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On conjugations of circle homeomorphisms with two break points

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Abstract. Let $f_i \in C^{2+\alpha}(S^1\backslash\{a_i, b_i\})$, $\alpha > 0$, $i = 1, 2$, be circle homeomorphisms with two break points a_i, b_i, that is, discontinuities in the derivative Df_i, with identical irrational rotation number ρ and $\mu_1([a_1, b_1]) = \mu_2([a_2, b_2])$, where μ_i are the invariant measures of f_i, $i = 1, 2$. Suppose that the products of the jump ratios of Df_1 and Df_2do not coincide, that is, $Df_1(a_1 - 0)/Df_1(a_1 + 0)\cdot Df_1(b_1 - 0)/Df_1(b_1 + 0) \neq Df_2(a_2 - 0)/Df_2(a_2 + 0) \cdot Df_2(b_2 - 0)/Df_2(b_2 + 0)$. Then the map ψ conjugating f_1 and f_2 is a singular function, that is, it is continuous on S^1, but $D\psi(x) = 0$ almost everywhere with respect to Lebesgue measure.

1. *Introduction*

Let f be an orientation preserving homeomorphism of the circle $S^1 \equiv \mathbb{R}/\mathbb{Z}$ with lift $\hat{f} : \mathbb{R} \to \mathbb{R}$, $\hat{f}$ continuous, strictly increasing and $\hat{f}(t + 1) = \hat{f}(t) + 1$, $t \in \mathbb{R}$. The circle homeomorphism f is then defined by $f(x) = \hat{f}(\hat{x}) \pmod 1$, $x \in S^1$, and $x \equiv \hat{x} + \mathbb{Z}$ with $\hat{x} \in [0, 1)$. In the following S^1 will be identified with $[0, 1)$ and $x \in S^1$ with $\hat{x} \in [0, 1)$. The interval $[x, y] \subset S^1$ then corresponds to the interval $[\hat{x}, \hat{y}] \subset [0, 2)$. If f is a circle diffeomorphism with irrational rotation number $\rho = \rho_f$ and $\log D\hat{f}$ is of bounded variation, then f is conjugate to the pure rotation f_ρ, that is, there exists an essentially unique homeomorphism φ of the circle with $f = \varphi^{-1} \circ f_\rho \circ \varphi$. This classical result of Denjoy [**2**] can be extended to circle homeomorphisms with break points. The exact statement of the corresponding theorem will be given later.

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It is well known that circle homeomorphisms f with irrational rotation number ρ_f admit a unique f-invariant probability measure μ_f. Since the conjugating map φ and the invariant measure μ_f are related by $\varphi(x) = \mu_f([0, x])$ (see [**7**]), regularity properties of the conjugating map φ imply corresponding properties of the density of the absolutely continuous invariant measure μ_f. This problem of smoothness of the conjugacy of smooth diffeomorphisms has come to be very well understood (see, for instance, [**1, 8–10, 14, 18**]).

An important class of circle homeomorphisms are homeomorphisms with break points, or P-homeomorphisms for short. In general their ergodic properties such as their invariant measures, their renormalizations and also their rigidity properties are rather different from those of diffeomorphisms (see [**8**, Ch. VI], [**12**], [**15**, Ch. I and IV]).

The class of *P-homeomorphisms* consists of orientation preserving circle homeomorphisms f whose lifts $\hat{f}$ are differentiable away from countably many points $\hat{b} \in BP(\hat{f}) \subset [0, 1)$, corresponding to the so-called break points $b \in BP(f) \subset S^1$ of f, at which left and right derivatives, denoted respectively by $D\hat{f}_-$ and $D\hat{f}_+$, exist, such that:

(i) there exist constants $0 < c_1 < c_2 < \infty$ with $c_1 < D\hat{f}(\hat{x}) < c_2$ for all $\hat{x} \in [0, 1)\backslash BP(\hat{f})$, $c_1 < D\hat{f}_-(\hat{b}) < c_2$ and $c_1 < D\hat{f}_+(\hat{b}) < c_2$ for all $\hat{b} \in BP(\hat{f})$;

(ii) $\log D\hat{f}$ has bounded variation in $[0, 1]$.

The ratio $\sigma_f(b) := D\hat{f}_-(\hat{b})/D\hat{f}_+(\hat{b})$ is called the *jump ratio* of f in $b \in BP(f)$. Denote by v the total variation $v = \mathrm{Var}_{[0,1]}(\log D\hat{f})$ of $\log D\hat{f}$ on $[0, 1]$.

General P-homeomorphisms with one break point were first studied by Khanin and Vul in [**11**]. Among other results, it was proved by these authors that their renormalizations approximate fractional linear transformations. Piecewise linear (PL) orientation preserving circle homeomorphisms with break points are the simplest examples in the class of P-homeomorphisms. They show up in many other areas of mathematics such as group theory and homotopy theory, and in logic via the Thompson group and its generalizations (see [**16**]). The invariant measures of PL homeomorphisms were first studied by Herman in [**8**], those of general P-homeomorphisms with one break point by Dzhalilov and Khanin in [**3**]. Their main result is the following theorem.

THEOREM 1.1. *Let f be a P-homeomorphism with one break point b. If the rotation number ρ_f is irrational and $f \in C^{2+\varepsilon}(S^1\backslash\{b\})$ for some $\varepsilon > 0$, then the f-invariant probability measure μ_f is singular with respect to Lebesgue measure μ_L on S^1, that is, there exists a measurable subset $A \subset S^1$ such that $\mu_f(A) = 1$ and $\mu_L(A) = 0$.*

Liousse got in [**13**] the same result for 'generic' PL circle homeomorphisms with several break points whose rotation number is irrational and of bounded type. In a next step Dzhalilov and Liousse [**4**] and Dzhalilov *et al* [**5**] studied another class of circle homeomorphisms with two break points. Their main result in [**5**] is the following theorem.

THEOREM 1.2. *Let f be a P-homeomorphism satisfying the following conditions:*

(a) *the rotation number $\rho = \rho_f$ of f is irrational;*

(b) *f has two break points b_1, b_2 with $\sigma_f(b_1) \cdot \sigma_f(b_2) \neq 1$;*

(c) *$D\hat{f}$ is absolutely continuous on every connected interval of $[0, 1]\backslash\{\hat{b}_1, \hat{b}_2\}$ and $D^2\hat{f} \in L^1([0, 1], d\mu_L)$.*

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Then the f-invariant probability measure μ_f is singular with respect to Lebesgue measure μ_L.

In the following we refer to the smoothness condition (c) in Theorem 1.2 on f as the Katznelson–Ornstein (KO) condition.

The above theorems show that, for a sufficiently piecewise smooth circle homeomorphism f with irrational rotation number and one or two break points, the map conjugating f and f_ρ is singular. Consider next the regularity properties of the conjugating map between two class P-homeomorphisms with one or two break points and coinciding irrational rotation numbers. The case of one break point with coinciding jump ratios, the so-called rigidity problem, was studied in great detail by Khanin and Khmelev in [**12**] and by Teplinskii and Khanin in [**17**].

If $\rho = [k_1, k_2, \ldots, k_n, \ldots]$ is the continued fraction expansion of the irrational rotation number ρ, define the sets

$$M_o = \{\rho : \forall n \in \mathbb{N}\ \exists C > 0 : k_{2n-1} \le C\},$$

$$M_e = \{\rho : \forall n \in \mathbb{N}\ \exists C > 0 : k_{2n} \le C\}.$$

Then the main result of [**17**] is the following theorem.

THEOREM 1.3. (Teplinskii and Khanin) *Let $f_i \in C^{2+\alpha}(S^1\backslash\{b_i\})$, $i = 1, 2$, be P-homeomorphisms each with one break point b_i. Assume that:*

(1) *their rotation numbers $\rho(f_i)$, $i = 1, 2$, are irrational and coincide, that is, $\rho(f_1) = \rho(f_2) = \rho$, $\rho \in \mathbb{R}^1\backslash\mathbb{Q}$;*

(2) *their jump ratios $\sigma_i = \sigma_{f_i}(b_i)$, $i = 1, 2$, coincide, that is, $\sigma_1 = \sigma_2 = \sigma$.*

Then the map ψ conjugating the homeomorphisms f_1 and f_2 is a C^1-diffeomorphism of the circle if either $\sigma > 1$ and $\rho \in M_o$ or $\sigma < 1$ and $\rho \in M_e$.

In the case of non-coinciding jump ratios Dzhalilov *et al* [**6**] proved the following theorem.

THEOREM 1.4. *Let $f_i \in C^{2+\alpha}(S^1\backslash\{b_i\})$, $i = 1, 2$, be P-homeomorphisms each with one break point b_i. Assume that:*

(1) *their rotation numbers ρ_i, $i = 1, 2$, are irrational and coincide, that is, $\rho_1 = \rho_2 = \rho$, $\rho \in \mathbb{R}^1\backslash\mathbb{Q}$;*

(2) *their jump ratios $\sigma_{f_i}(b_i)$, $i = 1, 2$, are positive and do not coincide.*

Then the homeomorphism ψ conjugating f_1 and f_2 is a singular function, that is, ψ is continuous on S^1 and $D\psi(x) = 0$ almost everywhere with respect to Lebesgue measure μ_L.

