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HOLONOMIC ÉTALE SHEAVES ARE CONSTRUCTIBLE

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Abstract Building on Beilinson's work, 'constructible sheaves are holonomic', we introduce the notion of holonomicity for étale sheaves, without assuming a priori constructibility. We establish the converse of Beilinson's result, showing that holonomic sheaves are indeed constructible. This can be seen as an étale analogue of Kashiwara's theorem on holonomic $\mathcal{D}_X$-modules.

For a coherent $\mathcal{D}_X$-module $\mathcal{M}$ on a complex manifold X, its singular support $SS\mathcal{M}$ is defined as a closed conical subset of the cotangent bundle [10], [7]. Inspired by the analogy between $\mathcal{D}_X$-modules on complex manifolds and étale sheaves on algebraic varieties, Beilinson proved that every constructible sheaf $\mathcal{F}$ on a smooth scheme X over a field has a singular support $SS\mathcal{F}$ and that its irreducible components have the same dimension as X in [3]. This means that constructible sheaves are holonomic as the title of his article says although he did not explicitly define the holonomicity.

Following his construction, we introduce the holonomicity for étale sheaves, without assuming a priori that they are constructible. We prove conversely that holonomic sheaves are in fact constructible. One may consider this as an étale analogue of a theorem of Kashiwara [9] asserting that for a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ on a complex manifold X, the $\mathbb{C}_X$-module $\mathcal{E}xt^i_X(\mathcal{M},\mathcal{O}_X)$ is constructible for every i, which is an important ingredient of the Riemann–Hilbert correspondence. We expect a similar notion of holonomicity for étale sheaves on rigid varieties could be defined and would give a good substitute for the deficient notion of constructibility in this context.

Keywords: singular support; constructible sheaves; holonomic sheaves

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The contents of the sections are summarized as follows. In the brief Section 1, we recall the fact that a closed conical subset of a vector bundle is determined by its base and its projectivization. In Sections 2 and 3, we study two properties of morphisms relative to a closed conical subset C of the cotangent bundle, largely following [3]. The notion of C-transversality applies when C is a subset for the target of a morphism $h\colon W \to X$, while C-acyclicity applies when C is a subset for the source of a morphism $f\colon X \to Y$. Both notions enter the definition of the micro support of a sheaf $\mathcal{F}$; in [3], they are both referred to as transversality. In Section 4, we extend the definition of micro support given in [3] to sheaves that are not necessarily constructible; in [3], constructibility is assumed. In Section 5, we prove the existence of the singular support of $\mathcal{F}$, defined as the smallest closed conical subset on which $\mathcal{F}$ is micro supported, without assuming constructibility. The proof follows that of [3] and relies on the Radon transform. Finally, in Section 6, we prove the main result: if $\dim SS\mathcal{F} \leqq \dim X$ and if every fibre of $\mathcal{F}$ is constructible, then $\mathcal{F}$ itself is constructible. The proof proceeds by induction on $\dim X$ and ultimately reduces to the fact that a sheaf micro supported on the zero section is locally constant.

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1. Closed conical subsets

Definition 1.1. Let X be a scheme and V be a vector bundle on X. We say that a closed subset C of V is conical if it is stable under the action of the multiplicative group $\mathbf{G}_m$.

The intersection B of a closed conical subset C with the 0-section, regarded as a closed subset of X, is called the base of C. The projectivization $\mathbf{P}(C) \subset \mathbf{P}(V)$ is defined to be the image of the complement $C - B$ by the canonical projection $V - X \to \mathbf{P}(V)$ from the complement of the 0-section.

Lemma 1.2. *Let X be a scheme and C be a closed conical subset of a vector bundle V on X.*

1. *C equals the union of the base B and the inverse image of the projectivization $\mathbf{P}(C)$ by $V - X \to \mathbf{P}(V)$.*
2. *$U = \{x \in X \mid C_x = \varnothing\} \subset U' = \{x \in X \mid C_x \subset \{0\}\}$ are open subsets of X.*

Proof. 1. Since C is conical, $C - B$ equals the inverse image of $\mathbf{P}(C)$.

2. U is the complement of B and U' is the complement of the image of $\mathbf{P}(C)$ by the projection $\mathbf{P}(V) \to X$. □

2. C-transversality

Let k be a field and X a smooth scheme over k. The covariant vector bundle T^*X associated to the locally free $\mathcal{O}_X$-module $\Omega^1_{X/k}$ is called the cotangent bundle of X. For

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a smooth subscheme Z of X, the conormal bundle $T^*_Z X$ associated to the conormal sheaf $N_{Z/X}$ is canonically identified with $\mathrm{Ker}(T^*X \times_X Z \to T^*Z)$. In particular for $Z = X$, the 0-section of T^*X is denoted by $T^*_X X$.

Lemma 2.1. *Let X be a smooth scheme over k. Let $C \subset T^*X$ be a closed conical subset satisfying $\dim C \leqq \dim X$. Then there exists a dense open scheme $U \subset X$ such that the restriction C_U is a subset of the 0-section $T^*_U U$.*

Proof. Since $\dim C \leqq \dim X$, the projectivization $\mathbf{P}(C)$ has dimension strictly less than $\dim X$. Hence it follows from Lemma 1.2. □

Definition 2.2 [3, 1.2]. Let $h\colon W \to X$ be a morphism of smooth schemes over a field k and C be a closed conical subset of the cotangent bundle T^*X.

1. Let w be a point of W. We say that h is C-transversal at w, if the intersection of the inverse image $C_w \subset T^*X \times_X w$ of C with the kernel $\mathrm{Ker}(T^*X \times_X w \to T^*W \times_W w)$ inside $T^*X \times_X w$ is a subset $\{0\}$.
2. We say that h is C-transversal if h is C-transversal at every point of W.

The condition that h is C-transversal means that the intersection of the inverse image $h^*C \subset T^*X \times_X W$ of C with the kernel $\mathrm{Ker}(T^*X \times_X W \to T^*W)$ inside $T^*X \times_X W$ is a subset of the 0-section.

Lemma 2.3. *Let $h\colon W \to X$ be a morphism of smooth schemes over a field k and C be a closed conical subset of the cotangent bundle T^*X.*

1. [3, Example 1.2 (i)] *h is smooth if and only if h is T^*X-transversal.*
2. *If C is a subset of the 0-section, then h is C-transversal.*
3. *If C' is a closed conical subset of T^*X containing C as a subset and if h is C'-transversal, then h is C-transversal. Consequently, if h is smooth, then h is C-transversal.*
4. [3, Lemma 1.2 (i)] *The subset $\{w \in W \mid h$ is C-transversal at $w\}$ is an open subset of W.*
5. *Let $Z \subset X$ be a closed subscheme smooth over k and assume that C is the conormal bundle $T^*_Z X \subset T^*X$. Then the following conditions are equivalent:*

(1) *h is C-transversal.*

(2) *$V = Z \times_X W$ is smooth over k and the codimension of the immersion $V \to W$ equals that of $Z \to X$.*

Proof. 1. h is T^*X-transversal if and only if $W \times_X T^*X \to T^*W$ is an injection. This means that $h^*\Omega^1_{X/k} \to \Omega^1_{W/k}$ is a locally split injection.

2. and 3. Clear from the definition and 1.

4. It suffices to apply Lemma 1.2.2 to the closed conical subset $h^*C \cap \mathrm{Ker}(W \times_X T^*X \to T^*W))$ of $W \times_X T^*X$.

5. Condition (1) means that $T^*_Z X \times_X W \to T^*W$ is an injection. In other words, $\mathcal{N}_{Z/X} \otimes_{\mathcal{O}_Z} \mathcal{O}_V$ is locally a direct summand of $\Omega^1_{W/k} \otimes_{\mathcal{O}_W} \mathcal{O}_V$. This is equivalent to condition (2). □

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In the next lemma, we regard the closed subset h^*C as a closed subscheme.

Lemma 2.4 [3, Lemma 1.2 (ii)]. *Let $h\colon W \to X$ be a morphism of smooth schemes over k and let $C \subset T^*X$ be a closed conical subset. Assume that $h\colon W \to X$ is C-transversal. Then, $W \times_X T^*X \to T^*W$ is finite on h^*C.*

Under the assumption in Lemma 2.4, we define a closed conical subset $h^\circ C \subset T^*W$ to be the image of h^*C. Recall that h^*C is the inverse image of $C \subset T^*X$ by $T^*X \times_X W \to T^*X$.

Lemma 2.5. *Let $g\colon U \to X$ be a morphism of smooth schemes over k and let $C \subset T^*X$ be a closed conical subset. Assume that g is C-transversal. Then for a morphism $h\colon W \to U$ of smooth schemes over k, the following conditions are equivalent:*

(1) *$g \circ h$ is C-transversal.*
(2) *h is $g^\circ C$-transversal.*

Proof. We consider the composition $(g\circ h)^*C \subset T^*X \times_X W \to T^*U \times_U W \to T^*W$. The assumption implies that the intersection $(g\circ h)^*C \cap \mathrm{Ker}(T^*X \times_X W \to T^*U \times_U W)$ is a subset of the 0-section. Since $h^*(g^\circ C) \subset T^*U \times_U W$ is the image of $(g\circ h)^*C$, the intersection $(g\circ h)^*C \cap \mathrm{Ker}(T^*X \times_X W \to T^*W)$ is a subset of the 0-section if and only $h^*(g^\circ C) \cap \mathrm{Ker}(T^*U \times_U W \to T^*W)$ is a subset of the 0-section. □

Definition 2.6. Let $g\colon X' \to X$ be a morphism of smooth schemes over k.

1. Let $C' \subset T^*X'$ be a closed conical subset and let $B' \subset X'$ be the base of C'. We say that g is proper on B' if the restriction of g on B' regarded as a closed subscheme of X' is proper.

Assume that g is proper on B'. We define a closed conical subset $g_\circ C' \subset T^*X$ to be the image by $X' \times_X T^*X \to T^*X$ of the inverse image of C' by the canonical morphism $X' \times_X T^*X \to T^*X'$.

2. Let $h\colon W \to X$ be a morphism of smooth schemes over k and $V' \subset V = X' \times W$ be an open subscheme. We say that h and g are transversal on V' if V' is smooth over k and if the morphism $\mathcal{O}_W \otimes^L_{\mathcal{O}_X} \mathcal{O}_Z \to \mathcal{O}_V$ is an isomorphism on V'.

