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On maximal subgroups of ample groups

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Abstract. The paper is concerned with maximal subgroups of the ample (better known as topological full) groups of homeomorphisms of totally disconnected compact metrizable topological spaces. We describe all maximal subgroups that are stabilizers of finite sets. Under certain assumptions on the ample group (including minimality), we describe all maximal subgroups that are stabilizers of closed sets or stabilizers of partitions into clopen sets. In particular, our results apply to the ample groups associated with Cantor minimal systems and to some Higman–Thompson groups.

Key words: ample group, topological full group, maximal subgroup, Cantor minimal system
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1. *Introduction*

In this paper, we study maximal subgroups of the ample groups (the latter are better known as the topological full groups (TFGs)). Let us begin with an overview of maximal and related to them weakly maximal subgroups.

A proper subgroup H of a group G is *maximal* if there is no group placed between H and G in the lattice of subgroups of G. Maximal subgroups play a very important role in group theory. The task of describing all maximal subgroups of a group G is equivalent to the task of classifying all primitive actions of G (an action of the group on a set X is *primitive* if it preserves no non-trivial equivalence relations on X). Indeed, if H is a maximal subgroup of G, then the natural action of G on the cosets of H is primitive. Conversely, if $\alpha : G \curvearrowright X$ is a primitive action on a set X that is not a singleton, then the point stabilizers $\mathrm{St}_\alpha(x)$, $x \in X$ are maximal subgroups of the group G. Maximal subgroups can also arise as the stabilizers of sets or collections of sets (e.g., partitions).

The problem of describing all maximal subgroups of a given group attracted much attention. Among the most remarkable results here is the complete solution of this problem for finite symmetric groups (based on the O'Nan–Scott theorem), which was obtained in the 1980s as one of the first applications of the classification of finite simple groups (see, e.g., the book [**15**, §8.5]). There is also an understanding on how to approach this problem

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for general finite groups (as outlined by Aschbacher and Scott [3]). Much less is known about maximal subgroups of infinite groups. One may be interested in whether such a group has a maximal subgroup of infinite index. For countable groups, this question is closely related to the question about *primitivity* of the group, that is, the existence of a faithful primitive action. Another question is to determine how many maximal subgroups are there, say, whether a given countable group has a *continuum* (that is, an uncountable set of cardinality $2^{\aleph_0}$) of maximal subgroups. If an infinite group does not have many maximal subgroups, it makes sense to look for *weakly maximal subgroups*, which are subgroups of infinite index maximal with respect to this property. Note that in the case a countable group G is finitely generated, any subgroup of finite index in G is finitely generated as well, which implies that there are at most countably many such subgroups. More importantly, it follows that any proper subgroup of G is contained in a maximal subgroup, while any subgroup of infinite index is contained in a weakly maximal subgroup.

A substantial progress in the late 1970s was achieved by Margulis and Soifer [41–43]. Answering a question of Platonov, they obtained fundamental results on maximal subgroups of finitely generated linear groups. The first of their results is that any such group admits a maximal subgroup of infinite index if and only if it is not virtually solvable. Another result is that the free non-abelian group F_n on $n \geq 2$ generators has a continuum of maximal subgroups. These results were extended by Gelander and Glasner [20] to general countable linear groups. Later, Gelander and Meiri [22] showed that each of the groups $SL_n(\mathbb{Z})$, $n \geq 3$ has a continuum of maximal subgroups. For more on these and related topics, see the survey [21].

A completely different story is told by the groups of branch type. The class of the *branch groups* was introduced in [32] in relation to the study of just infinite groups (these are infinite groups in which every proper quotient is finite) and as an abstract model behind the family of groups $\mathcal{X} = \{G_\omega\}$, $\omega \in \{0, 1, 2\}^{\mathbb{N}}$ constructed in [30]. This family mostly consists of groups of intermediate growth (faster than polynomial, but slower than exponential). Two notable representatives of the family are the groups $\mathcal{G}_{(012)^\infty}$ (the 'first' Grigorchuk group) and $\mathcal{G}_{(01)^\infty}$ (the Grigorchuk–Erschler group) given by periodic sequences $(012)^\infty = 012012\ldots$ and $(01)^\infty = 0101\ldots$. These two groups were also at the root of studies of the class of *self-similar groups* (see the book [49]). Among recent discoveries is the fact that they are related to substitution subshifts generated by primitive substitutions, in particular, the period doubling and Morse subshifts (see [28, 29, 44]). These subshifts are important representatives of the Cantor minimal systems (the latter are important in our study).

The question of Hartly from 1993 about the existence of maximal subgroups of infinite index in the group $\mathcal{G}_{(012)^\infty}$ was answered in the negative by Pervova [52]. This result had a big impact on the study of $\mathcal{G}_{(012)^\infty}$ and led to establishing such properties as subgroup separability (also called the LERF property) and decidability of the generalized word problem (see [34]). In [53], Pervova extended her result to some branch groups in a family that was named the GGS (Grigorchuk–Gupta–Sidki) groups in the book [9]. The groups considered by Pervova are torsion groups (that is, all elements are of finite order) and, presently, there are no known examples of finitely generated torsion branch groups admitting a maximal subgroup of infinite index. Recently, Francoeur and Thillaisundaram

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[19] extended Pervova's result to all GGS groups, including non-torsion groups. However, Bondarenko [13] constructed an example of a non-torsion branch group with a maximal subgroup of infinite index. Later, Francoeur and Garrido [18] discovered that, in fact, the group $\mathcal{G}_{(01)^\infty}$ has maximal subgroups of infinite index. There are only countably many of those and they are all described. Also, it was shown that the non-torsion iterated monodromy groups of the tent map (a special case of some groups first introduced by Šunić in [57] as 'siblings of the Grigorchuk group') have exactly countably many maximal subgroups of infinite index (those are described up to conjugacy). Presently, there are no known examples of finitely generated branch groups with uncountably many maximal subgroups. Another relevant fact is that the branch groups cannot act quasi-2-transitively on infinite sets (see [17]).

The *weakly branch groups* are a far going generalization of the class of branch groups. Their introduction was initiated by the studies around the Basilica group $\mathcal{B}$ invented in [35] as a group generated by a 3-state automaton over the binary alphabet. Bartholdi and Virag [8] proved amenability of $\mathcal{B}$, which produced the first example of an amenable but not subexponentially amenable group (answering the question from [31] about existence of such groups). Later, it was discovered that $\mathcal{B}$ is isomorphic to the iterated monodromy group of the polynomial $z^2 - 1$ [5, 49]. In the paper [16], Francoeur extended techniques of Pervova to a large class of weakly branch groups, which allowed him not only to prove that any maximal subgroup in a branch group is itself branch, but also to prove that the Basilica group has maximal subgroups of infinite index and hence is primitive (as each proper quotient of $\mathcal{B}$ is virtually nilpotent).

The next class of infinite groups whose maximal subgroups attracted the attention of researchers are the *Higman–Thompson-type groups*. Savchuk [55, 56] showed that all orbits of the action of Thompson's group F on the interval $(0, 1)$ are primitive, which implies that the point stabilizers $\mathrm{St}_F(x)$, $x \in (0, 1)$ are maximal subgroups of infinite index. The associated Schreier coset graphs are quasi-isometric to a tree. It was also shown that $\mathrm{St}_F(x)$ is not finitely generated if x is irrational while being finitely generated for $x \in \mathbb{Z}[1/2]$. This study was extended by Golan and Sapir [25–27], Aiello and Nagnibeda [1, 2], and others. Among other results, Golan and Sapir produced the first example of a maximal subgroup of F that does not arise as the stabilizer of a point. Another Thompson's group V is considered by Belk *et al* in [11] where an uncountable family of maximal subgroups of V is produced. Moreover, it is shown that there are uncountably many pairwise non-isomorphic maximal subgroups of V. In [10], the same four authors proved that Thompson's group T is yet another maximal subgroup of the group V.

Weakly maximal subgroups of finitely generated infinite groups can often play a role similar to that of maximal subgroups of infinite index. For instance, they are useful in the study of profinite groups [4]. As was observed in [6, 7], stabilizers of points on the boundary of the rooted tree on which a weakly branch group acts are weakly maximal subgroups. In the case of branch groups from the family $\mathcal{X}$, the associated Schreier coset graphs have surprisingly simple, linear geometric structure. Almost all of them are quasi-isometric to the Cayley graph of $\mathbb{Z}$. When taking into account edge labels, the structure becomes more complicated, but still controlled. For the group $\mathcal{G}_{(012)^\infty}$, for example, the graphs represent the so-called aperiodic order (see [28]).

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Branch groups, weakly branch groups, and groups of Higman–Thompson type are subclasses of the class of *micro-supported groups*. Elements of this class are groups G acting faithfully on a topological space X in such a way that for any non-empty open subset $U \subset X$, the rigid stabilizer $\mathrm{RiSt}_G(U)$ (also referred to as the local subgroup and denoted G_U), which consists of elements acting trivially on $X \setminus U$, is non-trivial. Also, all these groups represent an even wider class of *dynamically defined groups*, the object of a fast developing new area on the border between theory of dynamical systems and group theory. A recent book by Nekrashevych [51] is an excellent source of information on methods and results of this direction of mathematics.

A rich source of dynamically defined groups are groups that will be called in this paper the *ample groups*. The idea of amplification (or saturation) in dynamics and group theory is quite simple. Given a topological space X and a group G of its homeomorphisms, one can enlarge G to a group $\mathsf{F}(G)$ by adding those homeomorphisms of X that act locally as elements of G. We call the group $\mathsf{F}(G)$ the *full amplification* of G. The group G is called *ample* if $\mathsf{F}(G) = G$. Note that $\mathsf{F}(\mathsf{F}(G)) = \mathsf{F}(G)$ so that the group $\mathsf{F}(G)$ is always ample. This idea works best when X is a Cantor set or, more generally, a totally disconnected compact metrizable space. This is because the topology on such a space is generated by clopen (that is, both closed and open) sets. Clopen sets allow the cutting of all homeomorphisms in G into pieces, then new homeomorphisms can be constructed, as a jigsaw puzzle, out of those pieces.

In what follows, all topological spaces are assumed to be totally disconnected, compact, and metrizable. There are many situations when continuous group actions on such spaces arise so that the above construction can be used. For example, any countable group G acts naturally by permutations on the space A^G, where A is a finite set with more than one element. The actions of G on closed invariant subsets of A^G are now a popular area of study. The group of automorphisms of an infinite, locally finite tree acts naturally on the boundary of the tree. Any countable group G acts by conjugation on the space $\mathrm{Sub}(G)$ of its subgroups, where the topology on $\mathrm{Sub}(G)$ is induced by the product topology on 2^G. This list can go on.

The notion of an ample group was introduced by Krieger in [40]. The groups considered in [40] are locally finite (that is, every finitely generated subgroup is finite) and so this notion has seen limited use being applied only to locally finite groups. We would like to extend it to arbitrary groups of homeomorphisms replacing the common notion of a TFG. The latter originated in the theory of Cantor minimal systems. A *Cantor minimal system* (X, f) consists of a Cantor set X and a minimal homeomorphism $f : X \to X$. In the paper [24], Glasner and Weiss associated to (X, f) two groups, the *full group* $[f]$ and the *finite full group* $[[f]]$ (notation is from [23, 45], respectively). The full group $[f]$ consists of all homeomorphisms of X that leave invariant every orbit of f. It is relevant to the study of orbit equivalence. Any map $g \in [f]$ can be given by a formula $g(x) = f^{n(x)}(x)$, $x \in X$ for some function $n : X \to \mathbb{Z}$. Since the minimal homeomorphism f has no periodic points, it follows that the function n is unique and its level sets are closed. The finite full group $[[f]]$ consists of those $g \in [f]$ for which the function n is continuous or, equivalently, takes only finitely many values. In that case, all level sets of n are clopen. Informally, elements of $[f]$ are 'pointwise' elements of the cyclic group $\langle f \rangle$, while elements of

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$[[f]]$ are 'piecewise' elements of $\langle f\rangle$. It is easy to observe that $[[f]] = \mathsf{F}(\langle f\rangle)$, the full amplification of the cyclic group generated by f. The group $[[f]]$ was renamed the TFG in [23]. An important property of amplification is that the ample group $\mathsf{F}(G)$ is countable whenever G is countable. In particular, the TFG $[[f]]$ is always countable whereas the full group $[f]$ is not.

