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# Quadratic complexes. II

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## 1. *Introduction*

A quadratic complex $Q$ is the set of lines in 3-dimensional projective space $\mathbb{P}^3$ given by a single non-trivial quadratic equation in the Plücker coordinates. The lines of the complex which pass through a fixed point of $\mathbb{P}^3$ are, in general, the generators of a quadric cone; this cone degenerates for the points of a subvariety $K$ of $\mathbb{P}^3$. Thus one can associate with $Q$ a fibration with base $\mathbb{P}^3 - K$ and fibre isomorphic to $\mathbb{P}^1$, and ask whether this fibration is associated with an algebraic vector bundle of rank 2. When the base field is $\mathbb{C}$ and $Q$ is non-singular, the answer is negative; this was proved some years ago by Narasimhan and Ramanan ((8), proposition 8·1), and has the consequence that there is no universal family of stable vector bundles of rank 2 and degree 0 over a curve of genus 2.

The object of this note is to consider what happens when $Q$ is singular. The answer turns out to depend on certain geometrical properties of $Q$ (see Theorem 1 in §2 for a precise statement). The proofs involve a study of the geometry of $Q$, and in particular of the space $V$ of lines on $Q$. In the non-singular case, this variety has been studied previously and is known to be isomorphic to the Jacobian of a curve of genus 2 (7, 9, 13). Here we investigate $V$ by a method which is essentially that of ((9), §4); the calculations are more complicated, but we are able to prove that $V$ is irreducible in certain cases (see Theorem 2 in §2). In the course of this, we introduce a branched double covering of $\mathbb{P}^1$ which plays a similar role to that of the curve of genus 2 in the non-singular case (see §5).

The result of Narasimhan and Ramanan is restricted to the case where the base field is $\mathbb{C}$, but can easily be extended to an arbitrary algebraically closed field by replacing $H^2$ by Pic throughout the argument; in fact, the argument remains valid whenever $Q$ is locally factorial. This is not, however, sufficient for our results (see §3), which are proved by purely geometrical arguments. Note also that we do not appeal to the classification of quadratic complexes; apart from avoiding the necessity of considering large numbers of different cases, this has the advantage that our results are valid in any characteristic. (The classification for characteristic different from 2 is described briefly in the appendix, which also contains more details of some of the geometrical features of $Q$ which are used in the main body of the paper.)

Many questions remain to be settled. For instance, we would like to obtain a topological version of Theorem 1 analogous to the theorem of (10); this can certainly be done in some cases using ((11), theorem 1, corollary 2). One would also like to know whether all complexes of type I (see §2) are locally factorial; this seems quite likely

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and again is true in some cases. Perhaps the most interesting problem is to investigate further the branched double covering $C$ of $\mathbb{P}^1$ introduced in §5. As we have already noted, in the non-singular case $C$ is a curve of genus 2, and the geometry of $Q$ is intimately connected with certain moduli spaces of vector bundles over $C$; it seems possible that some of this theory might extend to the singular case.

Except in the appendix, we work throughout over an arbitrary algebraically closed field. We denote by $G(r,s)$ the Grassmannian of $r$-dimensional subspaces of the projective space $\mathbb{P}^s$, and write also $G$ for $G(1,3)$. We assume a knowledge of the standard properties of $G$ and of quadric hypersurfaces (see, for example, (4)); a summary of some of the most important properties of $G$ is given at the beginning of §2. We do not assume any knowledge of the theory of pencils of quadrics.

## 2. *Statement of results*

We embed the Grassmannian $G$ in the usual way as a non-singular quadric hypersurface in $\mathbb{P}^5$ using the Plücker coordinates. The points of $G$ which correspond to lines through a particular point $x$ in $\mathbb{P}^3$ then form a plane $\alpha_x$; similarly the points which correspond to lines lying in a plane $\pi$ in $\mathbb{P}^3$ form a plane $\beta_\pi$. We therefore have two systems of planes on $G$, which we call $\alpha$-planes and $\beta$-planes; these are the only planes on $G$. Two distinct planes of the same system meet in one point, while $\alpha_x \cap \beta_\pi = \emptyset$ unless $x \in \pi$, in which case $\alpha_x \cap \beta_\pi$ is a line. Conversely, every line on $G$ is contained in precisely one plane of each system.

We are particularly concerned here with the $\alpha$-planes on $G$. It is easy to see that the plane $\alpha_x$ corresponding to the point $x = (x_0, x_1, x_2, x_3)$ is given by the equations

$$x_i p_{jk} + x_j p_{ki} + x_k p_{ij} = 0 \quad (i,j,k \text{ all distinct}). \tag{1}$$

(There are four essentially different equations in (1), but the system has rank 3 for every $x$.) We write $Z$ for the subvariety of $\mathbb{P}^3 \times \mathbb{P}^5$ defined by (1), so that

$$Z = \{(x,z) \in \mathbb{P}^3 \times G : x \text{ lies on the line corresponding to } z\}.$$

Now let $Q$ be the intersection of $G$ with another quadric hypersurface in $\mathbb{P}^5$; we regard $Q$ as the base variety of a pencil $\{Q_\lambda : \lambda \in \mathbb{P}^1\}$ of quadric hypersurfaces, one member of which is the non-singular quadric $G$, and refer to $Q$ as a *quadratic complex*. (Note that $Q$ always has dimension 3.)

We write

$$Y = Z \cap (\mathbb{P}^3 \times Q),$$

so that $Y$ is defined as a subvariety of $\mathbb{P}^3 \times \mathbb{P}^5$ by (1) together with a single quadratic equation in the coordinates of $\mathbb{P}^5$. It follows that $Y$ is a conic bundle with degeneration over $\mathbb{P}^3$ in the sense of ((11), definition 4).

We denote by $p$ the natural projection of $Y$ on $\mathbb{P}^3$, so that

$$p^{-1}(x) = \{x\} \times (\alpha_x \cap Q),$$

and write

$$K = \{x \in \mathbb{P}^3 : \alpha_x \cap Q \text{ is not a non-singular conic}\},$$

$$Y_1 = p^{-1}(K).$$

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It is easy to see that $K$ is a subvariety of $\mathbb{P}^3$, and that $Y - Y_1$ is a conic bundle over $\mathbb{P}^3 - K$. (In the notation of (11), $\mathbb{P}^3$ becomes $X$ and $K$ becomes $X_1$.)

The main object of this paper is to decide when this conic bundle comes from a vector bundle. For this we need to divide the set of quadratic complexes into a number of types as follows:

Type I. $Q$ contains no planes and no quadric surfaces.

Type II. $Q$ contains a plane but no quadric surfaces.

Type III. $Q$ contains a quadric surface (possibly reducible).

For complexes of type II, a further subdivision is needed. We shall see in §4 that a complex $Q$ of type II contains exactly one plane; we say that $Q$ is of type II$\alpha$ or II$\beta$ according as to whether this plane is an $\alpha$-plane or a $\beta$-plane on $G$.

We can now state

THEOREM 1. *The conic bundle $Y - Y_1$ over $\mathbb{P}^3 - K$ comes from a vector bundle if and only if $Q$ is of type* II$\alpha$ *or* III.

THEOREM 2. *If $Q$ is of type* I, *then*

(i) *the variety $V$ of lines on $Q$ is irreducible and of dimension* 2;

(ii) *there exists an $\alpha$-plane on $G$ which meets $Q$ in two distinct lines.*

### 3. *Some preliminary remarks*

The key to this paper is the close connection between Theorems 1 and 2. To see this, recall that any line $l$ on $Q$ is contained in a unique $\alpha$-plane on $G$. It follows that one can define a map $q\colon V \to \mathbb{P}^3$ by the condition

$$l \subset \alpha_{q(l)}.$$

It is easy to see that $q$ is regular and that its image is $K$. The map

$$V \to K \times G(1,5)\colon l \to (q(l), l),$$

is now an isomorphism of $V$ onto the subvariety of $K \times G(1,5)$ denoted in ((11), §5) by the same symbol $V$. Thus, if we write

$$K_2 = \{x \in \mathbb{P}^3\colon \alpha_x \cap Q \text{ has rank } \leqslant 1\}$$

$$V_2 = q^{-1}(K_2),$$

the map

$$q\colon V - V_2 \to K - K_2$$

is precisely the double covering of ((11), §5) associated with the conic bundle $Y - Y_1$.

For $Q$ of type I, Theorem 1 will now follow from Theorem 2 by ((11), theorem 2, corollary), provided that the set of non-factorial points of $Y$ has codimension $\geqslant 3$. (Note that $K \neq K_2$ by Theorem 2 (ii).) Now the projection of $Y$ onto $Q$ makes $Y$ into a $\mathbb{P}^1$-bundle over $Q$; in fact it is just the restriction to $Q$ of the canonical $\mathbb{P}^1$-bundle



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over $G$. Hence it is sufficient to show that the set of non-factorial points of the 3-dimensional variety $Q$ is finite; this follows from

LEMMA 3·1. *If $Q$ is of type* I *or* II, *then*

(i) *every singular member of the pencil $\{Q_\lambda\}$ has rank* 5;

(ii) *$Q$ has finitely many singular points.*

*Proof.* (i) Suppose $Q_\lambda$ has rank $\leqslant 4$ for some $\lambda$. Then $Q_\lambda$ is the join of a line to a quadric surface, and therefore contains a 3-dimensional subspace $\Pi$ of $\mathbb{P}^5$. But then $Q$ contains the quadric surface $\Pi \cap G$, so that $Q$ is of type III.