If $i\colon Z \to X$ is a closed immersion of smooth schemes over k and if $C \subset T^*Z$ is a closed conical subset, then $i_\circ C \subset T^*X$ is the inverse image of C by the canonical surjection $T^*X|_Z \to T^*Z$ regarded as a subset of T^*X. Condition (2) in Lemma 2.3.5 means that the morphism $h\colon W \to X$ is transversal to the immersion $i\colon Z \to X$.

Lemma 2.7. *Let $g\colon X' \to X$ be a morphism of smooth schemes over k and let $C' \subset T^*X'$ be a closed conical subset. Assume that g is proper on the base $B' \subset X'$ of C' and define $C = g_\circ C'$. Let $h\colon W \to X$ be a morphism of smooth schemes over k. Assume that h is $g_\circ C'$-transversal.*

1. *There exists an open neighbourhood $V' \subset V = X' \times_X W$ of the inverse image $\mathrm{pr}_1^{-1}(B')$ such that g and h are transversal on V'.*
2. *Let $V' \subset V$ be an open subset as in 1. Then, the restriction of the projection $h'\colon V' \to X'$ is C'-transversal and $h^\circ g_\circ C'$ equals $g'_\circ h'^\circ C'$ for $g'\colon V' \to W$.*

Proof. 1. By decomposing h into $W \to X \times W \to X$, it suffices to consider the case where h is an immersion by Lemma 2.5. Since the immersion $h\colon W \to X$ is $g_\circ C'$-transversal,

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the intersection $\mathrm{Ker}(T^*X \times_X X' \to T^*X') \times_{X'} V \cap \mathrm{Ker}(T^*X \times_X W \to T^*W) \times_W V$ is a subset of the 0-section on a neighbourhood V' of $\mathrm{pr}_1^{-1}(B') \subset V$. Namely, the restriction of $(T^*X \times_X X' \to T^*X') \times_{X'} V'$ on $\mathrm{Ker}(T^*X \times_X W \to T^*W) \times_W V'$ is a locally split injection and the assertion follows.

2. Similarly as in the proof of 1, we may assume that $h\colon W \to X$ is an immersion. Then, further, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^*_W X \times_W V' & \longrightarrow & T^*X \times_X V' & \longrightarrow & T^*W \times_W V' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T^*_{V'} X' & \longrightarrow & T^*X'|_{V'} & \longrightarrow & T^*V' & \longrightarrow & 0 \end{array}$$

of exact sequences where the left vertical arrow is an isomorphism. Hence the C-transversality of h implies the C'-transversality of $h'\colon V' \to X'$ and the equality $h^\circ g_\circ C' = g'_\circ h'^\circ C'$ follows. □

Lemma 2.8. *Let X be a smooth scheme over a field k and let $W \subset Z \subset X$ be closed subschemes smooth over k. Let $s\colon T^*Z \to T^*X|_Z$ be a linear section of the canonical surjection $T^*X|_Z \to T^*Z$.*

1. *Locally on a neighbourhood of $W \subset X$, there exists a closed subscheme $V \subset X$ smooth over k such that $W \to Z \times_X V$ is an isomorphism,* $\mathrm{codim}_X V = \mathrm{codim}_Z W$ *and $T^*_V X|_W \subset T^*X|_W$ equals the image $s(T^*_W Z)$ by the section s.*

2. *Let $C \subset T^*X|_Z$ be a closed conical subset. Let $h\colon W \to Z$ be the closed immersion and let $h'\colon V \to X$ be a closed immersion as in 1. If h is $s^{-1}(C)$-transversal, then h' is C-transversal on a neighbourhood of the image of W.*

Proof. 1. Let $\mathcal{I} \subset \mathcal{O}_Z$ and $\mathcal{J} \subset \mathcal{O}_X$ be the ideal sheaves defining $W \subset Z$ and $W \subset X$. Then the section $s\colon T^*Z \to T^*X|_Z$ induces a section $T^*_W Z \to T^*_W X$ on sub vector bundles, namely $\mathcal{I}/\mathcal{I}^2 \to \mathcal{J}/\mathcal{J}^2$. Locally on W, we take a basis of $s(\mathcal{I}/\mathcal{I}^2) \subset \mathcal{J}/\mathcal{J}^2$. By taking its lifting to $\mathcal{J}$, we define a closed subscheme $V \subset X$. Then, for the conormal sheaf, we have an isomorphism $\mathcal{N}_{V/X} \otimes_{\mathcal{O}_V} \mathcal{O}_W \to \mathcal{N}_{W/Z}$. Hence after shrinking V if necessary, V is smooth over k and $\mathrm{codim}_X V = \mathrm{codim}_Z W$. The isomorphism $\mathcal{N}_{V/X} \otimes_{\mathcal{O}_V} \mathcal{O}_W \to \mathcal{N}_{W/Z}$ means the equality $T^*_V X|_W = s(T^*_W Z)$.

2. Since $V \cap Z = W$, the equality $T^*_V X|_W = s(T^*_W Z)$ implies that the section s induces a bijection $s^{-1}(C) \cap \mathrm{Ker}(T^*Z|_W \to T^*W) \to C \cap \mathrm{Ker}(T^*X|_V \to T^*V)$. □

3. *C*-acyclicity

Definition 3.1 [3, 1.2]. Let X be a smooth scheme over a field k and C be a closed conical subset of the cotangent bundle T^*X.

1. We say that a morphism $f\colon X \to Y$ of smooth schemes over k is C-acyclic if the inverse image of C by the canonical morphism $X \times_Y T^*Y \to T^*X$ is a subset of the 0-section.

2. Let $h\colon W \to X$ and $f\colon W \to Y$ be morphisms of smooth schemes over k. We say that (h, f) is C-acyclic if $h\colon W \to X$ is C-transversal and if $f\colon W \to Y$ is $h^\circ C$-acyclic.

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We say that (h,f) is universally C-acyclic over k if for every morphism $g\colon Y'\to Y$ of smooth schemes over k and for the commutative diagram

$$\begin{array}{ccccc} & & W' & \xrightarrow{f'} & Y' \\ & \swarrow^{h'} & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xleftarrow{h} & W & \xrightarrow{f} & Y \end{array} \tag{3.1}$$

with cartesian square, there exists a neighbourhood $U'\subset W'$ of the inverse image $h'^{-1}(B)$ of the base B of C such that g is transversal to f on U' and the pair (h',f') is C-acyclic on U'.

To avoid using the same terminology for different notions, we modified Beilinson's original terminology in [3].

Lemma 3.2. *Let X be a smooth scheme over a field k and C be a closed conical subset of T^*X.*

1. *Let $h\colon W\to X$ and $f\colon W\to Y$ be morphisms of smooth schemes over k. We identify $T^*(X\times Y)$ with $T^*X\times T^*Y$ and regard $C\times T^*Y$ as a closed conical subset of $T^*(X\times Y)$. Then, the following conditions are equivalent:*

(1) *The pair (h,f) is C-acyclic.*
(2) *The morphism $(h,f)\colon W\to X\times Y$ is $C\times T^*Y$-transversal.*

If these equivalent conditions are satisfied, then the morphism $f\colon W\to Y$ is smooth on a neighbourhood of $h^{-1}(B)$.

2. *Let $f\colon X\to Y$ be a morphism of smooth schemes over k and assume that C is the 0-section T^*_XX. Then $f\colon X\to Y$ is C-acyclic if and only if f is smooth.*

Proof. 1. (1)⇒(2): Let (a,b) be a point of the intersection $\big(h^*C\times_W(T^*Y\times_Y W)\big)\cap \mathrm{Ker}((T^*X\times_X W)\times_W(T^*Y\times_Y W)\to T^*W)$. Then, $b\in T^*Y\times_Y W$ is in the inverse image of $-dh(a)\in h^\circ C$ and the $h^\circ C$-acyclicity of f implies that $b=0$. Further the C-transversality of h implies $a=0$.

(2)⇒(1): Since the injection $T^*X\times_X W\to(T^*X\times_X W)\times_W(T^*Y\times_Y W)$ to the first factor maps $h^*C\cap\mathrm{Ker}(T^*X\times_X W\to T^*W)$ to $\big(h^*C\times_W(T^*Y\times_Y W)\big)\cap\mathrm{Ker}((T^*X\times_X W)\times_W(T^*Y\times_Y W)\to T^*W)$, the $C\times T^*Y$-transversality of (h,f) implies the C-transversality of h. If $b\in T^*Y\times_Y W$ is in the inverse image of $dh(a)\in h^\circ C$ for $a\in h^*C$, then $(-a,b)$ is contained in $\big(h^*C\times_W(T^*Y\times_Y W)\big)\cap\mathrm{Ker}((T^*X\times_Y W)\times_W(T^*Y\times_Y W)\to T^*W)$ and is 0.

If $f\colon W\to Y$ is $h^\circ C$-acyclic, then $\mathrm{Ker}(W\times_Y T^*Y\to T^*W)|_{h^{-1}(B)}$ is a subset of the 0-section.

2. The morphism f is T^*_XX-acyclic if and only if f is T^*Y-transversal. Hence the assertion follows from Lemma 2.3.1. □

Lemma 3.3. *Let $h\colon W\to X$ and $f\colon W\to Y$ be morphisms of smooth schemes over k and C be a closed conical subset of T^*X.*

1. [3, Example 1.2 (i)] *The following conditions are equivalent:*
(1) *(h,f) is T^*_XX-acyclic.*
(2) *$f\colon W\to Y$ is smooth.*

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2. [3, Example 1.2 (ii)] *The following conditions are equivalent:*
(1) (h,f) *is* T^*X*-acyclic.*
(2) $(h,f)\colon W \to X \times Y$ *is smooth.*

Proof. 1. Since h is $T^*_X X$-transversal by Lemma 2.3.2 and $h^\circ T^*_X X = T^*_W W$, condition (1) means that $f\colon W \to Y$ is $T^*_W W$-acyclic. This is equivalent to (2) by Lemma 3.2.2.
2. By Lemma 3.2.1, condition (1) means that $(h,f)\colon W \to X \times Y$ is $T^*X \times T^*Y$-transversal. This is equivalent to (2) by Lemma 2.3.1. □

Lemma 3.4. *Let* $g\colon U \to X$ *be a smooth morphism of smooth schemes over* k *and let* $C \subset T^*X$ *be a closed conical subset. Let* $g^\circ C$ *be as defined before Lemma 2.5. Then for a pair of morphisms* $h\colon W \to U$ *and* $f\colon W \to Y$ *of smooth schemes over* k*, the following conditions are equivalent:*

