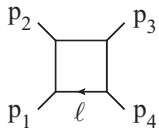
Loop amplitudes and generalized unitarity 6

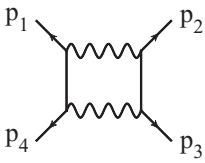
In Part I of the book, we focused exclusively on tree-level amplitudes, the leading contributions in the perturbative expansion of a scattering process. The loop-corrections are of course highly relevant, both in particle physics applications and for our understanding of the mathematical structure of the S-matrix. Here, in Part II of the book, we explore the structure of loop amplitudes.

Feynman rules require momentum conservation at each vertex of a Feynman diagram. At tree-level, this fixes all momenta of the internal lines in terms of the external momenta. At loop-level, momentum conservation leaves one momentum undetermined per loop and one must integrate over all such unfixed momenta. Thus in D-dimensions, one has a D-dimensional loop-integral for each loop. For example, in ϕ^3 theory one of the diagrams that contributes to the 1-loop 4-point amplitude is



$$= \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^2(\ell-p_1)^2(\ell-p_1-p_2)^2(\ell+p_4)^2}\,. \tag{6.1}$$

As another example, consider a diagram of the 1-loop correction to the 4-point all-scalar amplitude in scalar QED:



$$= e^4 \int \frac{d^D\ell}{(2\pi)^D} \frac{[(\ell-2p_1)\cdot(\ell+2p_2)][(\ell-p_1+p_4)\cdot(\ell+p_2-p_3)]}{\ell^2(\ell-p_1)^2(\ell-p_1-p_2)^2(\ell+p_4)^2}\,, \tag{6.2}$$

where e is the electric charge of the scalar. Note that the integral (6.2) has loop-momentum dependence in the numerator.

In general, an L-loop amplitude can be written schematically as

$$\mathcal{A}_n^{L\text{-loop}} = \sum_j \int \left(\prod_{l=1}^{L} \frac{d^D\ell_l}{(2\pi)^D}\right) \frac{1}{S_j} \frac{n_j\, c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}\,, \tag{6.3}$$

where the sum runs over all possible L-loop Feynman diagrams j. For each diagram, the ℓ_l denote the L loop-momenta, α_j label the propagators, and S_j is the symmetry factor associated with the diagram. The kinematic numerator factors n_j are polynomials of Lorentz-invariant contractions of external- and loop-momenta and polarization vectors (or

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other external wavefunctions). The constants c_j capture the information about couplings and gauge group factors.

The color-structure we have worked with for super Yang–Mills amplitudes is rather simple: each color-ordered amplitude is associated with a single trace of gauge group generators. At loop-level, more intricate trace-structures arise. In this book, however, we will be concerned with the ***planar limit*** of Yang–Mills amplitudes. In this context, the planar limit means that the amplitudes have only one trace-structure, just as the tree amplitudes. They are called ***planar amplitudes***. The color-ordered Feynman rules of Section 2.5 apply and the diagrams that contribute to an amplitude are precisely those in which no lines cross.

At loop-level, it is useful to distinguish the following three objects:

1. The ***loop-integrand*** is the rational function inside the loop-momentum integration.
2. The ***loop-integral*** is the combination of the integrand and the loop-momentum integration measure: this is a formal object, since we have not specified the integration region of the loop-momentum or addressed divergences.
3. The ***loop amplitude*** is the result of carrying out the loop-integrations in the loop-integral. If we integrate over physical momentum space $\mathbb{R}^{1,3}$, the integral may have infrared (IR) and/or ultraviolet (UV) divergences. We need to regulate such divergences in order to make the integrated result, the loop amplitude, well-defined.

Our focus will mainly be on the loop-integrand and the loop-integral. However, it is useful to understand the analytic structure of the loop amplitude, since this reveals which constraints the loop-integrand, and hence integral, must satisfy.

It is not surprising that integrals of loop amplitudes can be divergent. They integrate a rational function – the loop-integrand – over L copies of $\mathbb{R}^{1,3}$ and this includes regions where the loop-momentum is infinite $|\ell| \to \infty$ as well as regions where integrand-denominator may vanish; both cases give potential divergences. We discuss the UV and IR divergences in Section 6.1. After regulating the amplitude and separating out the divergences, the finite pieces of the amplitude are typically not just rational functions. For massless lower-loop amplitudes, one sometimes encounters functions that can be expressed in the form of iterative integrals or products of iterative integrals. For example,

$$\log(x) = \int_0^x \frac{dt}{t}\,, \quad \log(x)\log(y) = \left(\int_0^x \frac{dt}{t}\right)\left(\int_0^y \frac{ds}{s}\right), \quad \mathrm{Li}_2(x) = \int_0^x \frac{dt}{t}\int_0^t \frac{du}{1-u}\,. \tag{6.4}$$

The last function, Li_2, is a dilogarithm. It is the $q = 2$ special case of the polylogarithm Li_q. Starting with the familiar logarithm $\mathrm{Li}_1(x) = -\log(1-x)$, the polylogarithms are defined iteratively as

$$\mathrm{Li}_q(x) = \int_0^x dt\, \frac{\mathrm{Li}_{q-1}(t)}{t}\,. \tag{6.5}$$

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Such functions are usually referred to as ***transcendental functions***. They are associated with a ***degree of transcendentality***, given by the dimension of the integral involved. The three functions in (6.4) are of transcendentality 1, 2, and 2, while it is r for the polylog Li_r. Note that in this categorization, π has transcendentality 1, since $i\pi = \log(-1)$.

The analytic structure of loop amplitudes is clearly more complicated than for tree amplitudes: where tree amplitudes are simple rational functions with only simple poles, the loop-integrations typically give rise to generalized logarithms, as in (6.4), and other special functions. These have branch cuts in addition to poles. The well-understood pole structure of tree amplitudes was instrumental for developing the on-shell recursion relations (Chapter 3), so at first sight it looks challenging to develop a similar "on-shell" approach for loop amplitudes. Nonetheless, the analytic structure of the loop-integrands can be exploited to reconstruct the amplitude from lower-order on-shell data. The purpose of this and the following two chapters is to show how to do this.

In the present chapter, we begin with a brief description of the physics of loop-divergences. We then focus on the widely applicable and very successful unitarity method. Chapter 7 studies BCFW recursion for the loop-integrands and in Chapter 8 we discuss Leading Singularities and on-shell diagrams.

6.1 UV and IR divergences

If a loop-integral diverges as the loop-momentum becomes large, the loop-diagram is said to have an ***ultraviolet (UV) divergence***. At large loop-momentum, the denominators of the propagators behave as ℓ^2, so a given integrand behaves as

$$\int d^D\ell \frac{\ell^N}{(\ell^2)^P} \sim \int d|\ell| \frac{|\ell|^{N+D-1}}{|\ell|^{2P}}, \tag{6.6}$$

where D is the number of spacetime dimensions, P is the number of propagators, and N counts the powers of loop-momenta in the numerator. With $D + N - 2P = 0, 1, 2, \ldots$, the integral is logarithmically, linearly, quadratically divergent, etc. To regulate the UV behavior of the integral, one can introduce an explicit UV cutoff Λ, but it is often more convenient to analytically continue the spacetime to $4 - 2\epsilon$ dimensions, and use ϵ as a regularization parameter. This is ***dimensional regularization***, which is widely used in quantum field theory due to the appealing feature that it preserves gauge invariance.

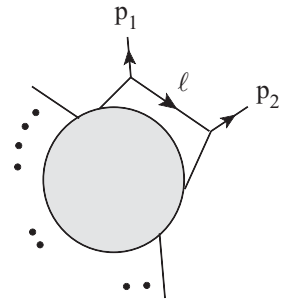
If the amplitude has UV divergences, this is telling us that, at short distances, the theory needs to be modified either by changing the constants in the original action or by adding new local operators to the action. The way in which the theory is modified depends crucially on the form of the divergence. Since the loop-integral has a definite mass dimension, the divergence must be multiplied by a kinematic invariant with the corresponding dimension. If this invariant is the tree amplitude, then we can simply modify the constants, such as overall coupling factors, of the tree amplitude to absorb this divergence. This modification can be translated into the modification of the coupling constants and masses in the Lagrangian description of the theory. If, on the other hand, the kinematic invariant in front of the

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loop-divergence is not proportional to the tree amplitude, then this means that one has to modify the theory by adding new operators to the action such that the divergences can be absorbed.