In the present paper we will extend this result to circle homeomorphisms with coinciding irrational rotation numbers each having two break points. Our main result is the following theorem.

THEOREM 1.5. *Let $f_i \in C^{2+\alpha}(S^1\backslash\{a_i, b_i\})$, $i = 1, 2$, be P-homeomorphisms each with two break points a_i, b_i. Assume that:*

(1) *their rotation numbers $\rho(f_i)$, $i = 1, 2$, are irrational and coincide, that is, $\rho(f_1) = \rho(f_2) = \rho$, $\rho \in \mathbb{R}^1\backslash\mathbb{Q}$;*

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(2) *the products of their jump ratios* $\sigma_{f_i}(a_i) \cdot \sigma_{f_i}(b_i)$ *do not coincide, that is,* $\sigma_{f_1}(a_1) \cdot \sigma_{f_1}(b_1) \neq \sigma_{f_2}(a_2) \cdot \sigma_{f_2}(b_2)$;

(3) $\mu_1([a_1, b_1]) = \mu_2([a_2, b_2])$, *where* μ_i *is the invariant probability measure of* f_i, $i = 1, 2$.

Then the map ψ *conjugating* f_1 *and* f_2 *is singular.*

2. *Preliminaries and notation*

Consider an orientation preserving circle homeomorphism f with lift $\hat{f}$ and irrational rotation number $\rho = \rho_f$. If the rotation number ρ has the continued fraction expansion $\rho = [k_1, k_2, \ldots, k_n, \ldots] = 1/(k_1 + 1/(k_2 + \cdots + 1/(k_n + \cdots)))$ its convergents p_n/q_n, $n \in \mathbb{N}$, are defined by $p_n/q_n = [k_1, k_2, \ldots, k_n]$. Then the denominators q_n satisfy the well-known recursion relation $q_{n+1} = k_{n+1}q_n + q_{n-1}$, $n \geq 1$, $q_0 = 1$, $q_1 = k_1$.

For an arbitrary point $x_0 \in S^1$, define $\Delta_0^{(n)}(x_0)$ as the closed interval in S^1 with endpoints x_0 and $x_{q_n} = f^{q_n}(x_0)$, such that, for n odd, x_{q_n} is to the left of x_0, and for n even, it is to its right with respect to the orientation induced from the real line. Denote by $\Delta_i^{(n)}(x_0) := f^i(\Delta_0^{(n)}(x_0))$, $i \geq 1$, the iterates of the interval $\Delta_0^{(n)}(x_0)$ under f. It is well known that the set $\xi_n(x_0)$ of intervals with mutually disjoint interiors defined as

$$\xi_n(x_0) = \{\Delta_i^{(n-1)}(x_0), 0 \leq i < q_n\} \cup \{\Delta_j^{(n)}(x_0), 0 \leq j < q_{n-1}\} \tag{1}$$

determines a partition of the circle for any n. The partition $\xi_n(x_0)$ is called the nth *dynamical partition* of the point x_0 with *generators* $\Delta_0^{(n-1)}(x_0)$ and $\Delta_0^{(n)}(x_0)$. Obviously, the partition $\xi_{n+1}(x_0)$ is a refinement of the partition $\xi_n(x_0)$: indeed, the intervals of order n belong to $\xi_{n+1}(x_0)$ and each interval $\Delta_i^{(n-1)}(x_0) \in \xi_n(x_0)$, $0 \leq i < q_n$, is partitioned into $k_{n+1} + 1$ intervals belonging to $\xi_{n+1}(x_0)$ such that

$$\Delta_i^{(n-1)}(x_0) = \Delta_i^{(n+1)}(x_0) \cup \bigcup_{s=0}^{k_{n+1}-1} \Delta_{i+q_{n-1}+sq_n}^{(n)}(x_0).$$

Recall the following definition introduced in [**9**].

Definition 2.1. An interval $I = (x, y) \subset S^1$ is q_n-small and its endpoints x, y are q_n-close if the intervals $f^i(I)$, $0 \leq i < q_n$, are disjoint.

It is clear that the interval (x, y) is q_n-small if, depending on the parity of n, either $y \prec x \preceq f^{q_{n-1}}(y) \prec y$ or $f^{q_{n-1}}(x) \preceq y \prec x \prec f^{q_{n-1}}(x)$ in the order induced from the real line.

Then we can show the following lemma.

LEMMA 2.2. *Let* f *be a* P*-homeomorphism with a finite number of break points* b_i, $i = 1, 2, \ldots, m$, *and irrational rotation number* ρ. *Assume that* $x, y \in S^1$ *are* q_n*-close and* $b_i \notin \{f^j(x), f^j(y), 0 \leq j < q_n, \}$, $i = 1, 2, \ldots, m$. *Then for any* $0 \leq k < q_n$ *the following inequality holds:*

$$e^{-v} \leq \frac{D\hat{f}^k(\hat{x})}{D\hat{f}^k(\hat{y})} \leq e^v, \tag{2}$$

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where v is the total variation of $\log D\hat{f}$ *on* $[0, 1]$ *and* $\hat{x}$, $\hat{y}$ *are the lifts of* x, y *to the interval* $[0, 1)$.

Proof. Take any two q_n-close points $x, y \in S^1$ and $0 \le k < q_n$. Denote by I the open interval with endpoints x and y. Because the intervals $f^s(I)$, $0 \le s < k$, are disjoint, we obtain

$$|\log D\hat{f}^k(\hat{x}) - \log D\hat{f}^k(\hat{y})| \le \sum_{s=0}^{k-1} |\log D\hat{f}(\hat{f}^s(\hat{x})) - \log D\hat{f}(\hat{f}^s(\hat{y}))| \le v,$$

from which inequality (2) follows immediately. □

The following lemma can be proved easily using Lemma 2.2.

LEMMA 2.3. *Let f be a P-homeomorphism with a finite number of break points b_i, $i = 1, 2, \ldots, m$, and irrational rotation number ρ. If $x_0 \in S^1$, $n \ge 1$ and $b_i \notin \{f^j(x_0), 0 \le j < q_n\}$ for $i = 1, 2, \ldots, m$, then*

$$e^{-v} \le \prod_{i=0}^{q_n-1} D\hat{f}(\hat{f}^i(\hat{x}_0)) \le e^v. \tag{3}$$

Inequality (3) is called the *Denjoy inequality*. The proof of Lemma 2.3 is as for circle diffeomorphisms (see, for instance, [**10**]). Using Lemma 2.3, it can be shown that the intervals of the dynamical partition $\xi_n(x_0)$ in (1) have exponentially small length. Indeed, one finds the following corollary.

COROLLARY 2.4. *Let $\Delta^{(n)}$ be an arbitrary element of the dynamical partition $\xi_n(x_0)$. Then*

$$l(\Delta^{(n)}) := \mu_L(\Delta^{(n)}) \le \text{const}\, \lambda^n, \tag{4}$$

where $\lambda = (1 + e^{-v})^{-1/2} < 1$.

From Corollary 2.4 it follows that the trajectory of every point $x \in S^1$ is dense in S^1. This together with monotonicity of the homeomorphism f implies the following theorem.

THEOREM 2.5. *Suppose that a homeomorphism f satisfies the conditions of Lemma 2.3. Then f is topologically conjugate to the linear rotation f_ρ.*

In the following discussion we have to compare different intervals. For this we use the following definition.

Definition 2.6. Let $C > 1$. We call two intervals of S^1 *C-comparable* if the ratio of their lengths is in $[C^{-1}, C]$.

COROLLARY 2.7. *Suppose that the homeomorphism f satisfies the conditions of Lemma 2.3. Then for any interval $I \subset S^1$ the intervals I and $f^{q_n}(I)$ are e^v-comparable. If the interval I is q_n-small then $l(f^i(I)) <$ const λ^n for all $i = 0, 1, \ldots, q_n - 1$.*

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3. *The cross-ratio tools*

Let us first recall two definitions given below.