(ii) It follows easily from the Jacobian criterion that a singular point of $Q$ must be a singular point of some member of the pencil $\{Q_\lambda\}$. This pencil has only finitely many singular members, and, by (i), each of these has only one singular point.

*Remark.* Condition (i) in fact characterises complexes of type I or II. Note also that $Q$ always possesses at least one singular member; this follows from the fact that the condition for a quadric to be singular is given by the vanishing of a single homogeneous polynomial of positive degree in the coefficients. This is standard in characteristic different from 2, and remains valid in characteristic 2, though the proof requires some care.

For $Q$ of type II, the variety $V$ is reducible, so Theorem 2 fails in this case. However we shall use a similar method to prove Theorem 1 for $Q$ of type II$\beta$; in the course of this we shall describe the components of $V$ in this case (see §7). The proof of Theorem 1 for $Q$ of type II$\alpha$ or III is more elementary, and uses another result from (11) (see §8).

One should remark here that all types of quadratic complex exist; for this, see the Appendix. The fact that the behaviour of $Y - Y_1$ is different for types II$\alpha$ and II$\beta$ is interesting, since the two types cannot be distinguished as subvarieties of $\mathbb{P}^5$. In particular, in these cases $Q$ cannot be locally factorial; otherwise, as remarked in §1, the method of (8) would show that $Y - Y_1$ cannot come from a vector bundle.

If $Q$ is the union of two hyperplane sections of $G$ (not necessarily distinct), then $Q$ is certainly of type III and $K = \mathbb{P}^3$; thus Theorem 1 holds trivially and Theorem 2 is not applicable. We shall exclude this case from further consideration; since every subvariety of $G$ is cut by a hypersurface in $\mathbb{P}^5$, it follows that this is equivalent to assuming that $Q$ is irreducible and not contained in any hyperplane.

We finish this section with a further lemma.

LEMMA 3·2. *Every component of $V$ has dimension $\geqslant 2$.*

*Proof.* The subvariety $K$ of $\mathbb{P}^3$ is defined locally (and even globally) by a single polynomial; hence every component of $K$ has dimension $\geqslant 2$. We prove that $V$ has dimension $\geqslant 2$ at every point $l$ of $V$. For this we write

$$V_3 = \{l \in V : \alpha_{q(l)} \subset Q\}$$

(this notation agrees with that of ((11), §5) and consider the following three cases.

(i) If $l \in V - V_2$, then $q$ is an unramified double covering in the neighbourhood of $l$; so the dimension of $V$ at $l$ is equal to the dimension of $K$ at $q(l)$.

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(ii) If $l \in V_2 - V_3$, then $q^{-1}(q(l)) = \{l\}$; the conclusion of (i) now follows from the fact that $q$ is proper and has image $K$.

(iii) If $l \in V_3$, then $q^{-1}(q(l))$ is a plane.

The required conclusion follows in every case.

*Remark.* We shall see in §4 that dim $V = 2$ for $Q$ of type I or II. In fact, this holds unless $Q$ contains a complete plane of singular points, in which case dim $V = 3$. When the characteristic is different from 2, this happens only if $Q$ has Segre symbol [(222)] (see Appendix).

## 4. *The lines on Q*

In this section we obtain some results concerning the existence of lines on $Q$. We start with a simple but useful lemma.

LEMMA 4·1. *Any two skew lines on Q have a common transversal lying entirely on Q.*

*Proof.* The 3-dimensional subspace of $\mathbb{P}^5$ spanned by the two skew lines meets $Q$ in the intersection of two quadric surfaces. If these quadric surfaces coincide or have a plane in common, the result is clear. Otherwise, consider the corresponding pencil of quadric surfaces. The singular members of this pencil cannot have rank 3 since all the lines on a quadric surface of rank 3 pass through the vertex. Hence the pencil must contain a plane-pair; in this case it is easy to list the possible intersections and check the result in each case.

As a first consequence of this lemma, we can prove a result promised in §2.

COROLLARY. *If Q is of type* II, *then Q contains exactly one plane, which passes through the vertices of all singular members of the pencil* $\{Q_\lambda\}$.

*Proof.* Suppose that $Q$ is of type II. By Lemma 3·1 (i), every singular member of $\{Q_\lambda\}$ has rank 5, and any plane on such a quadric passes through its vertex. It follows in particular that any two planes on $Q$ meet. Hence, if $Q$ contains two planes of opposite systems on $G$, they must meet in a line; but then the union of these planes is a quadric surface lying on $Q$, and this contradicts the assumption that $Q$ is of type II.

There remains the possibility that $Q$ contains two planes $\pi_0$, $\pi_1$ of the same system on $G$. In this case $\pi_0$ and $\pi_1$ meet in a unique point $z$, which must be the vertex of the unique singular quadric $Q_0$ of the pencil $\{Q_\lambda\}$. We can choose lines $l$, $n$ lying in $\pi_0$, $\pi_1$ respectively and not passing through $z$; by the lemma, $l$ and $n$ have a common transversal $m$ on $Q$. The plane which contains $m$ and belongs to the opposite system of planes on $G$ from $\pi_0$ and $\pi_1$ meets each of $\pi_0$, $\pi_1$ in a line and therefore passes through $z$. This plane is therefore the join of $m$ and $z$ and lies entirely on $Q_0$, and hence entirely on $Q$. So $Q$ contains a plane of each system on $G$, and the first part of the proof applies.

LEMMA 4·2. *For any point z of Q, there exists a line on Q passing through z. If z is non-singular, does not lie in any plane contained in Q, and is not the vertex of a 2-dimensional quadric cone contained in Q, the number of such lines is finite* ($\leqslant 4$). *In the exceptional cases, the lines on Q through z form a 1-dimensional subvariety of* $G(1, 5)$.



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*Proof.* Suppose first that $z$ is non-singular, and let $\Pi$ be the tangent space to $Q$ at $z$; clearly any line on $Q$ passing through $z$ lies in $\Pi$. Moreover $\Pi \cap Q$ is a cone with vertex $z$ on the intersection of two conics (one or both of which may be singular). This intersection consists of at least one and at most 4 points unless the two conics have a line in common or coincide. The result follows easily.

If $z$ is singular, note that all non-singular members of the pencil $\{Q_\lambda\}$ have the same tangent space $L$ at $z$, and that any line on $Q$ through $z$ lies in $L$. Moreover $L \cap Q$ is a cone with vertex $z$ on the intersection of two quadric surfaces. Since $Q$ is an irreducible variety of dimension 3 and is not contained in $L$, it follows that this intersection has dimension 1. Hence the variety of lines on $Q$ through $z$ has dimension 1.

COROLLARY 1. *If $Q$ is of type* I *or* II, *then* $\dim V = 2$.

COROLLARY 2. *For any line $l$ on $Q$, let $V_l$ be the subvariety of $V$ consisting of those lines on $Q$ which meet $l$. If $Q$ is of type* I, *or if $Q$ is of type* II *and $l$ is not contained in a plane on $Q$, then* $\dim V_l \leqslant 1$.

*Proofs.* Both corollaries follow from Lemmas 4·2 and 3·1(ii), the corollary to Lemma 4·1 and the fact that $Q$ contains no quadric surfaces.

COROLLARY 3. (Theorem 2(ii)) *If $Q$ is of type* I, *there exists an $\alpha$-plane on $G$ which meets $Q$ in two distinct lines.*

*Proof.* By Lemma 4·2 and Corollaries 1 and 2, there exist lines $l$, $n$ on $Q$ such that $l$ does not pass through any singular point and $n$ is skew to $l$. By Lemma 4·1, there exists a line $m$ on $Q$ which meets both $l$ and $n$. If $l$ and $m$ lie in the same $\alpha$-plane on $G$, the result is proved. Otherwise, the two $\alpha$-planes on $G$ containing $l$ and $m$ respectively both pass through the intersection $z$ of $l$ and $m$ and span a 4-dimensional subspace $L$ of $\mathbb{P}^5$. Since $z$ is non-singular, $L$ is not contained in the tangent space to $Q$ at $z$; hence at least one of the two $\alpha$-planes fails to touch $Q$ at $z$, and thus meets $Q$ in $l$ or $m$ together with another line not passing through $z$. This completes the proof.

We finish this section with an extension of Lemma 4·2, Corollary 2. For this, let $l$ be a line on $Q$, and let $V_l'$ be the subset of $V_l$ consisting of those $m \in V_l$ for which $l$ and $m$ do not both lie in the same plane on any *singular* member of the pencil $\{Q_\lambda\}$; it is clear that $V_l'$ is an open subset of $V_l$ and that $l \in V_l - V_l'$. We write also $U_l$ for the subset

$$\{n \in V : n \in V_m \quad \text{for some} \quad m \in V_l - V_l'\} \cup \{n \in V : n \in V_m - V_m' \quad \text{for some} \quad m \in V_l\}$$

of $V$; again it is quite straightforward to check that $U_l$ is a subvariety of $V$. Clearly $U_l \supset V_l$.

LEMMA 4·3. *Suppose that $Q$ is of type* I *and that $l$ does not pass through any singular point of $Q$. Then* $\dim U_l \leqslant 1$.

*Proof.* Suppose that $Q$ and $l$ satisfy the hypotheses of the lemma. By Lemma 3·1(i), every singular quadric of the pencil $\{Q_\lambda\}$ has rank 5; hence for each such quadric there is a unique plane lying on the quadric and containing $l$. Since no plane lies entirely on $Q$, it follows that $V_l - V_l'$ is finite. Moreover, by Lemma 4·2, Corollary 2, $\dim V_m \leqslant 1$ for each $m \in V_l - V_l'$. So the first component of $U_l$ has dimension $\leqslant 1$.