The procedure of removing UV-divergences from loop amplitudes is called ***renormalization***. If the UV-divergences of all loop amplitudes for a given theory can be removed by adding a *finite* number of terms to the action (including zero), then the theory is said to be ***renormalizable***. Renormalization is an extremely important subject in quantum field theory. It is crucial for making physical predictions from gauge theories and the techniques of renormalization are well-tested in the remarkable and successful comparisons of theoretical predictions with experimental results.

Another type of singularity is ***infrared (IR) divergences***. They can come from two sources: soft and collinear divergences. To facilitate the discussion, let us consider an integral with two adjacent massless legs:



. (6.7)

The relevant part of the integrand is

$$\int \frac{d^D\ell}{(\ell+p_1)^2\ell^2(\ell-p_2)^2}\,. \tag{6.8}$$

Let us consider light-cone coordinates

$$p^+ \equiv p^0 + p^1, \quad p^- \equiv p^0 - p^1, \quad \vec{p}_\perp \equiv (p^2, p^3, \ldots, p^{D-1})\,. \tag{6.9}$$

Then $p^2 = -p^+p^- + p_\perp^2$. We can always choose the coordinates in such a way that p_1 is in the p^- direction. To simplify the discussion, we choose p_2 to be in the p^+ direction; this is not general but the result of the discussion will be. The loop-integral is now (dropping overall factors)

$$\int \frac{d\ell^+\, d\ell^-\, d^{D-2}\ell_\perp}{(\ell^2 - 2\ell^+p_1^-)\,\ell^2\,(\ell^2 - 2\ell^-p_2^+)}\,. \tag{6.10}$$

Let us consider the integration region where the loop-momentum becomes soft, $\ell^\mu \to 0$. Practically, we scale the integral to this region by changing integration variable from ℓ^+ to t as

$$\ell^+ \to t\,, \qquad \ell^- \to t\,\ell^-\,, \qquad \vec{\ell}_\perp \to t\,\vec{\ell}_\perp\,. \tag{6.11}$$

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All terms in ℓ^2 scale homogeneously as t^2 which is subleading to $\ell.p_i$ for small t. Thus, for t near zero, the integral (6.10) gives

$$\int \frac{t^{D-1}\,dt}{t^4} = \int \frac{dt}{t^{5-D}}\,. \tag{6.12}$$

In $D = 4$, this integral is logarithmic divergent near $t = 0$.[1] We can regulate it by formally continuing the spacetime dimensions to $4 - 2\epsilon$, and one now has:

$$\int \frac{t^{3-2\epsilon}\,dt}{t^4} = \int \frac{dt}{t^{1+2\epsilon}} = \frac{1}{2\epsilon} + O\big(\epsilon^0\big)\,. \tag{6.13}$$

This divergence is called a ***soft divergence***, since it occurs when the loop-momentum is "softer" than the external legs.

Next, consider a limit in which the loop-momentum becomes collinear with p_1: we scale the components to align ℓ with the p^- direction,

$$\ell^+ \to t^2\,, \qquad \ell^- \to \ell^-\,, \qquad \vec{\ell}_\perp \to t\,\vec{\ell}_\perp\,. \tag{6.14}$$

Again, ℓ^2 scales as t^2, so for small t we have that $(\ell^2 - 2\ell^+ p_1^-)$ scales as t^2 and that $(\ell^2 - 2\ell^- p_2^+)$ goes as t^0 at leading order. Thus the overall behavior of the integral at small t is again (6.13), so it goes as $1/\epsilon$ in $4-2\epsilon$ dimensions. This is a ***collinear divergence***, it corresponds to the limit where the loop-momentum becomes collinear with one of the external momenta.

The soft divergences can occur simultaneously with collinear divergences, so in dimensional regularization $D = 4-2\epsilon$, the ℓ-integration of the diagram (6.7) behaves at leading order as $1/\epsilon^2$. For an L-loop amplitude, the leading IR divergences are usually of order $1/\epsilon^{2L}$. Soft and collinear limits for amplitudes in massless gauge theories have been studied since the late 1970s and this is an entire subject on its own; see for example the brief outline in [61] and references therein.

The appearance of IR divergences does not correspond to a need to modify our original theory. Rather, it is a consequence of a subtlety in defining scattering amplitudes for massless particles. To define a scattering amplitude, one must have a well-defined notion of asymptotic states, i.e. what exactly are the particles that we are scattering? If identical massless particles become collinear, it is difficult to distinguish them from a single massless particle: a single particle with definite light-like momentum is indistinguishable from hundreds of soft particles whose momenta are collinear and near zero. This ambiguity for asymptotic massless states is reflected in the IR divergences of loop amplitudes as well as in the singularities of the soft and collinear limits of the tree-level amplitude (Section 2.8).

The tree and loop soft and collinear singularities are intimately related, as can be seen from the computation of the scattering cross-section. The cross-section of a 4-point process for four gluons in pure Yang–Mills theory involves the norm-squared product of gluon

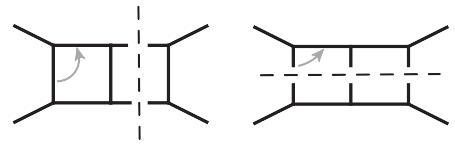
[1] It is also divergent for $t \to \infty$. But in that limit, our approximation where the loop-momentum is small compared to the external momenta is no longer valid.

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amplitudes, appropriately summed over states. Organizing the perturbative expansion in powers of the gauge coupling g, we find

$$\frac{d\sigma}{d\Omega} \propto \sum_i \langle g_1 g_2 | S | i \rangle \langle i | S | g_3 g_4 \rangle = \underbrace{\text{tree}\ \text{tree}}_{g^4} + \underbrace{\text{1–L}\ \text{tree}}_{g^6} + \underbrace{\text{tree}\ \text{tree}}_{g^6} + O(g^8) \,. \tag{6.15}$$

The sum $\sum_i$ represents the sum over all on-shell multi-particle states and S is the S-matrix. The dashed line in each diagram represents a sum over the particle species of the on-shell internal leg. The 4-gluon tree and 1-loop amplitudes are order g^2 and g^4, respectively, while the 5-gluon tree-amplitude is order g^3. Hence the product of the 4-gluon tree and 1-loop amplitudes contributes at the same order, namely g^6, as the product of two 5-gluon tree amplitudes. Now diagrammatically, one can see that the collinear divergence of the 1-loop 4-gluon amplitude in one diagram can indeed be regarded as the collinear divergence of the 5-point tree amplitude looked at in another way:



(6.16)

The collinear divergences of the 5-point tree amplitudes "conspire" to cancel the IR divergence of the 4-point 1-loop amplitudes to yield a finite *cross-section*. A simple toy model example of such cancellations can be found in [62, 63].

The preceding discussion illustrates that the IR divergences of loop amplitudes are intimately tied to the soft and collinear behavior of tree amplitudes. Historically, this has served as an inspirational starting point for the development of tree-level recursion relations [13] discussed in Section 3. Note that if a scattering process is absent at tree-level and is non-vanishing only at loop-level, then the leading-order result cannot have IR singularities; it would have no tree-level counterpart to cancel its IR divergences in the cross-section. For example, the all-minus or single-plus gluon amplitudes vanish at tree-level, as we have seen in (2.112), but in pure Yang–Mills theory they are non-vanishing at 1-loop order. It can be shown by explicit calculation that the results for these gluon amplitudes are free of IR divergences, and in fact are simple rational functions e.g. [64, 65]

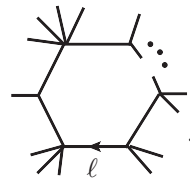
$$\text{pure Yang–Mills:} \qquad A_n^{\text{1-loop}}\left[1^- 2^- \ldots n^-\right] = \frac{\sum [i_1 i_2]\langle i_2 i_3\rangle [i_3 i_4]\langle i_4 i_1\rangle}{[12][23]\cdots[n1]} \,, \tag{6.17}$$

where the sum in the numerator is over $1 \le i_1 < i_2 < i_3 < i_4 \le n$ and we have dropped overall normalization factors.