Definition 3.1. The *cross-ratio* $\mathrm{Cr}(a_1, a_2, a_3, a_4)$ of four strictly ordered points $a_i \in \mathbb{R}$, $i = 1, 2, 3, 4$, is defined as

$$\mathrm{Cr}(a_1, a_2, a_3, a_4) = \frac{(a_2 - a_1)(a_4 - a_3)}{(a_3 - a_1)(a_4 - a_2)}.$$

Definition 3.2. The *cross-ratio distortion* $\mathrm{Dist}(a_1, a_2, a_3, a_4; f)$ of four strictly ordered points $a_i \in \mathbb{R}$, $i = 1, 2, 3, 4$, with respect to a strictly increasing function f on $\mathbb{R}$ is defined as

$$\mathrm{Dist}(a_1, a_2, a_3, a_4; f) = \frac{\mathrm{Cr}(f(a_1), f(a_2), f(a_3), f(a_4))}{\mathrm{Cr}(a_1, a_2, a_3, a_4)}.$$

For $k \geq 3$, let $\hat{z}_i \in [a, a+1] \subset \mathbb{R}$, $i = 1, \ldots, k$, be the lifts of the points $z_i \in S^1$, $i = 1, \ldots, k$, with $z_1 \prec z_2 \prec \cdots \prec z_k \prec z_1$ such that $\hat{z}_1 < \hat{z}_2 < \cdots < \hat{z}_k$. The vector $(\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_k) \in \mathbb{R}^k$ is called the *lifted vector* of $(z_1, z_2, \ldots, z_k) \in (S^1)^k$. Consider a circle homeomorphism f with lift $\hat{f}$. We define the cross-ratio distortion of (z_1, z_2, z_3, z_4) with respect to f by $\mathrm{Dist}(z_1, z_2, z_3, z_4; f) := \mathrm{Dist}(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4; \hat{f})$ where $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ is the lifted vector of (z_1, z_2, z_3, z_4). We need the following lemma.

LEMMA 3.3. (See [**3**]) *Suppose that f is a P-homeomorphism with a finite number of break points and $f \in C^{2+\alpha}(S^1 \setminus BP(f))$ for some $\alpha > 0$. Consider any four points $z_i \in S^1$, $i = 1, 2, 3, 4$, with $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$ and $[z_1, z_4] \subset S^1 \setminus BP(f)$. Then*

$$|\mathrm{Dist}(z_1, z_2, z_3, z_4; f) - 1| \leq K |\hat{z}_4 - \hat{z}_1|^{1+\alpha}$$

for some positive constant K depending only on f.

Next we consider the case where the interval $[z_1, z_4]$ contains one break point b of the homeomorphism f. We estimate the distortion of its cross-ratio when b lies outside the middle interval $[z_2, z_3]$. For this we define for $z_i \in S^1$, $i = 1, 2, 3, 4$, with $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$ and $b \in [z_1, z_2] \cup [z_3, z_4]$ the following quantities:

$$\alpha := \hat{z}_2 - \hat{z}_1, \quad \beta := \hat{z}_3 - \hat{z}_2, \quad \gamma := \hat{z}_4 - \hat{z}_3, \quad \tau := \hat{z}_2 - \hat{b}, \quad \xi := \frac{\beta}{\alpha},$$

$$\zeta := \frac{\tau}{\alpha}, \quad \eta := \frac{\beta}{\gamma}, \quad \vartheta := \frac{\hat{b} - \hat{z}_3}{\gamma}.$$

LEMMA 3.4. *Assume that f is a P-homeomorphism with a finite number of break points and $f \in C^2(S^1 \setminus BP(f))$. Choose points $z_i \in S^1$, $i = 1, 2, 3, 4$, with $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$ such that f has one single break point b in $[z_1, z_2] \cup [z_3, z_4]$. Then:*

(i) $$\left| \mathrm{Dist}(z_1, z_2, z_3, z_4; f) - \frac{[\sigma_f(b) + (1 - \sigma_f(b))\zeta](1 + \xi)}{\sigma_f(b) + (1 - \sigma_f(b))\zeta + \xi} \right| \leq K_1 |\hat{z}_4 - \hat{z}_1|$$
if $b \in [z_1, z_2]$;

(ii) $$\left| \mathrm{Dist}(z_1, z_2, z_3, z_4; f) - \frac{[\sigma_f(b) + (1 - \sigma_f(b))\vartheta](1 + \eta)}{\sigma_f(b) + (1 - \sigma_f(b))\vartheta + \eta} \right| \leq K_1 |\hat{z}_4 - \hat{z}_1|$$
if $b \in [z_3, z_4]$

for some positive constant K_1 depending only on f.

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Proof. We prove only the first assertion. The second one can be proved similarly. Obviously

$$\hat{f}(\hat{z}_2) - \hat{f}(\hat{z}_1) = D\hat{f}_+(\hat{b})(\hat{z}_2 - \hat{b}) + \frac{D^2\hat{f}(\theta_1)(\hat{z}_2 - \hat{b})^2}{2} + D\hat{f}_-(\hat{b})(\hat{b} - \hat{z}_1) + \frac{D^2\hat{f}(\theta_2)(\hat{b} - \hat{z}_1)^2}{2}$$

and

$$\hat{f}(\hat{z}_3) - \hat{f}(\hat{z}_1) = D\hat{f}_+(\hat{b})(\hat{z}_3 - \hat{b}) + \frac{D^2\hat{f}(\theta_3)(\hat{z}_3 - \hat{b})^2}{2} + D\hat{f}_-(\hat{b})(\hat{b} - \hat{z}_1) + \frac{D^2\hat{f}(\theta_4)(\hat{b} - \hat{z}_1)^2}{2},$$

for some $\theta_1 \in (\hat{b}, \hat{z}_2)$, $\theta_2 \in (\hat{z}_1, \hat{b})$, $\theta_3 \in (\hat{b}, \hat{z}_3)$, $\theta_4 \in (\hat{z}_1, \hat{b})$.

Using the last two relations, it is easy to show that

$$\frac{\hat{f}(\hat{z}_2) - \hat{f}(\hat{z}_1)}{\hat{f}(\hat{z}_3) - \hat{f}(\hat{z}_1)} = \frac{\sigma_f(b) + (1 - \sigma_f(b))\zeta + O(\alpha)}{G(\zeta, \xi) + O(\alpha + \beta)} \times \frac{\alpha}{\alpha + \beta} \tag{5}$$

where $G(\zeta, \xi) = (\sigma_f(b) + (1 - \sigma_f(b))\zeta + \xi)/(1 + \xi)$ and $\xi > 0$. It is clear that $\min\{1, \sigma_f(b)\} \le \sigma_f(b) + (1 - \sigma_f(b))\zeta \le \max\{1, \sigma_f(b)\}$ and $\min\{1, \sigma_f(b)\} \le G(\zeta, \xi) \le 1 + \max\{1, \sigma_f(b)\}$. The last two inequalities together with (5) imply that

$$\frac{\hat{f}(\hat{z}_2) - \hat{f}(\hat{z}_1)}{\hat{f}(\hat{z}_3) - \hat{f}(\hat{z}_1)} : \frac{\alpha}{\alpha + \beta} = \frac{[\sigma_f(b) + (1 - \sigma_f(b))\zeta](1 + \xi)}{\sigma_f(b) + (1 - \sigma_f(b))\zeta + \xi} + O(\alpha + \beta). \tag{6}$$

Since $\hat{f} \in C^2([\hat{z}_2, \hat{z}_4])$, we get

$$\frac{\hat{f}(\hat{z}_4) - \hat{f}(\hat{z}_3)}{\hat{f}(\hat{z}_4) - \hat{f}(\hat{z}_2)} : \frac{\gamma}{\gamma + \beta} = 1 + O(\gamma + \beta). \tag{7}$$

Relations (6) and (7) imply the first assertion of Lemma 3.4. □

4. *Proof of Theorem 1.5*

For the proof of Theorem 1.5 we need several lemmas which we formulate next. Their proofs will be given later. Consider two copies of the circle S^1 and homeomorphisms f_i each with two break points a_i, b_i, $i = 1, 2$, and the same irrational rotation number ρ. Assume that f_1 and f_2 satisfy the conditions of Theorem 1.5.

Let φ_1 and φ_2 be maps conjugating f_1 and f_2 with the pure rotation f_ρ: $\varphi_1 \circ f_1 = f_\rho \circ \varphi_1$ and $\varphi_2 \circ f_2 = f_\rho \circ \varphi_2$. It is easy to check that the map $\psi = \varphi_2^{-1} \circ \varphi_1$ conjugates f_1 and f_2, that is,

$$\psi(f_1(x)) = f_2(\psi(x)) \tag{8}$$

for all $x \in S^1$. By assumption of Theorem 1.5, $\sigma_{f_1}(a_1) \cdot \sigma_{f_1}(b_1) \ne \sigma_{f_2}(a_2) \cdot \sigma_{f_2}(b_2)$. Assume without loss of generality that $\sigma_{f_1}(a_1) \ne \sigma_{f_2}(a_2)$. Since φ_i, $i = 1, 2$, is unique up to an additive constant we can choose φ_i, $i = 1, 2$, such that $\varphi_1(a_1) = a_1$ and $\varphi_2^{-1}(a_1) = a_2$

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and hence $\psi(a_1) = a_2$. Then, by assumption of Theorem 1.5, $\psi(b_1) = b_2$. Recall that the length of an interval $[a, b] \subset S^1$ is defined by

$$l([a, b]) := \mu_L([a, b]) = \begin{cases} \hat{b} - \hat{a} & \text{if } 0 \le \hat{a} < \hat{b} < 1, \\ 1 + \hat{b} - \hat{a} & \text{if } 0 \le \hat{b} < \hat{a} < 1. \end{cases}$$

Definition 4.1. Let $R_1 > 1$ and $\varepsilon > 0$ be constants. The points $x_0, z_i \in S^1$ with $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$ satisfy conditions $(C_{R_1,\varepsilon})$ if:

(a) $R_1^{-1} l([z_2, z_3])\sqrt{\varepsilon} \le l([z_1, z_2]) \le R_1 l([z_2, z_3])\sqrt[4]{\varepsilon}$;

(b) $R_1^{-1} l([z_2, z_3]) \le l([z_3, z_4]) \le R_1 l([z_2, z_3])$;

(c) $\max_{1\le i\le 4} l([x_0, z_i]) \le R_1 l([z_2, z_3])$.