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For the second component, let $m \in V_l$. Then $\dim (V_m - V'_m) \leqslant 1$, and $V_m - V'_m$ is finite unless $m$ passes through a singular point of $Q$. Since there are only finitely many lines $m \in V_l$ which do pass through a singular point (in fact, one for each singular point), this gives the required result.

*Remark.* The inequalities in Lemma 4·2, Corollary 2 and Lemma 4·3 can be replaced by equalities. In fact $\dim V_l = 1$ unless every point of $l$ is a singular point of $Q$ or $l$ lies in a plane on $Q$; in the exceptional cases, $\dim V_l = 2$. As regards $U_l$, the proof of Lemma 4·3 remains valid if $Q$ is of type II and $l$ does not meet the unique plane on $Q$. In all other cases, one can show that $\dim U_l \geqslant 2$; in particular, if $l$ passes through a singular point of $Q$, then $U_l = V$.

## 5. *Another double covering*

Let $W$ be the subvariety of $\mathbb{P}^1 \times G(2, 5) \times V$ defined by

$$W = \{(\lambda, \pi, l) : \pi \text{ lies on } Q_\lambda \text{ and contains } l\}.$$

We denote by $p_1, p_2, p_3$ the projections of $W$ on the three factors, and put

$$W_l = p_3^{-1}(l),$$

for any $l \in V$. We write also

$$\Sigma = \{\lambda \in \mathbb{P}^1 : Q_\lambda \text{ is singular}\}$$

$$W' = p_1^{-1}(\mathbb{P}^1 - \Sigma),\ W'_l = W_l \cap W'.$$

Our main object in this section is to prove

THEOREM 3. *$W'_l$ is reducible if and only if $Q$ contains a plane.*

*Remark.* Note that this conclusion is independent of the line $l$. In fact, we shall show in the course of proving Theorem 3 that the isomorphism class of $W'_l$ is independent of $l$ (see Lemma 5·2); this is not true of $W_l$ or even of the closure of $W'_l$ in $W$.

LEMMA 5·1. *The natural projection*

$$p_{13} : W' \to (\mathbb{P}^1 - \Sigma) \times V$$

*is an unramified double covering.*

*Proof.* If $Q_\lambda$ is non-singular and $l \in V$, the line $l$ lies on precisely one plane of each system on $Q_\lambda$; so every fibre of $p_{13}$ consists of exactly two points. (This is, of course, not sufficient to prove the lemma, since we are allowing arbitrary characteristic, and, even in characteristic 0, the variety $V$ need not be normal.)

For any 3-dimensional subspace $\Pi$ of $\mathbb{P}^5$, one can define an open subset $U$ of $V$ by the condition

$$l \in U \Leftrightarrow l \text{ does not meet } \Pi.$$

Since the open sets of this type cover $V$ and the conclusion of the lemma is local with respect to $V$, it is sufficient to prove the lemma for

$$p_{13} : p_{13}^{-1}((\mathbb{P}^1 - \Sigma) \times U) \to (\mathbb{P}^1 - \Sigma) \times U.$$

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For this, let $L_1$, $L_2$ be distinct hyperplanes in $\mathbb{P}^5$ containing $\Pi$. We can define regular maps $z_i: U \to L_i$ by taking $z_i(l)$ to be the unique point in which $l$ meets $L_i$; note that $z_1(l) \neq z_2(l)$. We can identify $p_{13}^{-1}((\mathbb{P}_1 - \Sigma) \times U)$ with the subvariety $W''$ of

$$(\mathbb{P}^1 - \Sigma) \times \Pi \times U$$

defined by

$$W'' = \{(\lambda, z, l): \text{the plane joining } l \text{ to } z \text{ lies on } Q_\lambda\}.$$

Moreover a point $(\lambda, z, l)$ of $(\mathbb{P}^1 - \Sigma) \times \Pi \times U$ belongs to $W''$ if and only if

(*a*) $z$ lies on $Q_\lambda$

(*b*) $z$ is conjugate to $z_1(l)$ and $z_2(l)$ with respect to $Q_\lambda$.

Thus $W''$ is defined by one quadratic and two linear equations in $z$ (with coefficients which depend algebraically on $l$ and $\lambda$). Since every fibre of $p_{13}$ consists of two points, this completes the proof.

For each $l \in V$, the double covering of Lemma 5·1 restricts to give an unramified double covering

$$p_1|W_l': W_l' \to \mathbb{P}^1 - \Sigma.$$

We shall prove that the isomorphism class of this covering is independent of $l$ by giving an alternative, and more canonical, construction for $W_l'$. (If we knew that $V$ was connected, this fact would be immediate; but we do not know this, and in any case the new construction is of interest in its own right.)

Let $W^*$ denote the subvariety of $\mathbb{P}^1 \times G(2, 5)$ defined by

$$W^* = \{(\lambda, \pi): \pi \text{ lies on } Q_\lambda\}.$$

The projection $W^* \to \mathbb{P}^1$ can be factorized in the form

$$W^* \to C \xrightarrow{f} \mathbb{P}^1,$$

where $W^* \to C$ has connected fibres and $f$ is a finite regular map (see, for example, ((3), chapter III, corollary 11·5)). Composing the projection

$$W \to W^*: (\lambda, \pi, l) \to (\lambda, \pi),$$

with $W^* \to C$, we obtain a regular map $g: W \to C$ such that $f \circ g = p_1$. This restricts to give regular maps

$$g_l: W_l' \to f^{-1}(\mathbb{P}^1 - \Sigma)$$

such that

$$f \circ g_l = p_1|W_l'.$$

LEMMA 5·2. *$g_l$ is an isomorphism.*

*Proof.* For any $\lambda \in \mathbb{P}^1 - \Sigma$, the fibre $f^{-1}(\lambda)$ consists of two points corresponding to the two systems of planes on $Q_\lambda$. Since $f \circ g_l = p_1|W_l'$, it follows from Lemma 5·1 that $g_l$ is bijective and finite and that its differential is everywhere injective. So $g_l$ is an isomorphism as required (see, for example, ((15), chapter II, §5·5, lemma).

*Remark.* It is easy to check that $f^{-1}(\lambda)$ is a single point for $\lambda \in \Sigma$. It therefore follows from Lemmas 5·1 and 5·2 that $C$ is a branched double covering of $\mathbb{P}^1$ whose branch

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locus is precisely $\Sigma$. When $Q$ is non-singular and the characteristic is different from 2, this covering is a non-singular curve of genus 2 and $f$ is the branched double covering given by the canonical divisor class. This association between quadratic complexes and curves of genus 2 has been known for a long time, although it seems to be difficult to find precise references. There is also a more general link between non-singular pencils of quadrics in $\mathbb{P}^{2n+1}$ and hyperelliptic curves of genus $n$ (see **(1, 12, 13)**); note, for example, that ((13), §4, lemma 1) is a version of our Lemma 5·2.

*Proof of Theorem* 3. Any plane lying on $Q$ determines a section of $f$, and hence a decomposition of $f^{-1}(\mathbb{P}^1 - \Sigma)$ into two irreducible components. It follows at once from Lemma 5·2 that $W_l'$ is reducible when $Q$ contains a plane.

Now suppose that $Q$ does not contain a plane. We divide the proof into several steps.

*Step* 1. We consider first the case in which the pencil $\{Q_\lambda\}$ has a singular member $Q_0$ whose vertex does not meet $Q$. Any such $Q_0$ necessarily has rank 5; for if it had rank $\leqslant 4$, its vertex would have dimension $\geqslant 1$, and would therefore meet $G$, and hence also $Q$. If $z_0$ denotes the vertex of $Q_0$, then the join of $l$ and $z_0$ is the only plane on $Q_0$ which contains $l$. Hence $W_l$ has just one point lying over the point $\lambda_0$ of $\mathbb{P}^1$ corresponding to $Q_0$; since $p_1|W_l$ is proper and $W_l'$ is an unramified double covering of $\mathbb{P}^1 - \Sigma$ by Lemma 5·1, it follows that it is sufficient to prove that $W_l$ is non-singular at this point.

To see this, we choose a 3-dimensional subspace $\Pi$ of $\mathbb{P}^5$ passing through $z_0$ but not meeting $l$. As in the proof of Lemma 5·1, we can identify $W_l$ with the subvariety of $\mathbb{P}^1 \times \Pi$ consisting of the points $(\lambda, z)$ for which

($a$) $z$ lies on $Q_\lambda$

($b$) $z$ is conjugate to two fixed points $z_1$, $z_2$ of $l$ with respect to $Q_\lambda$.

To show that $W_l$ is non-singular at $(\lambda_0, z_0)$, it is therefore sufficient to show that the varieties defined by the three equations corresponding to ($a$) and ($b$) meet transversely at this point.

We consider first equation ($a$). We can choose local coordinates $(\mu, x_1, x_2, x_3)$ at $(\lambda_0, z_0)$, where $\mu$ is an affine coordinate in $\mathbb{P}^1$ and $x_1$, $x_2$, $x_3$ are affine coordinates in $\Pi$. The equation ($a$) then takes the form

$$q + \mu q' = 0,$$

where $q = 0$ is a local equation for $Q_0$ and is therefore a homogeneous quadratic in $x_1, x_2, x_3$. Moreover, since by assumption $z_0 \notin Q$, we have $q' \neq 0$ at $z_0$; it follows that the variety defined by ($a$) is non-singular at $(\lambda_0, z_0)$ and has tangent space given by $\mu = 0$, or equivalently by $\lambda = \lambda_0$.

