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Differentiable positive definite kernels and Lipschitz continuity

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Abstract

Reade[**11**] has shown that positive definite kernels $K(x, t)$ which satisfy a Lipschitz condition of order α on a bounded region have eigenvalues which are asymptotically $O(1/n^{1+\alpha})$. In this paper we extend this result to positive definite kernels whose symmetric derivative $K_{rr}(x, t) \equiv \partial^{2r}K(x, t)/\partial x^r \partial t^r$ is in Lip_α and establish $\lambda_n(K) = O(1/n^{2r+1+\alpha})$. If $\partial K_{rr}/\partial t$ is in Lip_α, the anticipated asymptotic estimate is also derived.

The proofs use a well-known result of Chang[**2**], recently rederived by Ha[**5**], and estimates based upon finite rank approximations to the kernels in question. In these latter estimates we employ the familiar piecewise linear 'hat' basis functions of approximation theory.

1. *Introduction*

Let the kernel $K(x, t)$ belong to $\mathscr{L}^2[0, 1]^2$ and satisfy $K(t, x) = \overline{K(x, t)}$ for almost all x, t with $0 \leqslant x, t \leqslant 1$. Assume that the compact Hermitian operator defined on $\mathscr{L}^2[0, 1]$ by

$$Kf(x) \equiv \int_0^1 K(x, t)f(t)\,dt$$

is also positive definite, i.e.

$$\int_0^1 \int_0^1 K(x, t)\overline{f(x)}f(t)\,dx\,dt \geqslant 0$$

for all $f \in \mathscr{L}^2[0, 1]$. In such a case the spectrum of K consists of a sequence $\{\lambda_n(K)\}$ of positive eigenvalues which converges to zero.

If the eigenvalues of K are arranged in decreasing order of size and repeated according to their multiplicities, then asymptotic estimates for these eigenvalues can be derived under various smoothness assumptions on the kernel $K(x, t)$. For example, if $K(x, t)$ is continuous on $[0, 1]^2$ the classic result of Mercer (see [**3**], p. 197; [**15**], p. 245) implies that

$$\lambda_n(K) = o\left(\frac{1}{n}\right)$$

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as $n \to \infty$. Indeed, if $K(x, t) \in C^p[0, 1]^2$, we now know that

$$\lambda_n(K) = o\left(\frac{1}{n^{p+1}}\right)$$

as $n \to \infty$ (see [12, 13, 5]).

Recently Reade[11] has established that the eigenvalues of positive definite continuous kernels which are Lipschitz continuous with exponent α satisfy

$$\lambda_n(K) = O\left(\frac{1}{n^{1+\alpha}}\right)$$

as $n \to \infty$. In the sequel we show that a comparable result is valid in the case when a higher-order derivative of the kernel satisfies such a Lipschitz condition.

This paper, and those of Reade[11–14] and Ha[5], belong to a long series of expositions dating back to Fredholm's fundamental contributions in 1903 which contain results relating asymptotic estimates for eigenvalues (or singular values) to kernel smoothness. These are precise analogues of results relating similar estimates for Fourier coefficients to function smoothness (see [4] for a survey of some of this literature). Surprisingly, it appears that this work with positive definite kernels represents another instance in which the kernel results actually predate the Fourier series results. Comparable conclusions do exist for Fourier sine and/or cosine series, notably Lorentz's theorem (see [7], p. 148; see also [1], §7). But Oehring's investigations [9, 10] are, as far as we know, the first consideration of these questions for classical exponential Fourier series.

2. *Preliminaries*

In our estimates we will utilize the symmetric integral operator R of finite rank on $[0, 1]$ generated by the separable kernel

$$R(x, t) \equiv \sum_{n=0}^{N} \psi_n(x)\,\psi_n(t) \quad (0 \leqslant x, t \leqslant 1)$$

where ψ_0, ψ_n for $1 \leqslant n < N$ and ψ_N are defined as follows:

$$\psi_0(x) \equiv \begin{cases} N^{\frac{3}{2}}(1/N - x) & \text{if } 0 \leqslant x \leqslant 1/N \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_n(x) \equiv \begin{cases} N^{\frac{3}{2}}(x - (n-1)/N) & \text{if } (n-1)/N \leqslant x \leqslant n/N \\ N^{\frac{3}{2}}((n+1)/N - x) & \text{if } n/N \leqslant x \leqslant (n+1)/N \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_N(x) \equiv \begin{cases} N^{\frac{3}{2}}(x - (N-1)/N) & \text{if } (N-1)/N \leqslant x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