For $x_0 \in S^1$ with lift $\hat{x}_0$ in $[0, 1)$, define $d(\hat{x}_0) := \min\{\hat{x}_0, (1 - \hat{x}_0)\}$.

LEMMA 4.2. *Assume that the lift $\hat{\psi}$ of the conjugating map ψ has a positive derivative $D\hat{\psi}(\hat{x}_0) = \omega$ at the point $\hat{x}_0 \in [0, 1)$ and let $R_1 > 1$ be a constant. Then there exists a constant $C_2 = C_2(\omega, R_1)$ such that for any $\varepsilon > 0$ there exists $\delta = \delta(\hat{x}_0, \varepsilon) \in (0, d(\hat{x}_0))$ such that, for all $z_i \in S^1$ with $\hat{z}_i \in (\hat{x}_0 - \delta, \hat{x}_0 + \delta)$, $i = 1, 2, 3, 4$, satisfying the conditions $(C_{R_1,\varepsilon})$, one has:*

$$\text{(i)} \quad \frac{l([z_1, z_2])}{l([z_2, z_3])}(1 - C_2\sqrt{\varepsilon}) \le \frac{l[\psi(z_1), \psi(z_2)]}{l[\psi(z_2), \psi(z_3)]} \le \frac{l([z_1, z_2])}{l([z_2, z_3])}(1 + C_2\sqrt{\varepsilon});$$

$$\text{(ii)} \quad \frac{l([z_3, z_4])}{l([z_2, z_3])}(1 - C_2\varepsilon) \le \frac{l[\psi(z_3), \psi(z_4)]}{l[\psi(z_2), \psi(z_3)]} \le \frac{l([z_3, z_4])}{l([z_2, z_3])}(1 + C_2\varepsilon).$$

LEMMA 4.3. *Suppose that the lift $\hat{\psi}$ has a positive derivative $D\hat{\psi}(\hat{x}_0) = \omega$ at the point $\hat{x}_0 \in [0, 1)$ and let $R_1 > 1$ be a constant. Then there exists a constant $R_2 = R_2(\omega, R_1)$ such that, for any $\varepsilon > 0$, there exists $\delta = \delta(\hat{x}_0, \varepsilon) \in (0, d(\hat{x}_0))$ such that, for all $z_i \in S^1$ with $\hat{z}_i \in (\hat{x}_0 - \delta, \hat{x}_0 + \delta)$, $i = 1, 2, 3, 4$, satisfying the conditions $(C_{R_1,\varepsilon})$, one has:*

$$|\text{Dist}(z_1, z_2, z_3, z_4; \psi) - 1| \le R_2\sqrt{\varepsilon}. \tag{9}$$

The main idea for proving that the map ψ conjugating f_1 and f_2 is a singular function is to construct a quadruple of points z_i, $i = 1, 2, 3, 4$, for which the ratio of the distortions $\text{Dist}(z_1, z_2, z_3, z_4; f_1^{q_n})$ and $\text{Dist}(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4); f_2^{q_n})$ stays away from 1.

For this purpose assume that $D\hat{\psi}(\hat{x}_0) = \omega > 0$ for the lift $\hat{x}_0 \in [0, 1)$ of a point $x_0 \in S^1$. Without loss of generality, we can choose n to be odd. Then $\Delta_0^{(n)}(z) = [f_1^{q_n}(z), z]$ and $\Delta_0^{(n-1)}(z) = [z, f_1^{q_{n-1}}(z)]$ for any point z of the circle. Consider the nth dynamical partition $\xi_n(x_0)$ of the point $x_0 \in S^1$ defined by the homeomorphism f_1. Only one interval of the partition $\xi_n(x_0)$ covers the break point a_1. Hence there exists a unique point $\overline{a}_1$ with either $\overline{a}_1 \in \Delta_0^{(n-1)}(x_0)$ and $f_1^l(\overline{a}_1) = a_1$ for some $0 \le l < q_n$, or $\overline{a}_1 \in \Delta_0^{(n)}(x_0)$ and $f_1^l(\overline{a}_1) = a_1$ for some $0 \le l < q_{n-1}$. We call the point $\overline{a}_1$ the *q_n-preimage* of the break point a_1 in $\Delta_0^{(n-1)}(x_0) \cup \Delta_0^{(n)}(x_0)$. There exists a unique point t_0 such that $\overline{a}_1$ is the *middle point* of the interval $[t_0, f_1^{q_{n-1}}(t_0)]$ (see Figure 1). Consider now the nth dynamical partitions $\xi_n(t_0)$ of the point t_0 defined by f_1 on the first circle and $\zeta_n(\psi(t_0))$ of the point $\psi(t_0)$ defined by f_2 on the second circle. For each $n \ge 1$, define

$$\Delta_i^{(n)}(t_0) := f_1^i(\Delta_0^{(n)}(t_0)), \quad C_i^{(n)}(\psi(t_0)) := f_2^i(C_0^{(n)}(\psi(t_0))), \quad 0 \le i < q_{n+1},$$

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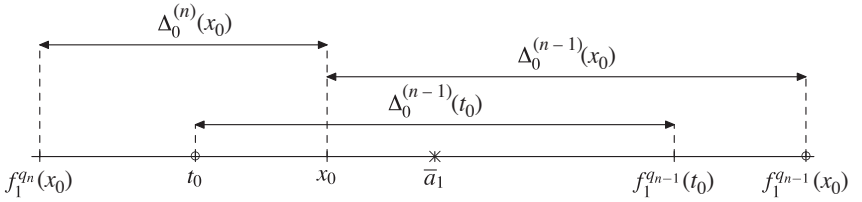


FIGURE 1. The point $\overline{a}_1 = f_1^{-l}(a_1)$ belongs to the interval $[f_1^{q_n}(x_0),\, f_1^{q_{n-1}}(x_0)]$ and is the middle point of $[t_0,\, f_1^{q_{n-1}}(t_0)]$.

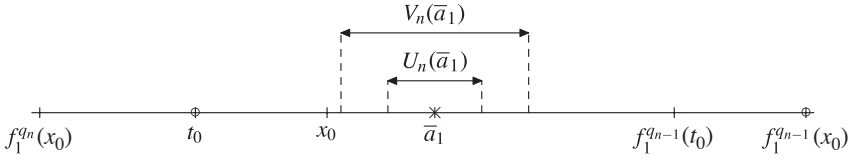


FIGURE 2. The intervals $U_n(\overline{a}_1)$ and $V_n(\overline{a}_1)$ are $\sqrt{\varepsilon}$ and $\sqrt[4]{\varepsilon}$ comparable with $[t_0,\, f_1^{q_{n-1}}(t_0)]$, respectively.

where $\Delta_0^{(n)}(t_0)$ (respectively, $C_0^{(n)}(\psi(t_0))$) are the initial intervals of order n of the points t_0 (respectively, $\psi(t_0)$) determined by f_1 (respectively, f_2). By definition,

$$\xi_n(t_0) = \{\Delta_i^{(n-1)}(t_0),\, 0 \le i < q_n\} \cup \{\Delta_j^{(n)}(t_0),\, 0 \le j < q_{n-1}\},$$
$$\zeta_n(\psi(t_0)) = \{C_i^{(n-1)}(\psi(t_0)),\, 0 \le i < q_n\} \cup \{C_j^{(n)}(\psi(t_0)),\, 0 \le j < q_{n-1}\}.$$

Since the common rotation number ρ of f_1 and f_2 is irrational, the order of the points on the orbit $\{f_1^k(x_0),\, k \in \mathbb{Z}^1\}$ on the first circle will be precisely the same as that for the orbit $\{f_2^k(\psi(x_0)),\, k \in \mathbb{Z}^1\}$ on the second circle. This together with the relation $\psi(f_1(x)) = f_2(\psi(x))$ for $x \in S^1$ implies that

$$\psi(\Delta_i^{(n-1)}(t_0)) = C_i^{(n-1)}(\psi(t_0)), \quad 0 \le i < q_n,$$
$$\psi(\Delta_j^{(n)}(t_0)) = C_j^{(n)}(\psi(t_0)), \quad 0 \le j < q_{n-1}.$$

Denote by $\overline{b}_1$ the q_n-preimage of the second break point b_1 of f_1 in $\Delta_0^{(n-1)}(t_0) \cup \Delta_0^{(n)}(t_0)$, such that $f_1^p(\overline{b}_1) = b_1$ for some $0 \le p < q_n$. The ψ-preimages of the points $\overline{a}_1$ and $\overline{b}_1$ lie in $C_0^{(n-1)}(\psi(t_0)) \cup C_0^{(n)}(\psi(t_0))$. Using relation (5), we get

$$f_2^l(\psi(\overline{a}_1)) = f_2^{l-1}(f_2(\psi(\overline{a}_1))) = f_2^{l-1}(\psi(f_1(\overline{a}_1))) = \cdots = \psi(f_1^l(\overline{a}_1)) = \psi(a_1) = a_2.$$

Similarly, one can show that $f_2^p(\psi(\overline{b}_1)) = b_2$.