On the other hand, the restrictions of equations ($b$) to $\{\lambda_0\} \times \Pi$ are linear, and must be independent since $W_l$ meets $\{\lambda_0\} \times \Pi$ in only one point. It follows that the varieties defined by these equations meet transversely at $(\lambda_0, z_0)$, and that the two tangent spaces and $\lambda = \lambda_0$ are all transverse there. This completes the proof in this case.

*Step* 2. Since $\{Q_\lambda\}$ always has at least one singular member $Q_0$, it remains to consider the case in which the vertex of $Q_0$ meets $Q$ at a point $z_0$. The point $z_0$ is then certainly a singular point of $Q$, and the non-singular members of $\{Q_\lambda\}$ all have the same tangent space $L$ at $z_0$. Moreover the pencil meets $L$ in the cone on a pencil $\{Q_\lambda'\}$ of quadric surfaces; we denote by $Q'$ the base variety of this pencil.

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Now let $l$ be a line on $Q$ which passes through $z_0$. This line projects to a point $x$ of $Q'$, and, for any $\lambda \in \mathbb{P}^1 - \Sigma$, the planes on $Q_\lambda$ through $l$ project precisely to the lines on $Q'_\lambda$ through $x$. Hence $W_l'$ can be identified with the subvariety

$$\{(\lambda, m): m \text{ lies on } Q'_\lambda \text{ and contains } x\}$$

of $(\mathbb{P}^1 - \Sigma) \times G$.

If the pencil $\{Q'_\lambda\}$ has a singular member whose vertex does not meet $Q'$, we can argue as in Step 1 to show that $W_l'$ is irreducible. Although we have placed a restriction on $l$ in this argument, this does not affect the result in view of Lemma 5·2.

*Step* 3. We are now left with the possibility that $\{Q'_\lambda\}$ does not satisfy this last condition. In this case $\{Q'_\lambda\}$ has a singular member whose vertex meets $Q'$ at a point $x_0$, and we can take $l$ to be the join of $z_0$ and $x_0$. We denote by $\Pi'$ the join of $z_0$ to the common tangent space of the non-singular members of $\{Q'_\lambda\}$ at $x_0$. The 3-dimensional subspace $\Pi'$ then touches every quadric $Q_\lambda$ at $z_0$ and $x_0$, and hence at every point of $l$; it must therefore coincide with the polar subspace of $l$ with respect to any non-singular member of the pencil. Hence

$$(\lambda, \pi) \in W_l' \Rightarrow \pi \subset \Pi',$$

so that $p_2(W_l')$ is contained in the subvariety of $G(2, 5)$ consisting of the planes which contain $l$ and are contained in $\Pi'$. This subvariety is irreducible of dimension 1 (it is isomorphic to $\mathbb{P}^1$); moreover $p_2|W_l'$ is injective since $Q$ does not contain a plane, and $W_l'$ is pure of dimension 1 by Lemma 5·1. It follows at once that $W_l'$ is irreducible; as in Step 2 this holds for any $l$ on $Q$ by Lemma 5·2.

This completes the proof of Theorem 3.

*Remark.* When the characteristic is different from 2, one can use the classification of quadratic complexes (see Appendix) to show that Steps 1 and 2 are sufficient to cover all cases. However this is not apparent without using the classification, and in any case Step 3 is necessary in characteristic 2.

## 6. *Proof of Theorem* 2 *and Theorem* 1*: Type I*

For $Q$ of type I, we have already shown that Theorem 2 implies Theorem 1 (see §3); we have also proved Theorem 2 (ii) and shown that dim $V = 2$ (Lemma 4·2, Corollaries 1 and 3). It remains therefore to prove that $V$ is irreducible; we shall do this by a method which is essentially the same as that used in ((9), §4) in the non-singular case to show that $V$ is isomorphic to the Jacobian of $X$. (The same line of argument is used in ((13), §4).) We shall suppose throughout that $Q$ is of type I, and continue with the notations of §§4, 5.

We begin by defining a regular map

$$r: W \to V.$$

For this, let $(\lambda, \pi, l) \in W$. Since $Q$ contains no planes, we have

$$\pi \cap Q = l \cup m,$$

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where $m$ is a line on $Q$ (which may coincide with $l$ in special cases); we define

$$r(\lambda, \pi, l) = m.$$

It is quite straightforward to check that $r$ is regular.

LEMMA 6·1. *For any line $l$ on $Q$,*

$$V - U_l \subset r(W' \cap p_3^{-1}(V_l')).$$

*Proof.* If $n \in V - U_l$, then $l$ and $n$ are skew; hence, by Lemma 4·1, there exists a third line $m$ on $Q$ which meets both $l$ and $n$, in other words $m \in V_l \cap V_n$. Since $n \notin U_l$, we have in fact $m \in V_l' \cap V_n'$; the plane $\pi$ spanned by $m$ and $n$ therefore lies on $Q_\lambda$ for some $\lambda \in \mathbb{P}^1 - \Sigma$, and

$$n = r(\lambda, \pi, m) \in r(W' \cap p_3^{-1}(V_l')).$$

LEMMA 6·2. *For any line $l$ on $Q$,*

$$V_l' \subset r(W_l') \subset V_l.$$

*Proof.* If $m \in V_l'$, then the plane $\pi$ spanned by $l$ and $m$ lies on $Q_\lambda$ for some $\lambda \in \mathbb{P}^1 - \Sigma$, and $m = r(\lambda, \pi, l)$. The second inclusion is obvious.

Now suppose that $l$ is a line on $Q$ not passing through any singular point. By Lemmas 3·2 and 4·3, the open set $V - U_l$ is dense in $V$; hence, by Lemma 6·1, so is $r(W' \cap p_3^{-1}(V_l'))$. It follows from this and Lemma 6·2 that $V_l'$ is a non-empty open subset of $r(W_l')$, and is therefore irreducible by Theorem 3. The same theorem implies that the restriction of the double covering of Lemma 5·1 to $(\mathbb{P}^1 - \Sigma) \times U$ is non-trivial for any non-empty subset $U$ of $V$. Hence $W' \cap p_3^{-1}(V_l')$ is irreducible; since $r(W' \cap p_3^{-1}(V_l'))$ is dense in $V$, it follows that $V$ is also irreducible. This completes the proof of Theorem 1 for $Q$ of type I.

*Remarks.* (i) Note that in the last part of the proof it is not sufficient to observe that every fibre of $p_3 | W'$ is irreducible, because this map is not proper. It is also not sufficient to note that the double covering of Lemma 5·1 is non-trivial over $(\mathbb{P}^1 - \Sigma) \times V_l'$; if $V_l'$ is not normal, this does not imply that the total space of the covering is irreducible.

(ii) The regular map $r$ can be defined when $Q$ is not of type I, provided that we restrict to the open subset

$$\{(\lambda, \pi, m) \in W : \pi \text{ does not lie on } Q\}.$$

The two lemmas remain true.

## 7. *Proof of Theorem 1: Type II$\beta$*

In this section we suppose throughout that $Q$ is of type II$\beta$, and let $\pi_0$ be the unique $\beta$-plane contained in $Q$. In this case, $V$ is reducible, but we can use the methods of §6 to analyse $V$ and hence to prove Theorem 1.

Note first that, by Lemma 4·2, Corollary 1, we know that $\dim V = 2$; it follows that $\dim K = 2$ and hence that the plane in $P^3$ corresponding to $\pi_0$ is a component $K^a$ of $K$. We denote by $V^a$ the part of $V$ lying over $K^a$ and by $V^b$ the union of the components of $V$ which are not contained in $V^a$. We shall show that $V^b$ is irreducible and is not

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contained in $V_2$; Theorem 1 follows by applying ((11), theorem 2, corollary) with $X = P^3 - K^a$. (The set of non-factorial points of $Y$ has codimension $\geqslant 3$ by Lemma 3·1 (ii).)

*Remark.* It is easy to see that $V^a$ consists precisely of the lines on $Q$ which meet $\pi_0$ and that it has two components, each of which maps isomorphically to $K^a$. In fact, the lines in $\pi_0$ clearly form one component of $V^a$; to prove that there is a second component, it is sufficient to show that there exists a line meeting $\pi_0$ but not contained in it. This follows easily from Lemmas 4·1 and 4·2.

We begin by proving that $V^b$ is not contained in $V_2$.

LEMMA 7·1. *There exists an $\alpha$-plane on $G$ which meets $Q$ in two distinct lines, both skew to $\pi_0$.*

*Proof.* We show first that there exists a line on $Q$ which is skew to $\pi_0$. For this, let $l$ be a line which belongs to $V^a$ but is not contained in $\pi_0$ and does not pass through any singular point: such lines exist by Lemmas 3·1 (ii) and 4·2 and the remark preceding this lemma. Let $\pi$ be the $\beta$-plane on $G$ which contains $l$; since $\pi \neq \pi_0$, the two planes meet in exactly one point $z$, namely the point in which $l$ meets $\pi_0$. Clearly $\pi_0$ is contained in the tangent space to $Q$ at $z$, and this space has dimension 3 since $z$ is non-singular. Hence $\pi$ does not touch $Q$ at $z$, and therefore meets $Q$ in $l$ and a line $m$ which is skew to $\pi_0$.

We know by Lemma 4·2, Corollary 2 that $\dim V_m \leqslant 1$. It follows that there exists a line $n$ on $Q$ which meets but does not lie in $\pi_0$, does not pass through any singular point, and is skew to $m$. By Lemma 4·1, the lines $m$ and $n$ have a common transversal $m_1$ on $Q$. Moreover $m_1$ is skew to $\pi_0$, for otherwise the plane spanned by $n$ and $m_1$ would meet $Q$ in three lines.