These piecewise linear functions ψ_n are the triangular 'hat' functions familiar to investigators in approximation theory and numerical analysis. In modern parlance they are essentially unnormalized B-splines of order 2 (see, for example, [16], pp. 112 ff.) and possess a number of useful, and easily verified, properties:

Property 1. $\|\psi_n\|^2 \equiv \int_0^1 \psi_n^2(x)\,dx = \frac{2}{3}$ if $0 < n < N$ and $\|\psi_0\|^2 = \|\psi_N\|^2 = \frac{1}{3}$.

Property 2. $(\psi_n, \psi_m) \equiv \int_0^1 \psi_n(x)\,\psi_m(x)\,dx = 0$ if $|n-m| \geqslant 2$ and $(\psi_n, \psi_{n+1}) = \frac{1}{6}$ for $0 \leqslant n \leqslant N-1$.

Property 3. $\sum_{n=0}^{N} \psi_n(x) = \sqrt{N}$ for $0 \leqslant x \leqslant 1$.

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Property 4. $\int_0^1 \psi_n(x)\,dx = 1/\sqrt{N}$ if $0 < n < N$ and $\int_0^1 \psi_0(x)\,dx = \int_0^1 \psi_N(x)\,dx = 1/2\sqrt{N}$.

Property 5. $\sum_{n=0}^{N} n\psi_n(x) = xN^{\frac{3}{2}}$ for $0 \leqslant x \leqslant 1$.

Property 6.

$$\int_0^1 x\psi_n(x)\,dx = \begin{cases} 1/6N^{\frac{3}{2}} & \text{if } n = 0 \\ n/N^{\frac{3}{2}} & \text{if } 0 < n < N \\ (3N-1)/6N^{\frac{3}{2}} & \text{if } n = N. \end{cases}$$

As a consequence of Properties 3–6, we have

Property 7.

$$\sum_{n=0}^{N} \psi_n(x) \int_0^1 \psi_n(t)\,(t-x)\,dt = 1/6N^{\frac{3}{2}}[\psi_0(x)\,(1+3Nx) - \psi_N(x)\,(1+3N-3Nx)]$$

for $0 \leqslant x \leqslant 1$.

For an arbitrary positive integer N the eigenvalues of R are given by the eigenvalues of the symmetric tridiagonal matrix $A \equiv (a_{ij})$ with entries $a_{ij} = (\psi_i, \psi_j)$ for $1 \leqslant i, j \leqslant N$, i.e.

$$A = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & 0 & 0 & & & & \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & & & & \\ 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ & & & & 0 & 0 & \frac{1}{6} & \frac{1}{3} \end{pmatrix}.$$

By classical results (see [**8**], p. 11, for example) all the eigenvalues of this matrix are in the union of the intervals $[\frac{1}{6}, \frac{1}{2}]$ and $[\frac{1}{3}, 1]$. Thus all of the eigenvalues are positive and less than or equal to 1. Hence R is a positive definite operator which satisfies $R \leqslant I$, the identity operator.

3. *Subsidiary estimates*

Assume that L is the compact operator on $\mathscr{L}^2[0, 1]$ generated by the square-integrable kernel $L(x, t)$, $0 \leqslant x, t \leqslant 1$, i.e.

$$Lf(x) \equiv \int_0^1 L(x, t) f(t)\,dt.$$

From Reade [**12**, **13**] and Ha [**5**], if $L(x, t)$ is continuous, Hermitian, and positive definite, then L has only positive eigenvalues and, by Mercer's theorem, is of trace class. It follows that if $L^{\frac{1}{2}}$ is the unique positive square root of L, then the operator $L^{\frac{1}{2}}RL^{\frac{1}{2}}$, formed using the finite rank operator defined in the previous section, is Hermitian and has a continuous kernel. Moreover $L^{\frac{1}{2}}RL^{\frac{1}{2}} \leqslant L$, and thus $L - L^{\frac{1}{2}}RL^{\frac{1}{2}}$ has a trace norm

$$\|L - L^{\frac{1}{2}}RL^{\frac{1}{2}}\|_{tr} = \int_0^1 L(x, x)\,dx - \int_0^1 \int_0^1 \sum_{n=0}^{N} \psi_n(x)\, L(x, t)\, \psi_n(t)\,dx\,dt. \tag{1}$$