For $\varepsilon > 0$ define the two neighbourhoods U_n, V_n of the point $\overline{a}_1 \in S^1$ as

$$U_n(\overline{a}_1) = \{z \in S^1 : \hat{z} \in (\hat{\overline{a}}_1 - \delta_n,\, \hat{\overline{a}}_1 + \delta_n) \text{ with } \delta_n = \tfrac{1}{4} l([t_0,\, f_1^{q_{n-1}}(t_0)])\sqrt{\varepsilon}\},$$
$$V_n(\overline{a}_1) = \{z \in S^1 : \hat{z} \in (\hat{\overline{a}}_1 - \gamma_n,\, \hat{\overline{a}}_1 + \gamma_n) \text{ with } \gamma_n = \tfrac{1}{2} l([t_0,\, f_1^{q_{n-1}}(t_0)])\sqrt[4]{\varepsilon}\}.$$

It is clear that $U_n(\overline{a}_1) \subset V_n(\overline{a}_1) \subset [t_0,\, f_1^{q_{n-1}}(t_0)]$ (see Figure 2).

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Then two cases are possible: either $\overline{b}_1 \in U_n(\overline{a}_1)$ or $\overline{b}_1 \notin U_n(\overline{a}_1)$, that is, $\overline{b}_1 \in [f^{q_n}(t_0),\ f^{q_{n-1}}(t_0)]\backslash U_n(\overline{a}_1)$. Consider first the case $\overline{b}_1 \in U_n(\overline{a}_1)$. If $\hat{\overline{b}}_1$ lies on the left-hand side of the point $\hat{\overline{a}}_1$, we define

$$\begin{aligned} \hat{z}_1 &:= \hat{\overline{a}}_1 - \tfrac{1}{2}l([t_0,\ f^{q_{n-1}}(t_0)])\sqrt[4]{\varepsilon}, \quad \hat{z}_2 := \hat{\overline{a}}_1, \\ \hat{z}_3 &:= \hat{\overline{a}}_1 + \tfrac{1}{4}l([t_0,\ f^{q_{n-1}}(t_0)]), \quad \hat{z}_4 := \hat{\overline{a}}_1 + \tfrac{1}{2}l([t_0,\ f^{q_{n-1}}(t_0)]), \end{aligned} \tag{10}$$

corresponding to the points $z_i \in S^1$, $i = 1, 2, 3, 4$, with $z_2 = \overline{a}_1$ and $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$. If, on the other hand, $\hat{\overline{b}}_1$ is on the right-hand side of $\hat{\overline{a}}_1$, we define

$$\begin{aligned} \hat{z}_1 &:= \hat{\overline{a}}_1 - \tfrac{1}{2}l([t_0,\ f^{q_{n-1}}(t_0)]), \quad \hat{z}_2 := \hat{\overline{a}}_1 - \tfrac{1}{4}l([t_0,\ f^{q_{n-1}}(t_0)]), \\ \hat{z}_3 &:= \hat{\overline{a}}_1, \quad \hat{z}_4 := \hat{\overline{a}}_1 + \tfrac{1}{2}l([t_0,\ f^{q_{n-1}}(t_0)])\sqrt[4]{\varepsilon}, \end{aligned} \tag{11}$$

corresponding to the points $z_i \in S^1$, $i = 1, 2, 3, 4$, with $z_3 = \overline{a}_1$ and $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$. In the following we consider only the first case, the second one can be handled similarly.

LEMMA 4.4. *Suppose that the circle homeomorphism f_1 satisfies the conditions of Lemma 2.3. Let $\delta > 0$ be the constant determined by Lemma 4.2 and let, for large enough n, the points $\hat{z}_i$, $i = 1, 2, 3, 4$, defined in (10) be the lifts of the points $z_i \in S^1$, $i = 1, 2, 3, 4$. Then the triple of intervals $[z_s, z_{s+1}]$, $s = 1, 2, 3$, has the following properties:*

(1) $[z_1, z_4]$, $[f_1^{q_n}(z_1),\ f_1^{q_n}(z_4)] \subset U_\delta(x_0) = \{z \in S^1 : \hat{x} \in (\hat{x}_0 - \delta,\ \hat{x}_0 + \delta)\}$;

(2) *the intervals* $[z_s, z_{s+1}]$, $[f_1^{q_n}(z_s),\ f_1^{q_n}(z_{s+1})]$, $s = 1, 2, 3$, *satisfy conditions* $(C_{R_1,\varepsilon})$ *for some constant $R_1 > 1$ depending only on the variation v of* $\log Df_1$.

LEMMA 4.5. *Assume that the circle homeomorphisms f_i, $i = 1, 2$, satisfy the conditions of Theorem 1.5. Let $z_i \in S^1$, $i = 1, 2, 3, 4$, be the points defined in Lemma 4.4. Then the following inequalities hold for sufficiently large n:*

$$|\mathrm{Dist}(z_1, z_2, z_3, z_4;\ f_1^{q_n}) - \sigma_{f_1}(a_1) \cdot \sigma_{f_1}(b_1)| \le R_2\sqrt[4]{\varepsilon}, \tag{12}$$

$$|\mathrm{Dist}(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4);\ f_2^{q_n}) - \sigma_{f_2}(a_2) \cdot \sigma_{f_2}(b_2)| \le R_2\sqrt[4]{\varepsilon}, \tag{13}$$

where the positive constant $R_2 = R_2(R_1,\ f_1,\ f_2)$ does not depend on ε.

After these preparations we can now proceed to the proof of Theorem 1.5.

Proof of Theorem 1.5. Let f_1 and f_2 be circle homeomorphisms satisfying the conditions of Theorem 1.5. The lift $\hat{\psi}(\hat{x})$ of the conjugating map $\psi(x)$ is a continuous and monotone increasing function on R^1. Hence $\hat{\psi}(\hat{x})$ has a finite derivative $D\hat{\psi}(x)$ almost everywhere (with respect to Lebesgue measure) on R^1. Recall that $D\hat{\psi}(\hat{x} + 1) = D\hat{\psi}(\hat{x})$ for each $\hat{x} \in R^1$ where the derivative $D\hat{\psi}(\hat{x})$ is defined. It is enough to show that $D\hat{\psi}(\hat{x}) = 0$ for almost all points $\hat{x}$ of the interval $[0, 1)$. Suppose that $D\hat{\psi}(\hat{x}_0) = \omega > 0$ for some point $\hat{x}_0 \in [0, 1)$ corresponding to the point $x_0 \in S^1$. Choose an $\varepsilon > 0$ and the points $z_i \in S^1$, $i = 1, 2, 3, 4$, with lifted vector $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ as defined in (10). Then, by the second assertion of Lemma 4.4, the intervals $[z_s, z_{s+1}]$, $[f_1^{q_n}(z_s),\ f_1^{q_n}(z_{s+1})]$, $s = 1, 2, 3$, satisfy conditions $(C_{R_1,\varepsilon})$ for some constant $R_1 > 1$ depending only on the variation v of $\log Df_1$.

Lemma 4.3 next implies that

$$|\mathrm{Dist}(z_1, z_2, z_3, z_4;\ \psi) - 1| \le R_2\sqrt{\varepsilon} \tag{14}$$

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and

$$|\mathrm{Dist}(f_1^{q_n}(z_1),\, f_1^{q_n}(z_2),\, f_1^{q_n}(z_3),\, f_1^{q_n}(z_4);\, \psi) - 1| \le R_2\sqrt{\varepsilon}. \tag{15}$$

Hence

$$\left|\frac{\mathrm{Dist}(f_1^{q_n}(z_1),\, f_1^{q_n}(z_2),\, f_1^{q_n}(z_3),\, f_1^{q_n}(z_4);\, \psi)}{\mathrm{Dist}(z_1,\, z_2,\, z_3,\, z_4;\, \psi)} - 1\right| \le R_3\sqrt{\varepsilon}, \tag{16}$$

where the constant $R_3 > 0$ does not depend on ε and n.