If $m$ and $m_1$ both belong to the same $\alpha$-plane, we are finished. Otherwise the two $\alpha$-planes containing $m$ and $m_1$ respectively cannot both touch $Q$ at the point of intersection of $m$ and $m_1$. (This point is non-singular since it does not belong to $\pi_0$; see the corollary to Lemma 4·1.) Hence at least one of these $\alpha$-planes meets $Q$ in two distinct lines; moreover this plane contains a line skew to the $\beta$-plane $\pi_0$, and therefore cannot meet $\pi_0$. This completes the proof of the lemma.

It remains to prove that $V^b$ is irreducible, which we do by adapting the argument of §6. The basic point is that $\pi_0$ allows us to distinguish the two systems of planes on the non-singular members of the pencil $\{Q_\lambda\}$. For this, we note, as in the proof of Theorem 3, that $\pi_0$ induces a decomposition of $f^{-1}(\mathbb{P}^1 - \Sigma)$ into two irreducible components. The inverse images of these components in $W'$ are subvarieties $W'^\alpha$ and $W'^\beta$, and are clearly defined as subsets of $W'$ by

$$W'^\alpha = \{(\lambda, \pi, l) \in W' : \pi \text{ and } \pi_0 \text{ belong to opposite systems on } Q_\lambda\}$$

$$W'^\beta = \{(\lambda, \pi, l) \in W' : \pi \text{ and } \pi_0 \text{ belong to the same system on } Q_\lambda\}.$$

LEMMA 7·2. *The natural projections*

$$W'^\alpha \to (\mathbb{P}^1 - \Sigma) \times V \quad \text{and} \quad W'^\beta \to (\mathbb{P}^1 - \Sigma) \times V$$

*are isomorphisms.*

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*Proof.* The projections are bijective since every line on $Q$ lies in exactly one plane of each system on each non-singular member of the pencil $\{Q_\lambda\}$. The result now follows from Lemma 5·1.

Now let $l$ be a line lying in $\pi_0$. If $m \in V_l'$, then $m$ does not lie in $\pi_0$; hence, by Remark (ii) at the end of §6, the regular map $r$ is defined on $W'^\beta \cap p_3^{-1}(V_l')$. We have the following analogue of Lemma 6·1.

LEMMA 7·3. *For any line $l$ on $Q$ which lies in $\pi_0$, but does not pass through a singular point,*

$$r(W'^\beta \cap p_3^{-1}(V_l')) = V - V^a.$$

*Proof.* For any $(\lambda, \pi, m) \in W'^\beta$ with $m \in V_l'$, we have $\pi \cap Q = m \cup r(\lambda, \pi, m)$.

Since $\pi$ belongs to the same system of planes as $\pi_0$, it meets $\pi_0$ in exactly one point $z$, namely the point in which $m$ meets $l$. Since the tangent space to $Q$ at $z$ has dimension 3 and contains $\pi_0$, it does not contain $\pi$. It follows that $r(\lambda, \pi, m)$ does not pass through $z$ and therefore does not meet $\pi_0$. Hence $r(\lambda, \pi, m) \in V - V^a$.

Conversely, suppose that $n \in V - V^a$. Then $n$ is skew to $\pi_0$ and therefore certainly skew to $l$; hence, by Lemma 4·1, there exists a line $m$ on $Q$ which meets both $l$ and $n$. For $\lambda$ in $\Sigma$, it follows from Lemma 3·1 (i) that $\pi_0$ is the unique plane on $Q_\lambda$ which contains $l$; hence $m \in V_l'$. Moreover the plane $\pi$ spanned by $m$ and $n$ lies on $Q_\lambda$ for some $\lambda$, and meets $\pi_0$ in a unique point, namely the point in which $m$ meets $l$. Since this point is non-singular, it follows that $\lambda \in \mathbb{P}^1 - \Sigma$, and then $\pi$ and $\pi_0$ must belong to the same system of planes on $Q$. Thus

$$n = r(\lambda, \pi, m) \in r(W'^\beta \cap p_3^{-1}(V_l')).$$

In view of Lemmas 7·2 and 7·3, it is now sufficient to prove that $V_l'$ is irreducible for any $l$ satisfying the hypothesis of Lemma 7·3. For this we note that $r$ is defined on the whole of $W'^\alpha$, and, by Lemma 7·2,

$$W'^\alpha \cap W_l' \cong \mathbb{P}^1 - \Sigma$$

for any line $l$ on $Q$. Moreover $V_l'$ is non-empty by Lemmas 7·1 and 7·3; the required result therefore follows from

LEMMA 7·4. *Let $l$ be a line on $Q$ which lies in $\pi_0$. Then*

$$V_l' \subset r(W'^\alpha \cap W_l') \subset V_l.$$

*Proof.* The proof is the same as that of Lemma 6·2; note that the plane $\pi$ in the proof meets $\pi_0$ in a line and therefore belongs to the opposite system of planes on $Q_\lambda$.

This completes the proof of Theorem 1 for $Q$ of type II$\beta$.

## 8. *Proof of Theorem* 1: *Types II$\alpha$ and III*

We now complete the proof of Theorem 1 by showing that the conic bundle $Y - Y_1$ does come from a vector bundle when $Q$ is of type II$\alpha$ or III. By ((11), proposition 2), it is sufficient to show that in these cases $Y - Y_1$ has a section.

For $Q$ of type II$\alpha$, we note that there exists a point $a \in \mathbb{P}^3$ such that $\alpha_a \subset Q$. Certainly

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$a \in K$; so, for every point $x \in \mathbb{P}^3 - K$, there is a unique line containing $x$ and $a$, and this line belongs to the complex $Q$. We denote the corresponding point of $Q$ by $s(x)$; the map

$$\mathbb{P}^3 - K \to Y - Y_1 : x \mapsto (x, s(x))$$

defines the required section.

For $Q$ of type III, we can choose a 3-dimensional subspace $\Pi$ of $\mathbb{P}^5$ such that $\Pi \cap G \subset Q$. For any $x \in \mathbb{P}^3$, we then have

$$\Pi \cap \alpha_x \subset Q,$$

and $\Pi \cap \alpha_x$ is certainly a point, a line or a plane. In the last two cases $\alpha_x \cap Q$ cannot be a non-singular conic; hence, for any $x \in \mathbb{P}^3 - K$, we can define $s(x)$ to be the unique point in which $\Pi$ meets $\alpha_x$. The required section is again given by

$$x \mapsto (x, s(x)).$$

*Remark.* In fact, for $Q$ of type III, one can find three sections $s_1$, $s_2$, $s_3$ of $Y - Y_1$ such that $s_1(x)$, $s_2(x)$, $s_3(x)$ are distinct points for all $x \in \mathbb{P}^3 - K$. This implies that the conic bundle $Y - Y_1$ is actually trivial.

## *Appendix*

In this appendix, we work throughout over an algebraically closed field $k$ of characteristic different from 2. We describe the classification of quadratic complexes, and then determine some of the geometrical features of each class of complex. The main features studied are the planes and quadric surfaces on $Q$, and the number of components of the variety $V$. (One should note that the classical papers all restrict attention to the case $k = \mathbb{C}$, but the methods are valid under our more general assumption. We have not attempted to bring our terminology into line with the classical terminology for quadratic complexes; thus, for example, when we speak of singular points, lines, etc. of $Q$, we are referring to features of $Q$ regarded as a subvariety of $\mathbb{P}^5$.)

The classification of quadratic complexes is derived from Weierstrass' classification (17) of pairs of quadratic forms. The idea of using Weierstrass' results to obtain information on quadratic complexes is due to Klein (6), but a complete list, including equations for $K$ and a discussion of various geometrical properties, was first obtained by Weiler. However, Weiler's paper contains a considerable number of errors; the accounts given in Jessop's book ((5), chapter XI) and Zindler's Encyklopädie article are therefore to be preferred. A synthetic treatment has been given by Sturm. (There is also a detailed account due to Segre (14); unfortunately I have been unable to obtain a copy, but the summary in Zindler ((19), pp. 1115–1122) suggests that his approach is close to ours.)

The singular points of $Q$ are known classically as the *double lines* of the complex, and are tabulated in Weiler (18), Sturm (16) and Zindler (19). Weiler's table contains many errors and omissions, and Sturm also has several mistakes; Zindler's table ((19), pp. 1128–1131, column 6) is accurate apart from the case [(33)], where the entry is incomplete. The double lines are not included in Jessop's table; they are discussed in the text, but with some omissions.

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The planes on $Q$ do not seem to have been studied in detail by the classical authors, but the planes contained in $K$ are listed by all the authors mentioned in the previous paragraph. Again Weiler and Sturm have many errors, while Zindler is correct, except that, for the complexes which exist in two dual forms, only one of the forms is described in detail. Jessop omits one plane in the cases [(22)11], [(32)1], [(22)2] and [(42)].

The quadric surfaces in $Q$ are known classically as *linear congruences* contained in $Q$. The way in which these arise is well known (see, for example, Jessop ((5), pp. 198, 199), but their degeneration does not seem to have been studied in detail.

The variety $V$ does not figure in any of the classical accounts.