The remarkable result proved by Giordano, Putnam, and Skau [23] is that the (isomorphism class of) TFG $[[f]]$ is an almost complete invariant of dynamics of a Cantor minimal system (X, f). To be precise, if (X, f) and (Y, g) are two Cantor minimal systems such that the groups $[[f]]$ and $[[g]]$ are isomorphic, then the systems are *flip conjugate*, which means that for some homeomorphism $\phi : X \to Y$, we have $\phi f \phi^{-1} = g$ or g^{-1}. As there is a continuum of pairwise non-flip-conjugate Cantor minimal systems (e.g., systems with different entropies), one gets a continuum of pairwise non-isomorphic TFGs. The result is based on two fundamental facts about TFGs. For simplicity, let us assume that $X = Y$. The first fact, established in [23], is that if the TFGs $[[f]]$ and $[[g]]$ are isomorphic, then any isomorphism between them is implemented by a conjugation in the group $\mathrm{Homeo}(X)$ of all homeomorphisms of X. This property of TFGs seems to be characteristic for various kinds of micro-supported groups (see the book [51, §2.2]). The second fact, derived from an older result of Boyle, is that if the ample groups $[[f]]$ and $[[g]]$ are the same, then the cyclic groups $\langle f\rangle$ and $\langle g\rangle$ are conjugate in the group $[[f]] = [[g]]$.

The systematic study of group-theoretic properties of the TFG $[[f]]$ associated with a Cantor minimal system (X, f) was initiated by Matui [45]. He showed that the commutator subgroup of $[[f]]$ is simple. In the case where f is a minimal subshift over a finite alphabet, the commutator subgroup is finitely generated (but not finitely presented). The group $[[f]]$ admits a non-trivial homomorphism onto $\mathbb{Z}$ and the kernel of that homomorphism is a product of two locally finite subgroups (the so-called factorization property). Further, the group $[[f]]$ has the property LEF (local embeddability into finite groups), see [33]. Any finite group and the free abelian group of infinite rank embed into $[[f]]$. Every branch group from the family $\mathcal{X}$ mentioned before embeds into some TFG [44]. The remarkable result obtained by Juschenko and Monod [37] (confirming a conjecture of Grigorchuk and Medynets [33]) states that any TFG associated with a Cantor minimal system is amenable. Note that these groups are not elementary amenable, which means that they cannot be obtained from finite and abelian groups using certain natural operations that preserve amenability. This is the second known type of amenable but not elementary amenable groups, the first type being the groups of intermediate growth.

The goal of this paper is to initiate the systematic study of maximal subgroups of general ample groups. Our results are mostly reminiscent of the classification of maximal subgroups in finite symmetric groups. Let us recall that all subgroups of a finite symmetric group are split into three classes: (i) *intransitive subgroups* (those that leave invariant a non-trivial subset); (ii) *imprimitive subgroups* (transitive subgroups that leave invariant a non-trivial partition); and (iii) *primitive subgroups* (the remaining ones). Maximal subgroups in the first two classes are easy to describe. Namely, intransitive maximal subgroups are stabilizers of certain subsets, while imprimitive maximal subgroups are stabilizers of certain partitions. Primitive maximal subgroups form a number of subclasses described by the O'Nan–Scott theorem.

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When dealing with the ample groups, arbitrary subsets and partitions should be replaced by closed subsets and partitions into closed subsets. Similarly, transitivity is replaced with minimality (which is the absence of non-trivial closed invariant subsets). An ample group $\mathcal{G} \subset \mathrm{Homeo}(X)$ acting minimally on the topological space X is an analog of a finite symmetric group (in the case where X is finite, $\mathcal{G}$ is exactly the symmetric group). So, we split all subgroups of the group $\mathcal{G}$ into three classes: (I) subgroups leaving invariant a nontrivial closed set; (II) subgroups acting minimally on X, but leaving invariant a non-trivial partition into closed sets; and (III) topologically primitive subgroups (the remaining ones). Furthermore, we distinguish three subclasses in the first class: (I_1) subgroups leaving invariant a non-empty finite set; (I_2) subgroups leaving invariant an infinite, nowhere dense closed set; and (I_3) subgroups leaving invariant a non-trivial clopen set. For general subgroups, these subclasses need not be disjoint or cover the entire class (I), but as far as the maximal subgroups are concerned, the subclasses (I_1), (I_2), and (I_3) form a partition of the class (I). In class (II), we distinguish one subclass (II_0) consisting of subgroups leaving invariant a non-trivial partition into clopen sets (or, equivalently, a finite partition into closed sets).

We proceed to the description of our main results. The first of them provides a characterization of those maximal subgroups of ample groups that are stabilizers of closed sets. Namely, they are exactly maximal subgroups in the class (I).

THEOREM 1.1. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X. Suppose H is a maximal subgroup of $\mathcal{G}$ that does not act minimally on X. Then, $H = \mathrm{St}_{\mathcal{G}}(Y)$, the stabilizer of some closed set $Y \subset X$ different from the empty set and X. Moreover, the induced action of $\mathrm{St}_{\mathcal{G}}(Y)$ on Y is minimal.*

The next theorem already gives a continuum of maximal subgroups in any ample group acting without finite orbits on a Cantor set. All those subgroups belong to the subclass (I_1) of the above classification.

THEOREM 1.2. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that has no finite orbits. Suppose Y is a finite non-empty subset of X. Then, the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ is a maximal subgroup of $\mathcal{G}$ if and only if Y is contained in a single orbit of $\mathcal{G}$.*

For a more general and more detailed result on the stabilizers of finite sets, see Theorem 6.11 below. Theorem 1.2 imposes very modest conditions on the ample group $\mathcal{G}$ (Theorem 6.11 imposes no conditions at all). To treat the stabilizers of infinite closed sets, we need stronger assumptions. Namely, $\mathcal{G}$ has to act minimally on X and to possess another property that we call *Property E* (entanglement): for any clopen sets $U_1, U_2 \subset X$ that overlap, the local subgroup (that is, rigid stabilizer) $\mathcal{G}_{U_1 \cup U_2}$ is generated by its subgroups $\mathcal{G}_{U_1}$ and $\mathcal{G}_{U_2}$.

THEOREM 1.3. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on a Cantor set X and has Property E. Suppose $Y \subset X$ is an infinite closed set that is nowhere dense in X. Then, the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ is a maximal subgroup of $\mathcal{G}$ if and only if it acts minimally when restricted to Y.*

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Proposition 7.2 below shows that Theorem 1.3 provides uncountably many examples of maximal subgroups of the ample group $\mathcal{G}$ (different from the stabilizers of finite sets).

If an ample group $\mathcal{G}$ acts minimally on a Cantor set X and the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ of a closed set Y is a maximal subgroup of $\mathcal{G}$, then $\mathrm{St}_{\mathcal{G}}(Y)$ acts minimally on Y. This implies that the set Y is either finite, or infinite and nowhere dense, or clopen. The first two cases are covered by Theorems 1.2 and 1.3. It remains to consider the stabilizers of clopen sets.

THEOREM 1.4. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on a Cantor set X and has Property E. Suppose U is a clopen set different from the empty set and X. Then, $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$, the stabilizer of the partition $X = U \sqcup (X \setminus U)$, is a maximal subgroup of $\mathcal{G}$. If U cannot be mapped onto $X \setminus U$ by an element of $\mathcal{G}$, then $\mathrm{St}_{\mathcal{G}}(U) = \mathrm{St}_{\mathcal{G}}(U, X \setminus U)$; otherwise, $\mathrm{St}_{\mathcal{G}}(U)$ is a subgroup of index 2 in $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$.*

In the case where the clopen set U can be mapped onto $X \setminus U$, the stabilizer $\mathrm{St}_{\mathcal{G}}(U)$ is not a maximal subgroup of $\mathcal{G}$. However, $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$ is the only group placed between $\mathrm{St}_{\mathcal{G}}(U)$ and $\mathcal{G}$ in the lattice of subgroups of $\mathcal{G}$.

In addition to the stabilizers of clopen sets, Theorem 1.4 also treats the stabilizers of partitions into two clopen sets. Our next result covers the stabilizers of partitions into three or more clopen sets.

THEOREM 1.5. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on a Cantor set X and has Property E. Suppose $X = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k$ is a partition of X into at least three non-empty clopen sets. Then, the stabilizer $\mathrm{St}_{\mathcal{G}}(U_1, U_2, \ldots, U_k)$ of the partition is a maximal subgroup of $\mathcal{G}$ if and only if its induced action on the set $\{U_1, U_2, \ldots, U_k\}$ is transitive.*

Our last result on maximal subgroups provides a characterization of those maximal subgroups of ample groups that are the stabilizers of partitions into clopen sets.

THEOREM 1.6. *Let X be a Cantor set and $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group with Property E. Suppose H is a maximal subgroup of $\mathcal{G}$ that acts minimally on X and contains a local subgroup $\mathcal{G}_U$ for some non-empty clopen set U. Then, $H = \mathrm{St}_{\mathcal{G}}(U_1, U_2, \ldots, U_k)$ for some partition $X = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k$ into non-empty clopen sets. Moreover, the partition is unique, it consists of at least two sets, and the induced action of H on the set $\{U_1, U_2, \ldots, U_k\}$ is transitive.*

Theorems 1.3, 1.4, 1.5, and 1.6 require the ample group to have Property E. One class of ample groups with this property is the TFGs associated with Cantor minimal systems.

THEOREM 1.7. *For any minimal homeomorphism f of a Cantor set, the ample group $\mathsf{F}(\langle f \rangle)$ has Property E.*

The TFGs of Cantor minimal systems are part of a larger class of ample groups with Property E. For example, the cyclic group $\langle f \rangle$ in Theorem 1.7 can be replaced by any countable elementary amenable group acting minimally and freely on the Cantor set. Another quite different example of an ample group with Property E is Thompson's group V as well as its various generalizations, for example, the Higman–Thompson groups $V_{n,r}$

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(see, e.g., [12]) and the Brin–Thompson groups nV (see [14]). Yet another class of ample groups with Property E is the (topological) *full groups* of minimal, purely infinite étale groupoids considered by Matui in [47, 48]. See §8 for more details.

It would be interesting to find more examples of the ample groups with Property E. More generally, it would be interesting to find out which of our results extend to other classes of micro-supported groups (in particular, the classes mentioned above).

For the ample groups acting minimally on a Cantor set and having Property E, our results describe all maximal subgroups in the subclasses (I_1), (I_2), (I_3), and (II_0). Presumably, the other maximal subgroups in the class (II) are also stabilizers of partitions into closed sets. In the case of finite symmetric groups, there is a large variety of primitive maximal subgroups. Maximal subgroups of Thompson's group V found in [11] include a continuum of topologically primitive subgroups. As for the ample groups associated with Cantor minimal systems (or, more generally, obtained by amplifying countable amenable groups acting minimally on the Cantor set), no interesting examples of such subgroups are found so far. This leads us to formulate the following bold conjecture.

Conjecture 1.8. Let f be a minimal homeomorphisms of a Cantor set X. Suppose H is a maximal subgroup of the ample group $\mathcal{G} = \mathsf{F}(\langle f \rangle)$. Then, exactly one of the following statements holds true.

(i) The subgroup H is the stabilizer of a closed set $Y \subset X$ different from X and the empty set. Moreover, the induced action of H on Y is minimal. Furthermore, the set Y cannot be mapped onto $X \setminus Y$ by elements of the group $\mathcal{G}$.

(ii) The subgroup H is the stabilizer of a partition $\mathcal{P} = \{Y_\alpha\}_{\alpha \in \mathcal{A}}$ of X into closed sets different from the partition into points and the trivial partition $\{X\}$. Moreover, the partition $\mathcal{P}$ is unique and the induced action of H on the factor space $X/\mathcal{P}$ is minimal.

(iii) The subgroup H is a normal subgroup of finite prime index in $\mathcal{G}$. Moreover, H contains the commutator subgroup of $\mathcal{G}$.

The paper is organized as follows. In §2, we describe the concept of amplification and define the ample groups. In §3, we define various stabilizers associated to a group of homeomorphisms, including the local subgroups of ample groups. In §4, we introduce generalized permutations and, in particular, generalized 2-cycles. The generalized 2-cycles are used in most constructions in our paper. In §5, we introduce a number of useful properties that groups of homeomorphisms can have, including Property E. Section 6 is devoted to the study of maximal subgroups of ample groups. In that section, we prove slightly generalized versions of Theorems 1.1, 1.2, 1.3, 1.4, 1.5, and 1.6 (respectively Theorems 6.6, 6.13, 6.18, 6.19, 6.20, and 6.21). In §7, we describe a construction of closed, nowhere dense sets and groups of homeomorphisms acting on them that allows provision of a wealth of examples for Theorem 1.3. In §8, we first discuss Property E in detail, then present examples of ample groups with that property. In particular, we prove Theorem 1.7 (Theorem 8.8).