(1) $(g \circ h, f)$ *is* C*-acyclic.*
(2) (h,f) *is* $g^\circ C$*-acyclic.*

Proof. By Lemma 3.2.1, the conditions are equivalent to the conditions: (1) $(g \times 1_Y) \circ (h,f)$ is $C \times T^*Y$-transversal, and (2) (h,f) is $(g \times 1_Y)^\circ (C \times T^*Y)$-transversal, respectively. Hence they are equivalent by Lemma 2.5. □

Lemma 3.5. *Let* $g\colon X' \to X$ *be a morphism of smooth schemes over* k *and let* $C' \subset T^*X'$ *be a closed conical subset. Assume that* g *is proper on the base* $B' \subset X'$ *of* C' *and define* $g_\circ C'$ *as in Definition 2.6. Let* $f\colon X \to Y$ *be a morphism of smooth schemes over* k*. Then, the following conditions are equivalent:*

(1) f *is* $g_\circ C'$*-acyclic.*
(2) $f \circ g$ *is* C'*-acyclic.*

Proof. Since $g_\circ C' \subset T^*X$ is defined as the image of the inverse image of C' by $T^*X \times_X X' \to T^*X'$, either of the conditions means that the inverse image of C' by $T^*Y \times_Y X' \to T^*X'$ is a subset of the 0-section. □

Lemma 3.6. *Let*

$$\begin{array}{ccccc} X & \xleftarrow{h} & W & \xrightarrow{f} & Y \\ & & & \searrow^{f'} & \downarrow{\scriptstyle g} \\ & & & & Y' \end{array}$$

be a commutative diagram of smooth schemes of finite type over k*. Let* $C \subset T^*X$ *be a closed conical subset. Assume that* g *is smooth. If* (h,f) *is* C*-acyclic, then* (h,f') *is* C*-acyclic.*

Proof. Since the inverse image of $h^\circ C$ by $T^*Y \times_Y W \to T^*W$ is a subset of the 0-section and since $T^*Y' \times_{Y'} Y \to T^*Y$ is an injection, the inverse image of $h^\circ C$ by $T^*Y' \times_{Y'} W \to T^*W$ is a subset of the 0-section. □

Lemma 3.7. *Let* $g\colon X' \to X$ *be a morphism of smooth schemes over* k*. Let* $C' \subset T^*X'$ *be a closed conical subset and assume that* g *is proper on the base* B' *of* C'*. Let* $C = g_\circ C'$*.*

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Let

$$\begin{array}{ccccc} X' & \xleftarrow{h'} & W' & & \\ {\scriptstyle g}\downarrow & & {\scriptstyle g'}\downarrow & \searrow^{f'} & \\ X & \xleftarrow{h} & W & \xrightarrow{f} & Y \end{array}$$

be a commutative diagram of separated morphisms of schemes of finite type over k with cartesian square. Assume that W and Y are smooth over k and that (h,f) is C-acyclic.

Then, there exists an open neighbourhood $U' \subset W'$ of $h'^{-1}(B')$ smooth over k on which (h',f') is C'-acyclic.

Proof. By Lemma 2.7, there exists an open neighbourhood $U' \subset W'$ of $h'^{-1}(B')$ such that g and h are transversal on U' and $h^\circ g_\circ C' = g'|_{U'\circ} h'^\circ C'$. Since f is $h^\circ g_\circ C'$-acyclic, f' is $h'^\circ C'$-acyclic by Lemma 3.5. □

Lemma 3.8. *Let $h\colon W \to X$ and $f\colon W \to Y$ be morphisms of smooth schemes of finite type over k. Let $C \subset T^*X$ be a closed conical subset. If (h,f) is C-acyclic, then (h,f) is universally C-acyclic.*

Proof. Let $g\colon Y' \to Y$ be a morphism of smooth schemes over k and consider the commutative diagram (3.1). By Lemma 3.2.1, $f\colon W \to Y$ is smooth on a neighbourhood $U \subset W$ of $h^{-1}(B)$. Hence the morphism g is transversal to f on the inverse image $U' \subset W'$ of U. To show that (h',f') is C-acyclic on U', it suffices to show that

$$\begin{aligned} &\mathrm{Ker}((T^*X \times_X W) \times_W (T^*Y \times_Y W) \to T^*W) \times_W U' \\ &\to \mathrm{Ker}((T^*X \times_X U') \times_{U'} (T^*Y' \times_{Y'} U') \to T^*U') \end{aligned}$$

is an isomorphism further by Lemma 3.2.1. By the factorization $Y' \to Y \times Y' \to Y$, it suffices to prove the cases where $Y' \to Y$ is smooth and $Y' \to Y$ is an immersion respectively. Since the morphism $f\colon W \to Y$ is transversal to $g\colon Y' \to Y$ in the second case, the assertion follows. □

4. Micro support

Let $f\colon X \to S$ be a morphism of schemes. Let $x \to X$ and $t \to S$ be geometric points and let $S_{(s)}$ be the strict localization of S at the image $s = f(x)$ of x. A specialization $s \leftarrow t$ means a lifting of $t \to S$ to $t \to S_{(s)}$.

In this article, we assume that a prime number ℓ is invertible on the considered scheme X and let Λ denote a finite field of characteristic ℓ. By abuse of terminology, an object $\mathcal{F}$ of $D^+(X,\Lambda)$ will be called a sheaf and it is called constructible if the cohomology sheaf $\mathcal{H}^q\mathcal{F}$ is constructible for every $q \in \mathbf{Z}$ and is 0 except for finitely many q. We say that a sheaf $\mathcal{F}$ is locally constant if every cohomology sheaf $\mathcal{H}^q\mathcal{F}$ is locally constant.

Definition 4.1. Let $f\colon X \to S$ be a morphism of schemes and $\mathcal{F}$ be a sheaf on X. We say that f is locally acyclic relatively to $\mathcal{F}$ or $\mathcal{F}$-acyclic for short if for each geometric point $x \to X$ and $t \to S$ and each specialization $s = f(x) \leftarrow t$, the canonical morphism $\mathcal{F}_x \to R\Gamma(X_{(x)} \times_{S_{(s)}} t, \mathcal{F})$ is an isomorphism.

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We say that f is universally $\mathcal{F}$-acyclic, if for every morphism $S' \to S$, the base change of f is locally acyclic relatively to the pull-back of $\mathcal{F}$.

The local acyclicity is an étale local property on X by definition.

For geometric points s,t of S and a specialization $t \to S_{(s)}$, let $i\colon X_s \to X \times_S S_{(s)}$ and $j\colon X_t \to X \times_S S_{(s)}$ denote the canonical morphisms. Then, the local acyclicity is equivalent to the condition that the canonical morphism $i^*\mathcal{F}_{X\times_S S_{(s)}} \to i^*j_*j^*\mathcal{F}_{X\times_S S_{(s)}}$ is an isomorphism for each s,t and $s \leftarrow t$. In fact, the morphism $\mathcal{F}_x \to R\Gamma(X_{(x)} \times_{S_{(s)}} t, \mathcal{F})$ in Definition 4.1 is the stalk at x of $i^*\mathcal{F}_{X\times_S S_{(s)}} \to i^*j_*j^*\mathcal{F}_{X\times_S S_{(s)}}$.

Theorem 4.2. *Let X be a scheme and $\mathcal{F}$ be a sheaf on X.*

1. *We consider the following conditions:*

(1) *$\mathcal{F}$ is locally constant.*

(2) *Every smooth morphism $f\colon X \to Y$ is $\mathcal{F}$-acyclic.*

(3) *The identity $1_X\colon X \to X$ is $\mathcal{F}$-acyclic.*

We have (1)⇒(2)⇒(3). If X is a locally noetherian scheme and if, for every geometric point x of X, the stalk $\mathcal{F}_x$ is constructible, then the three conditions are equivalent.

2. [4, Corollaire 2.16] *If X is a scheme over k, then the morphism $X \to \operatorname{Spec} k$ is universally $\mathcal{F}$-acyclic.*

Proof. 1. The implication (1)⇒(2) is the local acyclicity of smooth morphisms [2, Théorème 2.1] and (2)⇒(3) is clear. The implication (3)⇒(1) is [1, Proposition 2.13 (i)]. □

Lemma 4.3. *Let*

$$\begin{array}{ccc} X' & & \\ {\scriptstyle g}\downarrow & \searrow^{f'} & \\ X & \xrightarrow{f} & Y \end{array}$$

be a commutative diagram of schemes of finite type over k. Let $\mathcal{F}'$ be a constructible sheaf on X' and $B' \subset X'$ be its support. Assume that g is proper on B' and let $\mathcal{F} = g_*\mathcal{F}'$. We consider the following conditions:

(1) *f is $\mathcal{F}$-acyclic over S.*

(2) *f' is $\mathcal{F}'$-acyclic over S.*

We have the implication (2)⇒(1). If g is finite on B, then these conditions are equivalent.

Proof. Let s,t be geometric points of Y and $t \to Y_{(s)}$ be a specialization and consider the cartesian diagram

$$\begin{array}{ccccc} X'_s & \xrightarrow{i'} & X' \times_Y Y_{(s)} & \xleftarrow{j'} & X'_t \\ {\scriptstyle g'_s}\downarrow & & \downarrow{\scriptstyle g'_{Y_{(s)}}} & & \downarrow{\scriptstyle g'_t} \\ X_s & \xrightarrow{i} & X \times_Y Y_{(s)} & \xleftarrow{j} & X_t \end{array}$$

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where the vertical arrows are induced by g'. Since g is assumed proper on the support B' of $\mathcal{F}'$, the vertical arrows in the commutative diagram

$$\begin{array}{ccc} g'_{s*}i'^{*}\mathcal{F}'_{X'\times_Y Y_{(s)}} & \longrightarrow & g'_{s*}i'^{*}j'_{*}j'^{*}\mathcal{F}'_{X'\times_Y Y_{(s)}} \\ \uparrow & & \uparrow \\ i^{*}\mathcal{F}_{X\times_Y Y_{(s)}} & \longrightarrow & i^{*}j_{*}j^{*}\mathcal{F}_{X\times_Y Y_{(s)}} \end{array}$$

are isomorphisms. If (2) is satisfied, then the upper horizontal arrow is an isomorphism. Hence the lower horizontal arrow is an isomorphism and (1) holds. Conversely, if g is finite on B and if the lower horizontal arrow is an isomorphism, the upper horizontal arrow is an isomorphism before taking g'_{s*} and (1) holds. □

Definition 4.4. Let X be a smooth scheme over a field k. Let $\mathcal{F}$ be a sheaf on X and let $C \subset T^*X$ be a closed conical subset. We say that $\mathcal{F}$ is micro supported on C, if for every C-acyclic pair (h, f) of morphisms $h\colon W \to X$ and $f\colon W \to Y$ of smooth schemes over k, the morphism f is $h^*\mathcal{F}$-acyclic.

