, in SER, there is a maximum element and $\mathbb{R}^{\mathbb{N}}/c_0$ is a minimal element with respect to Borel reducibility. Hjorth's dichotomy below $\mathbb{R}^{\mathbb{N}}/\ell_1$ (see Corollary 5.6 of [11]) implies that $\mathbb{R}^{\mathbb{N}}/\ell_1$ is another minimal element in SER. If we restrict attention on two main classes of spaces X whose unit vector basis is symmetric, i.e., Orlicz sequence spaces and Lorentz sequence spaces, we always have either $\mathbb{R}^{\mathbb{N}}/\ell_1\leq_B \mathbb{R}^{\mathbb{N}}/X$ or $\mathbb{R}^{\mathbb{N}}/c_0\leq_B \mathbb{R}^{\mathbb{N}}/X$. Therefore, our first question is:

QUESTION 6.2. *Let (x_n) be a symmetric basis of X, does either $\mathbb{R}^{\mathbb{N}}/\ell_1\leq_B E(X,(x_n))$ or $\mathbb{R}^{\mathbb{N}}/c_0\leq_B E(X,(x_n))$ hold?*

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We say two elements $E, F \in \mathrm{SER}$ are incompatible in SER, if no element in SER can be Borel reducible to both E and F. It is well known that $\mathbb{R}^{\mathbb{N}}/\ell_1$ and $\mathbb{R}^{\mathbb{N}}/c_0$ form an incompatible pair. As Ma [17] indicated, Farah [6] potentially proved that, for any α-Tsirelson space T_α, $\mathbb{R}^{\mathbb{N}}/T_\alpha$ is incompatible with either $\mathbb{R}^{\mathbb{N}}/\ell_1$ or $\mathbb{R}^{\mathbb{N}}/c_0$. Furthermore, whenever $\alpha \neq \beta$, the relations $\mathbb{R}^{\mathbb{N}}/T_\alpha$ and $\mathbb{R}^{\mathbb{N}}/T_\beta$ are incompatible.

Let (x_n) be an unconditional basis of a Banach space X. James [12] proved that, if (x_n) is not boundedly complete, then there exists a block basis (u_k) of (x_n) such that $\mathrm{coef}(X,(u_k)) = c_0$ (see also [16], Theorem 1.c.10). In the light of Corollary 3.3, we get a dichotomy that, for an unconditional basis (x_n) of X, exactly one of the following holds:

(i) $E(X,(x_n)) \leq_B \mathbb{R}^{\mathbb{N}}/\ell_\infty$,
(ii) $\mathbb{R}^{\mathbb{N}}/c_0 \leq_B E(X,(x_n))$.

This dichotomy cannot be generalized to conditional basis, since $\mathbb{R}^{\mathbb{N}}/\mathrm{bv}_0$ can serve as a counterexample. This is because clause (i) is equivalent to saying that $\mathrm{coef}(X,(x_n))$ is F_σ, while clause (ii) implies it is $\mathbf{\Pi}^0_3$-complete, and bv_0 is $D_2(\mathbf{\Sigma}^0_2)$. Thus we ask two related questions:

QUESTION 6.3.

(I) *Is there a basis* (x_n) *of* X *such that* $\mathrm{coef}(X,(x_n))$ *is* $\mathbf{\Delta}^0_3$ *but not* $D_2(\mathbf{\Sigma}^0_2)$?
(II) *For any basis* (x_n) *of* X, *if* $\mathrm{coef}(X,(x_n))$ *is a* $\mathbf{\Pi}^0_3$*-complete set in* $\mathbb{R}^{\mathbb{N}}$, *does* $\mathbb{R}^{\mathbb{N}}/c_0 \leq_B E(X,(x_n))$?

We can use an unconditional basis (x_n) of X to generate an ideal on $\mathbb{N}$. Denote $\mathcal{I}(X,(x_n)) = \{A \subseteq \mathbb{N} : \sum_{n\in A} x_n \text{ converges}\}$. It is clear that $P(\mathbb{N})/\mathcal{I}(X,(x_n)) \leq_B E(X,(x_n))$. Let (x_n) and (y_n) be unconditional bases of X and Y respectively, and suppose $E(X,(x_n)) \leq_B E(Y,(y_n))$. Applying Lemma 4.2 on $\{0,1\}^{\mathbb{N}}$, we can find a subsequence (x_{n_k}) of (x_n) and a block basis (u_k) of (y_n) such that $\mathcal{I}(X,(x_{n_k})) = \mathcal{I}(Y,(u_k))$. Furthermore, for any block basis (v_k) of (x_n), we can find a subsequence of (v_k) and a block basis of (y_n) such that they generate the same ideal. Therefore, we may consider the following question:

QUESTION 6.4. *Let* (x_n) *be an unconditional basis of* X. *To what extent can ideals generated by block bases of* (x_n) *determine the equivalence relation* $E(X,(x_n))$?

It is worth noting that block bases of the unit bases of any space ℓ_p (in fact, any Orlicz sequence space) generate the same class of ideals, though they generate totally different equivalence relations with respect to Borel reducibility.

When (x_n) is a conditional basis of X, $\mathcal{I}(X,(x_n))$ is not an ideal in general. A powerful alternative tool is

$$\mathcal{S}(X,(x_n)) = \{\epsilon \in \{-1,1\}^{\mathbb{N}} : \sum_n \epsilon(n)x_n \text{ converges}\}.$$

In fact, it is already used in proofs of theorems 4.4 and 5.11. Another useful tool is

$$\mathcal{J}(X,(x_n)) = \{A \subseteq \mathbb{N} : \sum_{n\in A} x_n \text{ is unconditionally convergent}\}$$

(see e.g., [1, 5]). Note that $\mathcal{J}(X,(x_n))$ is always an ideal.

Besides unconditional bases, we are also interested in H.I. spaces (hereditarily indecomposable Banach spaces). Gowers [10] proved that any basis of a Banach

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space has a block basis which is either unconditional or a basis of an H.I. subspace. Then another interesting question is:

QUESTION 6.5. *Let (x_n) be an unconditional basis of X, (y_n) a basis of an H.I. space Y. Is it possible that $E(X,(x_n))$ and $E(Y,(y_n))$ are compatible in* SER?

In contrast, it is well known that, in this situation, no infinite dimensional Banach space can embed into both X and Y.

§7. **Acknowledgements.** Research partially supported by the National Natural Science Foundation of China (Grant No. 11371203). The main results of this article were conducted while I visited Nanyang Technological University and Institute for Mathematical Sciences of the National University of Singapore. The article was written while I visited the University of North Texas. I would like to thank Guohua Wu, Chi Tat Chong, and Su Gao for invitations of these visits. I am grateful to Rui Liu, Xin Ma, Zhi Yin, Minggang Yu, and Yukun Zhang for conversations in seminars. I also thank Su Gao and Bunyamin Sari for useful suggestions. Special thanks are due to anonymous referees for earlier versions of this article for carefully reading and helpful suggestions, especially on the improvement of language.

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https://doi.org/10.1017/jsl.2017.67 Published online by Cambridge University Press