2. *Amplification*

Let X be a topological space. We denote by $C(X, X)$ the set of all continuous maps $f : X \to X$. This set is a semigroup (and a monoid) with respect to the composition of maps.

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The identity function $\mathrm{id}_X : X \to X$ is the identity element. Homeomorphisms of X are invertible elements of $C(X, X)$. The set $\mathrm{Homeo}(X)$ of all homeomorphisms is a group.

We assume that the topological space X is compact, metrizable, and totally disconnected. These conditions imply that the topology is generated by *clopen* (that is, both closed and open) sets. The main example is a Cantor set, but a finite set with the discrete topology is an example as well. If Y is a non-empty closed subset of X, then Y as the topological space with the induced topology also satisfies those conditions.

Several properties of the topological space X are used repeatedly throughout the paper. First of all, any open neighborhood of a point $x \in X$ contains a clopen neighborhood of the same point x. Second, for any distinct points $x, y \in X$, there exists a clopen neighborhood of x that does not contain y. It easily follows by induction that for any finite number of distinct points in X, we can choose pairwise disjoint clopen neighborhoods.

PROPOSITION 2.1. *Given a subset $S \subset C(X, X)$ and an element $f \in C(X, X)$, the following conditions are equivalent:*

(i) *f is locally an element of S, which means that for any point $x \in X$, there exists an open neighborhood U_x of x and an element $g_x \in S$ such that f coincides with g_x on U_x: $f|_{U_x} = g_x|_{U_x}$;*

(ii) *f is piecewise an element of S, which means that there exist clopen sets $V_1, \dots, V_k$ forming a partition of X and elements $h_1, h_2, \dots, h_k \in S$ such that $f|_{V_i} = h_i|_{V_i}$ for $1 \le i \le k$.*

Proof. The implication (ii) $\Longrightarrow$ (i) is trivial. Let us prove that (i) $\Longrightarrow$ (ii). Assume condition (i) holds. Without loss of generality, we may further assume that the neighborhoods U_x, $x \in X$ are clopen. The sets U_x form an open cover of the compact space X. Hence, there are finitely many points $x_1, x_2, \dots, x_k \in X$ such that the sets U_{x_i}, $1 \le i \le k$ form a subcover. Now, let $V_1 = U_{x_1}$ and $V_i = U_{x_i} \setminus (U_{x_1} \cup \cdots \cup U_{x_{i-1}})$ for $i = 2, 3, \dots, k$. Then, the sets $V_1, V_2, \dots, V_k$ form a partition of X. Since each U_{x_i}, $1 \le i \le k$ is clopen, it follows that the sets V_i, $1 \le i \le k$ are clopen as well. Let $h_i = g_{x_i}$ for $1 \le i \le k$. Clearly, f coincides with h_i on V_i for $1 \le i \le k$. Thus, condition (ii) holds. □

Definition 2.2. Given a set $S \subset C(X, X)$, let $\widehat{\mathsf{F}}(S)$ denote the set of all continuous maps in $C(X, X)$ that are locally elements of S (that is, satisfy the two conditions in Proposition 2.1). Given a set $S \subset \mathrm{Homeo}(X)$, let $\mathsf{F}(S) = \widehat{\mathsf{F}}(S) \cap \mathrm{Homeo}(X)$.

Clearly, $S \subset \widehat{\mathsf{F}}(S)$ for any $S \subset C(X, X)$. If $S \subset \mathrm{Homeo}(X)$, then $S \subset \mathsf{F}(S) \subset \widehat{\mathsf{F}}(S)$.

LEMMA 2.3. *We have $\widehat{\mathsf{F}}(\widehat{\mathsf{F}}(S)) = \widehat{\mathsf{F}}(S)$ for any $S \subset C(X, X)$ and $\mathsf{F}(\mathsf{F}(S)) = \mathsf{F}(S)$ for any $S \subset \mathrm{Homeo}(X)$.*

Proof. As the inclusion $\widehat{\mathsf{F}}(S) \subset \widehat{\mathsf{F}}(\widehat{\mathsf{F}}(S))$ is obvious, we need to prove that $\widehat{\mathsf{F}}(\widehat{\mathsf{F}}(S)) \subset \widehat{\mathsf{F}}(S)$. Suppose $f \in \widehat{\mathsf{F}}(\widehat{\mathsf{F}}(S))$ and consider an arbitrary point $x \in X$. Since f is locally an element of $\widehat{\mathsf{F}}(S)$, there exists an open neighborhood U of x and a map $g \in \widehat{\mathsf{F}}(S)$ such that $f|_U = g|_U$. Since g is locally an element of S, there exists an open neighborhood W of x and a map $h \in S$ such that $g|_W = h|_W$. Then, f coincides with h on $U \cap W$, which is an open neighborhood of x. We conclude that f is locally an element of S.

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In the case $S \subset \mathrm{Homeo}(X)$, the inclusion $\mathsf{F}(S) \subset \mathsf{F}(\mathsf{F}(S))$ is obvious. Since $\mathsf{F}(S) \subset \widehat{\mathsf{F}}(S)$, it follows that $\widehat{\mathsf{F}}(\mathsf{F}(S)) \subset \widehat{\mathsf{F}}(\widehat{\mathsf{F}}(S))$. By the above, $\widehat{\mathsf{F}}(\widehat{\mathsf{F}}(S)) = \widehat{\mathsf{F}}(S)$. Therefore, $\mathsf{F}(\mathsf{F}(S)) = \widehat{\mathsf{F}}(\mathsf{F}(S)) \cap \mathrm{Homeo}(X) \subset \widehat{\mathsf{F}}(S) \cap \mathrm{Homeo}(X) = \mathsf{F}(S)$. Thus, $\mathsf{F}(\mathsf{F}(S)) = \mathsf{F}(S)$. □

LEMMA 2.4. *If $S \subset C(X, X)$ is a semigroup, then $\widehat{\mathsf{F}}(S)$ is also a semigroup. If $S \subset \mathrm{Homeo}(X)$ is a group, then $\mathsf{F}(S)$ is also a group.*

Proof. Suppose $S \subset C(X, X)$ is a semigroup and consider any two maps $f, g \in \widehat{\mathsf{F}}(S)$. Given a point $x \in X$, there exists an open neighborhood U of x and a map $\tilde{g} \in S$ such that $g|_U = \tilde{g}|_U$. Further, there exists an open neighborhood W of the point $g(x)$ and a map $\tilde{f} \in S$ such that $f|_W = \tilde{f}|_W$. Then, the composition fg coincides with $\tilde{f}\tilde{g}$, which belongs to S, on $g^{-1}(W) \cap U$, which is an open neighborhood of x. Hence, fg is locally an element of S. Thus, $\widehat{\mathsf{F}}(S)$ is a semigroup.

Now, assume $S \subset \mathrm{Homeo}(X)$ is a group. By the above, $\widehat{\mathsf{F}}(S)$ is a semigroup. Then, $\mathsf{F}(S)$ is also a semigroup as it is the intersection of two semigroups $\widehat{\mathsf{F}}(S)$ and $\mathrm{Homeo}(X)$. It remains to show that for any homeomorphism $f \in \mathsf{F}(S)$, the inverse f^{-1} is also in $\mathsf{F}(S)$. Take an arbitrary point $x \in X$. Since f is locally an element of S, there exists an open neighborhood U of the point $f^{-1}(x)$ and a map $g \in S$ such that $f|_U = g|_U$. Then, f^{-1} coincides with g^{-1}, which belongs to S, on $f(U)$, which is an open neighborhood of x. Thus, f^{-1} is locally an element of S. □

Definition 2.5. Given subgroups G and $\widetilde{G}$ of $\mathrm{Homeo}(X)$, we say that $\widetilde{G}$ *amplifies* G if $G \subset \widetilde{G} \subset \mathsf{F}(G)$. The group $\mathsf{F}(G)$ is called the *full amplification* of G. In the case $\mathsf{F}(G) = G$, the group G is called *ample* or *fully amplified*.

For any group $G \subset \mathrm{Homeo}(X)$, it follows from Lemmas 2.3 and 2.4 that $\mathsf{F}(G)$ is an ample group.

Definition 2.6. Given a group $G \subset \mathrm{Homeo}(X)$, for any $x \in X$, we denote by $\mathrm{Orb}_G(x)$ the *orbit* of the point x under the natural action of the group: $\mathrm{Orb}_G(x) = \{g(x) \mid g \in G\}$. We refer to any set of the form $\mathrm{Orb}_G(x)$ as an orbit of the group G.

An important property of the amplification is that this procedure enlarges a group while not enlarging orbits of the natural action. In particular, the ample group $\mathsf{F}(G)$ has the same orbits as the group G.

3. *Stabilizers and local subgroups*

Any group $G \subset \mathrm{Homeo}(X)$ acts naturally on the topological space X. This action induces several other actions. In this section, we define various stabilizers related to those actions. Note that the stabilizers of any group action on any set are subgroups of the acting group.

Definition 3.1. The *stabilizer* $\mathrm{St}_G(x)$ of a point $x \in X$ under the action of the group G consists of those elements of G that fix x: $\mathrm{St}_G(x) = \{g \in G \mid g(x) = x\}$.

The action of the group G on the set X induces an action on subsets of X.

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Definition 3.2. The *(set) stabilizer* $\mathrm{St}_G(Y)$ of a set $Y \subset X$ under the action of the group G consists of those elements of G that map Y onto itself. The *pointwise stabilizer* $\mathrm{St}^\star_G(Y)$ of Y consists of those elements of $\mathrm{St}_G(Y)$ that fix every point of the set Y. The *rigid stabilizer* $\mathrm{RiSt}_G(Y)$ of Y consists of those elements of $\mathrm{St}_G(Y)$ that fix every point not in Y.

Clearly, the stabilizer $\mathrm{St}_G(x)$ of a point $x \in X$ coincides with the stabilizer $\mathrm{St}_G(\{x\})$ of the one-point set $\{x\}$ as well as with $\mathrm{St}^\star_G(\{x\})$. For any subset $Y \subset X$, the individual stabilizer $\mathrm{St}^\star_G(Y)$ is the intersection of all point stabilizers $\mathrm{St}_G(y)$, $y \in Y$.

The pointwise stabilizer can be interpreted as the usual stabilizer of an induced action of G on ordered subsets of X. Consider the set of all pairs $(Y, \prec)$, where $Y \subset X$ and $\prec$ is an order on Y. The group G acts naturally on this set. Clearly, the stabilizer of a pair $(Y, \prec)$ always contains $\mathrm{St}^\star_G(Y)$. In the case $\prec$ is a well ordering on the subset Y, it is easy to observe that the stabilizer of $(Y, \prec)$ is exactly $\mathrm{St}^\star_G(Y)$.

For any invertible map $f : X \to X$, we have $f(Y) = Y$ if and only if $f(X \setminus Y) = X \setminus Y$. Hence, $\mathrm{St}_G(Y) = \mathrm{St}_G(X \setminus Y)$. It follows that $\mathrm{RiSt}_G(Y) = \mathrm{St}^\star_G(X \setminus Y)$.

The set stabilizer $\mathrm{St}_G(Y)$ acts naturally on the set Y. The pointwise stabilizer $\mathrm{St}^\star_G(Y)$ is the kernel of that action. Therefore, it is a normal subgroup of $\mathrm{St}_G(Y)$. Similarly, $\mathrm{RiSt}_G(Y)$ is a normal subgroup of $\mathrm{St}_G(X \setminus Y) = \mathrm{St}_G(Y)$.

LEMMA 3.3. *Suppose $G \subset \mathrm{Homeo}(X)$ is an ample group. Then, for any subset $Y \subset X$, the stabilizers $\mathrm{St}_G(Y)$, $\mathrm{St}^\star_G(Y)$, and $\mathrm{RiSt}_G(Y)$ are also ample groups.*

Proof. Since the set stabilizer $\mathrm{St}_G(Y)$ is a subgroup of G, its full amplification $\mathsf{F}(\mathrm{St}_G(Y))$ is a subgroup of $\mathsf{F}(G) = G$. Take any map $f \in \mathsf{F}(\mathrm{St}_G(Y))$. Then, f is piecewise an element of $\mathrm{St}_G(Y)$, which implies that $f(Y) \subset Y$. Since $\mathrm{St}_G(Y) = \mathrm{St}_G(X \setminus Y)$, it also follows that $f(X \setminus Y) \subset X \setminus Y$. As f is invertible, we obtain $f(Y) = Y$ so that $f \in \mathrm{St}_G(Y)$. Thus, the group $\mathrm{St}_G(Y)$ is ample.