It is useful for the following discussion to consider a set of basic ***scalar integrals*** and their integrated results in dimensional regularization. The 1-loop scalar integrals $I_m^{(i)}$ are

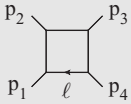
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the m-gon Feynman diagrams that appear in ϕ^n-theory:

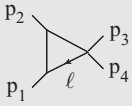


(6.18)

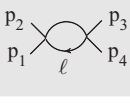
In the scalar integrals, the dependence on loop- and external momenta is contained solely in the propagators. We categorize 1-loop scalar integrals as ***bubbles***, ***triangles***, ***boxes***, etc. if they have 2, 3, 4, etc. propagators. These integrals are denoted by $I_{2;n}(K_i, K_j)$, $I_{3;n}(K_i, K_j, K_k)$, and $I_{4;n}(K_i, K_j, K_k, K_l)$, with the subscript indicating the number of corners of the diagrams as well as the number of external legs involved. K_i is the sum of external momenta at each corner. The 4-point box, triangle, and bubble scalar integrals are



$$= I_{4;4}(p_1, p_2, p_3, p_4) = \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^2(\ell - p_1)^2(\ell - p_1 - p_2)^2(\ell + p_4)^2},$$



$$= I_{3;4}(p_1, p_2, p_3 + p_4) = \int \frac{d^D\ell}{(2\pi)^4} \frac{1}{\ell^2(\ell - p_1)^2(\ell - p_1 - p_2)^2}, \quad (6.19)$$



$$= I_{2;4}(p_1 + p_2, p_3 + p_4) = \int \frac{d^D\ell}{(2\pi)^4} \frac{1}{\ell^2(\ell - p_1 - p_2)^2}.$$

We met the scalar box integral in (6.1). In $D = 4$, power counting shows that the box and triangle integrals $I_{4;n}$ and $I_{3;n}$ cannot have UV divergences, only IR divergences due to the massless corners. The exceptions are the cases in which all corners are massive, i.e. all $K_i^2 \neq 0$; such massive triangle and box integrals are IR finite. The bubble integrals $I_{2;n}$ have UV divergences, but no IR divergences because both corners are massive, $K_i^2 \neq 0$. The scalar bubble integral with a massless corner vanishes in dimensional regularization. This is because the integral has mass-dimension 2, yet there is no dimensionful quantity to give it dimension since the only invariant entering the integral is $K_i^2 = 0$.

Setting $D = 4 - 2\epsilon$, the integrated results of the 4-point box, triangle, and bubble integrals are

$$I_{4;4}(p_1, p_2, p_3, p_4) = \frac{\gamma_\Gamma}{su}\left(\frac{2}{\epsilon^2}\left[\left(-\mu^{-2}s\right)^{-\epsilon} + \left(-\mu^{-2}u\right)^{-\epsilon}\right] - \ln^2\left(\frac{s}{u}\right) - \pi^2\right) + O(\epsilon),$$

$$I_{3;4}(p_1, p_2, p_3 + p_4) = \frac{\gamma_\Gamma}{\epsilon^2}(-\mu^{-2}s)^{-1-\epsilon} + O(\epsilon), \quad (6.20)$$

$$I_{2;4}(p_1 + p_2, p_3 + p_4) = \gamma_\Gamma\left(\frac{1}{\epsilon} - \ln(-\mu^{-2}s) + 2\right) + O(\epsilon).$$

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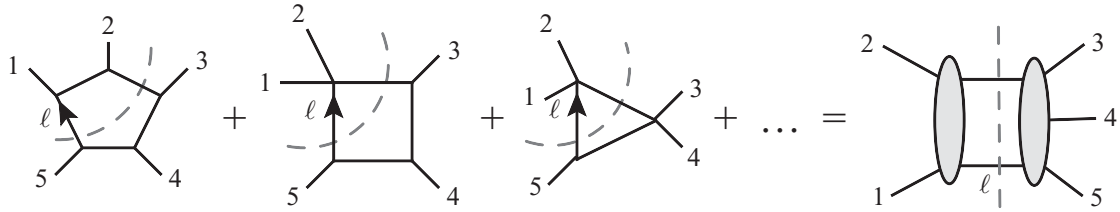


Fig. 6.1 The sum of residues from all Feynman diagrams with propagators ℓ^2 and $(\ell - p_1 - p_2)^2$ on-shell must give the product of two tree amplitudes.

Here, $\gamma_\Gamma = \Gamma(1+\epsilon)\Gamma^2(1-\epsilon)/\,\Gamma(1-2\epsilon)$ and μ is the regularization scale. These results and their n-point generalizations can be found in [66]. The 1-loop box and triangle integrals are $1/\epsilon^2$, as expected from the combination of the soft and collinear divergences. The $1/\epsilon$ divergence in the bubble integral is a UV divergence.

The scalar box, triangle, and bubble integrals provide a basis for 1-loop amplitudes in $D=4$ dimensions. We will see how this works next when we study the unitarity method.

6.2 Unitarity method

We begin with an example. Consider the color-ordered planar n-point 1-loop gluon amplitude in pure Yang–Mills theory. Suppose we identify[2] the loop-momentum ℓ such that in each Feynman diagram, ℓ is the momentum that flows between legs 1 and n, as indicated for $n = 5$ in Figure 6.1. Then we can unambiguously collect all the distinct Feynman diagrams under one integral,

$$\int d^D\ell \sum_j J_j \,. \tag{6.21}$$

The integrands J_j take the form indicated in (6.3). To compute the full amplitude we need to integrate ℓ over $\mathbb{R}^4$ (after Wick rotation from $\mathbb{R}^{1,3}$), but let us focus on the subplane where the loop-momentum satisfies the two ***cut conditions***

$$\ell^2 \;=\; (\ell - p_1 - p_2)^2 \;=\; 0\,. \tag{6.22}$$

On this subplane, integrands of the form

$$J_j = \frac{1}{S_j}\frac{c_j n_j}{\dots(\ell^2)\dots(\ell - p_1 - p_2)^2\dots} \tag{6.23}$$

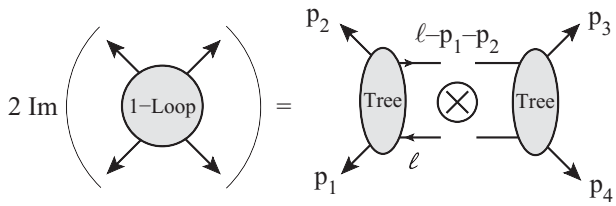
become singular. The singularity corresponds to a kinematic configuration where the two propagators, $1/\ell^2$ and $1/(\ell - p_1 - p_2)^2$, go on-shell. The sum of the residues from all such integrands must be equivalent to the product of two *on-shell* tree amplitudes, as shown schematically in Figure 6.1.

[2] More about this choice in Section 7.1.

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This observation is very useful for constraining the loop-integrand: if the enemy gives us a rational function and claims that it is the 1-loop integrand of some (unitary) theory, we can apply cuts, such as (6.22), and test the claim by checking if the function factorizes correctly into products of tree amplitudes. This way, our knowledge of tree amplitudes can be recycled into information about the loop-integrand!

The operation of taking loop propagators on-shell is called a ***unitarity cut***. It originates from the unitary constraint of the S-matrix. To see how, recall that unitarity requires $S^\dagger S = 1$. Writing $S = 1 + iT$, where T represents the interacting part of the S-matrix, unitarity requires $-i(T - T^\dagger) = T^\dagger T$. If we examine this constraint order by order in perturbation theory, it tells us that the imaginary part of the T-matrix at a given order is related to the product of lower-order results. In particular, the imaginary part of the 1-loop amplitude is given by a product of two tree amplitudes. This is illustrated by the diagram



(6.24)

The product of two tree amplitudes on the RHS involves a sum over all possible on-shell states that can "cross" the cut. Only states from the physical spectrum of the theory are included in this sum. In gauge theory, Feynman diagram calculations of loop amplitudes require ghosts in the loops: the purpose of the ghosts is to cancel unphysical modes in the loops. In contrast, in a unitarity cut (6.24) we restrict the loop-momenta to be on-shell and only physical modes are included in the two on-shell amplitudes on the RHS of (6.24).

The cutting rules also include integrals of any remaining freedom in the loop-momentum after imposing the cut constraints, such as (6.22), and momentum conservation. The integral over all allowed kinematic configurations, with respect to the diagram (6.24), can be written as

$$\int d^D\ell\ \delta_+\big(\ell^2\big)\,\delta_+\big((\ell - p_1 - p_2)^2\big)\,. \tag{6.25}$$

The subscript $_+$ means that we are choosing the solution to the on-shell condition that has a positive time component, $\ell^0 > 0$, i.e. it is associated with a particle (as opposed to an anti-particle) crossing the cut. Note that (6.25) just replaces the two cut propagators with their on-shell conditions, exactly as we did in Figure 6.1.