## (A) *Classification of quadratic complexes*

We start by describing briefly the classification of pencils $\{Q_\lambda\}$ of quadric hypersurfaces in $\mathbb{P}^{n-1}$ which contain non-singular members. (For more details, see Jessop (5) or, for a complete account which includes the singular case ((4), chapter XIII).) For any such pencil, we can choose symmetric $n \times n$ matrices $A, B$ defining distinct members of the pencil and with $A$ non-singular. One can show that two such pairs $A, B$ and $A', B'$ are equivalent under the action of $GL(n)$ if and only if the $k[t]$-valued matrices $B - tA$ and $B' - tA'$ have the same elementary divisors. Note that $t - \lambda$ divides $\det(B - tA)$ if and only if $B - \lambda A$ is singular. Corresponding to each such value of $\lambda$, we therefore obtain a sequence of elementary divisors

$$(t-\lambda)^{r_1}, \ldots, (t-\lambda)^{r_k}: r_1 \geqslant r_2 \geqslant \ldots \geqslant r_k \geqslant 1.$$

We associate to $\lambda$ the symbol

$$\mathbf{r} = (r_1 \ldots r_k), \tag{1}$$

and call $k$ the *corank* of $\mathbf{r}$ and

$$\mu(\mathbf{r}) = \Sigma r_i$$

the *multiplicity* of $\mathbf{r}$. When $k = 1$, we usually write $r$ for $(r)$.

Combining the symbols (1) corresponding to the various zeroes of $\det(B - tA)$, we obtain a symbol

$$[\mathbf{r}^{(1)} \ldots \mathbf{r}^{(m)}], \tag{2}$$

where each element $\mathbf{r}^{(j)}$ has the form (1) and

$$\Sigma \mu(\mathbf{r}^{(j)}) = n. \tag{3}$$

Up to order of the elements, (2) depends only on the projective equivalence class of the pencil $\{Q_\lambda\}$ and is called the *Segre symbol* of $\{Q_\lambda\}$. All symbols of the form (2) satisfying the condition (3) can occur, and one can write down canonical forms for the corresponding pencils. (Note that we are allowing the symbol $[(1 \ldots 1)]$, which corresponds to the case in which all $Q_\lambda$ are identical; this is convenient for an induction argument in section (C).)

It follows from this description that the elements $\mathbf{r}^{(j)}$ of the Segre symbol of $\{Q_\lambda\}$ correspond to the singular members of the pencil. In fact, for $k \leqslant n-1$, an element of

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Table 1. *Contribution to singular set of Q made by each singular member of the pencil* $\{Q_\lambda\}$

| Element of Segre symbol | Vertex | Vertex $\cap$ $Q$ | Singularity type |
|---|---|---|---|
| 1 | point | $\phi$ | — |
| 2, 3, 4, 5, 6 | point | point | $A_1$–$A_5$ |
| (11) | line | 2 points | $2A_1$ |
| (21), (31), (41), (51) | line | point | $A_3$, $D_4$, $D_5$, $D_6$ |
| (22), (32), (42), (33) | line | line | — |
| (111) | plane | conic | — |
| (211), (311), (411) | plane | 2 lines | — |
| (221), (321) | plane | line | — |
| (222) | plane | plane | — |

corank $k$ corresponds to a singular member of rank $n-k$, in other words a quadric whose vertex has dimension $k-1$. (The only case with $k = n$ is [(1 ... 1)].) If $k = n-1$, this quadric is a (repeated) plane, and the base variety $Q$ of the pencil is a (repeated) quadric; if $k = n-2$, the singular quadric is the union of two planes, and $Q$ is the union of two quadrics. In all other cases, provided $n \geqslant 5$, the variety $Q$ is an irreducible quartic.

We now turn to the case $n = 6$ and suppose that $G$ belongs to the pencil $\{Q_\lambda\}$, so that $Q$ is a quadratic complex (except in the case [(1 ... 1)]). If the Segre symbol of $Q$ contains an element of corank $\geqslant 4$, then $Q$ is the union of two linear complexes; as in §3, we exclude this case, and are left with 49 possibilities for the Segre symbol. One slight complication arises since there are two essentially different (and dual) ways in which a given non-singular quadric in $\mathbb{P}^5$ can be identified with $G$; in the notation of §2, we have to decide which system of planes corresponds to the $\alpha$-planes on $G$ and which to the $\beta$-planes. It turns out that in most cases $G$ has an automorphism which fixes $Q$ and interchanges the two systems of planes. In fact, this fails to be true precisely when every number in the Segre symbol is even (Frobenius(2)). We shall see later (Table 2) that in each such case $Q$ contains exactly one plane of one of the two systems and a different number of planes of the other system; we distinguish the two cases by adding $\alpha$ or $\beta$ to the Segre symbol depending on which system has exactly one plane lying on $Q$. (This is compatible with the notation of §2.) There are just six Segre symbols to which this applies, namely

$$[222],\ [42],\ [6],\ [(22)2],\ [(42)] \quad \text{and} \quad [(222)],$$

so we obtain a total of 55 classes of quadratic complex. (Our $\alpha$, $\beta$ correspond to (ii), (i) respectively in Jessop(5). Weiler uses $(A)$, $(B)$, but his notation is not consistent with ours; in fact $(A)$ corresponds to $\alpha$ for [42] and [(42)] and to $\beta$ for the other symbols.)

### (B) *Singular points of* Q

It follows at once from the Jacobian criterion that a point of $Q$ is singular if and only if it is a singular point of $Q_\lambda$ for some $\lambda$; moreover no point can be singular on two

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distinct members of the pencil. Thus the singular set $\Sigma Q$ of $Q$ is a union of subsets, one corresponding to each singular quadric $Q_\lambda$. The contribution of any such quadric to $\Sigma Q$ is just the intersection of its vertex with $Q$, and is determined by the corresponding element of the Segre symbol of $Q$. The only case in which $Q$ is non-singular is the generic case [1 ... 1].

This approach is valid for any $n$. For $n = 6$ and corank $\leqslant 3$, the various cases are listed in Table 1; note in particular that the singular set of $Q$ has dimension 2 in the case [(222)] and dimension $\leqslant 1$ in all other cases. The isolated singularities are all simple and of the types indicated in the last column.

### (C) *Planes on* Q

We begin with the following general result.

PROPOSITION. *Let $Q$ be the base of a pencil of quadrics in $P^{2r+1}$, the general member of which is non-singular. Then $Q$ contains a linear subspace of dimension $r$ if and only if every element of the Segre symbol of $Q$ has even multiplicity.*

*Proof.* For $r = 0$, there are just three possible Segre symbols [11], [2] and [(11)], and $Q$ is empty in the first case and non-empty in the other two.

Suppose, therefore, that $r \geqslant 1$ and that the result is true for pencils in $P^{2r-1}$. It is easy to check that a subspace of dimension $r$ on $Q$ must meet the vertex of every singular member of $\{Q_\lambda\}$, and must therefore pass through a singular point of $Q$. If $z$ is a particular singular point and $L$ is the common tangent space of the non-singular members of the pencil at $z$, then $\{Q_\lambda\}$ meets $L$ in the cone on a pencil $\{Q'_\lambda\}$ of quadrics in $\mathbb{P}^{2r-1}$; let $Q'$ denote the base of this pencil. An $r$-dimensional subspace lying on $Q$ and passing through $z$ is necessarily the join of $z$ to an $(r-1)$-dimensional subspace on $Q'$. The required result now follows from the inductive hypothesis and the fact that the Segre symbol of $Q'$ has the same multiplicities as that of $Q$, except that the multiplicity of the element corresponding to the singular point $z$ is reduced by 2.

The method of proof of this proposition can be used to obtain information about the number of planes lying on a quadratic complex $Q$ (or, more generally, the number of $r$-dimensional subspaces lying on the base of a pencil of quadrics in $\mathbb{P}^{2r+1}$). In particular, if the multiplicities of the Segre symbol of $Q$ are 2, 2, 2 or 2, 4, then the pencil $\{Q_\lambda\}$ contains a singular member whose vertex meets $Q$ in either one or two points (corresponding to an element 2 or (11)); moreover every plane on $Q$ passes through precisely one of these points. Using the classification of pencils of quadric surfaces (see, for example, (4), ch. XIII, §11), one can calculate both the number of planes on $Q$ and their incidence properties, and fill in the relevant columns of Table 2. (The required information on pencils of quadric surfaces can, of course, be obtained by a similar method.)

The same method works when $Q$ has Segre symbol [6] or ([51)]. The cases [(42)] and [(33)] need more care, since $Q$ then possesses a line $l$ of singular points. However it is easy to calculate the planes on $Q$ containing $l$; there is one such in the first case, and two in the second. Any other plane $\pi$ on $Q$ must meet $l$, and therefore lies together with $l$ in a 3-dimensional subspace $\Pi$. Moreover $\Pi$ lies on the singular member $Q_0$ of