In this section, using the nature and properties of the piecewise-linear 'hat' functions ψ_n, we estimate the right-hand side of the expression (1), and draw appropriate conclusions, in two cases:

(i) the kernel $L(x, t)$ is not only continuous but is also in Lip_α, i.e.

$$|L(x, t+h) - L(x, t)| < |h|^\alpha A(x)$$

where $0 < a \leqslant 1$;

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(ii) $L(x, t)$ is continuously differentiable and $\partial L(x, t)/\partial t$ is in Lip_α, i.e.

$$\left|\frac{\partial L(x, t+h)}{\partial t} - \frac{\partial L(x, t)}{\partial t}\right| < |h|^\alpha A(x)$$

where $0 < \alpha \leqslant 1$.

In each case $A(x)$ is a generic non-negative, integrable function on $[0, 1]$. Case (i) has, of course, been considered previously by Reade [11]. We include our new proof for the sake of completeness.

THEOREM 1. *Let the compact operator L on $\mathscr{L}^2[0, 1]$ be generated by the continuous, Hermitian, positive definite kernel $L(x, t)$, $0 \leqslant x, t \leqslant 1$. If $L(x, t)$ is in Lip_α where $0 < \alpha \leqslant 1$, then the eigenvalues of L satisfy*

$$\lambda_n(L) = O\left(\frac{1}{n^{1+\alpha}}\right)$$

as $n \to \infty$.

Proof. The right-hand side of (1) can be rewritten as

$$\int_0^1 L(x, x)\left[1 - \sum_{n=0}^{N} \psi_n(x) \int_0^1 \psi_n(t)\, dt\right] dx - \sum_{n=0}^{N} \int_0^1 \int_0^1 \psi_n(x)\,[L(x, t) - L(x, x)]\,\psi_n(t)\, dt\, dx. \quad (2)$$

By virtue of Properties 3, 4 of the previous section, the first term of (2) becomes

$$\int_0^1 L(x, x)\left[\frac{\psi_0(x) + \psi_N(x)}{2\sqrt{N}}\right] dx = \frac{N}{2}\int_0^{1/N} L(x, x)\left(\frac{1}{N} - x\right) dx + \frac{N}{2}\int_{(N-1)/N}^1 L(x, x)\left(x - \frac{N-1}{N}\right) dx = O\left(\frac{1}{N}\right)$$

as $N \to \infty$, since $L(x, x)$ is continuous. Owing to the Lipschitz condition on $L(x, t)$ the second term in (2) is bounded above, in absolute value, by

$$\int_0^{1/N}\int_0^{1/N} \psi_0(x)\,\psi_0(t)\,|t - x|^\alpha A(x)\, dt\, dx + \sum_{n=1}^{N-1} \int_{(n-1)/N}^{(n+1)/N}\int_{(n-1)/N}^{(n+1)/N} \psi_n(x)\,\psi_n(t)\,|t - x|^\alpha A(x)\, dt\, dx + \int_{(N-1)/N}^1\int_{(N-1)/N}^1 \psi_N(x)\,\psi_N(t)\,|t - x|^\alpha A(x)\, dt\, dx$$

$$\leqslant \frac{1}{N^\alpha}\left[\frac{1}{2}\int_0^{1/N} A(x)\, dx + 2^\alpha \int_0^1 A(x)\, dx + \frac{1}{2}\int_{(N-1)/N}^1 A(x)\, dx\right] = O\left(\frac{1}{N^\alpha}\right)$$

as $N \to \infty$ since A is integrable. Here again we have used Properties 3 and 4.