The imaginary part of the amplitude probes its branch cut structure, hence the unitarity cut allows us to relate the "pole structure" of the integrand with the "branch cut structure" of the loop-integral. One can reconstruct the integrand by analyzing different sets of unitarity cuts. The unitarity cuts can also involve more than two "cut" lines, i.e. several internal lines taken on-shell, such that the loop amplitude factorizes into multiple lower-loop on-shell amplitudes. An N-line cut simply means that N internal lines are taken on-shell. Reconstruction of the full loop amplitude from systematic application of unitarity cuts is called the ***generalized unitarity method*** [67]. It has been applied to a wide range of scattering

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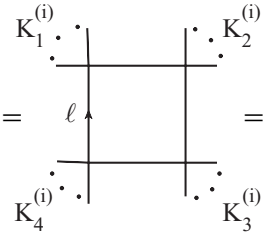
problems, from next-to-leading order precision QCD predictions to the ultraviolet behavior of perturbative supergravity theories. The generalized unitarity method [67] deserves much more attention than we are able to offer here. Our introduction to the unitarity method covers just the minimum needed for you to see the idea and appreciate its power. The method has been reviewed extensively and you can learn more about it and its applications to supersymmetric as well as non-supersymmetric theories in the reviews [68–71]. Next, we discuss its implementation at 1-loop level.

6.3 1-loop amplitudes from unitarity

The information from unitarity cuts can be utilized most efficiently if we know, a priori, a complete basis of integrals that can appear in the scattering amplitudes. It can be shown [72–76] that in D-dimensions all 1-loop amplitudes can be rewritten as a sum of m-gon 1-loop scalar integrals I_m for $m = 2, 3, \ldots, D$:

$$A^{\text{1-loop}} = \sum_i C_D^{(i)} I_D^{(i)} + \sum_j C_{D-1}^{(j)} I_{D-1}^{(j)} + \cdots + \sum_k C_2^{(k)} I_2^{(k)} + \text{rational terms}\,. \tag{6.26}$$

Here, $C_m^{(i)}$ are kinematics-dependent coefficients for the m-gon scalar integrals $I_m^{(i)}$ that were introduced towards the end of Section 6.1. The 4-point box, triangle, and bubble scalar integrals (6.19) were evaluated in (6.20). The n-point box integral $I_4^{(i)}$ takes the form (henceforth dropping the subscript n)

$I_4^{(i)}$ = 

$$= \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^2\big(\ell - K_1^{(i)}\big)^2\big(\ell - K_1^{(i)} - K_2^{(i)}\big)^2\big(\ell + K_4^{(i)}\big)^2}\,, \tag{6.27}$$

where $(K_1^{(i)}, K_2^{(i)}, K_3^{(i)}, K_4^{(i)})$ are sums of the external momenta at each corner. The label i indicates a particular choice of distributing the external lines on the four subamplitudes, i.e. different 4-line cuts.

It may seem surprising that all 1-loop integrals can be written in terms of a basis of scalar 1-loop integrals as in (6.26), since integrals obtained from Feynman diagrams generally involve numerators that depend on loop-momentum. The simplification is that one can use the external momenta to form a basis for any vectors in the integrals. For example, a 4-point 1-loop "vector integral" can be expanded as

$$\int d^D\ell \frac{\ell^\mu}{\ell^2\big(\ell - p_1\big)^2\big(\ell - p_1 - p_2\big)^2\big(\ell + p_4\big)^2} = a_1 p_1^\mu + a_2 p_2^\mu + a_3 p_3^\mu\,. \tag{6.28}$$

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Here we are simply using the fact that the integral must be a Lorentz vector, and since by momentum conservation there are only three linearly independent momentum vectors in the 4-particle input, the answer must be a linear combination thereof. The coefficients in (6.28) are solved using the ***Passarino–Veltman reduction*** procedure [76]. The idea is simply to dot in various combinations of external momenta and simplify the result. For example, dotting p_1 into (6.28), the numerator of the integral becomes

$$(\ell \cdot p_1) = \frac{1}{2}\left[(\ell + p_1)^2 - \ell^2\right]. \tag{6.29}$$

The two terms of the RHS each cancel a loop-propagator, leaving behind two scalar integrals. Then (6.28) gives

$$\int d^D\ell \left[\frac{1}{2\ell^2(\ell + p_1 + p_2)^2(\ell - p_4)^2} - \frac{1}{2(\ell + p_1)^2(\ell + p_1 + p_2)^2(\ell - p_4)^2}\right] = -\frac{1}{2}(a_2 s + a_3 t). \tag{6.30}$$

Repeating the above procedure, one can obtain three linear equations for a_1, a_2, and a_3. Solving this linear system is straightforward and gives the vector integral (6.28) as a linear combination of scalar integrals.

▶ **Exercise 6.1**

Only scalar triangle integrals appear on the LHS of (6.30). Show that scalar box integrals appear if we contract (6.28) with p_2.

In general, since there are only D independent vectors in D-dimensions, the set of needed scalar integrals can be reduced to the set of scalar integrals shown in (6.26). Tensor-integrals can be simplified similarly. A more detailed discussion of integral reductions can be found in Section 4.2 of [77], also see [3].

The basis expansion (6.26) makes the task of computing the 1-loop amplitude a matter of determining the coefficients $C_m^{(i)}$. Since the scalar integrals have distinct propagator structures, only a subset contributes to a given unitarity cut. By applying multiple unitarity cuts, one obtains a set of linear equations that relate the $C_m^{(i)}$s to the results of the cuts, each computed as a product of tree amplitudes. Solving these linear equations gives us the coefficients $C_m^{(i)}$s as a combination of products of tree amplitudes. By (6.26), this determines the 1-loop amplitude, up to the possibility of rational terms that we discuss below.

The integral coefficients $C_m^{(i)}$ can be found systematically by organizing the calculation according to the number of propagators present in the scalar integrals $I_m^{(i)}$. In D-dimensions, the loop-momentum ℓ has D components, so one can find isolated solutions for ℓ such that all propagators in the scalar integral $I_D^{(i)}$ are on-shell: this corresponds to a D-line cut and this is the maximally possible cut in D dimensions. It is called a ***maximal cut*** and is discussed in much more detail in Chapter 8.

Since the cut constraints are quadratic in loop-momentum, there are two isolated solutions, denoted $\ell^{*(1)}$ and $\ell^{*(2)}$, to a maximal cut. The corresponding coefficient $C_D^{(i)}$ is

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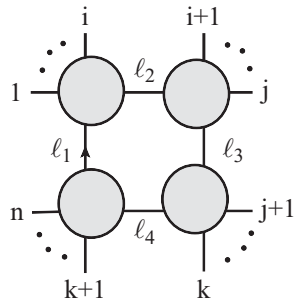


Fig. 6.2 1-loop box diagram with $K_1^{(i)} = p_1 + \cdots + p_i$, $K_2^{(i)} = p_{i+1} + \cdots + p_j$, $K_3^{(i)} = p_{j+1} + \cdots + p_k$ and $K_4^{(i)} = p_{k+1} + \cdots + p_n$. The corresponding box coefficient $C_4^{(i)}$ in (6.32) is the product of the four tree amplitudes at each corner.

completely determined by the product of D tree amplitudes as

$$C_D^{(i)} = \frac{1}{2} \sum_{\ell = \ell^{*(1)}, \ell^{*(2)}} A_{n_1}^{\text{tree}} \cdots A_{n_D}^{\text{tree}} . \tag{6.31}$$

Note that one averages over the two solutions, $\ell^{*(1)}$ and $\ell^{*(2)}$. At 1-loop, the relative weight between the two solutions can be determined by considering special integrands that integrate to zero. The associated maximal cut must also vanish and this fixes the above prescription. See [78] for a concise discussion.