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Table 2. *Planes on Q and K*

| | Planes on $Q$ | Plane-pairs* | Planes $\subset K$ |
|---|---|---|---|
| $[222]\alpha$ | $1\alpha$ | 0 | 0 |
| $\beta$ | $1\beta$ | 0 | 1 |
| [22(11)] | $1\alpha$, $1\beta$ | 1 | 1 |
| [2(11) (11)] | $2\alpha$, $2\beta$ | 4 | 2 |
| [(11) (11) (11)] | $4\alpha$, $4\beta$ | 12* | 4 |
| $[42]\alpha$ | $1\alpha$ | 0 | 0 |
| $\beta$ | $1\beta$ | 0 | 1 |
| [4(11)] | $1\alpha$, $1\beta$ | 1 | 1 |
| [(31)2] | $1\alpha$, $1\beta$ | 1 | 1 |
| [(31) (11)] | $2\alpha$, $2\beta$ | 4 | 2 |
| $[(22)2]\alpha$ | $1\alpha$, $2\beta$ ($\alpha$ contains singular line) | 2 | 3† |
| $\beta$ | $2\alpha$, $1\beta$ ($\beta$ contains singular line) | 2 | 1 |
| [(22) (11)] | $3\alpha$, $3\beta$ ($1\alpha$, $1\beta$ contain singular line) | 7* | 3 |
| [(211)2] | $1\alpha$, $1\beta$ (each contains a singular line) | 1 | 2† |
| [(211) (11)] | $2\alpha$, $2\beta$ (each contains a singular line) | 4 | 2 |
| $[6]\alpha$ | $1\alpha$ | 0 | 0 |
| $\beta$ | $1\beta$ | 0 | 1 |
| [(51)] | $1\alpha$, $1\beta$ | 1 | 1 |
| $[(42)]\alpha$ | $1\alpha$, $2\beta$ ($\alpha$ contains singular line) | 2 | 3† |
| $\beta$ | $2\alpha$, $1\beta$ ($\beta$ contains singular line) | 2 | 1 |
| [(33)] | $2\alpha$, $2\beta$ ($1\alpha$, $1\beta$ contain singular line) | 3* | 2 |
| [(411)] | $1\alpha$, $1\beta$ (each contains a singular line) | 1 | 2† |
| [(321)] | $1\alpha$, $1\beta$ (both contain singular line) | 1 | 1 |
| $[(222)]\alpha$ | $1\alpha$, $\infty^1\beta$ ($\alpha$ is singular plane) | $\infty^1$ | $K = \mathbb{P}^3$† |
| $\beta$ | $\infty^1\alpha$, $1\beta$ ($\beta$ is singular plane) | $\infty^1$ | 1 |
| [(22)11] | 0 | 0 | 1† |
| [(32)1] | 0 | 0 | 1† |
| [(211)11] | 0 | 0 | 2† |
| [(311)1] | 0 | 0 | 2† |
| [(221)1] | 0 | 0 | 1† |

* By a plane-pair, we mean a pair of planes with a common line, and thus forming a quadric surface on $Q$; the planes are necessarily of opposite systems on $G$. In the cases indicated, there exist pairings of $\alpha$- and $\beta$-planes which do not form plane-pairs.

† In these cases $K$ contains planes which do not correspond to $\beta$-planes on $Q$.

the pencil $\{Q_\lambda\}$, and therefore meets $Q$ in $\pi$ and another plane $\pi'$ containing $l$. On the other hand, since $Q_0$ is the join of $l$ to a non-singular quadric surface, each plane $\pi'$ gives rise to just two 3-dimensional subspaces $\Pi$ containing $\pi'$ and lying on $Q_0$. In case [(42)], this shows at once that there are two further planes on $Q$; for [(33)], there are again just two further planes, since the second plane $\pi''$ which contains $l$ counts as one of the two planes to which $\pi'$ gives rise.

Suppose finally that the Segre symbol of $Q$ has multiplicity 6, and that the singular quadric $Q_0$ has rank 3, so that every plane on $Q$ contains a singular line. In cases [(411)] and [(321)], there are only finitely many singular lines, and it is easy to determine the required planes. In case [(222)], there is a complete plane $\pi_0$ of singular points. Any other plane $\pi$ on $Q$ must meet $\pi_0$ in a line, and thus lies together with $\pi_0$

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in a unique 3-dimensional subspace $\Pi$ lying on $Q_0$. The set of such 3-dimensional subspaces is infinite, being parametrized by the base conic of the quadric $Q_0$ of rank 3; hence the planes on $Q$, other than $\pi_0$, also form a family parametrized by this conic.

*Remark.* By similar arguments, but without using the classification, one can show that a quadratic complex $Q$ contains infinitely many planes if and only if $Q$ has a plane of singular points. This is valid in any characteristic.

(D) *Planes contained in* K

It is clear that a plane $\pi$ in $\mathbb{P}^3$, for which $\beta_\pi$ lies on $Q$, is contained in $K$. The plane $\pi$ is also contained in $K$ if $\beta_\pi$ contains a singular line of $Q$, for in this case

$$x \in \pi \Rightarrow \alpha_x \cap \beta_\pi \text{ is a line}$$
$$\Rightarrow \alpha_x \text{ contains a singular point}$$
$$\Rightarrow \alpha_x \cap Q \text{ is singular.}$$

In the highly exceptional case $[(222)]\alpha$, the infinite family of planes $\pi$, for which $\beta_\pi$ lies on $Q$, fills $\mathbb{P}^3$; thus $K = \mathbb{P}^3$. In all other cases, however, the two methods described above yield all planes contained in $K$. To see this, we prove first

LEMMA. *Let $z \in Q$. The following conditions are equivalent*:

(i) *every $\alpha$-plane through $z$ has singular intersection with $Q$*;

(ii) *the point $z$ is singular, or lies in a $\beta$-plane on $Q$, or is the vertex of a quadric cone lying on $Q$.*

*Proof.* Two distinct $\alpha$-planes through $z$ meet only in $z$; so (i) certainly implies that $Q$ contains infinitely many lines through $z$. A slight refinement of the proof of Lemma 4·2 now shows that (i) $\Rightarrow$ (ii). Conversely, it is clear that (i) holds if $z$ is singular, or if $z$ lies in a $\beta$-plane on $Q$. If $z$ is the vertex of a quadric cone on $Q$, not more than two generators of this cone can lie in the same $\alpha$-plane. Hence there are infinitely many $\alpha$-planes through $z$ having singular intersection with $Q$. Since these $\alpha$-planes form a subvariety of the irreducible 1-dimensional variety of $\alpha$-planes through $z$, this implies (i).

Now suppose that $Q$ is not of class $[(222)]\alpha$ and that $\pi \subset K$. Then $Q$ contains only finitely many $\beta$-planes, and every point of $\beta_\pi \cap Q$ satisfies condition (i) of the lemma. If $\beta_\pi$ is not contained in $Q$, then only finitely many of these points can lie in $\beta$-planes on $Q$. Moreover, we shall see in section (E) that $Q$ contains only finitely many quadric cones with vertices at non-singular points. It follows that $\beta_\pi \cap Q$, which is pure of dimension 1, must be contained in $\Sigma Q$, and this is possible only if $\beta_\pi \cap Q$ is a singular line of $Q$. This completes the proof that, for $Q$ not of type $[(222)]\alpha$,

$$\pi \subset K \Leftrightarrow \beta_\pi \text{ lies on } Q \text{ or contains a singular line of } Q.$$

Note that a $\beta$-plane containing a singular line may lie on $Q$, but need not do so. Using the results of the previous section and the fact that every singular line of $Q$ lies in a unique $\beta$-plane, one can easily determine all the planes contained in $K$; in particular, there are five cases in which $K$ contains a plane but $Q$ does not (see Table 2).

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*Remarks.* (i) In all cases except [(222)]$\alpha$, there are only finitely many planes contained in $K$; hence $\dim K = 2$ and every plane contained in $K$ is a component of $K$. It follows that $\dim V = 2$ unless $Q$ is of class [(222)].

(ii) When the pencil $\{Q_\lambda\}$ has a singular member $Q_0$ of rank 3, we can easily determine $K$ completely. If $Q$ is of class [(222)]$\alpha$, we have already remarked that $K = \mathbb{P}^3$. Otherwise, we note that every line on $Q$ must meet the vertex of $Q_0$; hence $\alpha_x$ meets the vertex of $Q_0$ for every $x \in K$. It follows that $K$ is contained in the subvariety of $\mathbb{P}^3$ generated by the lines corresponding to the points in which the vertex of $Q_0$ meets $Q$. Thus $K$ is a non-singular quadric surface if $Q$ contains a singular conic, and a union of planes in the other cases.

### (E) *Quadric surfaces on* Q

Any quadric surface on $Q$ is contained in a unique 3-dimensional subspace $\Pi$ of $\mathbb{P}^5$; moreover $\Pi$ is contained in some member $Q_0$ of the pencil $\{Q_\lambda\}$, and $Q_0$ is necessarily singular and indeed of rank $\leqslant 4$. Conversely, if $Q_0$ is of rank 4, it is the join of a line to a quadric surface, and therefore contains two disjoint families of 3-dimensional subspaces, each family being parametrized by $\mathbb{P}^1$. Similarly, if $Q_0$ is of rank 3, it gives rise to just one such family, again parametrised by $\mathbb{P}^1$. It is now easy to write down the total number of families of quadric surfaces on any quadratic complex of type III (see Table 3, column 3); each family is parametrized by $\mathbb{P}^1$, and two distinct families have no members in common.

It remains to consider the possible degeneration of quadric surfaces on $Q$. Since no 3-dimensional subspace can lie on $G$ or meet $G$ in a (repeated) plane, the most degenerate form possible is a plane-pair. We know from Table 2 that there are finitely many such except in the case [(222)], for which it is easy to see that every quadric surface on $Q$ is a plane-pair.

There still remains the possibility that the general member of a family may be a cone. We show first that this implies that the vertex of every cone in the family is at a singular point of $Q$. For this, we can clearly suppose that the curve $A$ of vertices of the cones is not contained in the vertex of $Q_0$. Using the fact that each family of quadric surfaces on $Q$ is parametrized by $\mathbb{P}^1$ we obtain a rational map from $A$ to $\mathbb{P}^1$; geometrically this is given by projection from the vertex of $Q_0$, followed (when $Q_0$ has rank 4) by projection along one system of generators of the non-singular quadric surface which forms the base of the singular quadric $Q_0$. In characteristic 0, this map must be generically smooth; this means that the tangent to $A$ at a general point $z$ is not contained in the 3-dimensional subspace of $\mathbb{P}^5$ spanned by the corresponding quadric cone. If $z$ is non-singular, this is a contradiction, since it implies that the tangent space to $Q$ at $z$ has dimension 4.