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Combining these results into (1) we find that

$$\|L - L^{\frac{1}{2}} R L^{\frac{1}{2}}\|_{tr} = O\left(\frac{1}{N^{\alpha}}\right)$$

as $N \to \infty$. But R, and hence $L^{\frac{1}{2}} R L^{\frac{1}{2}}$, has rank N, and so by the Weyl-Courant minimax principle applied to the trace norm (see [12], p. 153, for example)

$$\sum_{n=N+1}^{\infty} \lambda_n(L) \leqslant \|L - L^{\frac{1}{2}} R L^{\frac{1}{2}}\|_{tr} = O\left(\frac{1}{N^{\alpha}}\right).$$

From this we immediately deduce that

$$\lambda_n(L) = O\left(\frac{1}{n^{1+\alpha}}\right)$$

as $n \to \infty$. ∎

THEOREM 2. *Let the compact operator L on $\mathscr{L}^2[0, 1]$ be generated by the continuously differentiable, Hermitian, positive definite kernel $L(x, t)$, $0 \leqslant x, t \leqslant 1$. Assume also that $L(0, 0) = 0 = L(1, 1)$. If $\partial L(x, t)/\partial t$ is in Lip_{α} where $0 < \alpha \leqslant 1$, then the eigenvalues of L satisfy*

$$\lambda_n(L) = O\left(\frac{1}{n^{2+\alpha}}\right)$$

as $n \to \infty$.

Proof. As in the proof of Theorem 1, the right-hand side of (1) can be rewritten as

$$\frac{N}{2}\int_0^{1/N} L(x, x)\left(\frac{1}{N} - x\right)dx + \frac{N}{2}\int_{(N-1)/N}^{1} L(x, x)\left(x - \frac{N-1}{N}\right)dx$$
$$- \sum_{n=0}^{N} \int_0^1 \int_0^1 \psi_n(x)\,[L(x, t) - L(x, x)]\,\psi_n(t)\,dt\,dx. \quad (3)$$

Since $L(x, x)$ is in $C^1[0, 1]$ and $L(0, 0) = 0 = L(1, 1)$, the first two terms are $O(1/N^2)$ as $N \to \infty$. Using the mean value theorem, the third term in (3) becomes

$$- \sum_{n=0}^{N} \int_0^1 \int_0^1 \psi_n(x)\,\psi_n(t)\,(t - x)\frac{\partial L(x, \zeta)}{\partial t}\,dt\,dx,$$

where ζ lies between x and t, and thus the right-hand side of (1) is

$$O\left(\frac{1}{N^2}\right) - \sum_{n=0}^{N} \int_0^1 \psi_n(x)\frac{\partial L(x, x)}{\partial t}\int_0^1 \psi_n(t)\,(t - x)\,dt\,dx$$
$$- \sum_{n=0}^{N} \int_0^1 \int_0^1 \psi_n(x)\,\psi_n(t)\,(t - x)\left[\frac{\partial L(x, \zeta)}{\partial t} - \frac{\partial L(x, x)}{\partial t}\right]dt\,dx. \quad (4)$$

By virtue of Property 7 of §2, when summation and integration over x are interchanged the second term in this new expression becomes

$$-\frac{1}{6N^{\frac{3}{2}}}\int_0^1 \frac{\partial L(x, x)}{\partial t}\,[\psi_0(x)\,(1 + 3Nx) - \psi_N(x)\,(1 + 3N - 3Nx)]\,dx$$

$$= -\frac{1}{6}\int_0^{1/N} \frac{\partial L(x, x)}{\partial t}\left(\frac{1}{N} - x\right)(1 + 3Nx)\,dx$$

$$+ \frac{1}{6}\int_{(N-1)/N}^{1} \frac{\partial L(x, x)}{\partial t}\left(x - \frac{N-1}{N}\right)(1 + 3N - 3Nx)\,dx$$

$$= O\left(\frac{1}{N^2}\right)$$

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as $N \to \infty$. The last term in (4) is bounded above, in absolute value, by

$$\int_0^{1/N}\int_0^{1/N} \psi_0(x)\,\psi_0(t)\,|t-x|^{1+\alpha}\,A(x)\,dt\,dx + \sum_{n=1}^{N-1}\int_{(n-1)/N}^{(n+1)/N}\int_{(n-1)/N}^{(n+1)/N} \psi_n(x)\,\psi_n(t)\,|t-x|^{1+\alpha}\,A(x)\,dt\,dx + \int_{(N-1)/N}^{1}\int_{(N-1)/N}^{1} \psi_N(x)\,\psi_N(t)\,|t-x|^{1+\alpha}\,A(x)\,dt\,dx,$$