▷ *Example.* Let us make (6.31) concrete. For an n-point 1-loop amplitude in $D = 4$, the coefficient of the box integral shown in Figure 6.2 is given by

$$C_4^{(\text{Fig. 6.2})} = \frac{1}{2} \sum_{\ell = \ell^{*(1)}, \ell^{*(2)}} \Bigg[\sum_{\text{states}} A_{n_1}\big[-\ell_1, 1, \ldots, i, \ell_2\big] \times A_{n_2}\big[-\ell_2, i+1, \ldots, j, \ell_3\big] \times A_{n_3}\big[-\ell_3, j+1, \ldots, k, \ell_4\big] \times A_{n_4}\big[-\ell_4, k+1, \ldots, n, \ell_1\big] \Bigg], \tag{6.32}$$

where $\sum_{\text{states}}$ indicates a state sum for each internal line $\ell_1 = \ell$, $\ell_2 = \ell - (p_1 + \cdots + p_i)$, $\ell_3 = \ell - (p_1 + \cdots + p_j)$, and $\ell_4 = \ell + (p_{k+1} + \cdots + p_n)$. The vectors $\ell^{*(1)}$ and $\ell^{*(2)}$ solve the on-shell conditions $\ell_1^2 = \ell_2^2 = \ell_3^2 = \ell_4^2 = 0$ of the 4-line cut. ◁

Coefficients $C_m^{(i)}$ with $m < D$ are not quite as simple to calculate, but they can be obtained systematically. After determining all D-gon coefficients, we treat $(D-1)$-cuts. Both $I_D^{(i)}$ and $I_{D-1}^{(j)}$ integrals can potentially contribute to $(D-1)$-cuts, but since we have already determined all the $C_D^{(i)}$s, we can unambiguously determine all $C_{D-1}^{(j)}$s. Similarly, all integral coefficients can be determined iteratively. This way, the generalized unitarity method offers a systematic way to determine the 1-loop amplitude in terms of tree amplitudes. Detailed discussions of extracting 1-loop integral coefficients in $D = 4$ can be found in [47, 79].

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▷ *Example.* As a concrete example, the integral coefficients for the fermion-loop correction of the 1-loop 4-gluon amplitude $A_4[1^+, 2^-, 3^+, 4^-]$ are

$$\begin{aligned} C_4(p_1, p_2, p_3, p_4) &= A_4^{\text{tree}} \frac{-u^2 s^2 (u^2 + s^2)}{t^4}, \\ C_3(p_1, p_2 + p_3, p_4) &= A_4^{\text{tree}} \frac{us(u^3 + us^2)}{t^4}, \\ C_2(p_1 + p_4, p_2 + p_3) &= A_4^{\text{tree}} \frac{2u(5s^2 + stu + 2u^2)}{6t^3}, \end{aligned} \tag{6.33}$$

where the integral coefficients are labeled in the same way as the scalar integrals in (6.19). All other integral coefficients are related to the above by cyclic permutation. ◁

▶ Exercise 6.2

Use the integrated results in (6.20) for the 4-point box, triangle, and bubble integrals, and the above integral coefficients to compute the final result for the fermion loop-correction to the 1-loop 4-gluon amplitude $A_4[1^+, 2^-, 3^+, 4^-]$. Show that there are only UV and no IR divergences. Explain why. What does the UV divergence indicate about renormalization of the theory at 1-loop level?

Of course, there is a big elephant in the room – you already met it in (6.26): it is the ***rational terms***. Rational terms are rational functions that do not possess branch cuts, so they are undetectable by unitarity cuts. The rational terms arise from the need to regularize the loop integrals. In dimensional regularization, the loop-momentum ℓ is really $(D-2\epsilon)$-dimensional. If we separate the loop-momentum into a D-dimensional part, $\ell^{(D)}$, and a (-2ϵ)-dimensional part, $\mu^{-2\epsilon}$, there can also be contributions from the μ-integrals. An example of an integrand that gives a branch-cut-free contribution is the $(D/2+1)$-gon scalar integral with μ^2 numerator: it integrates to a finite value

$$\int \frac{d\ell^{(D)} d\mu^{-2\epsilon}}{(2\pi)^{D-2\epsilon}} I_{D/2+1}[\mu^2] = -\frac{1}{(4\pi)^{D/2}} \frac{1}{(D/2)!} + O(\epsilon). \tag{6.34}$$

One cannot capture this from the ordinary unitarity cut since it is just a rational function (here, a constant). The unitarity cut forces the loop-momentum to be on-shell in D-dimensions and this implies $\mu^2 = 0$, so the contribution from the above integrand vanishes. On the other hand, if one considers unitarity cuts where the internal lines become massless in $D - 2\epsilon$ dimensions $(\ell^{(D)})^2 + \mu^2 = 0$, or equivalently *massive* in D-dimensions with mass $m^2 = \mu^2$, such terms *are* detectable. Thus rational terms can be reconstructed from unitarity cuts if we allow the states crossing the cut to be massive.

Rational terms are *absent* for supersymmetric Yang–Mills theories because supersymmetry cancellations ensure that the powers of loop-momentum in the integrals do not lead to rational terms after integral reduction. For non-supersymmetric theories – for example $\lambda\phi^4$, QED, QCD – rational terms are present and they are the most time-consuming ones to compute. We will not discuss this important issue in further detail, but simply refer you to

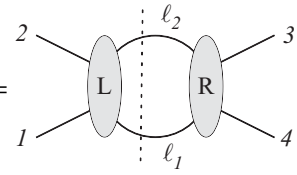
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[80, 81] and references therein. Instead, we illustrate the unitarity method by working out an explicit example.

▷ *Example.* In this example, we calculate the **4-point 1-loop superamplitude in $\mathcal{N}=4$ SYM** using the generalized unitarity method. We do so by first considering a 2-line cut and then infer from it which terms contribute in the integral basis expansion (6.26).

The s-channel unitarity cut is

$$\text{Cut}_s = \quad = \sum_{\text{states}} \mathcal{A}_4[-\ell_1, 1, 2, \ell_2] \times \mathcal{A}_4[-\ell_2, 3, 4, \ell_1]\,. \tag{6.35}$$



Using the analytic continuation (3.17), the tree-level MHV superamplitudes are

$$\begin{aligned}\mathcal{A}_4[-\ell_1, 1, 2, \ell_2] &= \frac{\delta^{(8)}(L)}{\langle \ell_1 1\rangle\langle 12\rangle\langle 2\ell_2\rangle\langle \ell_2\ell_1\rangle}\,,\\ \mathcal{A}_4[-\ell_2, 3, 4, \ell_1] &= \frac{\delta^{(8)}(R)}{\langle \ell_2 3\rangle\langle 34\rangle\langle 4\ell_1\rangle\langle \ell_1\ell_2\rangle}\,.\end{aligned} \tag{6.36}$$

The arguments of the Grassmann delta functions are $L = -|\ell_1\rangle\eta_{\ell_1} + |1\rangle\eta_1 + |2\rangle\eta_2 + |\ell_2\rangle\eta_{\ell_2}$, and $R = -|\ell_2\rangle\eta_{\ell_2} + |3\rangle\eta_3 + |4\rangle\eta_4 + |\ell_1\rangle\eta_{\ell_1}$. As in the case of tree-level recursion (see the discussion around (4.55)), the intermediate state sum is performed as an integration of the on-shell Grassmann variables η_{ℓ_i} associated with each internal line [36]. These integrals are easy to perform when we use $\delta^{(8)}(L)\,\delta^{(8)}(R) = \delta^{(8)}(L+R)\,\delta^{(8)}(R) = \delta^{(8)}(\tilde{Q})\,\delta^{(8)}(R)$. We find

$$\begin{aligned}\text{Cut}_s &= \frac{\delta^{(8)}(\tilde{Q})}{\langle 12\rangle\langle 34\rangle}\int d^4\eta_{\ell_1} d^4\eta_{\ell_2}\,\frac{\delta^{(8)}(R)}{\langle \ell_1 1\rangle\langle 2\ell_2\rangle\langle \ell_2\ell_1\rangle\langle \ell_2 3\rangle\langle 4\ell_1\rangle\langle \ell_1\ell_2\rangle}\\ &= -\frac{\delta^{(8)}(\tilde{Q})}{\langle 12\rangle\langle 34\rangle}\,\frac{\langle \ell_1\ell_2\rangle^2}{\langle \ell_1 1\rangle\langle 2\ell_2\rangle\langle \ell_2 3\rangle\langle 4\ell_1\rangle}\,.\end{aligned} \tag{6.37}$$

On the unitarity cut, one can convert the loop-momentum part of the denominator in the above expression into propagators:

$$\text{Cut}_s = \mathcal{A}_4^{\text{tree}}[1234] \times \left.\frac{-su}{(\ell_2+p_2)^2(\ell_1+p_4)^2}\right|_{\ell_1^2=\ell_2^2=0}\,. \tag{6.38}$$

► **Exercise 6.3**

Show that (6.38) follows from (6.37).