If $\mathcal{F}$ is micro supported on $C \subset C'$, then $\mathcal{F}$ is micro supported on C' by Lemma 3.2.1 and Lemma 2.3.3 (cf. [3, 1.3]).

Lemma 4.5. *Let X be a smooth scheme over a field k and let $\mathcal{F}$ be a sheaf on X.*

1. (cf. [3, Lemma 2.1 (iii)]) *Assume that for every geometric point x of X, the stalk $\mathcal{F}_x$ is constructible. Then the following conditions are equivalent:*

(1) *$\mathcal{F}$ is locally constant.*

(2) *$\mathcal{F}$ is micro supported on the 0-section $T^*_X X$.*

2. (cf. [3, Lemma 1.3]) *Every sheaf $\mathcal{F}$ is micro supported on T^*X.*

3. *Let $C \subset T^*X$ be a closed conical subset on which $\mathcal{F}$ is micro supported and $U \subset X$ be an open subset. If C_U is empty, then the restriction $\mathcal{F}_U$ is 0.*

Conversely, $\mathcal{F} = 0$ is micro supported on $\varnothing$.

4. *Let $h\colon W \to X$ and $f\colon W \to Y$ be morphisms of smooth schemes over k. Let $C \subset T^*X$ be a closed conical subset on which $\mathcal{F}$ is micro supported. If (h, f) is C-acyclic, then f is universally $h^*\mathcal{F}$-acyclic.*

Note that in the convention of [3], a sheaf is assumed constructible.

Proof. 1. By Lemma 3.3.1, a pair (h, f) of morphisms $h\colon W \to X$ and $f\colon W \to Y$ of smooth schemes over k, is $T^*_X X$-acyclic if and only if f is smooth.

(1)⇒(2): If f is smooth and $\mathcal{F}$ is locally constant, then f is $h^*\mathcal{F}$-acyclic by Theorem 4.2.1 (1)⇒(2).

(2)⇒(1): Since the pair $(1_X, 1_X)$ is $T^*_X X$-acyclic, the condition (2) implies that the identity 1_X is $\mathcal{F}$-acyclic. By Theorem 4.2.1 (3)⇒(1), this implies that $\mathcal{F}$ is locally constant.

2. Suppose that a pair (h, f) of morphisms $h\colon W \to X$ and $f\colon W \to Y$ of smooth schemes over k, is T^*X-acyclic. Then $(h, f)\colon W \to X \times Y$ is smooth by Lemma 3.3.2. Since the question is étale local on W, we may assume that $W = \mathbf{A}^n_{X\times Y}$. Then, we have a cartesian

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diagram

$$\begin{array}{ccccc} X & \longleftarrow & X\times Y & \longleftarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} k & \longleftarrow & Y & \longleftarrow & \mathbf{A}^n_Y. \end{array}$$

By Theorem 4.2.2, the morphism $W \to \mathbf{A}^n_Y$ is $\mathcal{F}_W$-acyclic for the pull-back $\mathcal{F}_W$ of $\mathcal{F}$. Since $\mathbf{A}^n_Y \to Y$ is smooth, the composition $W \to Y$ is also $\mathcal{F}_W$-acyclic by [8, Corollaire 2.7].

3. Since the pair $(1,0)$ of the identity $1\colon U \to U$ and the 0-mapping $U \to \mathbf{A}^1$ is C-acyclic, the constant morphism $0\colon U \to \mathbf{A}^1$ is $\mathcal{F}$-acyclic. Hence we have $\mathcal{F} = 0$.

If $\mathcal{F} = 0$, every pair (h,f) of morphisms $h\colon W \to X$ and $f\colon W \to Y$ of smooth schemes over k is $\mathcal{F}$-acyclic.

4. Since (h,f) is universally C-acyclic by Lemma 3.8, the assertion follows. □

Lemma 4.6 (cf. [3, Lemma 2.1 (iv)]). *Let X be a smooth scheme over a field k and $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ be a distinguished triangle of sheaves on X. Suppose that $\mathcal{F}$ and $\mathcal{F}'$ are micro supported on C and on C' respectively. Then $\mathcal{F}''$ is micro supported on $C'' = C \cup C'$.*

Proof. Let $h\colon W \to X$ and $f\colon W \to Y$ be morphisms of smooth schemes over k such that the pair (h,f) is C''-acyclic. Then, since $C, C' \subset C''$, the pair (h,f) is C-acyclic and C'-acyclic. Hence f is $h^*\mathcal{F}$-acyclic and $h^*\mathcal{F}'$-acyclic. By the distinguished triangle $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$, f is $h^*\mathcal{F}''$-acyclic. □

Lemma 4.7. *Let X be a smooth scheme over a field k and let $\mathcal{F}$ be a sheaf on X.*

1. (cf. [3, Lemma 2.2 (i)]) *Assume that $\mathcal{F}$ is micro supported on C. If a morphism $g\colon V \to X$ of smooth schemes over k is C-transversal, then the pull-back $g^*\mathcal{F}$ is micro supported on $g^\circ C$.*

2. *Let $(j_i\colon U_i \to X)_{i\in I}$ be an étale covering and let $C_i \subset T^*U_i$ be a family of closed conical subsets such that the pull-backs $\mathcal{F}_{U_i}$ are micro supported on C_i. We identify T^*U_i with $T^*X \times_X U_i$ and consider the image of $C_i \subset T^*U_i = T^*X \times_X U_i$ as a subset $\operatorname{Im}(C_i) \subset T^*X$. Then, $\mathcal{F}$ is micro supported on the closure $C = \overline{\bigcup_{i\in I} \operatorname{Im}(C_i)}$ of the union of the images.*

3. *Let $U \subset X$ be an open subset and $Z = X - U$ be the complement. Assume that $\mathcal{F}$ is micro supported on $C \subset T^*X$ and that the restriction $\mathcal{F}_U$ is micro supported on $C' \subset T^*U$. Then, $\mathcal{F}$ is micro supported on the union $\overline{C'} \cup C|_Z \subset T^*X$.*

Proof. 1. Let $h\colon W \to V$ and $f\colon W \to Y$ be morphisms of smooth schemes over k such that the pair (h,f) is $g^\circ C$-acyclic. Then, (gh,f) is C-acyclic by Lemma 3.4 (2)⇒(1). Hence f is locally acyclic relatively to $(gh)^*\mathcal{F} = h^*(g^*\mathcal{F})$.

2. Let $h\colon W \to X$ and $f\colon W \to Y$ be morphisms of smooth schemes over k such that the pair (h,f) is C-acyclic. Then, for every $i \in I$, the pair of $h_i\colon W_i = W \times_X U_i \to U_i$ and $f_i\colon W_i \to Y$ is $j_i^\circ C$-acyclic by Lemma 3.4 (2)⇒(1). Further it is C_i-acyclic since $C_i \subset j_i^\circ C$ by Lemma 2.3.3. Hence f_i is locally acyclic relatively to $h_i^*(\mathcal{F}_{U_i}) = (h^*\mathcal{F})_{W_i}$ for every $i \in I$. Since $(W_i \to W)_{i\in I}$ is an étale covering, f is $h^*\mathcal{F}$-acyclic.

3. Let $h\colon W \to X$ and $f\colon W \to Y$ be morphisms of smooth schemes over k and assume that (h,f) is $\overline{C'} \cup C|_Z$-acyclic. Then, (h,f) is $\overline{C'}$-acyclic on U and is C-acyclic on a neighbourhood V of Z. Hence f is $h^*\mathcal{F}$-acyclic on $X = U \cup V$. □

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Lemma 4.8 (cf. [3, Lemma 2.5]). *Let X be a smooth scheme over a field k and let $\mathcal{F}$ be a sheaf on X. Let $i\colon Z \to X$ be a closed immersion. Assume that $\mathcal{F} = i_*\mathcal{F}_Z$ is supported on Z and is micro supported on a closed conical subset $C \subset T^*X$.*

1. (cf. [3, Lemma 2.3]) *The sheaf $\mathcal{F}$ is micro supported on $C|_Z$.*
2. *Assume that Z is smooth over k and that $C = C|_Z$ is a subset of $T^*X|_Z$. Let $s\colon T^*Z \to T^*X|_Z$ be a section of the surjection $T^*X|_Z \to T^*Z$. Then, $\mathcal{F}_Z$ is micro supported on $s^{-1}(C)$. Define $C_Z \subset T^*Z$ to be the closure of the image of C by the surjection $T^*X|_Z \to T^*Z$. Then, $\mathcal{F}_Z$ is micro supported on C_Z.*
3. (cf. [3, Lemma 2.2 (ii)]) *Assume that $\mathcal{F}_Z$ is micro supported on a closed conical subset $C_Z \subset T^*Z$. Then $\mathcal{F}$ is micro supported on $i_\circ C_Z$.*

Proof. 1. Let $h\colon W \to X$ and $f\colon W \to Y$ be morphisms of smooth schemes over k such that the pair (h,f) is $C|_Z$-acyclic. Then, by Lemma 2.3.4, there exists an open neighbourhood U of Z such that the pair of $h_U\colon W \times_X U \to X$ and $f_U\colon W \times_X U \to Y$ is C-acyclic. Hence f_U is $(h^*\mathcal{F})_{W\times_X U}$-acyclic. For the complement $V = X - Z$, the restriction $(h^*\mathcal{F})_{W\times_X V}$ is 0 since $\mathcal{F} = i_*\mathcal{F}_Z$. Hence $f_V\colon W \times_X V \to Y$ is $(h^*\mathcal{F})_{W\times_X V}$-acyclic. Since $U \cup V = X$, f is $\mathcal{F}$-acyclic.

2. First, we show that $\mathcal{F}_Z$ is micro supported on $s^{-1}(C)$. Let $h\colon W \to Z$ and $f\colon W \to Y$ be morphisms of smooth schemes over k such that the pair (h,f) is $s^{-1}(C)$-acyclic. We show that we may assume h is an immersion. We consider the commutative diagram

$$\begin{array}{ccccccc} X \times W & \longleftarrow & Z \times W & & & & \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & {\scriptstyle \mathrm{pr}_1}\downarrow & \searrow & & & \\ X & \longleftarrow & Z & \longleftarrow & W & \longrightarrow & Y. \end{array}$$

For the section $\widetilde{s} = s \times 1\colon T^*Z \times T^*W = T^*(Z \times W) \to T^*X|_Z \times T^*W = T^*(X \times W)|_{Z\times W}$ extending s, we have $\mathrm{pr}_1^\circ(s^{-1}(C)) = \widetilde{s}^{-1}(\mathrm{pr}_1^\circ C)$. Since the projection $Z \times W \to Z$ is smooth, we may replace $h\colon W \to Z$, $X \supset Z$ and C by $(h,1)\colon W \to Z \times W$, $X \times W \supset Z \times W$ and $\mathrm{pr}_1^\circ C$, by Lemma 4.7.1. Thus we may assume that h is an immersion.