which, reasoning as before, is

$$O\left(\frac{1}{N^{1+\alpha}}\right)$$

as $N \to \infty$. It therefore follows that in this case

$$\sum_{n=N+1}^{\infty} \lambda_n(L) \leqslant \|L - L^{\frac{1}{2}}RL^{\frac{1}{2}}\|_{tr} = O\left(\frac{1}{N^{1+\alpha}}\right),$$

and hence

$$\lambda_n(L) = O\left(\frac{1}{n^{2+\alpha}}\right)$$

as $n \to \infty$. ∎

Remark. The artificial condition on the values of $L(x, t)$ at the corners, occasioned by our method of proof in this theorem, will be removed in the next section.

4. *The principal result*

We are now ready to move towards the main conclusion of this paper. We begin with two ready consequences of the Weyl-Courant minimax characterization of eigenvalues (see, for example, [3], p. 126 or [15], p. 238.)

LEMMA 1. *A Hermitian kernel that differs from a positive definite kernel by a degenerate (separable) kernel of rank N can have at most N negative eigenvalues.*

Proof. Let $M(x, t) = K(x, t) + K_N(x, t)$ where M is Hermitian, K is positive definite, and K_N is degenerate of rank N. (Note that K_N is necessarily Hermitian.) If we order both the positive eigenvalues λ_n^+ and negative eigenvalues λ_n^-, separately, by size (magnitude), the Weyl-Courant characterization of the eigenvalues implies that

$$\lambda_{n+m-1}^+(M) \leqslant \lambda_n^+(K) + \lambda_m^+(K_N) \tag{5}$$

and

$$\lambda_{n+m-1}^-(M) \geqslant \lambda_n^-(K) + \lambda_m^-(K_N). \tag{6}$$

Since $\lambda_{N+1}^-(K_N) = 0$, while $\lambda_n^-(K) = 0$ for all n, by hypothesis, then $\lambda_n^-(M) = 0$ for $n > N$. ∎

LEMMA 2. *If a Hermitian kernel $M(x, t)$ differs from a positive definite kernel $K(x, t)$ by a degenerate kernel, then $\lambda_n^+(M) = O(n^\gamma)$ if and only if $\lambda_n(K) = O(n^\gamma)$ as $n \to \infty$.*

Proof. The result may be viewed as a special case of the Lemma in [4], p. 154. Alternatively, complementing (5) we have

$$\lambda_{n+m-1}^+(K) \leqslant \lambda_n^+(M) - \lambda_m^-(K_N). \tag{7}$$

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If in both (5) and (7) we set $m = N+1$, then

$$\lambda^+_{n+N}(M) \leqslant \lambda_n(K) \quad \text{and} \quad \lambda_{n+N}(K) \leqslant \lambda^+_n(M). \quad \blacksquare$$

Given the positive definite Hermitian kernel $K(x, t)$, our principal result concerns the symmetric derivative

$$K_{rr}(x, t) \equiv \frac{\partial^{2r}}{\partial x^r \, \partial t^r} K(x, t).$$

If $K_{rr}(x, t)$ exists and is continuous on $[0, 1]^2$, then Kadota[6] has shown that $K_{rr}(x, t)$ is positive definite Hermitian on the fundamental domain and generates an integral operator of trace class. A well-known result of Chang (see [2], p. 26) relates the eigenvalues of $K(x, t)$ and $K_{rr}(x, t)$.

LEMMA 3 (Chang). $\lambda_{2n}(K) \leqslant$ const. $(n-r)^{-2r} \lambda_{n+1}(K_{rr})$ *for* $n > r$.

Remark. Ha in [5] has recently rederived this result and obtained a specific value for the constant.