Let us now consider the possible integrals from (6.26) that contribute to the s-channel cut: the integrals that contain the propagators with $\ell_1^2 = \ell^2$ and $\ell_2^2 = (\ell - p_1 - p_2)^2$ are the box-integral $I_4(p_1, p_2, p_3, p_4)$, the triangle-integrals $I_3(p_1, p_2, p_3 + p_4)$, and $I_3(p_3, p_4, p_1 + p_2)$, and the bubble-integrals $I_2(p_1 + p_2, p_3 + p_4)$ and $I_2(p_2 + p_3, p_4 + p_1)$. In each case, we have indicated the distribution of the external lines. The

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result (6.38) for Cut_s shows that there are two uncut propagators left after cutting ℓ_1 and ℓ_2, so this excludes the triangle- and bubble-integrals. Thus we conclude that only the box-integral is present, i.e. $\mathcal{A}_4^{\text{1-loop}}[1234] = C_4\, I_4(p_1, p_2, p_3, p_4)$. The box-coefficient C_4 is readily determined from (6.38), giving[3]

$$\mathcal{A}_4^{\text{1-loop}}[1234] = su\, \mathcal{A}_4^{\text{tree}}[1234]\, I_4(p_1, p_2, p_3, p_4)\,. \tag{6.39}$$

To make sure that (6.39) is the correct result for the amplitude, we examine other distinct cuts to see if there could be terms that vanish in the s-channel cut and were therefore not captured in our analysis. The only other available cut is the u-channel cut. (Color-ordering excludes the t-channel cut.) But since the RHS of (6.39) is invariant under cyclic permutations, it is guaranteed to produce the correct u-channel cut. Hence (6.39) is indeed the correct 1-loop 4-point amplitude for $\mathcal{N} = 4$ SYM. We discuss the evaluation of I_4 in dimensional regularization in Section 6.4. ◁

Working through the details in the example above, you will notice that the unitarity method does take some work and you may wonder how it compares with a brute-force 1-loop Feynman diagram calculation. The answer is that the unitarity method is superior, since it heavily reduces the number of diagrams needed and it avoids gauge obscurities. For the unitarity method, the input is gauge-invariant on-shell amplitudes. You might find it curious that the first computation of the 1-loop 4-gluon amplitude $A_4^{\text{1-loop}}$ in $\mathcal{N} = 4$ SYM was not done in QFT, but in string theory: in 1982, Green, Schwarz, and Brink [82] obtained $A_4^{\text{1-loop}}$ as the low-energy limit of the superstring scattering amplitude with four external gluon states.

We close this section with some general comments on the 1-loop integral expansion (6.26). The representation of 1-loop amplitudes in terms of scalar integrals provides an interesting categorization scheme in terms of whether or not particular classes of integrals – in 4d: boxes, triangles, bubbles, and rationals – appear or not. For example, we mentioned earlier that rational terms are absent in $\mathcal{N} > 0$ super Yang–Mills theory. One can ask which theories involve only box-integrals, i.e. no triangle- or bubble-integrals and no rational terms. In 4d, such so-called "no-triangle" theories include $\mathcal{N} = 4$ SYM [67], $\mathcal{N} = 8$ supergravity [83, 84], and $\mathcal{N} = 2$ SYM coupled to specific tensor matter fields [85]. For pure $\mathcal{N} = 6$ supergravity, only box- and triangle-integrals appear [86], while for pure $\mathcal{N} \leq 4$ supergravity all integrals in (6.26), including rational terms, appear.

One relevant aspect of the above analysis is that in 4d only the bubble integrals $I_2^{(i)}$ contain ultraviolet divergences. In dimensional regularization, all bubble integrals have a common leading $1/\epsilon$-term, and hence contribute $\frac{1}{\epsilon}\sum_i C_2^{(i)}$ to the amplitude. As a result, the beta function for a given theory vanishes at 1-loop order precisely when $\sum_i C_2^{(i)} = 0$. In fact, in a renormalizable theory, one must have $\sum_i C_2^{(i)} \sim A_n^{\text{tree}}$ and the proportionality constant is related to the 1-loop beta function [47, 85, 87, 88]. Note that even though bubble coefficients are non-trivial for pure $\mathcal{N} \leq 4$ supergravity theories, their sum $\sum_i C_2^{(i)}$ must vanish since the theory is known to be free of ultraviolet divergences at 1-loop order.

Next, we offer a quick survey of results for loop amplitudes in planar $\mathcal{N} = 4$ SYM.

[3] The minus sign compared with (6.38) comes from the $(-i)^2$ in the cut propagators.

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6.4 1-loop amplitudes in planar $\mathcal{N} = 4$ SYM

When we introduced the $\mathcal{N} = 4$ SYM theory in Section 4.4, we mentioned that this is a conformal theory, there is no running of the coupling. This means that there are no UV divergences in the on-shell scattering amplitudes at any loop-order. At 1-loop order, this can be seen from the fact that in $\mathcal{N} = 4$ SYM only the box-integrals contribute in the scalar integral basis expansion (6.26), i.e.

$$\mathcal{N} = 4 \text{ SYM in 4d:} \qquad A_n^{\text{1-loop}} = \sum_i C_{\text{box}}^{(i)} I_{\text{box}}^{(i)} \,. \tag{6.40}$$

Only bubble-integrals give UV divergences at 1-loop in 4d, so it is clear that the 1-loop amplitudes in $\mathcal{N} = 4$ SYM are UV finite. Loop amplitudes in $\mathcal{N} = 4$ SYM theory do have IR divergences and we will see more of them in the following.

We used the unitarity method in Section 6.3 to construct the 1-loop 4-point superamplitude in planar $\mathcal{N} = 4$ SYM. We found that the answer (6.39) could be written in terms of a single scalar box-integral:

$$\mathcal{A}_4^{\text{1-loop}}[1234] = su\, \mathcal{A}_4^{\text{tree}}[1234]\, I_4(p_1, p_2, p_3, p_4) \,. \tag{6.41}$$

The scalar box integral I_4, evaluated in dimensional regularization $D = 4 - 2\epsilon$, was given in (6.20). It follows that

$$\mathcal{A}_4^{\text{1-loop}}[1234] = \mathcal{A}_4^{\text{tree}}[1234] \left\{ \frac{2}{\epsilon^2} \Big[\big(-\mu^{-2} y_{13}^2 \big)^{-\epsilon} + \big(-\mu^{-2} y_{24}^2 \big)^{-\epsilon} \Big] - \ln^2 \left(\frac{y_{13}^2}{y_{24}^2} \right) - \pi^2 + O(\epsilon) \right\} , \tag{6.42}$$

where μ is the regularization scale and $y_{ij} = y_i - y_j$ are the zone-variables defined in (5.31). In terms of Mandelstam variables, we have $s = -y_{13}^2$ and $u = -y_{24}^2$.

Let us now use the notation

$$\mathcal{A}_{n;L}^{\text{N}^K\text{MHV}} = n\text{-point } L\text{-loop N}^K\text{MHV superamplitude of planar } \mathcal{N} = 4 \text{ SYM} \,, \tag{6.43}$$

with the color-ordering $12 \ldots n$ of external particles implicit. Since supersymmetry and $SU(4)$ R-symmetry[4] Ward identities hold at each loop-order, this decomposition of the loop amplitude is sensible and $\mathcal{A}_{n;L}^{\text{N}^K\text{MHV}}$ has Grassmann degree $4(K + 2)$.

It is convenient to factor out an MHV tree-level superamplitude and write the loop-expansion as

$$\mathcal{A}_{n;L}^{\text{N}^K\text{MHV}}(\epsilon) = \mathcal{A}_{n;0}^{\text{MHV}} \Big(\mathcal{P}_{n;0}^{\text{N}^K\text{MHV}} + \lambda\, \mathcal{P}_{n;1}^{\text{N}^K\text{MHV}}(\epsilon) + \cdots \Big) \,, \tag{6.44}$$

[4] The $SU(4)$ R-symmetry is non-anomalous [89, 90].

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where $\lambda \sim g^2 N$ is the t'Hooft coupling written in terms of the gauge coupling g and the rank of the gauge group $SU(N)$. We include the dependence on the ϵ-regulator explicitly in the loop amplitudes. At tree-level, we have

$$\mathcal{P}^{\text{MHV}}_{n;0} = 1 \quad \text{and} \quad \mathcal{P}^{\text{NMHV}}_{n;0} = \sum_{j=3}^{n-2} \sum_{k=j+2}^{n} R_{1jk}\,. \tag{6.45}$$

The NMHV result is given in terms of the dual superconformal invariants R_{1jk} defined in (4.83) and discussed further in Section 5.4.