If Y is the empty set or X, then one of the groups $\mathrm{St}^\star_G(Y)$ and $\mathrm{RiSt}_G(Y)$ is trivial while the other coincides with G, and both are ample. Now, assume Y is different from $\emptyset$ and X. By the above, for any point $x \in X$, the stabilizer $\mathrm{St}_G(x) = \mathrm{St}_G(\{x\})$ is an ample group. Therefore, the point stabilizer $\mathrm{St}^\star_G(Y)$ is the intersection of ample groups $\mathrm{St}_G(x)$, $x \in Y$, while the rigid stabilizer $\mathrm{RiSt}_G(Y)$ is the intersection of ample groups $\mathrm{St}_G(x)$, $x \notin Y$. It is easy to observe that for any collection of ample subgroups of $\mathrm{Homeo}(X)$, their intersection is ample as well. □

The action of the group G on subsets of X induces an action on collections of such subsets (that is, on the set of subsets of the set of subsets of X).

Definition 3.4. The *(collective) stabilizer* $\mathrm{St}_G(Y_1, Y_2, \dots, Y_k)$ of a collection of distinct sets $Y_i \subset X$, $1 \le i \le k$ under the action of the group G consists of those elements of G that map each set in the collection onto (the same or another) set in the collection. The *individual stabilizer* $\mathrm{St}^\bullet_G(Y_1, Y_2, \dots, Y_k)$ of the collection consists of those elements of $\mathrm{St}_G(Y_1, Y_2, \dots, Y_k)$ that map each set Y_i, $1 \le i \le k$ onto itself.

Clearly, the individual stabilizer $\mathrm{St}^\bullet_G(Y_1, Y_2, \dots, Y_k)$ is the intersection of the set stabilizers $\mathrm{St}_G(Y_i)$, $1 \le i \le k$. In the case where the collection consists of a single set

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Y, the notation $\mathrm{St}_G(Y)$ is not ambiguous since the collective stabilizer of the collection $\{Y\}$ coincides with the set stabilizer of the set Y.

The collective stabilizer $\mathrm{St}_G(Y_1, Y_2, \ldots, Y_k)$ acts naturally on the set $\{Y_1, Y_2, \ldots, Y_k\}$. The individual stabilizer $\mathrm{St}^{\bullet}_G(Y_1, Y_2, \ldots, Y_k)$ is the kernel of that action. Therefore, it is a normal subgroup of $\mathrm{St}_G(Y_1, Y_2, \ldots, Y_k)$.

Definition 3.5. Let $f \in \mathrm{Homeo}(X)$. A point $x \in X$ is called a *fixed point* of f if $f(x) = x$. The set of all fixed points of f is denoted $\mathrm{Fix}(f)$. Further, a point $x \in X$ is called a *support point* of f if the homeomorphism f does not coincide with the identity map in any neighborhood of x. The set of all support points of f is called the *support* of f and denoted $\mathrm{supp}(f)$.

By definition, the support of f is the complement of the interior of $\mathrm{Fix}(f)$. Equivalently, $\mathrm{supp}(f) = \overline{X \setminus \mathrm{Fix}(f)}$, the closure of the complement of $\mathrm{Fix}(f)$.

If two homeomorphisms f and g have disjoint supports then, clearly, they commute: $fg = gf$. In fact, f and g commute whenever $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ share no interior point.

Definition 3.6. Let G be a subgroup of $\mathrm{Homeo}(X)$ and U be a clopen subset of X. The *local subgroup* of the ample group $\mathsf{F}(G)$ associated to U, denoted $\mathsf{F}_U(G)$, consists of all maps $f \in \mathsf{F}(G)$ such that $\mathrm{supp}(f) \subset U$. In the case where the group G is already ample, we may use alternative notation G_U.

For any clopen set $U \subset X$ and homeomorphism $f : X \to X$, the condition $\mathrm{supp}(f) \subset U$ is equivalent to the condition $f(x) = x$ for all $x \notin U$. Hence, the local subgroup $\mathsf{F}_U(G)$ coincides with the rigid stabilizer $\mathrm{RiSt}_{\mathsf{F}(G)}(U)$. By Lemma 3.3, $\mathsf{F}_U(G)$ is an ample group.

4. *Generalized permutations*

For any integer $n \geq 1$, let S_n denote the *symmetric group* of all permutations on the set $\{1, 2, \ldots, n\}$.

Definition 4.1. Let U be a clopen subset of a topological space X. Suppose $f_1, f_2, \ldots, f_n$ are homeomorphisms of X such that the images $f_1(U), f_2(U), \ldots, f_n(U)$ are disjoint. Then, for any permutation $\pi \in \mathsf{S}_n$, we define a *generalized permutation* $\mu[U; f_1, f_2, \ldots, f_n; \pi] : X \to X$ by

$$\mu[U; f_1, f_2, \ldots, f_n; \pi](x) = \begin{cases} f_{\pi(i)}(f_i^{-1}(x)) & \text{if } x \in f_i(U),\ 1 \leq i \leq n, \\ x & \text{otherwise.} \end{cases}$$

The generalized permutation $\mu[U; f_1, f_2, \ldots, f_n; \pi]$ is a homeomorphism of X that permutes disjoint clopen sets $f_1(U), f_2(U), \ldots, f_n(U)$ while fixing the rest of the space. For any $x \in U$ and $i \in \{1, 2, \ldots, n\}$, the point $f_i(x)$ is mapped to $f_{\pi(i)}(x)$. The generalized permutation has finite order in the group $\mathrm{Homeo}(X)$, which is the same as the order of the permutation π. If the sets $f_1(U), f_2(U), \ldots, f_n(U)$ are not disjoint, then $\mu[U; f_1, f_2, \ldots, f_n; \pi]$ is not defined.

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LEMMA 4.2. *Suppose $U \subset X$ is a clopen set and $f_1, f_2, \ldots, f_n$ are homeomorphisms of X such that the images $f_1(U), f_2(U), \ldots, f_n(U)$ are disjoint. Then, the map $\Phi : \mathsf{S}_n \to \mathrm{Homeo}(X)$ given by $\Phi(\pi) = \mu[U; f_1, f_2, \ldots, f_n; \pi]$ is a group homomorphism.*

The proof of Lemma 4.2 is straightforward and we omit it.

Let G be a subgroup of $\mathrm{Homeo}(X)$ and $\mathsf{F}(G)$ be its full amplification. If homeomorphisms $f_1, f_2, \ldots, f_n$ belong to the group G, then any generalized permutation of the form $\mu[U; f_1, f_2, \ldots, f_n; \pi]$ belongs to $\mathsf{F}(G)$.

Definition 4.3. The *generalized symmetric group* $\mathsf{S}(G)$ over G is the subgroup of $\mathsf{F}(G)$ generated by all generalized permutations in $\mathsf{F}(G)$.

The group $\mathsf{S}(G)$ was introduced (in a more general context of étale groupoids) by Nekrashevych [**50**].

LEMMA 4.4. *The generalized symmetric group $\mathsf{S}(G)$ is a normal subgroup of the group $\mathsf{F}(G)$.*

Proof. It is enough to show that any conjugate of a generalized permutation in the group $\mathrm{Homeo}(X)$ is itself a generalized permutation. After that, the proof of the lemma is straightforward. Suppose a generalized permutation $g = \mu[U; f_1, f_2, \ldots, f_n; \pi]$ is defined. Then, $f_1(U), f_2(U), \ldots, f_n(U)$ are disjoint clopen sets. The support of g is the union of those $f_i(U)$ for which $\pi(i) \neq i$. For any $x \in U$ and $i \in \{1, 2, \ldots, n\}$, we have $g(f_i(x)) = f_{\pi(i)}(x)$. Now, consider an arbitrary homeomorphism $h \in \mathrm{Homeo}(X)$. It is easy to observe that $\mathrm{supp}(hgh^{-1}) = h(\mathrm{supp}(g))$. Hence, the support of hgh^{-1} is contained in the union of sets $hf_1(U), hf_2(U), \ldots, hf_n(U)$, which are disjoint clopen sets. Furthermore, for any $x \in U$ and $i \in \{1, 2, \ldots, n\}$, we have $hgh^{-1}(hf_i(x)) = hg(f_i(x)) = hf_{\pi(i)}(x)$. It follows that $hgh^{-1} = \mu[U; hf_1, hf_2, \ldots, hf_n; \pi]$. Thus, hgh^{-1} is a generalized permutation. □

LEMMA 4.5. *For any group $G \subset \mathrm{Homeo}(X)$, the generalized symmetric group $\mathsf{S}(G)$ is generated by all maps of the form $\mu[U; f_1, f_2, \ldots, f_n; \pi]$, where $f_1, f_2, \ldots, f_n \in \mathsf{F}(G)$.*

Proof. Let P be the set of all generalized permutations of the form $\mu[U; f_1, f_2, \ldots, f_n; \pi]$, where $f_1, f_2, \ldots, f_n \in \mathsf{F}(G)$. It follows from the definition that any element of P is piecewise an element of the group $\mathsf{F}(G)$. Therefore, P is contained in the group $\mathsf{F}(\mathsf{F}(G))$, which coincides with $\mathsf{F}(G)$ due to Lemma 2.3.

To prove the lemma, it is enough to show that any generalized permutation $g \in \mathsf{F}(G)$ is a product of elements of the set P. The map g is represented as $\mu[U; f_1, f_2, \ldots, f_n; \pi]$, where maps $f_1, f_2, \ldots, f_n$ need not be in $\mathsf{F}(G)$. Let $\pi = \sigma_1\sigma_2 \cdots \sigma_k$ be the decomposition of the permutation π as a product of disjoint cycles. Lemma 4.2 implies that $g = g_1g_2 \cdots g_k$, where $g_j = \mu[U; f_1, f_2, \ldots, f_n; \sigma_j]$, $1 \leq j \leq k$. We are going to show that each g_j belongs to P.

First, let us show that $g_j \in \mathsf{F}(G)$. For any $i \in \{1, 2, \ldots, n\}$, we have either $\sigma_j(i) = \pi(i)$ or $\sigma_j(i) = i$. In the former case, g_j coincides with g on the clopen set $f_i(U)$. In the latter case, g_j coincides with the identity map on $f_i(U)$. On the complement

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of the union of sets $f_i(U)$, $1 \le i \le n$, both g and g_j coincide with the identity map. Hence, g_j is piecewise an element of the set $\{\mathrm{id}_X, g\}$. Since $g \in \mathsf{F}(G)$, it follows that $g_j \in \mathsf{F}(\mathsf{F}(G)) = \mathsf{F}(G)$.

Recall that the permutation σ_j is a cycle: $\sigma_j = (i_1\, i_2 \ldots i_m)$, where $i_1, i_2, \ldots, i_m$ are distinct elements of $\{1, 2, \ldots, n\}$. Let $W_s = f_{i_s}(U)$, $1 \le s \le m$. Then, $W_1, W_2, \ldots, W_m$ are disjoint clopen sets and g_j coincides with the identity map away from them. We have $g_j(W_s) = W_{s+1}$ for $1 \le s \le m-1$ and $g_j(W_m) = W_1$. It follows by induction that $W_s = g_j^{s-1}(W_1)$ for $1 \le s \le m$. Since the cycle σ_j has order m in the group S_n, Lemma 4.2 implies that $g_j^m = \mathrm{id}_X$. As a consequence, $(g_j^{m-1})^{-1}$ coincides with g_j on W_m (as well as anywhere else on X). All this leads to an alternative representation of g_j as a generalized permutation:

$$g_j = \mu[W_1; \mathrm{id}_X, g_j, g_j^2, \ldots, g_j^{m-1}; (1\ 2 \ldots m)].$$

Since g_j and all its powers belong to the group $\mathsf{F}(G)$, we conclude that $g_j \in P$. □

Definition 4.6. Let U be a clopen subset of X. For any homeomorphism $f : X \to X$ such that the image $f(U)$ is disjoint from U, we define a *generalized 2-cycle* $\delta_{U;f} : X \to X$ by

$$\delta_{U;f}(x) = \begin{cases} f(x) & \text{if } x \in U, \\ f^{-1}(x) & \text{if } x \in f(U), \\ x & \text{otherwise.} \end{cases}$$

The generalized 2-cycles are a special case of the generalized permutations. Indeed, $\delta_{U;f} = \mu[U; \mathrm{id}_X, f; (1\ 2)]$. As a consequence, $\delta_{U;f} \in \mathsf{S}(G)$ whenever $f \in G$.