Since the assertion is local on X, by Lemma 2.8.1, we may extend the immersion $(h,f)\colon W \to Z \times Y$ to an immersion $(h',f')\colon V \to X \times Y$ transversal to the immersion $Z \times Y \to X \times Y$ such that the restriction $T^*_V(X \times Y)|_W$ of the conormal bundle is the image of $T^*_W(Z \times Y)$ by the section $s|_W \times 1\colon T^*Z|_W \times_W (T^*Y \times_Y W) \to T^*X|_W \times_W (T^*Y \times_Y W)$.

We consider the cartesian diagram

$$\begin{array}{ccc} W & \longrightarrow & Z \times Y \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \times Y. \end{array}$$

Since $T^*_V(X \times Y)|_W$ is the image of $T^*_W(Z \times Y)$ by the section, the $s^{-1}(C)$-acyclicity of $(h,f)\colon W \to Z \times Y$ implies the C-acyclicity of $(h',f')\colon V \to X \times Y$ by Lemma 3.2.1 and Lemma 2.8.2, after shrinking V if necessary. Since $\mathcal{F}$ is micro supported on C, this implies that $f'\colon V \to Y$ is $h'^*\mathcal{F}$-acyclic. This means that $f\colon W \to Y$ is $h^*\mathcal{F}_Z$-acyclic since $h'^*\mathcal{F}$ is

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the direct image of $h^*\mathcal{F}_Z$ by the closed immersion $W \to V$. Hence $\mathcal{F}_Z$ is micro supported on $s^{-1}(C)$.

We show that $\mathcal{F}_Z$ is micro supported on C_Z. Since the assertion is local on Z, we may assume that there exists a section $s\colon T^*Z \to T^*X|_Z$. Then, since $s^{-1}(C) \subset C_Z$, the assertion follows.

3. Let $h\colon W \to X$ and $f\colon W \to Y$ be morphisms of smooth schemes over k such that the pair (h,f) is $i_\circ C$-acyclic. By Lemma 4.5.3, the support of $\mathcal{F}$ is a subset of the base B of C. By Lemma 2.7 and Lemma 3.5, there exists an open neighbourhood $V' \subset V = Z \times_X W$ of the inverse image of B such that V' is smooth over k, that the pair of $h'\colon V' \to Z$ and the composition $f'\colon V' \to W \to Y$ is C-acyclic. Hence f' is $h'^*\mathcal{F}$-acyclic. Since V' contains the inverse image of the support of $\mathcal{F}$, the morphism f is $h^*i_*\mathcal{F}$-acyclic. □

Lemma 4.9. *Let $g\colon X' \to X$ be a morphism of smooth schemes over k. Assume that g is proper on the base of a closed conical subset $C' \subset T^*X'$. Let $\mathcal{F}'$ be a constructible sheaf on X'. If $\mathcal{F}'$ is micro supported on C', then $Rg_*\mathcal{F}'$ is micro supported on $C = g_\circ C'$.*

Proof. Let $h\colon W \to X$ and $f\colon W \to Y$ be morphisms of smooth schemes over k such that (h,f) is C-acyclic. Let

$$\begin{array}{ccc} X' & \xleftarrow{h'} & W' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ X & \xleftarrow{h} & W \end{array}$$

be the cartesian diagram. Then, by Lemma 3.7, there exists an open regular neighbourhood $U' \subset W' = W \times_X X'$ of the inverse image of B' where h is transversal to g and (h', fg') is C'-acyclic. Hence the assertion follows from Lemma 4.3. □

5. Singular support and Radon transform

Definition 5.1 [3, 1.3]. Let X be a smooth scheme over a field k and $\mathcal{F}$ be a sheaf on X. We say that a closed conical subset $C \subset T^*X$ is the singular support of $\mathcal{F}$ if for every closed conical subset $C' \subset T^*X$, the inclusion $C \subset C'$ is equivalent to the condition that $\mathcal{F}$ is micro supported on C'. If the singular support of $\mathcal{F}$ exists, it is denoted by $SS\mathcal{F}$.

Following Beilinson's argument in [3], we will prove the existence of singular support for any sheaf without assuming constructibility by reducing to the case where X is the projective space $\mathbf{P}^n$ and using the Radon transform in the case $X = \mathbf{P}^n$. For the sake of convenience and completeness, we give the detail.

Lemma 5.2. *Let X be a smooth scheme over k and let $\mathcal{F}$ be a sheaf on X.*

1. *Let $U \subset X$ be an open subscheme. Assume that $C \subset T^*X$ is the singular support of $\mathcal{F}$. Then, $C|_U$ is the singular support of $\mathcal{F}|_U$.*
2. *Let (U_i) be an open covering of X and $C_i \subset T^*U_i$ be the singular support of $\mathcal{F}|_{U_i}$. Then, $C = \bigcup_i C_i \subset T^*X$ is the singular support of $\mathcal{F}$.*

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Proof. 1. The restriction $\mathcal{F}|_U$ is micro supported on $C|_U$ by Lemma 4.7.1. We show that $C|_U$ is the smallest. Let $C' \subset T^*U$ be a closed conical subset on which $\mathcal{F}|_U$ is micro supported and $Z = X - U$ be the complement. Then $\mathcal{F}$ is micro supported on $\overline{C'} \cup C|_Z$ by Lemma 4.7.3. Since C is the smallest, we have $C \subset \overline{C'} \cup C|_Z$ and $C|_U \subset (\overline{C'} \cup C|_Z)|_U = C'$.

2. For every i,j, the restrictions $C_i|_{U_i \cap U_j}$ and $C_j|_{U_i \cap U_j}$ are the singular support of $\mathcal{F}|_{U_i \cap U_j}$ and are the same by 1. Hence the union $C = \bigcup_i C_i$ is a closed conical subset of T^*X. Since $C_i = C|_{U_i}$, $\mathcal{F}$ is micro supported on C by Lemma 4.7.1.

We show that C is the smallest. Let $C' \subset T^*X$ be a closed conical subset on which $\mathcal{F}$ is micro supported. Then, for each i, we have $C_i \subset C'|_{U_i}$. Hence we have $C \subset C'$. □

Proposition 5.3. *Let $i\colon X \to P$ be a closed immersion of smooth schemes over k and let $\mathcal{F}$ be a constructible sheaf on X. Let $C_P \subset T^*P$ be a closed conical subset and assume that C_P is the singular support of $i_*\mathcal{F}$. Then the following holds.*

1. *C_P is a subset of $T^*P|_X$.*
2. *Define $C \subset T^*X$ to be the closure of the image of C_P by the surjection $T^*P|_X \to T^*X$. Then C is the singular support $SS\mathcal{F}$ and we have $C_P = i_\circ C$.*

Proof. 1. Let $U = P - X$ be the complement. Since $\mathcal{F}|_U = 0$ is micro supported on $\varnothing$ by Lemma 4.5.3, $\mathcal{F}$ is micro supported on $C_P|_X \subset T^*P$ by Lemma 4.7.3. Since C_P is the smallest, we have $C_P = C_P|_X \subset T^*P|_X$.

2. By Lemma 4.8.2, $\mathcal{F}$ is micro supported on C. We show that C is the smallest. Assume that $\mathcal{F}$ is micro supported on a closed conical subset $C' \subset T^*X$. Then, since $i_*\mathcal{F}$ is micro supported on $i_\circ C'$ by Lemma 4.8.3, we have $C_P \subset i_\circ C'$. By taking the closure of the image by the surjection $T^*P|_X \to T^*X$, we obtain $C \subset C'$.

Since $i_*\mathcal{F}$ is micro supported on $i_\circ C$, we have $C_P \subset i_\circ C$. We show the other inclusion $i_\circ C \subset C_P$. Since the assertion is local, we may assume that there exists a section $s\colon T^*X \to T^*P|_X$ of the surjection $T^*P|_X \to T^*X$. Since $\mathcal{F}$ is micro supported on $s^{-1}(C_P)$ by Lemma 4.8.2, we have $C \subset s^{-1}(C_P)$. Hence we have $i_\circ C \subset C_P$. □

We recall the Radon transform and the Legendre transform. Let $V = \mathbf{A}^{n+1}$ and let $\mathbf{P} = \mathbf{P}(V)$ be the projective space of dimension n parametrizing lines in V. The dual projective space $\mathbf{P}^\vee = \mathbf{P}(V^\vee)$ is the moduli space of hyperplanes in $\mathbf{P}$.

By the exact sequence $0 \to \Omega^1_{\mathbf{P}/k}(1) \to \mathcal{O}_{\mathbf{P}} \otimes V^\vee \to \mathcal{O}_{\mathbf{P}}(1) \to 0$ of locally free $\mathcal{O}_{\mathbf{P}}$-modules, we define a closed subscheme $Q = \mathbf{P}(T^*\mathbf{P}) \subset \mathbf{P} \times \mathbf{P}^\vee$ of codimension 1. This equals the universal family of hyperplanes since it is defined by the tautological section $\Gamma(\mathbf{P} \times \mathbf{P}^\vee, \mathcal{O}(1) \boxtimes \mathcal{O}(1)) = V^\vee \otimes V$ corresponding to the identity $1 \in \mathrm{End}(V)$. Let $q\colon Q \to \mathbf{P}$ and $q^\vee\colon Q \to \mathbf{P}^\vee$ be the restrictions of the projections $\mathbf{P} \times \mathbf{P}^\vee \to \mathbf{P}$ and $\mathbf{P} \times \mathbf{P}^\vee \to \mathbf{P}^\vee$. By symmetry, $Q \subset \mathbf{P} \times \mathbf{P}^\vee$ is identified with $\mathbf{P}(T^*\mathbf{P}^\vee)$.