MAIN THEOREM. *Let the Hermitian kernel $K(x, t)$ be continuous and positive definite on $[0, 1]^2$ and assume that, for some positive integer r, the symmetric derivative $K_{rr}(x, t)$ exists and is continuous on $[0, 1]^2$ as well. If $K_{rr}(x, t)$ is in* Lip_α *on $[0, 1]^2$ where $0 < \alpha \leqslant 1$, then the eigenvalues of K satisfy*

$$\lambda_n(K) = O\left(\frac{1}{n^{2r+1+\alpha}}\right)$$

as $n \to \infty$. If $K_{rr}(x, t)$ is continuously differentiable and $\partial K_{rr}(x, t)/\partial t$ is in Lip_α *on $[0, 1]^2$, then*

$$\lambda_n(K) = O\left(\frac{1}{n^{2r+2+\alpha}}\right).$$

Proof. If $K_{rr}(x, t)$ is in Lip_α, identifying $K_{rr}(x, t)$ with the kernel $L(x, t)$ of Theorem 1 gives

$$\lambda_n(K_{rr}) = O\left(\frac{1}{n^{1+\alpha}}\right)$$

as $n \to \infty$. The desired conclusion immediately follows from Lemma 3.

If $K_{rr}(x, t)$ is in $C^1[0, 1]^2$ and $\partial K_{rr}(x, t)/\partial t$ is in Lip_α, we first form the continuously differentiable Hermitian kernel

$$\begin{aligned} M(x, t) \equiv {} & K_{rr}(x, t) + (x-1) K_{rr}(0, t) + (t-1) K_{rr}(x, 0) \\ & - xK_{rr}(1, t) - tK_{rr}(x, 1) + (x-1)(t-1) K_{rr}(0, 0) \\ & + (1-x) tK_{rr}(0, 1) + (1-t) xK_{rr}(1, 0) + xtK_{rr}(1, 1) \end{aligned}$$

and note that $M(0, t) = 0 = M(1, t)$. As a consequence of this latter property, the continuously differentiable eigenfunctions of M also vanish for $x = 0, 1$.

Although $M(x, t)$ may no longer be positive definite, by Lemma 1 it has at most a finite number of negative eigenvalues. Excluding these from consideration, the kernel $M^+(x, t)$ with the eigenfunction expansion

$$M^+(x, t) = \sum_{n=1}^{\infty} \lambda^+_n(M) \phi^+_n(x) \bar{\phi}^+_n(t)$$

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is therefore a positive definite kernel which differs from $M(x, t)$, and hence $K_{rr}(x, t)$, by at most a degenerate kernel. Thus, owing to Lemma 2, M^+ and K_{rr} have the same eigenvalue asymptotics. Moreover, by construction, $M^+(x, t)$ is continuously differentiable and not only is $\partial M(x, t)/\partial t$ and hence $\partial M^+(x, t)/\partial t$ in Lip_α on $[0, 1]^2$, but $M^+(0, 0) = 0 = M^+(1, 1)$ as well. Identifying $M^+(x, t)$ with the kernel $L(x, t)$ of Theorem 2, therefore, we find

$$\lambda_n(M^+) = O\left(\frac{1}{n^{2+\alpha}}\right).$$

In view of the observations above, use of Lemma 3 then yields

$$\lambda_n(K) = O\left(\frac{1}{n^{2r+2+\alpha}}\right). \quad \blacksquare$$

5. *Concluding remarks*

Like Mercer's classic theorem, the above result remains valid as long as all but a *finite* number of the eigenvalues of the given continuous Hermitian kernel are of the same sign. The main theorem represents the extension of the results of Reade [**12**, **13**] and Ha [**5**] to the Lip_α cases. If the positive definite kernel $K(x, t)$ is in $C^p[0, 1]^2$, then all pth order partial derivatives are continuous. The first part of the above theorem extends the case of even p, the second part the case of odd p.

The results are best possible, as well, in the sense that the exponent in the order relation cannot be improved. As appropriate examples we merely need to consider

$$K(x, t) = \sum_{n=1}^{\infty} \frac{e^{2in\pi(x-t)}}{n^{p+1+\alpha}} \quad (0 < \alpha < 1)$$

and, to cover the case of odd p,

$$K(x, t) = \sum_{n=1}^{\infty} \frac{\cos n\pi(x-t)}{n^{p+2} \ln^2 n} \quad (\alpha = 1).$$

In all cases $\partial^p K(x, t)/\partial x^{[p/2]} \partial t^{[(p+1)/2]}$ is in Lip_α on $[0, 1]^2$.

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