Recall that the symmetric group S_n is generated by all 2-cycles $(i\ j)$ in S_n.

LEMMA 4.7. *For any group $G \subset \mathrm{Homeo}(X)$, a generalized 2-cycle belongs to $\mathsf{F}(G)$ if and only if it is of the form $\delta_{U;f}$, where $f \in \mathsf{F}(G)$. The generalized symmetric group $\mathsf{S}(G)$ is generated by all generalized 2-cycles in $\mathsf{F}(G)$.*

Proof. If a generalized 2-cycle $g = \delta_{U;f}$ is defined and $f \in \mathsf{F}(G)$, then g is piecewise an element of the group $\mathsf{F}(G)$. Hence, it belongs to the group $\mathsf{F}(\mathsf{F}(G))$, which coincides with $\mathsf{F}(G)$ due to Lemma 2.3. Conversely, if $g = \delta_{U;f}$ is defined and $g \in \mathsf{F}(G)$, we observe that the generalized 2-cycle $\delta_{U;g}$ is also defined and $g = \delta_{U;g}$.

By Lemma 4.5, the group $\mathsf{S}(G)$ is generated by generalized permutations of the form $\mu[U; f_1, f_2, \ldots, f_n; \pi]$, where $f_1, f_2, \ldots, f_n \in \mathsf{F}(G)$. Those include all maps of the form $\delta_{U;f}$, $f \in \mathsf{F}(G)$. Whenever $\mu[U; f_1, f_2, \ldots, f_n; \pi]$ is defined, all maps $\mu[U; f_1, f_2, \ldots, f_n; \sigma]$, $\sigma \in \mathsf{S}_n$ are defined as well. Assuming $f_1, f_2, \ldots, f_n \in \mathsf{F}(G)$, they all belong to the group $\mathsf{F}(\mathsf{F}(G)) = \mathsf{F}(G)$. Moreover, $\mathsf{S}_n \ni \sigma \mapsto \mu[U; f_1, f_2, \ldots, f_n; \sigma]$ is a homomorphism of the group S_n to $\mathsf{F}(G)$ due to Lemma 4.2. Since the group S_n is generated by 2-cycles $(i\ j)$, it follows that the group $\mathsf{S}(G)$ is generated by generalized permutations of the form $\mu[U; f_1, f_2, \ldots, f_n; (i\ j)]$. It remains to show that each $\mu[U; f_1, f_2, \ldots, f_n; (i\ j)]$ is a generalized 2-cycle of prescribed form. First, we observe that $\mu[U; f_1, f_2, \ldots, f_n; (i\ j)] = \mu[U; f_i, f_j; (1\ 2)]$. Then, we observe that $\mu[U; f_i, f_j; (1\ 2)] = \delta_{f_i(U);f_j f_i^{-1}}$. Finally, we note that $f_j f_i^{-1} \in \mathsf{F}(G)$. □

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5. *Regularity properties*

In this section, we formulate several useful properties that a subgroup G of $\mathrm{Homeo}(X)$ may or may not have, and establish relations between them. When proving statements on the ample group $\mathsf{F}(G)$, we will usually need some of those properties. Let us begin with a very well-known property that will be required most of the time.

Definition 5.1. We say that a group $G \subset \mathrm{Homeo}(X)$ acts *minimally* on X (or that the natural action of G on X is *minimal*) if there are no closed sets invariant under the action other than the empty set and X itself.

A set $Y \subset X$ is *invariant* under a homeomorphism $f : X \to X$ if $f(Y) \subset Y$. It is invariant under the natural action of a group $G \subset \mathrm{Homeo}(X)$ if $g(Y) \subset Y$ for all $g \in G$. For any $g \in G$, the inverse map g^{-1} is in G as well. Assuming invariance of the set Y, we have $g(Y) \subset Y$ and $g^{-1}(Y) \subset Y$, which implies that, in fact, $g(Y) = Y$ for all $g \in G$.

For any point $x \in X$, the closure $\overline{\mathrm{Orb}_G(x)}$ of its orbit is a closed set invariant under the action of the group G. It easily follows that the action is minimal if and only if every orbit is dense in X.

LEMMA 5.2. *Suppose a topological space X is totally disconnected, compact, and metrizable. If there is a group $G \subset \mathrm{Homeo}(X)$ that acts minimally on X, then X is either finite or a Cantor set.*

Proof. If a point $x \in X$ is isolated, then the set $\{x\}$ is clopen. It follows that the set Y of all non-isolated points of X is closed. Any homeomorphism of X maps an isolated point to an isolated point. Hence, the set Y is invariant under the action of any group $G \subset \mathrm{Homeo}(X)$. If the action is minimal, then either Y is empty or $Y = X$. In the case $Y = X$, there are no isolated points. Then, X is a Cantor set due to Brouwer's characterization of Cantor sets. In the case where Y is empty, every point of X is isolated. Hence, the partition of X into points is a partition into clopen sets. Then, compactness of X implies that it is finite. □

The following technical statement is used several times in our paper.

LEMMA 5.3. *Let $U, U' \subset X$ be non-empty clopen sets. If a group $G \subset \mathrm{Homeo}(X)$ acts minimally on X, then for some $k \geq 1$, there exist disjoint clopen sets $V_1, V_2, \ldots, V_k$ and maps $g_1, g_2, \ldots, g_k \in G$ such that $U = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$ and $g_i(V_i) \subset U'$ for all i.*

Proof. Since the group G acts minimally, the orbit of any point $x \in U$ is dense in X. In particular, the orbit has a point in the non-empty clopen set U'. Hence, $g_x(x) \in U'$ for some map $g_x \in G$. Let $U_x = U \cap g_x^{-1}(U')$. Then, U_x is a clopen subset of U containing the point x, and $g_x(U_x) \subset U'$. The clopen sets U_x, $x \in U$ cover U. Since U is compact, there are finitely many points $x_1, x_2, \ldots, x_k \in U$ such that the sets U_{x_i}, $1 \leq i \leq k$ form a subcover. Let $V_1 = U_{x_1}$ and $V_i = U_{x_i} \setminus (U_{x_1} \cup \cdots \cup U_{x_{i-1}})$ for $i = 2, 3, \ldots, k$. Then, $V_1, V_2, \ldots, V_k$ are clopen sets that partition U. By construction, $g_{x_i}(V_i) \subset g_{x_i}(U_{x_i}) \subset U'$ for all i. □

The other properties formulated in this section are less (if at all) known, and we will introduce an initialism for each of them.

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Property UR. (Universal recurrence) For any open set $U \subset X$, any point $x \in U$, and any element $g \in G$, there exists an integer $k \geq 1$ such that $g^k(x) \in U$.

A point $x \in X$ is called *recurrent* relative to a homeomorphism $f : X \to X$ if the sequence $f(x), f^2(x), f^3(x), \ldots$ visits every neighborhood of x. Property UR means that every point of X is recurrent relative to every element of the group G.

Property NC. (No contraction) If $g(U) \subset U$ for some element $g \in G$ and open set $U \subset X$, then, in fact, $g(U) = U$.

Open sets could be replaced by closed sets in the formulation of Property NC. Indeed, a homeomorphism $f \in \mathrm{Homeo}(X)$ contracts an open set $U \subset X$ (that is, maps it onto a proper subset of U) if and only if the inverse map f^{-1} contracts the closed set $f(X \setminus U)$.

Let us mention two obvious cases in which no contraction of any sets, open or not, is possible. The first case is when the topological space X is finite. The second case is when every element of the group G has finite order (this actually includes the first case).

LEMMA 5.4. *Property NC is equivalent to Property UR.*

Proof. Assume that the group $G \subset \mathrm{Homeo}(X)$ has Property NC. Take any open set $U \subset X$ and element $g \in G$. Since g is a homeomorphism, the sets $g^{-1}(U), g^{-2}(U), \ldots$ are open as well and so is the union $W = U \cup g^{-1}(U) \cup g^{-2}(U) \cup \cdots$. Note that $g^{-1}(W) = g^{-1}(U) \cup g^{-2}(U) \cup g^{-3}(U) \cup \cdots$, which implies that $g^{-1}(W) \subset W$. Since $g^{-1} \in G$ and the group G has Property NC, we have $g^{-1}(W) = W$. As a consequence, any point $x \in U$ belongs to $g^{-k}(U)$ for some $k \geq 1$. Then, $g^k(x) \in U$. Thus, G has Property UR.

Conversely, assume that the group G does not have Property NC. Then, there exist an open set $U \subset X$ and an element $g \in G$ such that $g(U)$ is a proper subset of U. The inverse map $h = g^{-1}$ is also an element of G. Take any point $x \in U$ not in $g(U)$. The condition $g(U) \subset U$ implies that $h(X \setminus g(U)) \subset X \setminus U$ and $h(X \setminus U) \subset X \setminus U$. It follows that the sequence $h(x), h^2(x), h^3(x), \ldots$ never visits U. We conclude that the group G does not have Property UR. □

LEMMA 5.5. *Suppose that the group G does not have Property NC. Then, there exist an open set $U \subset X$ and an element $g \in G$ such that $g(U) \subset U$ and the set difference $U \setminus g(U)$ consists of a single point.*

Proof. Lemma 5.4 implies that the group G does not have Property UR. Hence, there exist an open set $W \subset X$, a point $x \in W$, and a map $h \in G$ such that the sequence $h(x), h^2(x), h^3(x), \ldots$ never visits W. The inverse map $g = h^{-1}$ is also an element of G. Let $S = \{h^k(x) \mid k \geq 1\}$ and $U = X \setminus \overline{S}$, the complement of the closure of S. Since g is a homeomorphism, it follows that $g(\overline{S}) = \overline{g(S)}$. It is easy to see that $g(S) = S \cup \{x\}$. Hence, $\overline{g(S)} = \overline{S} \cup \{x\}$. Consequently, $g(U) = X \setminus g(\overline{S}) = X \setminus (\overline{S} \cup \{x\}) = U \setminus \{x\}$. By construction, U is an open set. In addition, $W \subset U$ since W is an open set disjoint from S. In particular, $x \in U$. Thus, $g(U) \subset U$ and $U \setminus g(U)$ consists of a single point. □

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Note that by replacing open sets with clopen sets in the formulation of Property UR, we obtain an equivalent property. However, Lemma 5.5 suggests that having no contraction of clopen sets may not be enough to have Property NC.

Property SC. (Strong contraction) For any clopen set $U \subset X$ different from the empty set and X, there exist maps $f_1, f_2 \in G$ such that the images $f_1(U)$ and $f_2(U)$ are disjoint subsets of U.

In the case where the group G is ample, Property SC coincides with a property introduced by Matui [47] in a more general context of étale groupoids. Matui calls groupoids with this property *purely infinite* (see [47, Definition 4.9]). Just like in [47], we introduce another, closely related property.

Property UC. (Universal contraction) For any clopen sets $A, B \subset X$ different from the empty set and X, there exists a map $f \in G$ such that $f(A) \subset B$.

Properties SC and UC vacuously hold if the topological space X consists of a single point. Property UC also holds if X consists of two points and the group G is not trivial. Otherwise, either of the two properties implies the absence of isolated points in X, which requires X to be a Cantor set.

LEMMA 5.6. *Suppose X is a Cantor set. Then, for any group $G \subset \mathrm{Homeo}(X)$, Property UC implies minimality and Property SC. If the group G is ample, then conversely, minimality and Property SC imply Property UC.*

Proof. Assume a group $G \subset \mathrm{Homeo}(X)$ has Property UC. Since the topology on X is generated by clopen sets, Property UC clearly implies that every orbit of the group G is dense. Hence, G acts minimally on X. Since X is a Cantor set, any clopen set $U \subset X$ different from $\emptyset$ and X consists of more than one point. Therefore, it can be represented as a disjoint union of two non-empty clopen sets U_1 and U_2. Property UC implies that $f_1(U) \subset U_1$ and $f_2(U) \subset U_2$ for some maps $f_1, f_2 \in G$. Then, $f_1(U)$ and $f_2(U)$ are disjoint subsets of U. Thus, the group G has Property SC.

Conversely, assume the group G is ample, acts minimally on X, and has Property SC. For any clopen set $W \subset X$ different from $\emptyset$ and X, there exist maps $f_1, f_2 \in G$ such that the images $f_1(W)$ and $f_2(W)$ are disjoint subsets of W. Then, $f_1(W)$ is a proper subset of W. It follows that $f_1^{j+1}(W)$ is a proper subset of $f_1^j(W)$ for all $j \geq 0$. Let $D = W \setminus f_1(W)$. Then, $f_1^j(D) = f_1^j(W) \setminus f_1^{j+1}(W)$ for all $j \geq 0$, which implies that $D, f_1(D), f_1^2(D), \ldots$ are disjoint subsets of W.