The conormal bundle $L_Q = T^*_Q(\mathbf{P} \times \mathbf{P}^\vee) \subset (T^*\mathbf{P} \times T^*\mathbf{P}^\vee)|_Q$ is a line bundle. Since $1 \in \mathrm{End}(V) = V^\vee \otimes V$ regarded as a global section of $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$ is the bilinear form defining $Q \subset \mathbf{P} \times \mathbf{P}^\vee$, the morphism $N_{Q/(\mathbf{P}\times\mathbf{P}^\vee)} \to \Omega^1_{(\mathbf{P}\times\mathbf{P}^\vee)/\mathbf{P}^\vee} \otimes_{\mathcal{O}_{\mathbf{P}\times\mathbf{P}^\vee}} \mathcal{O}_Q = \Omega^1_{\mathbf{P}/k} \otimes_{\mathcal{O}_{\mathbf{P}}} \mathcal{O}_Q$ defines a tautological sub invertible sheaf on $Q = \mathbf{P}(T^*\mathbf{P})$. In other words, the tautological sub line bundle $L \subset T^*\mathbf{P} \times_{\mathbf{P}} Q$ is the image of L_Q by the first projection $\mathrm{pr}_1\colon (T^*\mathbf{P} \times T^*\mathbf{P}^\vee)|_Q \to T^*\mathbf{P} \times_{\mathbf{P}} Q$. By symmetry, the image by the second projection equals the tautological sub line bundle $L^\vee$ on $Q = \mathbf{P}(T^*\mathbf{P}^\vee)$.

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Since the conormal bundle L_Q is the kernel of the surjection $(T^*\mathbf{P}\times T^*\mathbf{P}^\vee)|_Q \to T^*Q$, the intersection $q^\circ T^*\mathbf{P}\cap q^{\vee\circ}T^*\mathbf{P}^\vee \subset T^*Q$ equals the image of the tautological bundle $L\subset T^*\mathbf{P}\times_{\mathbf{P}} Q$. By symmetry, the intersection also equals the image of the tautological bundle $L^\vee \subset T^*\mathbf{P}^\vee \times_{\mathbf{P}^\vee} Q$.

Let $C\subset T^*\mathbf{P}$ denote a closed conical subset. We define the Legendre transform $C^\vee \subset T^*\mathbf{P}^\vee$ to be $q^\vee_\circ q^\circ C$. We consider projectivizations $\mathbf{P}(C)\subset \mathbf{P}(T^*\mathbf{P})$ and $\mathbf{P}(C^\vee)\subset \mathbf{P}(T^*\mathbf{P}^\vee)$ as closed subsets of Q. Let $C^+\subset T^*\mathbf{P}$ denote the union of C and the 0-section.

Proposition 5.4. *Let $C\subset T^*\mathbf{P}$ be a closed conical subset. Let $E=\mathbf{P}(C)\subset Q=\mathbf{P}(T^*\mathbf{P})$ be the projectivization. Let $L_Q=T^*_Q(\mathbf{P}\times\mathbf{P}^\vee)\subset (T^*\mathbf{P}\times T^*\mathbf{P}^\vee)|_Q$ be the conormal line bundle.*

1. *The projectivization $E=\mathbf{P}(C)\subset Q$ is the complement of the largest open subset where $(q,q^\vee)$ is C-acyclic.*
2. *The Legendre transform $C^\vee$ equals the union of the image of $L|_E\subset q^\circ T^*\mathbf{P}\cap q^{\vee\circ}T^*\mathbf{P}^\vee\subset T^*Q$ and its base. We have $\mathbf{P}(C)=\mathbf{P}(C^\vee)$.*
3. *We have $C^{\vee\vee}\subset C^+$.*

Proof. 1. The kernel $\mathrm{Ker}((T^*\mathbf{P}\times T^*\mathbf{P}^\vee)|_Q\to T^*Q)$ equals the conormal bundle L_Q and the first projection induces an isomorphism $L_Q\to L\subset T^*\mathbf{P}\times_{\mathbf{P}}Q$ to the tautological bundle. By this isomorphism, the intersection $(C\times T^*\mathbf{P}^\vee)|_Q\cap L_Q$ is identified with $q^*C\cap L$. Hence $(q,q^\vee)$ is C-acyclic on $U\subset Q$ if and only if the restriction $(q^*C\cap L)|_U$ is a subset of the 0-section. Since the projectivization $E=\mathbf{P}(C)\subset\mathbf{P}(T^*\mathbf{P})$ equals $\mathbf{P}(q^*C\cap L)\subset\mathbf{P}(L)=Q$, the assertion follows.

2. The Legendre transform $C^\vee$ is the image of the intersection $q^\circ C\cap q^{\vee\circ}T^*\mathbf{P}^\vee\subset T^*Q$ by $q^{\vee\circ}T^*\mathbf{P}^\vee\to T^*\mathbf{P}^\vee$. We identify the intersection $q^\circ T^*\mathbf{P}\cap q^{\vee\circ}T^*\mathbf{P}^\vee\subset T^*Q$ with the tautological line bundle $L\subset T^*\mathbf{P}\times_{\mathbf{P}}Q$. Then, the intersection $q^\circ C\cap q^{\vee\circ}T^*\mathbf{P}^\vee\subset T^*Q$ is identified with $q^*C\cap L$. Since the projectivization $E=\mathbf{P}(C)\subset\mathbf{P}(T^*\mathbf{P})$ equals $\mathbf{P}(q^*C\cap L)\subset\mathbf{P}(L)=Q$, the closed conical subset $q^*C\cap L$ equals $L|_E$ outside the 0-section.

Since $L|_E$ is identified with $L^\vee|_E$ inside T^*Q, we have $\mathbf{P}(C^\vee)=\mathbf{P}(L^\vee|_E)=E\subset\mathbf{P}(T^*\mathbf{P}^\vee)=Q$.

3. By 2 and symmetry, we have $\mathbf{P}(C)=\mathbf{P}(C^\vee)=\mathbf{P}(C^{\vee\vee})$. Hence we have $C^{\vee\vee}\subset C^+$. □

Lemma 5.5. *Let $C\subset T^*\mathbf{P}$ be a closed conical subset and $E=\mathbf{P}(C)\subset Q=\mathbf{P}(T^*\mathbf{P})$ be its projectivization. Let*

$$\begin{array}{ccccc}
\mathbf{P} & \xleftarrow{q} & Q & \xleftarrow{h'} & Q_W \\
 & & {\scriptstyle q^\vee}\downarrow & & \downarrow{\scriptstyle q^\vee_W} \quad \searrow^{f'} \\
 & & \mathbf{P}^\vee & \xleftarrow{h} & W \xrightarrow{f} Y
\end{array}$$

be a commutative diagram of smooth schemes over k with cartesian square. Let $C^{\vee+}=C^\vee\cup T^*_{\mathbf{P}^\vee}\mathbf{P}^\vee\subset T^*\mathbf{P}^\vee$ be the union of the Legendre transform with the 0-section and suppose that (h,f) is $C^{\vee+}$-acyclic.

1. *The morphism f is smooth on W and the pair (qh',f') is C-acyclic on the complement Q_W-E_W.*
2. *On a neighbourhood of E_W, the pair (qh',f') is $T^*\mathbf{P}$-acyclic.*

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Proof. 1. Since (h,f) is $C^{\vee+}$-acyclic and $C^{\vee+}$ contains the 0-section, the morphism $f\colon W\to Y$ is smooth. By Proposition 5.4.1, $(q,q^\vee)$ is C-acyclic outside E. Hence $(qh',q_W^\vee)$ is C-acyclic outside E_W by Lemma 3.8 and (qh',f') is C-acyclic outside E_W by Lemma 3.6.

2. By the description of $C^\vee$ in Proposition 5.4.2 and by the open condition Lemma 2.3.4 and Lemma 3.2.1, the $C^\vee$-acyclicity of (h,f) implies the $T^*\mathbf{P}$-acyclicity of (qh',f') on a neighbourhood of $E_W\subset Q_W$. □

We define the naive Radon transform $R\mathcal{F}$ to be $q_*^\vee q^*\mathcal{F}$ and the naive inverse Radon transform $R^\vee\mathcal{G}$ to be $q_*q^{\vee *}\mathcal{G}$. Since we use only the naive Radon transform, we drop the adjective 'naive' in the sequel. We will refine in Proposition 5.9, after studying the difference between $R^\vee R\mathcal{F}$ and $\mathcal{F}$, the following elementary property.

Lemma 5.6. *Assume that $\mathcal{F}$ is micro supported on C.*

1. *The Radon transform $R\mathcal{F}$ is micro supported on $C^\vee$.*
2. *$R^\vee R\mathcal{F}$ is micro supported on C^+.*

Proof. 1. By Lemma 4.7.1 and Lemma 4.9, the Radon transform $R\mathcal{F}=q_*^\vee q^*\mathcal{F}$ is micro supported on the Legendre transform $C^\vee=q_\circ^\vee q^\circ C$.

2. By 1, $R^\vee R\mathcal{F}$ is micro supported on $C^{\vee\vee}\subset C^+$. □

Lemma 5.7. *We consider the commutative diagram*

$$\begin{array}{ccccc} Q\times_{\mathbf{P}^\vee}Q & \xrightarrow{i} & \mathbf{P}\times\mathbf{P}^\vee\times\mathbf{P} & & \\ & \searrow^{q\times q} & \downarrow{\scriptstyle \mathrm{pr}_{13}} & & \\ & & \mathbf{P}\times\mathbf{P} & \xleftarrow{\delta_{\mathbf{P}}} & \mathbf{P} \end{array}$$

where $\delta_{\mathbf{P}}\colon\mathbf{P}\to\mathbf{P}\times\mathbf{P}$ is the diagonal immersion.

1. *The closed immersion $i=((q,q^\vee),(q^\vee,q))$ induces isomorphisms*

$$R^s(q\times q)_*\Lambda_{Q\times_{\mathbf{P}^\vee}Q}\to\begin{cases}\Lambda(-t)[-2t] & \text{if } s=2t \text{ and } 0\leqslant t\leqslant n-2,\\ \delta_{\mathbf{P}*}\Lambda(-(n-1))[-2(n-1)] & \text{if } s=2(n-1),\\ 0 & \text{otherwise.}\end{cases}\tag{5.1}$$

2. *Let $p\colon\mathbf{P}\to\operatorname{Spec}k$ and $p^\vee\colon\mathbf{P}^\vee\to\operatorname{Spec}k$ denote the projections. Then, we have a distinguished triangle*

$$\to\tau_{\leqq 2(n-2)}(p\times p)^*p_*^\vee\Lambda\to(q\times q)_*\Lambda_{Q\times_{\mathbf{P}^\vee}Q}\to\delta_{\mathbf{P}*}\Lambda_{\mathbf{P}}(n-1)[2(n-1)]\to.\tag{5.2}$$

Proof. 1. The immersion i induces a morphism

$$(p\times p)^*p_*^\vee\Lambda=\mathrm{pr}_{13*}\Lambda\to(q\times q)_*\Lambda_{Q\times_{\mathbf{P}^\vee}Q}\tag{5.3}$$

and we have isomorphisms $R^sp_*^\vee\Lambda\to\Lambda(-t)[-2t]$ for $s=2t,0\leqslant t\leqslant n$ and $R^sp_*^\vee\Lambda=0$ otherwise. The restriction of the closed immersion $i\colon Q\times_{\mathbf{P}^\vee}Q\to\mathbf{P}\times\mathbf{P}^\vee\times\mathbf{P}$ on the diagonal $\mathbf{P}\subset\mathbf{P}\times\mathbf{P}$ is the sub $\mathbf{P}^{n-1}$-bundle $Q\subset\mathbf{P}\times\mathbf{P}^\vee$. On the complement $\mathbf{P}\times\mathbf{P}-\mathbf{P}$, $Q\times_{\mathbf{P}^\vee}Q$ is a sub $\mathbf{P}^{n-2}$-bundle. Hence (5.3) induces an isomorphism of cohomology

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sheaves except for degree $s = 2(n-1)$ and induces an isomorphism $\delta_{\mathbf{P}*}p^*R^{2(n-1)}p^\vee_*\Lambda \to R^{2(n-1)}(q\times q)_*\Lambda_{Q\times_{\mathbf{P}^\vee}Q}$.