But by definition,

$$\begin{aligned}&\mathrm{Dist}(f_1^{q_n}(z_1),\, f_1^{q_n}(z_2),\, f_1^{q_n}(z_3),\, f_1^{q_n}(z_4);\, \psi)\\ &\quad= \frac{\mathrm{Cr}(\psi(f_1^{q_n}(z_1)),\, \psi(f_1^{q_n}(z_2)),\, \psi(f_1^{q_n}(z_3)),\, \psi(f_1^{q_n}(z_4)))}{\mathrm{Cr}(f_1^{q_n}(z_1),\, f_1^{q_n}(z_2),\, f_1^{q_n}(z_3),\, f_1^{q_n}(z_4))}.\end{aligned}$$

Since ψ is conjugating f_1 and f_2, we can readily see that

$$\begin{aligned}&\mathrm{Cr}(\psi(f_1^{q_n}(z_1)),\, \psi(f_1^{q_n}(z_2)),\, \psi(f_1^{q_n}(z_3)),\, \psi(f_1^{q_n}(z_4)))\\ &\quad= \mathrm{Cr}(f_2^{q_n}(\psi(z_1)),\, f_2^{q_n}(\psi(z_2)),\, f_2^{q_n}(\psi(z_3)),\, f_2^{q_n}(\psi(z_4))).\end{aligned}$$

It now follows that

$$\begin{aligned}&\frac{\mathrm{Dist}(f_1^{q_n}(z_1),\, f_1^{q_n}(z_2),\, f_1^{q_n}(z_3),\, f_1^{q_n}(z_4);\, \psi)}{\mathrm{Dist}(z_1,\, z_2,\, z_3,\, z_4;\, \psi)}\\ &\quad= \frac{\mathrm{Cr}(\psi(f_1^{q_n}(z_1)),\, \psi(f_1^{q_n}(z_2)),\, \psi(f_1^{q_n}(z_3)),\, \psi(f_1^{q_n}(z_4)))}{\mathrm{Cr}(f_1^{q_n}(z_1),\, f_1^{q_n}(z_2),\, f_1^{q_n}(z_3),\, f_1^{q_n}(z_4))}\\ &\qquad\times \frac{\mathrm{Cr}(z_1,\, z_2,\, z_3,\, z_4)}{\mathrm{Cr}(\psi(z_1),\, \psi(z_2),\, \psi(z_3),\, \psi(z_4))}\\ &\quad= \frac{\mathrm{Cr}(f_2^{q_n}(\psi(z_1)),\, f_2^{q_n}(\psi(z_2)),\, f_2^{q_n}(\psi(z_3)),\, f_2^{q_n}(\psi(z_4)))}{\mathrm{Cr}(\psi(z_1),\, \psi(z_2),\, \psi(z_3),\, \psi(z_4))}\\ &\qquad: \frac{\mathrm{Cr}(f_1^{q_n}(z_1),\, f_1^{q_n}(z_2),\, f_1^{q_n}(z_3),\, f_1^{q_n}(z_4))}{\mathrm{Cr}(z_1,\, z_2,\, z_3,\, z_4)}\\ &\quad= \frac{\mathrm{Dist}(\psi(z_1),\, \psi(z_2),\, \psi(z_3),\, \psi(z_4);\, f_2^{q_n})}{\mathrm{Dist}(z_1,\, z_2,\, z_3,\, z_4;\, f_1^{q_n})}.\end{aligned}$$

Combining this with inequality (16), we get

$$\left|\frac{\mathrm{Dist}(\psi(z_1),\, \psi(z_2),\, \psi(z_3),\, \psi(z_4);\, f_2^{q_n})}{\mathrm{Dist}(z_1,\, z_2,\, z_3,\, z_4;\, f_1^{q_n})} - 1\right| \le R_3\sqrt{\varepsilon}. \tag{17}$$

But using Lemma 4.5 gives

$$\left|\frac{\mathrm{Dist}(\psi(z_1),\, \psi(z_2),\, \psi(z_3),\, \psi(z_4);\, f_2^{q_n})}{\mathrm{Dist}(z_1,\, z_2,\, z_3,\, z_4;\, f_1^{q_n})} - \frac{\sigma_{f_2}(a_2)\cdot\sigma_{f_2}(b_2)}{\sigma_{f_1}(a_1)\cdot\sigma_{f_1}(b_1)}\right| \le \mathrm{Const}\sqrt[4]{\varepsilon} \tag{18}$$

for sufficiently large n. This contradiction proves Theorem 1.5 in the first case.

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There remains the case where the point $\overline{b}_1$ belongs to the set $[f_1^{q_n}(t_0), f_1^{q_{n-1}}(t_0)]\backslash U_n(\overline{a}_1)$. Let $\hat{\overline{b}}_1$ lie on the left-hand side of the point $\hat{\overline{a}}_1$; the case of $\hat{\overline{b}}_1$ on the right-hand side of $\hat{\overline{a}}_1$ can be handled similarly. We define

$$\begin{aligned} \hat{z}_1 &:= \hat{\overline{a}}_1 - \tfrac{1}{4}l([t_0,\, f^{q_{n-1}}(t_0)])\sqrt{\varepsilon}, \quad \hat{z}_2 := \hat{\overline{a}}_1, \\ \hat{z}_3 &:= \hat{\overline{a}}_1 + \tfrac{1}{4}l([t_0,\, f^{q_{n-1}}(t_0)]), \quad \hat{z}_4 := \hat{\overline{a}}_1 + \tfrac{1}{2}l([t_0,\, f^{q_{n-1}}(t_0)]), \end{aligned} \tag{19}$$

which determine the points $z_i \in S^1$, $i = 1, 2, 3, 4$, with $z_2 = \overline{a}_1$ and $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$. The proof of Theorem 1.5 for the corresponding intervals $[z_s, z_{s+1}]$, $s = 1, 2, 3$, now proceeds exactly as in the previous case. This concludes the proof of Theorem 1.5. □

5. *Proofs of Lemmas 4.2–4.5*

Proof of Lemma 4.2. Suppose that the derivative $D\hat{\psi}(\hat{x}_0)$ exists and $D\hat{\psi}(\hat{x}_0) = \omega > 0$ for the lift $\hat{x}_0 \in [0, 1)$ of some point x_0 in S^1. By the definition of the derivative there exists, for any $\varepsilon > 0$, a number $\delta = \delta(x_0, \varepsilon) \in (0, d(x_0))$ such that, for all $\hat{x} \in (\hat{x}_0 - \delta, \hat{x}_0 + \delta)$,

$$\omega - \varepsilon < \frac{\hat{\psi}(\hat{x}) - \hat{\psi}(\hat{x}_0)}{\hat{x} - \hat{x}_0} < \omega + \varepsilon. \tag{20}$$

Now take four points $\hat{z}_i \in (\hat{x}_0 - \delta, \hat{x}_0 + \delta) \subset [0, 1)$ satisfying conditions $(C_{R_1,\varepsilon})$. We can assume without loss of generality that $[z_1, z_4] \subset U_\delta(x_0)$ and $z_1 \prec z_4 \prec x_0 \prec z_1$. Relation (20) then implies for $\hat{x} = \hat{z}_i$, $i = 1, 2, 3, 4$, that

$$(\omega - \varepsilon)(\hat{x}_0 - \hat{z}_i) < \hat{\psi}(\hat{x}_0) - \hat{\psi}(\hat{z}_i) < (\omega + \varepsilon)(\hat{x}_0 - \hat{z}_i).$$

This yields the following inequalities for $\hat{z}_s$, $s = 1, 2, 3$:

$$\begin{aligned} \omega - \varepsilon\frac{(\hat{x}_0 - \hat{z}_{s+1}) + (\hat{x}_0 - \hat{z}_s)}{\hat{z}_{s+1} - \hat{z}_s} &< \frac{\hat{\psi}(\hat{z}_{s+1}) - \hat{\psi}(\hat{z}_s)}{\hat{z}_{s+1} - \hat{z}_s} \\ &< \omega + \varepsilon\frac{(\hat{x}_0 - \hat{z}_{s+1}) + (\hat{x}_0 - \hat{z}_s)}{\hat{z}_{s+1} - \hat{z}_s}, \end{aligned} \tag{21}$$

and for $s = 1, 2$,

$$\begin{aligned} \omega - \varepsilon\frac{(\hat{x}_0 - \hat{z}_{s+2}) + (\hat{x}_0 - \hat{z}_s)}{\hat{z}_{s+2} - \hat{z}_s} &\le \frac{\hat{\psi}(\hat{z}_{s+2}) - \hat{\psi}(\hat{z}_s)}{\hat{z}_{s+2} - \hat{z}_s} \\ &\le \omega + \varepsilon\frac{(\hat{x}_0 - \hat{z}_{s+2}) + (\hat{x}_0 - \hat{z}_s)}{\hat{z}_{s+2} - \hat{z}_s}. \end{aligned} \tag{22}$$