Let us show that any clopen set $U \subset X$ different from $\emptyset$ and X can be mapped to its complement by an element of G. By the above, there exist a non-empty clopen set D and a map $f \in G$ such that $D, f(D), f^2(D), \ldots$ are disjoint subsets of $X \setminus U$. By Lemma 5.3, for some $k \geq 1$, there exist disjoint clopen sets $V_1, V_2, \ldots, V_k$ and maps $g_1, g_2, \ldots, g_k \in G$ such that $U = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$ and $g_i(V_i) \subset D$ for all i. For any i, $1 \leq i \leq k$, let $h_i = f^{i-1} g_i$. Then, $h_i \in G$ and $h_i(V_i) \subset f^{i-1}(D)$. Therefore, $h_1(V_1), h_2(V_2), \ldots, h_k(V_k)$ are disjoint subsets of $X \setminus U$. Note that the generalized 2-cycles $\delta_{V_i;h_i}$, $1 \leq i \leq k$ are well defined. Since the group G is ample, it contains all of

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them as well as their product $h = \delta_{V_1;h_1}\delta_{V_2;h_2}\cdots\delta_{V_k;h_k}$. As the sets $V_1, V_2, \ldots, V_k$ and $h_1(V_1), h_2(V_2), \ldots, h_k(V_k)$ are disjoint from one another, $h(V_i) = \delta_{V_i;h_i}(V_i) = h_i(V_i)$ for all i so that $h(V_i) \subset X \setminus U$ for all i. Then, $h(U) \subset X \setminus U$ as well.

Now, we can establish Property UC. Take any clopen sets $A, B \subset X$ different from the empty set and X. We need to show that A can be mapped to B by an element of the group G. First, consider the case where $A \cup B \neq X$. Then, $C = X \setminus (A \cup B)$ is a non-empty clopen set. By the above, $h_1(A \cup B) \subset C$ and $h_2(X \setminus B) \subset B$ for some maps $h_1, h_2 \in G$. Since $C \subset X \setminus B$, we obtain that $h_2h_1(A) \subset h_2h_1(A \cup B) \subset h_2(C) \subset h_2(X \setminus B) \subset B$. As for the case where $A \cup B = X$, we have $h(A) \subset X \setminus A$ for some $h \in G$. If $A \cup B = X$, then the complement $X \setminus A$ is contained in B. □

Property CI. (Cancellation of intersection) For any non-empty clopen sets $A, B, C \subset X$ such that C is disjoint from $A \cup B$, and the unions $A \cup C$ and $B \cup C$ are different from X, we have $f(A \cup C) = B \cup C$ for some $f \in G$ if and only if $h(A) = B$ for some $h \in G$.

Definition 5.7. Given non-empty clopen sets $D, E \subset X$, we say that D can be *absorbed* by E under the action of a group $G \subset \mathrm{Homeo}(X)$ if E is disjoint from D and $g(D \cup E) = E$ for some $g \in G$. Further, a non-empty clopen set is called *absorbable* under the action of G if it can be absorbed by another clopen set.

LEMMA 5.8. *Suppose a non-empty clopen set $D \subset X$ is absorbable under the action of a group $G \subset \mathrm{Homeo}(X)$ that has Property CI. Then, D can be absorbed by any clopen set $E \subset X \setminus D$ different from the empty set and $X \setminus D$.*

Proof. Since the set D is absorbable under the action of the group G, we have $f(D \cup E_0) = E_0$ for some $f \in G$ and clopen set E_0 disjoint from D. Note that E_0 is different from $\emptyset$ and $X \setminus D$. Consider an arbitrary clopen set $E \subset X \setminus D$ different from $\emptyset$ and $X \setminus D$. In the case where $E \cap E_0 \neq \emptyset$, Property CI first implies that $g(D \cup (E \cap E_0)) = E \cap E_0$ for some $g \in G$, then implies that $h(D \cup E) = E$ for some $h \in G$. Similarly, in the case where $E \cup E_0 \neq X \setminus D$, Property CI first implies that $g(D \cup (E \cup E_0)) = E \cup E_0$ for some $g \in G$, then implies that $h(D \cup E) = E$ for some $h \in G$. Finally, in the case where $E \cap E_0 = \emptyset$ and $E \cup E_0 = X \setminus D$, we have $f(E) = D \cup E$ so that $f^{-1}(D \cup E) = E$. Thus, D can be absorbed by E. □

LEMMA 5.9. *Any group $G \subset \mathrm{Homeo}(X)$ with Property CI satisfies the following two conditions.*

(CI-1) *For any disjoint nonempty clopen sets $A, B, C \subset X$, if $f(A \cup C) = B \cup C$ for some $f \in G$, then $h(A) = B$ for some $h \in G$.*

(CI-2) *For any disjoint non-empty clopen sets $D, E_0, E \subset X$, if $f(D \cup E_0) = E_0$ for some $f \in G$, then $h(D \cup E) = E$ for some $h \in G$.*

If the group G is ample, then conversely, conditions (CI-1) and (CI-2) imply Property CI.

Proof. Suppose a group $G \subset \mathrm{Homeo}(X)$ has Property CI. For any disjoint non-empty sets $A, B, C \subset X$, the unions $A \cup C$ and $B \cup C$ are both different from X. Hence, condition

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(CI-1) follows directly from Property CI. As for condition (CI-2), it easily follows from Lemma 5.8.

Now, let us prove that any ample group $G \subset \mathrm{Homeo}(X)$ satisfying conditions (CI-1) and (CI-2) has Property CI. Given non-empty clopen sets $A, B, C \subset X$, where C is disjoint from both A and B, $A \cup C \neq X$ and $B \cup C \neq X$, we need to show that the set $A \cup C$ can be mapped onto $B \cup C$ under the action of G if and only if the set A can be mapped onto B. This is trivial when $A = B$. Consider the case where both $A \setminus B$ and $B \setminus A$ are non-empty sets. If $f(A \cup C) = B \cup C$ or $f(A) = B$ for some $f \in G$, condition (CI-1) implies that $g(A \setminus B) = B \setminus A$ for some $g \in G$. Since the clopen sets $A \setminus B$ and $B \setminus A$ are disjoint, a generalized 2-cycle $h = \delta_{A \setminus B; g}$ is well defined. It belongs to the group G as G is ample. Since h coincides with the identity map on $A \cap B$ and on C, we obtain that $h(A) = B$ and $h(A \cup C) = B \cup C$.

It remains to consider the case where one of the sets A and B is a proper subset of the other. Without loss of generality, we may assume that B is a proper subset of A. Note that the set $B \cup C$ is disjoint from $A \setminus B$ and $(A \setminus B) \cup (B \cup C) = A \cup C \neq X$. Hence, X is a disjoint union of four non-empty clopen sets $A \setminus B$, B, C, and $X \setminus (A \cup C)$. Condition (CI-2) first implies that the set $A \setminus B$ can be absorbed by $B \cup C$ if and only if it can be absorbed by $X \setminus (A \cup C)$, then implies that $A \setminus B$ can be absorbed by $X \setminus (A \cup C)$ if and only if it can be absorbed by B. Consequently, $f(A \cup C) = B \cup C$ for some $f \in G$ if and only if $h(A) = B$ for some $h \in G$. □

LEMMA 5.10. *For any ample group, Property NC implies Property CI.*

Proof. Suppose a group $G \subset \mathrm{Homeo}(X)$ is ample and has Property NC. To establish Property CI, we are going to show that G satisfies conditions (CI-1) and (CI-2) in Lemma 5.9. Since the group G allows no contraction of open sets, it also allows no absorbable sets. Hence, condition (CI-2) holds vacuously. Let us verify condition (CI-1). Consider any disjoint non-empty clopen sets $A, B, C \subset X$ such that $f(A \cup C) = B \cup C$ for some $f \in G$. We need to find an element of G that maps A onto B. By Lemma 5.4, Property NC is equivalent to Property UR. Therefore, the group G has Property UR. As a consequence, for any point $x \in A$, the sequence $f(x), f^2(x), f^3(x), \ldots$ visits the set A. In particular, this sequence is not contained in C. Let $p(x)$ denote the least positive integer k such that $f^k(x) \notin C$. Since $f(A \cup C) = B \cup C$, it follows that $f^{p(x)}(x) \in B$. Likewise, Property UR implies that for any point $y \in B$, the sequence $f^{-1}(y), f^{-2}(y), f^{-3}(y), \ldots$ visits B and, hence, is not confined to C. Let $m(y)$ denote the least positive integer k such that $f^{-k}(y) \notin C$. Then, $f^{-m(y)}(y) \in A$. For any integer $k \geq 1$, let $A_k = \{x \in A \mid p(x) = k\}$ and $B_k = \{y \in B \mid m(y) = k\}$. The sets $A_1, A_2, A_3, \ldots$ form a partition of A, while the sets $B_1, B_2, B_3, \ldots$ form a partition of B. By construction, $f^k(A_k) \subset B_k$ and $f^{-k}(B_k) \subset A_k$ for all k, which implies that $f^k(A_k) = B_k$ for all k.

Now, we define a map $h : X \to X$ by $h(x) = f^{p(x)}(x)$ if $x \in A$, $h(x) = f^{-m(x)}(x)$ if $x \in B$, and $h(x) = x$ otherwise. For any $k \geq 1$, the map h coincides with f^k on A_k and with f^{-k} on B_k. Since $f^k(A_k) = B_k$ for all k, it follows that h is an involution. In addition, $h(A) = B$.

Notice that all sets of the form A_k and B_k are clopen. Indeed, A_k is the intersection of clopen sets A, $f^{-k}(B)$ and $f^{-i}(C)$, $1 \leq i \leq k-1$. Likewise, B_k is the intersection

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of clopen sets B, $f^k(A)$ and $f^i(C)$, $1 \le i \le k-1$. Since the set A, which is compact, is the disjoint union of clopen sets $A_1, A_2, A_3, \ldots$, all but finitely many of those sets have to be empty. Likewise, all but finitely many of the sets $B_1, B_2, B_3, \ldots$ are empty. As a consequence, the map h is piecewise an element of the group G. Since h is invertible, we conclude that h belongs to the group $\mathsf{F}(G)$, which coincides with G as G is ample. □

LEMMA 5.11. *For any ample group, Property UC implies Property CI.*

Proof. Suppose a group $G \subset \mathrm{Homeo}(X)$ is ample and has Property UC. To establish Property CI, we are going to show that the group G satisfies conditions (CI-1) and (CI-2) in Lemma 5.9. We begin with condition (CI-1). Take any disjoint non-empty clopen sets $A, B, C \subset X$ such that $f(A \cup C) = B \cup C$ for some $f \in G$. First, consider the case where the image $f(A)$ intersects B. Then, $A \cap f^{-1}(B)$ is a non-empty clopen set. Property UC implies that $g(C) \subset A \cap f^{-1}(B)$ for some $g \in G$. Let $O_1 = g(C)$ and $O_2 = f(O_1)$. Then, O_1 and O_2 are non-empty clopen subsets of respectively A and B. In particular, O_1 and O_2 are both disjoint from C. Therefore, the generalized 2-cycles $\delta_{C;g}$ and $\delta_{C;fg}$ are well defined. Since the group G is ample, it contains $\delta_{C;g}$ and $\delta_{C;fg}$ as well as the map $h = \delta_{C;fg} f \delta_{C;g}$. Let us find $h(A)$. The map $\delta_{C;g}$ fixes all points in $A \setminus O_1$ while O_1 is mapped onto C. Hence, $\delta_{C;g}$ maps A onto $(A \setminus O_1) \cup C = (A \cup C) \setminus O_1$. Further, f maps $(A \cup C) \setminus O_1$ onto $(B \cup C) \setminus O_2$. Finally, $\delta_{C;fg}$ coincides with the identity map on $B \setminus O_2$ while mapping C onto O_2. We conclude that $h(A) = B$.