2. This follows from the isomorphisms (5.1). □

Next, we consider the diagram

$$\begin{array}{ccccc} \mathbf{P} & \xleftarrow{\mathrm{pr}_1} & \mathbf{P}\times\mathbf{P} & \xleftarrow{q\times q} & Q\times_{\mathbf{P}^\vee}Q \\ & & {\scriptstyle \mathrm{pr}_2}\downarrow & & \\ & & \mathbf{P}. & & \end{array} \tag{5.4}$$

Proposition 5.8. 1. *We have a canonical isomorphism*

$$R^\vee R\mathcal{F} \to R\mathrm{pr}_{2*}\big(\mathrm{pr}_1^*\mathcal{F}\otimes R(q\times q)_*\Lambda_{Q\times_{\mathbf{P}^\vee}Q}\big). \tag{5.5}$$

2. *The isomorphism (5.5) induces a distinguished triangle*

$$\to \bigoplus_{q=0}^{n-2} p^*p_*\mathcal{F}(-q)[-2q] \to R^\vee R\mathcal{F} \to \mathcal{F}(-(n-1))[-2(n-1)] \to .$$

Proof. 1. By the cartesian diagram

$$\begin{array}{ccccc} \mathbf{P} & \xleftarrow{q} & Q & \xleftarrow{\mathrm{pr}_1} & Q\times_{\mathbf{P}^\vee}Q \\ & & {\scriptstyle q^\vee}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ & & \mathbf{P}^\vee & \xleftarrow{q^\vee} & Q \\ & & & & \downarrow{\scriptstyle q} \\ & & & & \mathbf{P}, \end{array}$$

we have $R^\vee R\mathcal{F} = Rq_*q^{\vee *}Rq^\vee_*q^*\mathcal{F}$. By the proper base change theorem, we have a canonical isomorphism $Rq_*q^{\vee *}Rq^\vee_*q^*\mathcal{F} \to R(q\circ \mathrm{pr}_2)_*(q\circ\mathrm{pr}_1)^*\mathcal{F}$. In the notation of (5.4), the latter is identified with $R(\mathrm{pr}_2\circ(q\times q))_*(\mathrm{pr}_1\circ(q\times q))^*\mathcal{F}$. This is identified with $R\mathrm{pr}_{2*}(\mathrm{pr}_{1*}\mathcal{F}\otimes R(q\times q)_*\Lambda_{Q\times_{\mathbf{P}^\vee}Q})$ by the projection formula.

2. This follows from the isomorphism (5.5) and the distinguished triangle (5.2). □

Proposition 5.9. *For a sheaf $\mathcal{F}$ on $\mathbf{P}$ and a closed conical subset $C\subset T^*\mathbf{P}$, we have implications (1)⇒(2)⇒(3)⇒(4).*

(1) *$\mathcal{F}$ is micro supported on C.*
(2) *$q^\vee$ is universally $q^*\mathcal{F}$-acyclic outside $E = \mathbf{P}(C)$.*
(3) *The Radon transform $R\mathcal{F}$ is micro supported on $C^{\vee +}$.*
(4) *$\mathcal{F}$ is micro supported on C^+.*

Proof. (1)⇒(2): The pair $(q,q^\vee)$ of $q\colon Q\to\mathbf{P}$ and $q^\vee\colon Q\to\mathbf{P}^\vee$ is C-acyclic outside $E=\mathbf{P}(C)$ by Proposition 5.4.1. Hence (1) implies that $q^\vee$ is universally $q^*\mathcal{F}$-acyclic on the complement $Q - E$ by Lemma 4.5.4.

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(2)⇒(3): Assume that a pair of morphisms $h\colon W \to \mathbf{P}^\vee, f\colon W \to Y$ is $C^{\vee +}$-acyclic and show that f is $h^* R\mathcal{F}$-acyclic. We consider the commutative diagram

$$\begin{array}{ccccc} \mathbf{P} & \xleftarrow{q} & Q & \xleftarrow{h'} & Q_W \\ & & {\scriptstyle q^\vee}\downarrow & & \downarrow{\scriptstyle q_W^\vee} \quad \searrow^{f'} \\ & & \mathbf{P}^\vee & \xleftarrow{h} & W \xrightarrow{f} Y \end{array}$$

with cartesian square as in Lemma 5.5. Since $q^\vee$ is proper, it suffices to show that f' is $h'^* q^* \mathcal{F}$-acyclic by Lemma 4.3.

By (2), $q_W^\vee$ is $(qh')^*\mathcal{F}$-acyclic on the complement $Q_W - E_W$. By Lemma 5.5.1, the morphism $f\colon W \to Y$ is smooth on W. Hence f' is $(qh')^*\mathcal{F}$-acyclic on $Q_W - E_W$ by [8, Corollaire 2.7].

By Lemma 5.5.2, the pair (qh', f') is $T^*\mathbf{P}$-acyclic on a neighbourhood of $E_W \subset Q_W$. Since $\mathcal{F}$ is micro supported on $T^*\mathbf{P}$ by Lemma 4.5.2, f' is $(qh')^*\mathcal{F}$-acyclic on a neighbourhood of E_W. Thus f' is $(qh')^*\mathcal{F}$-acyclic as required.

(3)⇒(4) By (3) and (1)⇒(3), $R^\vee R\mathcal{F}$ is micro supported on $(C^{\vee +})^{\vee +} = C^+$. Since $p^* p_* \mathcal{F}$ is micro supported on the 0-section $T^*_{\mathbf{P}}\mathbf{P}$, by the distinguished triangle in Proposition 5.8.2, $\mathcal{F}$ is also micro supported on C^+. □

Corollary 5.10. *Let $\mathcal{F}$ be a sheaf on $\mathbf{P}$. Let $E \subset Q = \mathbf{P}(T^*\mathbf{P})$ be the complement of the largest open subset on which $q^\vee$ is universally $q^*\mathcal{F}$-acyclic. Then the closed conical subset $C \subset T^*\mathbf{P}$ corresponding to the base $B = \operatorname{supp}\mathcal{F}$ and the projectivization $E \subset \mathbf{P}(T^*\mathbf{P})$ is the singular support $SS\mathcal{F}$ of $\mathcal{F}$.*

Proof. By Proposition 5.9 (2)⇒(4), $\mathcal{F}$ is micro supported on C^+. Hence $\mathcal{F}$ is micro supported on $C = C^+|_B$ by Lemma 4.8.1.

Assume that $\mathcal{F}$ is micro supported on $C' \subset T^*\mathbf{P}$. Then, by Proposition 5.9 (1)⇒(2), we have $\mathbf{P}(C') \supset E = \mathbf{P}(C)$ since E is the smallest. Since the base of C' contains $B = \operatorname{supp}\mathcal{F}$ as a subset, we have $C' \supset C$. □

Theorem 5.11. *Let X be a smooth scheme over a field k and $\mathcal{F}$ be a sheaf. Then the singular support $SS\mathcal{F}$ of $\mathcal{F}$ exists.*

Proof. By Lemma 5.2, the assertion is local on X. Hence, we may assume that X is affine and is a closed subscheme of $\mathbf{A}^n$. By Proposition 5.3, we may assume that $X = \mathbf{A}^n$. Further by Lemma 5.2, we may assume that $X = \mathbf{P}^n$. This case is proved in Corollary 5.10. □

Corollary 5.12. *Let X be a smooth scheme over a field k and $\mathcal{F}$ be a sheaf. Let k' be an extension of k and $\mathcal{F}'$ be the pull-back of $\mathcal{F}$ on the base change $X' = X_{k'}$. Then, we have $SS\mathcal{F}' \subset (SS\mathcal{F})_{k'}$ and if k' is an algebraic extension of k, we have $SS\mathcal{F}' = (SS\mathcal{F})_{k'}$.*

Proof. We may assume that $X = \mathbf{P}$ is a projective space and it suffices to show the inequality $E' \subset E_{k'}$ or the equality $E' = E_{k'}$ of the projectivizations $E = \mathbf{P}(SS\mathcal{F})$ and $E' = \mathbf{P}(SS\mathcal{F}')$. Since the complement $Q - E$ is the largest open subset on which

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$p^\vee\colon Q \to \mathbf{P}^\vee$ is universally $p^*\mathcal{F}$-acyclic and similarly for $Q_{k'} - E'$, we have $(Q - E)_{k'} \subset Q_{k'} - E'$. Further if k' is an algebraic extension, we have $(Q - E)_{k'} = Q_{k'} - E'$. □

Theorem 5.13 [3, 1.3]. *Let X be a smooth scheme over a field k. If $\mathcal{F}$ is a constructible sheaf, then every irreducible component of $SS\mathcal{F}$ has the same dimension as X.*

6. Holonomic sheaves are constructible

Definition 6.1. Let X be a smooth scheme over a field k and let $\mathcal{F}$ be a sheaf on X. We say that $\mathcal{F}$ is *holonomic* if the following conditions (1) and (2) are satisfied:

(1) There exists a closed conical subset $C \subset T^*X$ such that $\dim C \leqq \dim X$ and $\mathcal{F}$ is micro supported on C.

(2) For every geometric point x, the stalk $\mathcal{F}_x$ is constructible.

Since the singular support $SS\mathcal{F}$ exists in general, condition (1) is equivalent to $\dim SS\mathcal{F} \leqq \dim X$. To emphasize that we also impose condition (2), we say that $\mathcal{F}$ is *quasi-holonomic* if only the first condition (1) is satisfied, although we will not use this notion.