Since the points z_i, $i = 1, 2, 3, 4$, satisfy conditions $(C_{R_1,\varepsilon})$, it is easy to show that

$$\max_{1\le i\le 4}\left\{\frac{\hat{x}_0 - \hat{z}_i}{\hat{z}_2 - \hat{z}_1}\right\} \le R_1\frac{\hat{z}_3 - \hat{z}_2}{\hat{z}_2 - \hat{z}_1} \le \frac{R_1^2}{\sqrt{\varepsilon}}, \tag{23}$$

$$\max_{1\le i\le 4}\left\{\frac{\hat{x}_0 - \hat{z}_i}{\hat{z}_3 - \hat{z}_2}, \frac{\hat{x}_0 - \hat{z}_i}{\hat{z}_4 - \hat{z}_3}\right\} \le R_1^2. \tag{24}$$

Combining relations (21)–(24), we get

$$\omega - C_4\sqrt{\varepsilon} \le \frac{\hat{\psi}(\hat{z}_2) - \hat{\psi}(\hat{z}_1)}{\hat{z}_2 - \hat{z}_1} \le \omega + C_4\sqrt{\varepsilon}; \tag{25}$$

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for $l = 2, 3$, we get

$$\omega - C_4\varepsilon \le \frac{\hat\psi(\hat z_{l+1}) - \hat\psi(\hat z_l)}{\hat z_{l+1} - \hat z_l} \le \omega + C_4\varepsilon, \tag{26}$$

and for $s = 1, 2$,

$$\omega - C_4\varepsilon \le \frac{\hat\psi(\hat z_{s+2}) - \hat\psi(\hat z_s)}{\hat z_{s+2} - \hat z_s} \le \omega + C_4\varepsilon, \tag{27}$$

where the constant $C_4 > 0$ depends on R_1, ω, but does not depend on $l([\hat z_s, \hat z_{s+1}])$, $s = 1, 2, 3$, or on ε. Using the equality

$$\frac{\hat\psi(\hat z_{s+1}) - \hat\psi(\hat z_s)}{\hat\psi(\hat z_s) - \hat\psi(\hat z_{s-1})} : \frac{\hat z_{s+1} - \hat z_s}{\hat z_s - \hat z_{s-1}} = \frac{\hat\psi(\hat z_{s+1}) - \hat\psi(\hat z_s)}{\hat z_{s+1} - \hat z_s} \cdot \frac{\hat z_s - \hat z_{s-1}}{\hat\psi(\hat z_s) - \hat\psi(\hat z_{s-1})}$$

and relations (25)–(27), we get the assertions of Lemma 4.2. □

Proof of Lemma 4.3. Since

$$\begin{aligned}\mathrm{Dist}(z_l, z_2, z_3, z_4, \psi) &= \frac{\mathrm{Cr}(\hat\psi(\hat z_1), \hat\psi(\hat z_2), \hat\psi(\hat z_3), \hat\psi(\hat z_4))}{\mathrm{Cr}(\hat z_1, \hat z_2, \hat z_3, \hat z_4)} \\ &= \frac{\hat\psi(\hat z_2) - \hat\psi(\hat z_1)}{\hat z_2 - \hat z_1} \cdot \frac{\hat\psi(\hat z_4) - \hat\psi(\hat z_3)}{\hat z_4 - \hat z_3} \cdot \frac{\hat z_3 - \hat z_1}{\hat\psi(\hat z_3) - \hat\psi(\hat z_1)} \cdot \frac{\hat z_4 - \hat z_2}{\hat\psi(\hat z_4) - \hat\psi(\hat z_2)},\end{aligned}$$

inequalities (25)–(27) prove Lemma 4.3. □

Proof of Lemma 4.4. We assume n to be odd. Hence $f_1^{q_n}(z) \prec z \prec f_1^{q_{n-1}}(z) \prec f_1^{q_n}(z)$ for any point z on the circle S^1. The point $\overline{a}_1$ lies in the interval $[f_1^{q_n}(x_0), f_1^{q_{n-1}}(x_0)]$ and is the middle point of the interval $[t_0, f_1^{q_{n-1}}(t_0)]$. This and the structure of the orbits imply $x_{-3q_{n-1}} \prec f_1^{q_n}(t_0) \prec t_0 \prec f_1^{q_{n-1}}(t_0) \prec x_{3q_{n-1}}$. By construction, $[z_1, z_4] \subset [t_0, f_1^{q_{n-1}}(t_0)]$. Consequently, $[f_1^{q_n}(z_1), f_1^{q_n}(z_4)] \subset [f_1^{q_n}(t_0), f_1^{q_n+q_{n-1}}(t_0)]$. Summarizing, we therefore get

$$[z_1, z_4], [f_1^{q_n}(z_1), f_1^{q_n}(z_4)] \subset [x_{-3q_{n-1}}, x_{3q_{n-1}}]. \tag{28}$$

Obviously,

$$\begin{aligned}[x_{-3q_{n-1}}, x_{3q_{n-1}}] = [x_{-3q_{n-1}}, x_{-2q_{n-1}}] \cup [x_{-2q_{n-1}}, x_{-q_{n-1}}] \cup [x_{-q_{n-1}}, x_0] \\ \cup [x_0, x_{q_{n-1}}] \cup [x_{q_{n-1}}, x_{2q_{n-1}}] \cup [x_{2q_{n-1}}, x_{3q_{n-1}}].\end{aligned} \tag{29}$$

By Corollary 2.7, the intervals $[x, y]$, $[f_1^{q_{n-1}}(x), f_1^{q_{n-1}}(y)]$ and $[f_1^{-q_{n-1}}(x), f_1^{-q_{n-1}}(y)]$ are e^{v_1}-comparable for any $x, y \in S^1$. This together with equation (29) and Corollary 2.4 implies that

$$l([x_{-3q_{n-1}}, x_{3q_{n-1}}]) \le (1 + 5e^{3v_1}) l([x_0, x_{q_{n-1}}]) \le \mathrm{const}\, \lambda_1^n,$$

for a constant $\lambda_1 \in (0, 1)$. For sufficiently large n then obviously $[x_{-3q_{n-1}}, x_{3q_{n-1}}] \subset (x_0 - \delta, x_0 + \delta)$. This together with (28) implies the first assertion of Lemma 4.4.

Next we will prove the second assertion of Lemma 4.4 . By Corollary 2.7, the intervals $[z_s, z_{s+1}]$ and $[f^{q_n}(z_s), f^{q_n}(z_{s+1})]$ are e^{v_1}-comparable for all $s = 1, 2, 3$. Using

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the definition of the points z_s, $s = 1, 2, 3, 4$, and the Denjoy inequality, it is easy to verify that these intervals satisfy assumptions (a) and (b) of conditions $(C_{R_1,\varepsilon})$. Using the relations

$$[z_1, z_4], [f_1^{q_n}(z_1), f_1^{q_n}(z_4)] \subset [x_{-3q_{n-1}}, x_{3q_{n-1}}],$$

we get

$$\max_{1\le s\le 4}\{|\hat{x}_0 - \hat{z}_s|, |\hat{x}_0 - \hat{y}_s|\} \le l([x_{-3q_{n-1}}, x_{3q_{n-1}}]), \tag{30}$$

where $y_s := f_1^{q_n}(z_s)$ $s = 1, 2, 3, 4$. Now we want to compare the lengths of the intervals $[x_{-3q_{n-1}}, x_{3q_{n-1}}]$ and $[t_0, f_1^{q_{n-1}}(t_0)]$. Using the definition of t_0, it is easy to see that $x_{-2q_{n-1}} \prec t_0 \prec x_{2q_{n-1}}$. Applying $f_1^{sq_{n-1}}$, $s \in \mathbb{Z}$, to these relations, we get $x_{(s-2)q_{n-1}} \prec t_{sq_{n-1}} \prec x_{(s+2)q_{n-1}}$, $s \in \mathbb{Z}$. In particular, the last relations imply that $t_{-5q_{n-1}} \prec x_{-3q_{n-1}}, x_{3q_{n-1}} \prec t_{5q_{n-1}}$ and hence $[x_{-3q_{n-1}}, x_{3q_{n-1}}] \subset [t_{-5q_{n-1}}, t_{5q_{n-1}}]$. Consequently,

$$l[(x_{-3q_{n-1}}, x_{3q_{n-1}}]) \le l([t_{-5q_{n-1}}, t_{5q_{n-1}}]). \tag{31}$$

But the intervals $[t_{-5q_{n-1}}, t_{5q_{n-1}}]$ and $[t_0, f_1^{q_{n-1}}(t_0)]$ are $(1 + 2e^{v_1} + 2e^{2v_1} + 2e^{3v_1} + 2e^{4v_1} + e^{5v_1})$-comparable. This together with equation (30) implies that $l[(x_{-3q_{n-1}}, x_{3q_{n-1}}]) \le 10e^{5v_1} l([t_0, f_1^{q_{n-1}}(t_0)])$. Finally, we conclude that the points z_s, $s = 1, 2, 3, 4$, and $f_1^{q_n}(z_s)$, $s = 1, 2, 3, 4$, satisfy conditions $(C_{R_1,\varepsilon})$ with the constant $R_1 = 40e^{5v_1}$. This concludes the proof of Lemma 4.4. □

Proof of Lemma 4.5. Choose the points z_s, $s = 1, 2, 3, 4$, according to formulas (10) and consider the two sets of intervals $\{f_1^i[z_s, z_{s+1}]$, $0 \le i < q_n$, $s = 1, 2, 3\}$ and $\{f_2^i[\psi(z_s), \psi(z_{s+1})]$, $0 \le i < q_n$, $s = 1, 2, 3\}$. By the construction of the intervals $[z_s, z_{s+1}]$, $s = 1, 2, 3$, only the intervals $f_1^l([z_1, z_2])$ and $f_1^p([z_1, z_2])$ cover the break points a_1 and b_1 respectively, namely $a_1 = f_1^l(z_2)$, $b_1 \in f_1^p[z_1, z_2)$.