Now, consider the case where $f(A)$ is disjoint from B. In this case, $f(A) \subset C$ and also $f^{-1}(B) \subset C$. Property UC implies that $g_1(C) \subset A$ and $g_2(A) \subset B$ for some maps $g_1, g_2 \in G$. Let $O_1 = g_1(C)$, $O_2 = g_2(O_1)$, and $O_3 = f^{-1}(O_2)$. Then, O_1, O_2, and O_3 are non-empty clopen subsets of A, B, and C, respectively. In particular, O_1 and O_2 are both disjoint from C and O_3. Therefore, the generalized 2-cycles $\delta_{C;g_1}$, $\delta_{C;g_2g_1}$, and $\delta_{O_1;f^{-1}g_2}$ are well defined. The ample group G contains all these generalized 2-cycles as well as the map $h = \delta_{C;g_2g_1} f \delta_{O_1;f^{-1}g_2} \delta_{C;g_1}$. Let us find $h(A)$. The map $\delta_{C;g_1}$ fixes all points in $A \setminus O_1$ while O_1 is mapped onto C. Hence, $\delta_{C;g_1}$ maps A onto $(A \cup C) \setminus O_1$. Then, $\delta_{O_1;f^{-1}g_2}$ maps $(A \cup C) \setminus O_1$ onto $(A \cup C) \setminus O_3$. Further, f maps $(A \cup C) \setminus O_3$ onto $(B \cup C) \setminus O_2$. Finally, $\delta_{C;g_2g_1}$ coincides with the identity map on $B \setminus O_2$ while mapping C onto O_2. Thus, $h(A) = B$.

We proceed to verification of condition (CI-2). Consider any disjoint non-empty clopen sets $D, E_0, E \subset X$ such that $f(D \cup E_0) = E_0$ for some $f \in G$. Let $E_1 = f(D)$ and $E_2 = f(E_0)$. Then, E_1 and E_2 are non-empty clopen sets that partition E_0. Property UC implies that $g(E_0) \subset E$ for some $g \in G$. Let $U = g(E_0)$, $U_1 = g(E_1)$, and $U_2 = g(E_2)$. Then, U_1 and U_2 are non-empty clopen sets that partition U. Now, we define a continuous map $h : X \to X$ that is piecewise an element of the group G. The map h coincides with gf on D, with f^{-1} on E_0, with gfg^{-1} on U, and with the identity map everywhere else. We have $h(D) = g(E_1) = U_1$, $h(E_0) = D \cup E_0$ and $h(U) = gf(E_0) = g(E_2) = U_2$. Since the sets $h(D)$, $h(E_0)$, and $h(U)$ form a partition of $D \cup E_0 \cup U$, it follows that the map h is invertible and, hence, a homeomorphism. By construction, h belongs to the group $\mathsf{F}(G)$, which coincides with G as G is ample. Note that $h(D \cup U) = U_1 \cup U_2 = U$. In addition, $U \subset E$ and h fixes every point in $E \setminus U$. Therefore, $h(D \cup E) = E$. Thus, the set D can be absorbed by E. □

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LEMMA 5.12. *If a non-empty clopen set $U \subset X$ is absorbable under the action of a group $G \subset \mathrm{Homeo}(X)$, then for any $g \in G$, the set $g(U)$ is absorbable as well. If the group G has Property CI then, conversely, any two absorbable sets can be mapped onto each other by elements of G.*

Proof. If a clopen set U can be absorbed by a clopen set E under the action of a group $G \subset \mathrm{Homeo}(X)$, then for any $g \in G$, the set $g(U)$ can be absorbed by $g(E)$. Indeed, E is disjoint from U and $f(U \cup E) = E$ for some $f \in G$. Then, for any $g \in G$, the clopen set $g(E)$ is disjoint from $g(U)$, the map $h = gfg^{-1}$ belongs to G, and $h(g(U) \cup g(E)) = hg(U \cup E) = gf(U \cup E) = g(E)$.

Now, suppose the group G has Property CI and $U_1, U_2 \subset X$ are two absorbable sets. We need to show that $h(U_1) = U_2$ for some $h \in G$. First, consider the case where both U_1 and U_2 can be absorbed by the same clopen set E. In this case, $f_1(U_1 \cup E) = E$ and $f_2(U_2 \cup E) = E$ for some $f_1, f_2 \in G$. Then, $f_2^{-1} f_1 \in G$ and $f_2^{-1} f_1(U_1 \cup E) = U_2 \cup E$. Since E is disjoint from both U_1 and U_2, Property CI implies that $h(U_1) = U_2$ for some $h \in G$.

If one of the sets U_1 and U_2 contains the other, it follows from Lemma 5.8 that the smaller set can be absorbed by any clopen set that can absorb the larger one. Next, consider the case where neither of the sets U_1 and U_2 contains the other, and also $U_1 \cup U_2 \neq X$. Then, the clopen set $E = X \setminus (U_1 \cup U_2)$ is non-empty, and the unions $U_1 \cup E$ and $U_2 \cup E$ are both different from X. By Lemma 5.8, both U_1 and U_2 can be absorbed by E.

It remains to consider the case where $U_1 \cup U_2 = X$. Take any clopen set E that can absorb U_1. Then, E is disjoint from U_1 and $f(U_1 \cup E) = E$ for some $f \in G$. The set $U = f(U_1)$ is also absorbable. Indeed, $f(U \cup (E \setminus U)) = f(E) = E \setminus U$ so that U can be absorbed by $E \setminus U$. Since U is disjoint from U_1 and $U_1 \cup U_2 = X$, the set U_2 contains U. By the above, $h(U) = U_2$ for some $h \in G$. Then, $hf \in G$ and $hf(U_1) = U_2$. □

LEMMA 5.13. *For any ample group that acts minimally and admits an absorbable set, Property CI implies Property UC.*

Proof. Suppose $G \in \mathrm{Homeo}(X)$ is an ample group that acts minimally on X, admits an absorbable set, and has Property CI. We are going to show that the group G has the following property, which is a weakened form of Property SC: for any non-empty clopen set $W \subset X$, there exist a clopen set $W_0 \subset W$ and a map $f \in G$ such that $f(W_0)$ is a proper subset of W_0. After that, Property UC can be established in the same way as it was done in the proof of Lemma 5.6.

Take any non-empty clopen set $D \subset X$ absorbable under the action of the group G. Let E be any clopen set that can absorb D. Then, E is disjoint from D and $g(D \cup E) = E$ for some $g \in G$. The set $D_1 = g(D)$, which is a proper subset of E, is absorbable due to Lemma 5.12. Since the group G has Property CI, it follows from Lemma 5.8 that every non-empty clopen subset of D can absorb D_1, every non-empty clopen subset of D_1 can absorb D, and every non-empty clopen subset of $X \setminus (D \cup D_1)$ can absorb both D and D_1. Any non-empty clopen set $W \subset X$ intersects at least one of the sets D, D_1, and $X \setminus (D \cup D_1)$. Hence, W has a clopen subset W_0 that can absorb an absorbable set D_0,

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where $D_0 = D$ or D_1. Then, W_0 is disjoint from D_0 and $f(D_0 \cup W_0) = W_0$ for some $f \in G$. Since W_0 is a proper subset of $D_0 \cup W_0$, it follows that $f(W_0)$ is a proper subset of W_0. □

Combining Lemmas 5.11 and 5.13, we obtain that an ample group has Property UC if and only if it acts minimally, admits an absorbable set (or, equivalently, admits contraction of a clopen set), and has Property CI.

Question 5.14. Let G be an ample group that acts minimally on a Cantor set.

- Suppose the group G admits contraction of a clopen set. Is it possible that G does not have Property UC?
- Suppose the group G does not allow contraction of any clopen set. Is it possible that G does not have Property NC?

Let $C(X, \mathbb{Z})$ denote the set of all continuous functions $\phi : X \to \mathbb{Z}$. We regard $C(X, \mathbb{Z})$ as an additive group. For any clopen set $U \subset X$, the characteristic function χ_U, defined by $\chi_U(x) = 1$ if $x \in U$ and $\chi_U(x) = 0$ if $x \in X \setminus U$, belongs to $C(X, \mathbb{Z})$. Any function $\phi \in C(X, \mathbb{Z})$ takes only finitely many values and non-empty level sets of ϕ form a partition of X into clopen sets. It follows that the group $C(X, \mathbb{Z})$ is generated by characteristic functions of clopen sets.

Definition 5.15. Given a group $G \subset \mathrm{Homeo}(X)$, let K_G denote the subgroup of $C(X, \mathbb{Z})$ generated by all functions of the form $\phi - \phi \circ g$, where $\phi \in C(X, \mathbb{Z})$ and $g \in G$. The factor group $C(X, \mathbb{Z})/K_G$ is denoted $H_0(G)$ and called the zeroth *homology group* of G.

It is not hard to show that the homology group $H_0(G)$ is preserved when the group G is amplified. In particular, the homology group $H_0(\mathsf{F}(G))$ of the ample group $\mathsf{F}(G)$ is the same as $H_0(G)$.

Property HS. (Homologous sets can be mapped onto each other) Whenever the characteristic functions of two clopen sets $A, B \subset X$ different from the empty set and X represent the same homology class in $H_0(G)$, we have $g(A) = B$ for some $g \in G$.

If $g(A) = B$ for some clopen sets $A, B \subset X$ and $g \in G$, then we have $\chi_A - \chi_B = \chi_A - \chi_A \circ g^{-1} \in K_G$ so that χ_A and χ_B represent the same homology class in $H_0(G)$. Property HS means that the action of the group G on clopen subsets of X has as much transitivity as the homology group $H_0(G)$ allows.

An analog of Property HS was established by Matui for two classes of étale groupoids, for *almost finite* groupoids in [**46**] and for purely infinite groupoids in [**48**]. The authors are grateful to the anonymous referee for drawing our attention to these results. In [**48**], this property is called *cancellation* (see [**48**, Definition 2.11]). Our naming of Property CI is a tribute to that.

LEMMA 5.16. *Property HS implies Property CI.*

Proof. If A, B, and C are clopen subsets of X such that C is disjoint from both A and B, then $\chi_{A\cup C} - \chi_{B\cup C} = \chi_A - \chi_B$. Hence, for any group $G \subset \mathrm{Homeo}(X)$, the characteristic

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functions $\chi_{A\cup C}$ and $\chi_{B\cup C}$ represent the same homology class in $H_0(G)$ if and only if the characteristic functions χ_A and χ_B represent the same homology class in $H_0(G)$. It follows that Property HS implies Property CI. □

It seems plausible that for an ample group, Properties CI and HS are equivalent. However, the authors are not able to prove this yet.

Property E. (Entanglement) Whenever clopen sets U_1 and U_2 intersect, the local subgroup $\mathsf{F}_{U_1\cup U_2}(G)$ is generated by the union of local subgroups $\mathsf{F}_{U_1}(G)\cup\mathsf{F}_{U_2}(G)$.

Along with minimality, Property E will be very important for us in §6. The other properties introduced in this section are relevant only because they can help to establish Property E. We defer a detailed discussion of Property E to §8.

6. *Maximal subgroups of ample groups*

In this section, we study maximal subgroups of an ample group $\mathcal{G}\subset \mathrm{Homeo}(X)$, where the topological space X is compact, metrizable, and totally disconnected. We give examples of such subgroups and obtain some partial results on their classification. Most of the time, we require the group $\mathcal{G}$ to act minimally on X. Many results also require Property E (see §5).

Definition 6.1. A subgroup H of a group G is called *maximal* if $H\neq G$ and there is no subgroup K such that $H\subset K\subset G$, while $K\neq H$ and $K\neq G$.

The first class of subgroups to look for maximal subgroups are the stabilizers of closed sets (see Definition 3.2). Since $\mathrm{St}_{\mathcal{G}}(Y)=\mathrm{St}_{\mathcal{G}}(X\setminus Y)$ for any set $Y\subset X$, they are also the stabilizers of open sets.

LEMMA 6.2. *Let $\mathcal{G}\subset\mathrm{Homeo}(X)$ be an ample group that acts minimally on X. Suppose Y is a closed subset of X. Then, $\mathrm{St}_{\mathcal{G}}(Y)\neq\mathcal{G}$ unless Y is the empty set or $Y=X$.*

Proof. Clearly, the entire group $\mathcal{G}$ is the stabilizer of X and of the empty set. If Y is not empty, we can find a point $x\in Y$. Since the group $\mathcal{G}$ acts minimally on X, the orbit $\mathrm{Orb}_{\mathcal{G}}(x)$ is dense in X. If $Y\neq X$, then the open set $X\setminus Y$ is not empty. Hence, $\mathrm{Orb}_{\mathcal{G}}(x)$ has a point in $X\setminus Y$. This means that $g(x)\notin Y$ for some $g\in\mathcal{G}$. Then, $g\notin\mathrm{St}_{\mathcal{G}}(Y)$ so that $\mathrm{St}_{\mathcal{G}}(Y)\neq\mathcal{G}$. □

The following lemma and its proof describe a general construction that will be used repeatedly throughout this section.