Lemma 6.2. *Let X be a smooth scheme over a field k and let $\mathcal{F}$ be a sheaf on X.*

1. *Assume that $\mathcal{F}$ is holonomic. Then, there exists a dense open scheme $U \subset X$ such that $\mathcal{F}_U$ is locally constant.*

2. *Let $(U_i \to X)_{i\in I}$ be a finite family of étale morphisms and suppose that it is an étale covering of an open neighbourhood of the support of $\mathcal{F}$. Then the following conditions are equivalent:*

(1) *$\mathcal{F}$ is holonomic.*

(2) *$\mathcal{F}_{U_i}$ is holonomic for every $i \in I$.*

3. *Let $i\colon X \to P$ be a closed immersion of smooth schemes over k. Then, the following conditions are equivalent:*

(1) *$\mathcal{F}$ is holonomic.*

(2) *$i_*\mathcal{F}$ is holonomic.*

4. *Let $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ be a distinguished triangle of sheaves on X. If $\mathcal{F}$ and $\mathcal{F}'$ are holonomic, then $\mathcal{F}''$ is holonomic.*

Proof. 1. There exists a dense open scheme $U \subset X$ such that $\mathcal{F}_U$ is micro supported on the 0-section by Lemma 2.1 and an easy case of Lemma 4.7.1. By Lemma 4.5.1 (1)⇒(2), $\mathcal{F}_U$ is locally constant.

2. By Lemma 4.8.1, we may assume that $(U_i \to X)_{i\in I}$ is an étale covering.

(1)⇒(2): If $\mathcal{F}$ is micro supported on $C \subset T^*X$ and if $\dim C \leqq \dim X$, then $\mathcal{F}_{U_i}$ is micro supported on $C_{U_i} \subset T^*U_i$ by Lemma 4.7.1 and $\dim C_{U_i} \leqq \dim U_i$ for every $i \in I$.

(2)⇒(1): Suppose $\mathcal{F}_{U_i}$ is micro supported on $C_i \subset T^*U_i$ and $\dim C_i \leqq \dim U_i$ for every $i \in I$. Then, $\mathcal{F}$ is micro supported on $C = \overline{\bigcup_{i\in I} \mathrm{Im}(C_i)} \subset T^*X$ by Lemma 4.7.2. Since I is finite, we have $C = \bigcup_{i\in I} \overline{\mathrm{Im}(C_i)}$ and $\dim C = \max_{i\in I} \dim C_i \leqq \dim X$.

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3. (1)⇒(2): If $\mathcal{F}$ is micro supported on $C \subset T^*X$ and if $\dim C \leqq \dim X$, then $i_*\mathcal{F}$ is micro supported on $i_\circ C \subset T^*P$ by Lemma 4.8.3 and $\dim i_\circ C \leqq \dim P$.

(2)⇒(1): Suppose that the singular support $C_P = SSi_*\mathcal{F}$ satisfies $\dim C_P \leqq \dim P$. By Lemma 5.3.2, the singular support $C = SS\mathcal{F}$ exists and $C_P = i_\circ C$. Hence, we have $\dim C = \dim C_P - \operatorname{codim}_P X \leqq \dim X$.

4. Suppose that $\mathcal{F}$ and $\mathcal{F}'$ are micro supported on C and C' respectively and that $\dim C, \dim C' \leqq \dim X$. By Lemma 4.6, $\mathcal{F}''$ is micro supported on $C'' = C \cup C'$ and we have $\dim C'' \leqq \dim X$. □

Theorem 5.13 states that every constructible sheaf is holonomic as the title of the article [3] says. We have a partial converse.

Theorem 6.3. *Let X be a smooth scheme of finite type over a field k and $\mathcal{F}$ be a sheaf on X. If $\mathcal{F}$ is holonomic, then $\mathcal{F}$ is constructible.*

Without condition (2) in Definition 6.1, we have an obvious counterexample where $X = \operatorname{Spec} k$. Condition (2) is not enough for the converse since $\mathcal{F} = \bigoplus_{x:\text{closed points of } X} \Lambda_x$ satisfies (2) but its micro support is the whole T^*X.

Proof. We prove the theorem first assuming that k is perfect by induction on $\dim X$. If $X = \varnothing$, the assertion is clear. By Lemma 6.2.1, the largest open subset $U \subset X$ on which $\mathcal{F}$ is constructible is nonempty. To prove the equality $X = U$ by contradiction, we assume that the complement $Z = X - U$ is not empty. Since k is assumed perfect, after shrinking X if necessary, we may assume that Z is smooth over k.

Let $j\colon U \to X$ be the open immersion, $i\colon Z \to X$ be the closed immersion and consider the distinguished triangle $j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to$. Since $j_!j^*\mathcal{F}$ is constructible, it suffices to show that $i_*i^*\mathcal{F}$ is constructible. By Theorem 5.13, $j_!j^*\mathcal{F}$ is holonomic. Hence by Lemma 6.2.4, $i_*i^*\mathcal{F}$ is also holonomic. Since replacing $\mathcal{F}$ by $i_*i^*\mathcal{F}$ does not change U, we may assume that $\mathcal{F} = i_*\mathcal{F}_Z$ for $\mathcal{F}_Z = i^*\mathcal{F}$. Since $i_*\mathcal{F}_Z$ is holonomic, $\mathcal{F}_Z$ is also holonomic by Lemma 6.2.3. Since $\dim Z < \dim X$, $\mathcal{F}_Z$ is constructible. This contradicts the assumption that U is the largest.

We show the general case. Let k' be a perfect closure of k. Then, by Corollary 5.12, the pull-back $\mathcal{F}'$ on the base change $X_{k'}$ is holonomic. Hence by the perfect case, $\mathcal{F}'$ is constructible. Since the canonical morphism $X_{k'} \to X$ is a homeomorphism, $\mathcal{F}$ is also constructible. □

We may extend Definition 6.1 to sheaves on singular schemes as follows.

Lemma 6.4. *Let X be a scheme of finite type over a field k and $\mathcal{F}$ be a sheaf on X.*

1. *Let $i\colon X \to P$ and $i'\colon X \to Q$ be closed immersions to smooth schemes over k. Then, the following conditions are equivalent:*

(1) *$i_*\mathcal{F}$ is holonomic.*

(2) *$i'_*\mathcal{F}$ is holonomic.*

2. *The following conditions are equivalent:*

(1) *For every étale morphism $U \to X$ and every closed immersion $i\colon U \to P$ to a smooth scheme over k, $i_*(\mathcal{F}|_U)$ is holonomic.*

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(2) *There exists an étale covering* $(U_j \to X)_{j\in J}$ *and a family of closed immersions* $i'_j\colon U_j \to Q_j$ *to smooth schemes over* k*, such that* $i'_{j*}(\mathcal{F}|_{U_j})$ *is holonomic for every* j*.*

Proof. 1. It suffices to show (1)⇒(2). By replacing Q by $P\times Q$, we may assume that there exists a smooth morphism $Q\to P$ compatible with i and i'. Since i' induces a section of a smooth morphism $Q\times_P X\to X$ and since the question is local on X by Lemma 6.2.2, we may assume that there exists a closed subscheme Q' of Q containing X and étale over P by [6, Chapitre 0_{IV} (15.1.16) a) ⟺ c)] and [6, Théorème (17.6.1) a) ⟺ c′)]. By Lemma 6.2.3, replacing Q by Q', we may assume that there exists an étale morphism $h\colon Q\to P$ compatible with i and i'. Since $X\to Q$ induces an open immersion $X\to X\times_P Q$, after shrinking Q, we may assume that $X\to X\times_P Q$ is an isomorphism. If $i_*\mathcal{F}$ is micro supported on $C\subset T^*P$ and $\dim C\leqq \dim P$, then $i'_*\mathcal{F}$ is micro supported on $h^*C\subset T^*Q$ by Lemma 4.7.1 and $\dim h^*C=\dim C\leqq \dim P=\dim Q$.

2. The implication (1)⇒(2) is clear.

(2)⇒(1): Since X is quasi-compact, we may assume that J is finite. By [6, Corollaire (18.4.7)], for each $j\in J$, there exist an open covering $(U_{jh}\to U\times_X U_j)_{h\in H_j}$, closed immersions $i_{jh}\colon U_{jh}\to P_{jh}$ to schemes étale over P and closed immersions $i'_{jh}\colon U_{jh}\to Q_{jh}$ to schemes étale over Q_j. Then since $U_{jh}\to U_j\times_{Q_j} Q_{jh}$ is an open and closed immersion, $i'_{jh*}(\mathcal{F}|_{U_{jh}})$ is holonomic for every j,h by Lemma 6.2.2. Hence $i_{jh*}(\mathcal{F}|_{U_{jh}})$ is also holonomic by 1. Since $(P_{jh}\to P)_{j\in J,h\in H_j}$ is an étale covering of the support of an open neighbourhood of the image $i(U)$, $i_*(\mathcal{F}|_U)$ is holonomic by Lemma 6.2.2. □

Definition 6.5. Let $\mathcal{F}$ be a sheaf on a scheme X of finite type over a field k. We say that $\mathcal{F}$ is *holonomic* if the equivalent conditions in Lemma 6.4.2 are satisfied.

Proposition 6.6. *Let* $\mathcal{F}$ *be a sheaf on a scheme* X *of finite type over a field* k*. Then, the following conditions are equivalent:*

(1) $\mathcal{F}$ *is constructible.*
(2) $\mathcal{F}$ *is holonomic.*

Proof. Since the question is local on X, we may assume that there exists a closed immersion $i\colon X\to P$ to a smooth scheme over k. Then $i_*\mathcal{F}$ is constructible if and only if it is holonomic by Theorem 5.13 and Theorem 6.3. □

We have the following corollary since the corresponding properties are known for constructible sheaves [5].

Corollary 6.7. *Let* $\mathcal{F}$ *be a sheaf on a scheme* X *of finite type over a field* k*.*

1. *Let* $f\colon X\to Y$ *be a morphism of schemes of finite type over* k*. If* $\mathcal{F}$ *is holonomic, then* $f_*\mathcal{F}$ *is holonomic. Further if* f *is separated, then* $f_!\mathcal{F}$ *is holonomic.*
2. *Let* $h\colon W\to X$ *be a morphism of schemes of finite type over* k*. If* $\mathcal{F}$ *is holonomic, then* $h^*\mathcal{F}$ *is holonomic. Further if* h *is separated, then* $h^!\mathcal{F}$ *is holonomic.*

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