▶ **Exercise 6.4**

Why is it possible to factor out $\mathcal{A}^{\text{MHV}}_{n;0}$ even at loop-level?

The ϵ-regulator explicitly breaks the conformal and dual conformal symmetry. You can see that in the expression (6.42) for the 1-loop 4-point superamplitude: not even the finite part $O(\epsilon^0)$ respects dual conformal inversion (5.37):

$$I(y_{ij}^2) = \frac{y_{ij}^2}{y_i^2 y_j^2}\,. \tag{6.46}$$

So the raw output of the loop amplitudes does not entertain the ordinary or dual conformal symmetries of the $\mathcal{N}=4$ SYM theory. However, the IR divergences take a universal form that facilitates construction of IR-finite quantities that turn out to respect the symmetries.

At 1-loop order, the IR divergent part of $\mathcal{A}^{\text{N}^K\text{MHV}}_{n;1}$ is captured entirely by the MHV superamplitude in the sense that

$$\mathcal{A}^{\text{N}^K\text{MHV}}_{n;1}(\epsilon) = \mathcal{A}^{\text{N}^K\text{MHV}}_{n;0} \times \text{IRdiv}\big[\mathcal{P}^{\text{MHV}}_{n;1}(\epsilon)\big] + O(\epsilon^0)\,, \tag{6.47}$$

where

$$\text{IRdiv}\big[\mathcal{P}^{\text{MHV}}_{n;1}(\epsilon)\big] = \frac{1}{\epsilon^2}\sum_{i=1}^{n}(-\mu^{-2}\, y_{i,i+2}^2)^{-\epsilon}\,. \tag{6.48}$$

Note that for $n=4$, this reproduces the IR divergent terms in (6.42). The *finite* part of the 4-point MHV superamplitude is

$$\mathcal{F}^{\text{MHV}}_{4;1}(\epsilon) \equiv \mathcal{P}^{\text{MHV}}_{4;1}(\epsilon) - \text{IRdiv}\big[\mathcal{P}^{\text{MHV}}_{4;1}(\epsilon)\big] = -\ln^2\left(\frac{y_{13}^2}{y_{24}^2}\right) - \pi^2 + O(\epsilon)\,. \tag{6.49}$$

The universal form (6.47) of the 1-loop IR divergences implies that the ***ratio functions***[5]

$$\mathcal{R}^{\text{N}^K\text{MHV}}_{n;1}(\epsilon) \equiv \mathcal{P}^{\text{N}^K\text{MHV}}_{n;1}(\epsilon) - \mathcal{P}^{\text{N}^K\text{MHV}}_{n;0}\,\mathcal{P}^{\text{MHV}}_{n;1}(\epsilon) \tag{6.50}$$

[5] The RHS of (6.50) can be viewed as the $O(\lambda)$ term in the small λ expansion of the ratio $\mathcal{A}^{\text{N}^K\text{MHV}}_n/\mathcal{A}^{\text{MHV}}_n$.

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are IR finite. Moreover, it has been proposed [50, 91] that $\mathcal{R}_{n;1}^{\mathrm{N}^K\mathrm{MHV}}(0)$s are actually dual conformal invariant. This was shown at NMHV level for $n \leq 9$ in [50, 91] and for general n in [92, 93] using generalized unitarity. To give you a sense of the expressions, we present the result [91] for the ratio function for the 6-point 1-loop NMHV superamplitude. It is

$$\mathcal{R}_{6;1}^{\mathrm{NMHV}}(0) = \frac{1}{2}\Big(R_{146}\, V_{146} + \text{cyclic}\Big)\,, \tag{6.51}$$

where

$$V_{146} = -\log u_1\, \log u_2 + \frac{1}{2}\sum_{k=1}^{3}\Big[\log u_k\, \log u_{k+1} + \mathrm{Li}_2(1-u_k)\Big] - \frac{\pi^2}{6}\,. \tag{6.52}$$

The dilogarithm Li_2 was defined in (6.4). The u_is are dual conformal cross-ratios,

$$u_1 = \frac{y_{13}^2 y_{46}^2}{y_{14}^2 y_{36}^2}\,, \quad u_2 = \frac{y_{24}^2 y_{51}^2}{y_{25}^2 y_{41}^2}\,, \quad u_3 = \frac{y_{35}^2 y_{62}^2}{y_{36}^2 y_{52}^2}\,, \tag{6.53}$$

so each V_{ijk} is a dual conformal invariant, as you can see by applying dual inversion (6.46). The "+ cyclic" in (6.51) is the instruction to sum over the cyclic sum of the external state labels; note that V_{251} is just V_{146} with $u_1 \to u_2 \to u_3 \to u_1$.

Recalling that the BCFW recursion relations for the tree-level 6-point NMHV superamplitude only have three terms, you might be surprised to see six terms in the 1-loop result (6.51). However, in Exercise 5.15 we used the cyclically invariant 5-brackets $[i, j-1, j, k-1, k] = R_{ijk}$ to rewrite the tree-level superamplitude as $\mathcal{A}_{6;0}^{\mathrm{NMHV}}/\mathcal{A}_{6;0}^{\mathrm{MHV}} = \mathcal{P}_{6;0}^{\mathrm{NMHV}} = \frac{1}{2}(R_{146} + \text{cyclic})$. This was done in anticipation of the 1-loop ratio function (6.51), and now you see that (6.51) is just like the tree-level result but with each R_{ijk} dressed with a dual conformal invariant V_{ijk}. Adding loop-orders 0 and 1, we can therefore write 6-point ratio function

$$\mathcal{R}_{6}^{\mathrm{NMHV}}(0) = \frac{1}{2}\Big(R_{146}\big(1 + \lambda\, V_{146}\big) + \text{cyclic}\Big) + O(\lambda^2)\,. \tag{6.54}$$

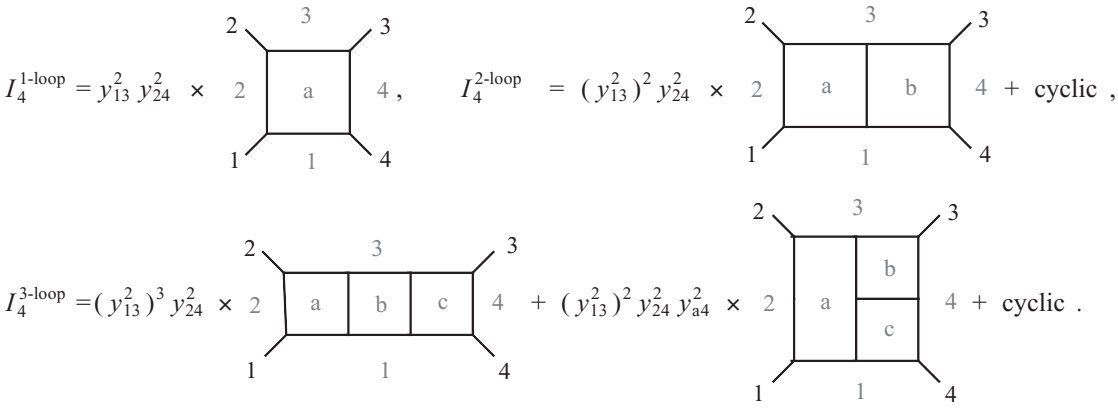
There are two properties worth noting about the 1-loop ratio function $\mathcal{R}_{6;1}^{\mathrm{NMHV}}(0)$:

- It is dual conformal invariant, but not dual *super*conformal invariant. For a discussion of this, see [94].
- V_{146} – and hence $\mathcal{R}_{6;1}^{\mathrm{NMHV}}(0)$ – has uniform transcendentality 2. This can be extended to the ϵ-dependent terms if ϵ is assigned transcendentality -1.

Both of these properties carry over to all $\mathcal{R}_{n;1}^{\mathrm{NMHV}}(0)$. At higher-loop order in *planar* $\mathcal{N} = 4$ SYM, the degree of transcendentality is expected to be uniformly $2L$.

At higher-point, there are more dual conformal invariant cross-ratios available than just the three u_is for $n = 6$. Consequently, the NMHV 1-loop ratio functions $\mathcal{R}_{n;1}^{\mathrm{NMHV}}(0)$ are more involved; however, they are all known explicitly and they take a similar form to (6.51). You can find the results for $\mathcal{R}_{n;1}^{\mathrm{NMHV}}(0)$ in [93].