Unfortunately this argument fails in finite characteristic, but one can still prove the required result by direct calculation in each case; this is tedious, but not difficult, and we leave the details to the reader. Note that, if the vertex of $Q_0$ contains a singular line $l$ of $Q$, then any cone in one of the families of quadric surfaces corresponding to $Q_0$ must contain $l$; hence any such cone has vertex on $l$. Thus the only cases that need

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Table 3. *Families of quadrics and components of V*

| Type | | Families of quadrics | $K$ | Components of $V$ |
|---|---|---|---|---|
| I | [111111], [21111], [3111], [2211], [411], [321], [51], [33] | 0 | Irred. | 1 |
| II$\alpha$ | [222]$\alpha$, [42]$\alpha$, [6]$\alpha$ | 0 | Irred. | 3 |
| II$\beta$ | [222]$\beta$, [42]$\beta$, [6]$\beta$ | 0 | Cubic + plane | 3 |
| III | [(11)1111], [(11)211], [(11)31], [(21)111], [(21)21], [(21)3], [(31)11], [(41)1] | 2 | Irred. | 2 |
| | [(11) (11)11], [(21) (11)1], [(21) (21)] | 4 | 2 quadrics | 4 |
| | [(11)22], [(11)4], [(31)2], [(51)] | 2 | Cubic + plane | 5 |
| | [(11) (11)2], [(31) (11)] | 4 | Quadric + 2 planes | 8 |
| | [(11) (11) (11)] | 6 | 4 planes | 12 |
| | [(22)11], [(32)1] | 2 | Quadric cone + plane | 3 |
| | [(22)2]$\alpha$, [(42)]$\alpha$ | 2 | 3 planes | 6 |
| | [(22)2]$\beta$, [(42)]$\beta$ | 2 | Quadric cone + plane | 6 |
| | [(22) (11)] | 4 | 3 planes | 9 |
| | [(33)] | 2 | 2 planes | 6 |
| | [(111)111], [(111)21], [(111)3] | 1 | Quadric | 1 |
| | [(111) (11)1], [(111) (21)] | 3 | Quadric | 2 |
| | [(111) (111)] | 2 | Quadric ($K = K_2$) | 1 |
| | [(211)11], [(311)1] | 1 | 2 planes | 2 |
| | [(211)2], [(411)] | 1 | 2 planes | 4 |
| | [(211) (11)] | 3 | 2 planes | 6 |
| | [(221)1] | 1 | Plane | 1 |
| | [(321)] | 1 | Plane | 3 |
| | [(222)]$\alpha$ | 1 | $\mathbb{P}^3$ | 2 |
| | [(222)]$\beta$ | 1 | Plane ($K = K_2$) | 2 |

further calculation are those for which the element of the Segre symbol corresponding to $Q_0$ is $(r1)$ for some $r \geqslant 1$ or $(111)$.

*Remark.* It would be nice to have a more geometrical argument for proving this result in finite characteristic. There are two points that suggest that a fairly subtle argument is needed. Firstly, an isolated quadric cone on $Q$ can have its vertex at a non-singular point. Secondly, the result itself fails in characteristic 2; for example, if $Q$ is given by

$$z_0 z_1 + z_3 z_5 + z_4^2 = z_2 z_5 + z_3 z_4 = 0,$$

then $Q$ has only three singular points, but contains a quadric cone with vertex at any point of the twisted cubic given parametrically by

$$z = (0, 0, \lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3).$$

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We can now calculate the families of quadric cones on $Q$ by looking at them from a different angle. In the first place, we can clearly restrict attention to the case in which $\Sigma Q$ has dimension 1. The lines of $Q$ through a point of $\Sigma Q$ form a cone whose base is a quartic curve (see the proof of Lemma 4·2). Direct calculations show that the cone of lines through a general point of any 1-dimensional component of $\Sigma Q$ contains a quadric cone when $Q$ has Segre symbol

$$[(22)(11)],\ [(33)],\ [(111)(111)],\ [(211)(11)] \quad \text{or} \quad [(321)],$$

and breaks up as the union of two quadric cones when $Q$ has Segre symbol

$$[(111)(11)1],\ [(111)(21)] \quad \text{or} \quad [(221)1].$$

In cases $[(111)(111)]$ and $[(211)(11)]$, the singular set has two 1-dimensional components, thus giving two families of quadric cones. For $[(221)1]$, there is only one family of quadric surfaces on $Q$, so certainly only one family of quadric cones. For $[(111)(11)1]$ and $[(111)(21)]$, however, it is easy to see that there are two separate families of quadric cones with the same vertices; it may be noted that these are the families corresponding to the elements (11), (21), respectively, of the Segre symbol, while the family corresponding to (111) has a non-singular quadric surface as its general member. Thus the families of quadric surfaces on $Q$ include two families of quadric cones in the cases

$$[(111)(11)1],\ [(111)(21)],\ [(111)(111)] \quad \text{and} \quad [(211)(11)],$$

and one family for

$$[(22)(11)],\ [(33)],\ [(221)1] \quad \text{and} \quad [(321)].$$

### (F) *Components of* V

It should be possible to use the methods of §§6 and 7 to obtain detailed information on the structure of $V$ for any quadratic complex $Q$. Here we simply compute the number of components of $V$ in each case using comparatively crude arguments; the results are listed in the last column of Table 3. For convenience, we assume known the structure of $K$ (see Table 3, column 4), although the only cases in which this is really needed are those in the very first group under type III. In all cases, we indicate the general line of argument, but leave detailed verifications to the reader.

For types I and II, the components of $V$ have already been computed in the main body of the paper. In Table 3, we have divided the complexes of type III into five major sections, each of which is further subdivided into groups. The first section consists of those complexes which have a finite number of singular points. Note that the maximum number of components of $V$ is equal to the number of $\alpha$-planes on $Q$ added to twice the number of components of $K$. To show that this number is always attained, we note that the families of quadric surfaces on $Q$ give rise to at least 2 and at most 4 components of $V$ in addition to those arising from planes on $Q$; a careful use of this fact gives the required results, although for the last three groups it may be easier to use a method similar to that used to prove Lemma 7·1.

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The second section consists of complexes for which the pencil $\{Q_\lambda\}$ has a singular member $Q_0$ of rank 4 whose vertex $l$ lies entirely on $Q$. For the first four groups, the two families of quadric surfaces corresponding to $Q_0$ give rise to 3 components of $V$. To see this, note that, for any such quadric surface, the generators of one system meet $l$, while those of the other do not. This might appear to split $V$ into 4 components, but in fact the results of section (E) show that the lines on $Q$ that meet $l$ give rise to only one component of $V$ (apart from those arising from planes). (The only alternative is that the lines through a general point of $l$ form two quadric cones, and this never happens.) In case [(33)], one of the two families of quadric surfaces consists entirely of cones; this reduces the number of components of $V$ obtained in this way to 2.

The third and fourth sections comprise complexes for which the pencil $\{Q_\lambda\}$ has a singular member $Q_0$ of rank 3, whose vertex meets $Q$ in a 1-dimensional variety. Here all lines on $Q$ meet the vertex of $Q_0$, and we can use the analysis in section (E) to determine the components of $V$. Ignoring the components which arise from planes on $Q$, we obtain one component of $V$ for each 1-dimensional component of $(\Sigma Q) \cap Q_0$ except in the cases [(111)(11)1] and [(111)(21)], each of which gives 2 components of $V$.

In the final case [(222)], every line on $Q$ lies in a plane on $Q$; it follows that $V$ has one component of dimension 3, and one of dimension 2.

Whenever $V$ has the maximum number of components, it follows at once that no component of $K$ can be contained in $K_2$. By looking at the lines through a particular singular point of $Q$ and the $\alpha$-planes in which they lie, one can show that the only cases in which a component of $K$ is contained in $K_2$ are [(111)(111)] and [(222)]$\beta$. (In these two cases $Q$ is given, respectively, by the tangents of a non-singular quadric surface, and by the lines which meet a non-singular conic.)

*Remark.* If $z$ is a singular point of $Q$, then the corresponding line in $P^3$ lies entirely on $K$. It follows that the image $K'$ in $P^3$ of the set of non-factorial points of $Y$ is contained in $K$. If $Q$ is of type II$\alpha$ or III, then, by Theorem 1 and ((11), theorem 2), the double covering

$$q\colon V - V_2 \to K - K_2,$$

is trivial over $K - (K_2 \cup K')$. When $Q$ has a finite number of singular points, so that $\dim K' \leqslant 1$, this means that $q$ is trivial over a dense open subset of $K - K_2$. If $Q$ has infinitely many singular points, this is not necessarily true, as one can see from Table 3; indeed in some cases $V$ is irreducible even though $K \neq K_2$. The cases in which $q$ is non-trivial over every dense open subset of $K - K_2$ are precisely those for which $K$ contains a plane not corresponding to a $\beta$-plane on $Q$, together with three of the six cases in which $Q$ has a singular conic.

The impetus for the work described here came from a letter from W. L. Edge commenting on my paper (10) and some discussion with M. S. Narasimhan and U. V. Desale when I visited the Tata Institute of Fundamental Research in 1975; my thanks are due to all concerned. I am also grateful to Miles Reid for supplying me with a copy of his thesis (13); our methods are basically similar to some of those in his §4.

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