Similarly, only the intervals $f_2^l[\psi(z_1), \psi(z_2)]$ and $f_2^p[\psi(z_1), \psi(z_2)]$ cover the break points a_2 and b_2 respectively, namely $a_2 = f_2^l(\psi(z_2))$ and $b_2 \in f_2^p[\psi(z_1), \psi(z_2))$. Next we compare the distortions $\mathrm{Dist}(z_1, z_2, z_3, z_4; f_1^{q_n})$ and $\mathrm{Dist}(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4); f_2^{q_n})$. We estimate only the first distortion; the second one can be estimated analogously. Rewriting it as

$$\begin{aligned}\mathrm{Dist}(z_1, z_2, z_3, z_4; f_1^{q_n}) = &\prod_{\substack{i=0\\ i\ne l,p}}^{q_n-1} \mathrm{Dist}(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1)\\ &\times \mathrm{Dist}(f_1^l(z_1), f_1^l(z_2), f_1^l(z_3), f_1^l(z_4); f_1)\\ &\times \mathrm{Dist}(f_1^p(z_1), f_1^p(z_2), f_1^p(z_3), f_1^p(z_4); f_1),\end{aligned} \tag{32}$$

we apply Lemma 3.3 to obtain

$$\begin{aligned}&\prod_{\substack{i=0\\ i\ne l,p}}^{q_n-1} \mathrm{Dist}(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1)\\ &\quad = \exp\left\{\sum_{\substack{i=0\\ i\ne l,p}}^{q_n-1} \log(1 + O(|[f_1^i(z_1), f_1^i(z_4)]|^{1+\alpha}))\right\}.\end{aligned} \tag{33}$$

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By construction, $[f_1^i(z_1),\ f_1^i(z_4)] \subset [f_1^i(t_0),\ f_1^i(f_1^{q_{n-1}}(t_0))]$ for $0 \le i < q_n$. By Corollary 2.4 the length of the last interval is bounded by const λ_1^n. Thus we get, for $0 \le i < q_n$,

$$l([f_1^i(z_1),\ f_1^i(z_4)]) \le \text{const}\ \lambda_1^n. \tag{34}$$

The interval $[t_0,\ f_1^{q_{n-1}}(t_0)]$ is q_n-small and hence

$$\sum_{i=0}^{q_n-1} l([f_1^i(z_1),\ f_1^i(z_4)]) \le 1. \tag{35}$$

Combining equations (33)–(35), we get

$$\begin{aligned}&\left|\prod_{\substack{i=0\\ i\ne l,p}}^{q_n-1} \mathrm{Dist}(f_1^i(z_1),\ f_1^i(z_2),\ f_1^i(z_3),\ f_1^i(z_4);\ f_1) - 1\right| \\ &\quad \le \text{const}\ \lambda_1^{n\alpha} \sum_{\substack{i=0\\ i\ne l,p}}^{q_n-1} l([f_1^i(z_1),\ f_1^i(z_4)]) \le \text{const}\ \lambda_1^{n\alpha}.\end{aligned} \tag{36}$$

Next we estimate the distortions

$$\mathrm{Dist}(f_1^l(z_1),\ f_1^l(z_2),\ f_1^l(z_3),\ f_1^l(z_4);\ f_1)$$

and

$$\mathrm{Dist}(f_1^p(z_1),\ f_1^p(z_2),\ f_1^p(z_3),\ f_1^p(z_4);\ f_1).$$

Define, for $0 \le m < q_n$, the length ratios

$$\vartheta(m) := \frac{l([f_1^m(z_2),\ f_1^m(z_3)])}{l([f_1^m(z_1),\ f_1^m(z_2)])}, \quad \tau(m) := \frac{l([f_1^m(\bar{b}_1),\ f_1^m(z_2)])}{l([f_1^m(z_1),\ f_1^m(z_2)])}. \tag{37}$$

Lemma 2.2 then implies the inequalities

$$e^{-v_1}\cdot\vartheta(0) \le \vartheta(m) \le e^{v_1}\cdot\vartheta(0), \quad e^{-v_1}\cdot\tau(0) \le \tau(m) \le e^{v_1}\cdot\tau(0). \tag{38}$$

Using the definitions of the points z_i, $i = 1, 2, 3$, we get

$$\vartheta(0) = \frac{1}{2\sqrt[4]{\varepsilon}}, \quad 0 \le \tau(0) \le \frac{\sqrt[4]{\varepsilon}}{4}. \tag{39}$$

Define next, for $x > 0$ and $0 \le t \le 1$, the functions $G(x, \sigma)$ and $F(x, t, \sigma)$ as

$$G(x,\sigma) := \frac{\sigma(1+x)}{\sigma+x}, \quad F(x,t,\sigma) := \frac{[\sigma+(1-\sigma)t](1+x)}{\sigma+(1-\sigma)t+x}. \tag{40}$$

Applying Lemma 3.4, we get

$$\begin{aligned}&|\mathrm{Dist}(f_1^l(z_1),\ f_1^l(z_2),\ f_1^l(z_3),\ f_1^l(z_4);\ f_1) - G(\vartheta(l), \sigma_{f_1}(a_1))| \\ &\quad \le K_2\, l([f_1^l(z_1),\ f_1^l(z_4)]),\end{aligned} \tag{41}$$

$$\begin{aligned}&|\mathrm{Dist}(f_1^p(z_1),\ f_1^p(z_2),\ f_1^p(z_3),\ f_1^p(z_4);\ f_1) - F(\vartheta(p), \tau(p), \sigma_{f_1}(b_1))| \\ &\quad \le K_2\, l([f_1^p(z_1),\ f_1^p(z_4)]).\end{aligned} \tag{42}$$

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The definitions of the functions G, F together with equations (37) and (38) imply that

$$|G(\vartheta(l), \sigma_{f_1}(a_1)) - G(\vartheta(0), \sigma_{f_1}(a_1))| \leq K_3 \sqrt[4]{\varepsilon}, \tag{43}$$

$$|F(\vartheta(p), \tau(p), \sigma_{f_1}(b_1)) - F(\vartheta(0), \tau(0), \sigma_{f_1}(b_1))| \leq K_3 \sqrt[4]{\varepsilon},$$

$$|G(\vartheta(0), \sigma_{f_1}(a_1)) - \sigma_{f_1}(a_1)| \leq K_3 \sqrt[4]{\varepsilon}, \tag{44}$$

$$|F(\vartheta(0), \tau(0), \sigma_{f_1}(b_1)) - \sigma_{f_1}(b_1)| \leq K_3 \sqrt[4]{\varepsilon},$$

where the constant K_3 is given by

$$K_3 = (\sigma_{f_1}(a_1)|1 - \sigma_{f_1}(a_1)| + \sigma_{f_1}(b_1)|1 - \sigma_{f_1}(b_1)|)(1 + e^{v_1}).$$

The last four equations imply that

$$|G(\vartheta(l), \sigma_{f_1}(a_1)) - \sigma_{f_1}(a_1)| \leq 2\, K_3 \sqrt[4]{\varepsilon}, \tag{45}$$

$$|F(\vartheta(p), \tau(p), \sigma_{f_1}(b_1)) - \sigma_{f_1}(b_1)| \leq 2\, K_3 \sqrt[4]{\varepsilon}. \tag{46}$$

Combining equations (31), (32), (35) and (40)–(46) we obtain, finally,

$$|\mathrm{Dist}(z_1, z_2, z_3, z_4;\, f_1^{q_n}) - \sigma_{f_1}(a_1) \cdot \sigma_{f_1}(b_1)| \leq R_2 \sqrt[4]{\varepsilon},$$

which proves the first inequality in Lemma 4.5. □

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