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The integrands of $\mathcal{N}=4$ SYM 4-point amplitude to 3-loop order. These are the unique scalar integrands that are dual conformal invariant. **Fig. 6.3**

6.5 Higher-loop amplitudes in planar $\mathcal{N}=4$ SYM

The generalized unitarity method can be applied successfully to higher-loop amplitudes, both at the planar and non-planar level; for a recent review see [68]. The application of unitarity is most efficient when a complete integral basis is available; for example, for 1-loop amplitudes in 4d, the basis consists of the scalar box-, triangle-, and bubble-integrals in (6.26).

Beyond 1-loop, there is not a complete understanding of the basis integrals for amplitudes in generic quantum field theories, although partial results have been achieved at 2-loops in the planar limit, see [78, 95–99]. One thing worth noting is that the integral basis is finite [100].

Without a given basis of integrals, one strategy is to construct the most general integral Ansatz that satisfies certain criteria, such as dimension-counting, and then use various integral identities to recast the Ansatz into a basis of independent integrals. Further symmetries, such as dual conformal invariance in planar $\mathcal{N}=4$ SYM, can be a strong handle on finding a complete integral basis. As an example, the diagrams in Figure 6.3 correspond to the only dual conformal invariant scalar integrals for the 4-point 1-, 2- and 3-loop integrands of planar $\mathcal{N}=4$ SYM. The coefficients of each integral are fixed by applying unitarity cuts [101, 102], so that the LHSs of the equations in Figure 6.3 are the full integrands for the 4-point 1-, 2- and 3-loop amplitudes in planar $\mathcal{N}=4$ SYM. The evaluation of these integrals leads to interesting results that we discuss next.

The analytical result [103] for the **2-loop 4-point amplitude** in planar $\mathcal{N}=4$ SYM was shown by Anastasiou, Bern, Dixon, and Kosower (ABDK) [101] to be expressible in terms of the 1-loop amplitude as

$$\mathcal{P}_{4;2}^{\text{MHV}}(\epsilon) = \frac{1}{2}\Big[\mathcal{P}_{4;1}^{\text{MHV}}(\epsilon)\Big]^2 + \mathcal{P}_{4;1}^{\text{MHV}}(2\epsilon)\, f^{(2)}(\epsilon) + C^{(2)} + O(\epsilon)\,, \tag{6.55}$$

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where the MHV tree factor is stripped off as in (6.44), $f^{(2)}(\epsilon) = -\zeta_2 - \zeta_3\,\epsilon - \zeta_4\,\epsilon^2$ and $C^{(2)} = -\zeta_2^2/2$. Here $\zeta_s = \sum_{k=1}^{\infty} k^{-s}$ is the Riemann zeta function; note $\zeta_2 = \frac{\pi^2}{6}$, $\zeta_3 \approx 1.202$, and $\zeta_4 = \frac{\pi^4}{90}$.

It is interesting that the 2-loop 4-point amplitude in planar $\mathcal{N} = 4$ SYM can be written in terms of the 1-loop result. But the plot thickens! By explicit calculation of the **3-loop 4-point amplitude** in planar $\mathcal{N} = 4$ SYM, Bern, Dixon, and Smirnov (BDS) [102] found that the iterative structure continues:

$$\mathcal{P}^{\rm MHV}_{4;3}(\epsilon) = -\frac{1}{3}\Big[\mathcal{P}^{\rm MHV}_{4;1}(\epsilon)\Big]^3 + \mathcal{P}^{\rm MHV}_{4;1}(\epsilon)\,\mathcal{P}^{\rm MHV}_{4;2}(\epsilon) + f^{(3)}(\epsilon)\,\mathcal{P}^{\rm MHV}_{4;1}(3\epsilon) + C^{(3)} + O(\epsilon)\,. \tag{6.56}$$

Here $f^{(3)}(\epsilon) = \frac{11}{2}\zeta_4 + O(\epsilon)$ and $C^{(3)}$ is a constant.

The 2- and 3-loop results indicate an exponentiation structure. This motivates the ***ABDK/BDS Ansatz for the full MHV superamplitude in*** $\mathcal{N} = 4$ ***SYM***:

$$\mathcal{P}^{\rm MHV(BDS)}_{n}(\epsilon) = \exp\left[\sum_{L=1}^{\infty} \lambda^L \left(f^{(L)}(\epsilon)\,\mathcal{P}^{\rm MHV}_{n;1}(L\epsilon) + C^{(L)} + O(\epsilon)\right)\right]. \tag{6.57}$$

This Ansatz is almost correct: keep reading! In the ABDK/BDS Ansatz, the functions $f^{(L)}$ are of the form $f^{(L)}(\epsilon) = f_0^{(L)} + \epsilon\, f_1^{(L)} + \epsilon^2 f_2^{(L)}$, and the constants $C^{(L)}$ and $f_{0,1,2}^{(L)}(\epsilon)$ are independent of the number of external legs n. In particular, at 1-loop order $f^{(1)}(\epsilon) = 1$ and $C^{(1)} = 0$, and at 2-loops the results for $f^{(2)}(\epsilon)$ and $C^{(2)}$ were given below (6.55).

▶ **Exercise 6.5**

Show that (6.57) reproduces the 4-point 2- and 3-loop expressions (6.55) and (6.56).

Of course, the way one would go about testing the ABDK/BDS exponentiation Ansatz (6.57) is by direct calculation of the n-point L-loop amplitudes at $L = 2, 3, \ldots$. But how many 2-loop amplitudes have you ever calculated? Yeah, it is not an easy task, nonetheless progress has been made. It has been shown numerically in [104, 105] that the exponentiation Ansatz correctly produces the **5-point 2-loop amplitude**. It is very interesting that something new happens at 6- and higher-point: while *the ABDK/BDS Ansatz matches the IR divergent structure, it does not fully produce the correct finite part. The ABDK/BDS Ansatz determines the finite part of the amplitude only up to a function of dual conformal cross-ratios of the external momenta.* This function is called the ***remainder function*** and it is defined as

$$\mathbf{r}_{n;L}(\epsilon) \;\equiv\; \mathcal{P}^{\rm MHV}_{n;L}(\epsilon) - \mathcal{P}^{\rm MHV(BDS)}_{n;L}(\epsilon)\,, \tag{6.58}$$

where $\mathcal{P}^{\rm MHV}_{n;L}(\epsilon)$ is the actual MHV L-loop amplitude and $\mathcal{P}^{\rm MHV(BDS)}_{n;L}(\epsilon)$ is the $O(\lambda^L)$ terms in the expansion of the exponential Ansatz (6.57). The remainder function does not show up for $n = 4, 5$ because in those cases there are no available conformal cross-ratios.

The first indication of the remainder function came from a strong coupling calculation by Alday and Maldacena [106] who proposed [107] to use the AdS/CFT correspondence

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to calculate $\mathcal{P}_n^{\text{MHV}}$. Subsequently, it was verified numerically that a remainder function is needed for the parity-even part of the **6-point 2-loop MHV amplitude** [108], whereas ABDK/BDS successfully determines the parity-odd part [109]. The analytic form of the remainder function $\mathbf{r}_{6;2}$ for the 6-point 2-loop MHV amplitude was calculated (as a hexagonal Wilson-loop) by Del Duca, Duhr, and Smirnov [110, 111]. The result, written in terms of the three dual conformal cross-ratios $u_{1,2,3}$ in (6.53), is a respect-inducing 17-page long sum of generalized polylogarithms; all terms have transcendentality 4. In an impressive application of a mathematical tool known as ***the Symbol***, Goncharov, Spradlin, Vergu, and Volovich [112] managed to simplify this complicated result for $\mathbf{r}_{6;2}$ to an expression that involves only regular polylogs – Li_s and log – and fits in just a few lines of LaTeX.

The simple answer [112] for $\mathbf{r}_{6;2}$ is an important step towards a better understanding of loop amplitudes in planar $\mathcal{N} = 4$ SYM. The Symbol is now being used to understand higher-loop amplitudes, however, there will be amplitudes in planar $\mathcal{N} = 4$ SYM involving integrals that the Symbol does not help with. Thus techniques are eventually needed beyond the Symbol.

We have reviewed the unitarity method and shown you how it allows us to construct L-loop amplitudes from *on-shell* lower-loop input. The approach explores the analytic structure of the loop-integrands in a somewhat different way from the recursive techniques you know from tree-level amplitudes. BCFW is available at the level of loop-integrands, and that is the subject of the next chapter.

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