LEMMA 6.3. *Let $\mathcal{G}\subset\mathrm{Homeo}(X)$ be an ample group. Suppose x and y are two different points in the same orbit of $\mathcal{G}$, and $Z\subset X$ is a closed set that contains neither x nor y. Then, there exists a map $f\in\mathcal{G}$ such that $f(x)=y$, $f(y)=x$, and $f(z)=z$ for all $z\in Z$.*

Proof. For any point $z\in Z$, we can find a clopen neighborhood U_z that contains neither x nor y. Since the set Z is closed and hence compact, it can be covered by finitely many clopen sets of the form U_z, $z\in Z$. Taking the union of all sets in that finite cover, we obtain

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a clopen set U containing Z and disjoint from $\{x, y\}$. Next, we can find disjoint clopen sets $V_x, V_y \subset X \setminus U$ such that $x \in V_x$ and $y \in V_y$. Since the points x and y are in the same orbit of the group $\mathcal{G}$, we have $y = g(x)$ for some $g \in \mathcal{G}$. Then, $W = V_x \cap g^{-1}(V_y)$ is a clopen neighborhood of x. The set W is disjoint from $g(W)$ as $W \subset V_x$, while $g(W) \subset V_y$. Therefore, the generalized 2-cycle $f = \delta_{W;g}$ is defined. Since $g \in \mathcal{G}$, the map f belongs to the group $\mathsf{F}(\mathcal{G}) = \mathcal{G}$. By construction, $f(x) = g(x) = y$ and $f(y) = g^{-1}(y) = x$. All points in the complement of the set $W \cup g(W)$ are fixed by f. Since $W \cup g(W) \subset V_x \cup V_y \subset X \setminus U$ and $Z \subset U$, it follows that $f(z) = z$ for all $z \in Z$. □

LEMMA 6.4. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X. Suppose Y_1 and Y_2 are distinct closed subsets of X. Then, $\mathrm{St}_{\mathcal{G}}(Y_1) = \mathrm{St}_{\mathcal{G}}(Y_2)$ if and only if $Y_1 \cup Y_2 = X$ and $Y_1 \cap Y_2 = \partial Y_1 \cap \partial Y_2$. Equivalent conditions are that $Y_1 = \overline{X \setminus Y_2}$ and $Y_2 = \overline{X \setminus Y_1}$.*

Proof. Recall that $\mathrm{St}_{\mathcal{G}}(X \setminus E) = \mathrm{St}_{\mathcal{G}}(E)$ for any set $E \subset X$. If $f : X \to X$ is a homeomorphism, then $f(\overline{E}) = \overline{f(E)}$. Hence, $f(\overline{E}) = \overline{E}$ whenever $f(E) = E$. As a consequence, $\mathrm{St}_{\mathcal{G}}(E) \subset \mathrm{St}_{\mathcal{G}}(\overline{E})$.

Assume that sets $Y_1, Y_2 \subset X$ satisfy conditions $Y_1 = \overline{X \setminus Y_2}$ and $Y_2 = \overline{X \setminus Y_1}$. By the above, $\mathrm{St}_{\mathcal{G}}(Y_2) = \mathrm{St}_{\mathcal{G}}(X \setminus Y_2) \subset \mathrm{St}_{\mathcal{G}}(\overline{X \setminus Y_2}) = \mathrm{St}_{\mathcal{G}}(Y_1)$. Likewise, $\mathrm{St}_{\mathcal{G}}(Y_1) \subset \mathrm{St}_{\mathcal{G}}(Y_2)$. Hence, $\mathrm{St}_{\mathcal{G}}(Y_1) = \mathrm{St}_{\mathcal{G}}(Y_2)$. Further, either of the two assumed conditions implies that $Y_1 \cup Y_2 = X$. The first condition also implies that no interior point of Y_2 belongs to Y_1. Likewise, the second of the two implies that no interior point of Y_1 belongs to Y_2. Therefore, any common point of Y_1 and Y_2 is their common boundary point: $Y_1 \cap Y_2 \subset \partial Y_1 \cap \partial Y_2$. Since the sets Y_1 and Y_2 are clearly closed, they contain their own boundaries so that $Y_1 \cap Y_2 = \partial Y_1 \cap \partial Y_2$. Conversely, assume $Y_1, Y_2 \subset X$ are closed sets such that $Y_1 \cup Y_2 = X$ and $Y_1 \cap Y_2 = \partial Y_1 \cap \partial Y_2$. Then, X is a disjoint union of three sets $Y_1 \setminus Y_2$, $Y_2 \setminus Y_1$, and $Y_1 \cap Y_2$. Moreover, the sets $Y_1 \setminus Y_2 = X \setminus Y_2$ and $Y_2 \setminus Y_1 = X \setminus Y_1$ are open. It follows that $Y_1 \setminus Y_2$ and $Y_2 \setminus Y_1$ are disjoint from both ∂Y_1 and ∂Y_2. Hence, $\partial Y_1 \cup \partial Y_2 \subset Y_1 \cap Y_2$, which implies that $\partial Y_1 = \partial Y_2 = Y_1 \cap Y_2$. Since $\partial Y_1 = \partial(X \setminus Y_1) = \partial(Y_2 \setminus Y_1)$, we obtain that $\overline{X \setminus Y_1} = \overline{Y_2 \setminus Y_1} = (Y_2 \setminus Y_1) \cup \partial(Y_2 \setminus Y_1) = (Y_2 \setminus Y_1) \cup (Y_1 \cap Y_2) = Y_2$. Likewise, $\overline{X \setminus Y_2} = Y_1$.

It remains to show that $\mathrm{St}_{\mathcal{G}}(Y_1) \neq \mathrm{St}_{\mathcal{G}}(Y_2)$ for any distinct closed sets $Y_1, Y_2 \subset X$ that do not satisfy at least one of the conditions $Y_1 \cup Y_2 = X$ and $Y_1 \cap Y_2 = \partial Y_1 \cap \partial Y_2$. First, we consider the case where $Y_1 \cup Y_2 \neq X$. Then, $U = X \setminus (Y_1 \cup Y_2)$ is a non-empty open set. Since $Y_1 \neq Y_2$, there is a point $x \in X$ that belongs to one of these sets but not to the other. We may assume without loss of generality that $x \in Y_1$ and $x \notin Y_2$. Since the group $\mathcal{G}$ acts minimally on X, the orbit $\mathrm{Orb}_{\mathcal{G}}(x)$ is dense in X. In particular, it has a point y in U. Clearly, $y \notin Y_1$ and $y \notin Y_2$. By Lemma 6.3, there exists a map $f \in \mathcal{G}$ such that $f(x) = y$ and $f(z) = z$ for all $z \in Y_2$. Then, $f \in \mathrm{St}^{\star}_{\mathcal{G}}(Y_2) \subset \mathrm{St}_{\mathcal{G}}(Y_2)$. However, $f \notin \mathrm{St}_{\mathcal{G}}(Y_1)$ as $x \in Y_1$, while $y \notin Y_1$. Therefore, $\mathrm{St}_{\mathcal{G}}(Y_1) \neq \mathrm{St}_{\mathcal{G}}(Y_2)$.

Now, consider the case where $Y_1 \cup Y_2 = X$, but there exists a point $x \in Y_1 \cap Y_2$ that is not a boundary point for at least one of the sets Y_1 and Y_2. We may assume without loss of generality that x is an interior point of Y_2. Furthermore, we may assume that $Y_1 \neq X$ as otherwise, $\mathrm{St}_{\mathcal{G}}(Y_2) \neq \mathcal{G} = \mathrm{St}_{\mathcal{G}}(Y_1)$ due to Lemma 6.2. Then, $X \setminus Y_1$ is a non-empty open set. Just as above, minimality of the action of the group $\mathcal{G}$ on X implies that the orbit $\mathrm{Orb}_{\mathcal{G}}(x)$ has a point y in $X \setminus Y_1$. Note that $X \setminus Y_1 \subset Y_2$ since $Y_1 \cup Y_2 = X$. Therefore,

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every point of $X \setminus Y_1$ (including y) is an interior point of Y_2. Hence, the closed set $\overline{X \setminus Y_2}$ contains neither x nor y. By Lemma 6.3, there exists a map $f \in \mathcal{G}$ such that $f(x) = y$ and $f(z) = z$ for all $z \in \overline{X \setminus Y_2}$. Then, $f \in \mathrm{St}_{\mathcal{G}}(X \setminus Y_2) = \mathrm{St}_{\mathcal{G}}(Y_2)$. However, $f \notin \mathrm{St}_{\mathcal{G}}(Y_1)$ as $x \in Y_1$, while $y \notin Y_1$. Thus, $\mathrm{St}_{\mathcal{G}}(Y_1) \neq \mathrm{St}_{\mathcal{G}}(Y_2)$. □

LEMMA 6.5. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X. Suppose Y_1 and Y_2 are non-empty closed subsets of X such that $Y_1 \subset Y_2$ and $Y_1 \neq Y_2$. Then, $\mathrm{St}_{\mathcal{G}}(Y_1) \neq \mathrm{St}_{\mathcal{G}}(Y_2)$.*

Proof. If $Y_2 = X$, then $\mathrm{St}_{\mathcal{G}}(Y_1) \neq \mathcal{G} = \mathrm{St}_{\mathcal{G}}(Y_2)$ due to Lemma 6.2. Otherwise, $Y_1 \cup Y_2 = Y_2 \neq X$. Then, $\mathrm{St}_{\mathcal{G}}(Y_1) \neq \mathrm{St}_{\mathcal{G}}(Y_2)$ due to Lemma 6.4. □

Now, we are ready to establish our first result on the classification of maximal subgroups of ample groups.

THEOREM 6.6. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X. Suppose H is a maximal subgroup of $\mathcal{G}$ that does not act minimally on X. Then, $H = \mathrm{St}_{\mathcal{G}}(Y)$ for some closed set $Y \subset X$ different from the empty set and X. Furthermore, if the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ of a closed set $Y \subset X$ is indeed a maximal subgroup of $\mathcal{G}$, then the induced action of $\mathrm{St}_{\mathcal{G}}(Y)$ on Y is minimal.*

Proof. Since the group H does not act minimally on X, it admits a closed invariant set Y different from the empty set and X. We have $h(Y) = Y$ for any $h \in H$. Hence, $H \subset \mathrm{St}_{\mathcal{G}}(Y)$. Lemma 6.2 implies that $\mathrm{St}_{\mathcal{G}}(Y) \neq \mathcal{G}$. As H is a maximal subgroup of $\mathcal{G}$, it follows that $H = \mathrm{St}_{\mathcal{G}}(Y)$.

Now, assume that a group $H = \mathrm{St}_{\mathcal{G}}(Y)$, where Y is a closed subset of X, is indeed a maximal subgroup of $\mathcal{G}$. To prove that the induced action of H on Y is minimal, we need to show that any closed set $Y_0 \subset Y$ invariant under it coincides with the empty set or Y. We have $h(Y_0) = Y_0$ for any $h \in H$. Hence, $H \subset \mathrm{St}_{\mathcal{G}}(Y_0)$. Note that $Y \neq X$ since $H \neq \mathcal{G}$. Then, $Y_0 \neq X$ as well. Assume that $Y_0 \neq \emptyset$. Just as above, Lemma 6.2 implies that $\mathrm{St}_{\mathcal{G}}(Y_0) \neq \mathcal{G}$ and then maximality of H implies that $H = \mathrm{St}_{\mathcal{G}}(Y_0)$. Since $\mathrm{St}_{\mathcal{G}}(Y_0) = \mathrm{St}_{\mathcal{G}}(Y)$ and $Y_0 \subset Y$, it follows from Lemma 6.5 that $Y_0 = Y$. □

The assumption that the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ of a closed set Y acts minimally when restricted to Y has implications for the set Y and for the group $\mathrm{St}_{\mathcal{G}}(Y)$.

LEMMA 6.7. *Let G be a subgroup of* $\mathrm{Homeo}(X)$. *Suppose Y is a closed subset of X such that the stabilizer $\mathrm{St}_G(Y)$ acts minimally when restricted to Y. Then, the set Y is either clopen or nowhere dense in X.*

Proof. We have $h(Y) = Y$ for any $h \in \mathrm{St}_G(Y)$. Since h is a homeomorphism of the ambient space X, it maps interior points of the set Y to its interior points, and boundary points of Y to its boundary points. Therefore, the boundary ∂Y, which is a closed subset of Y, is invariant under the induced action of the group $\mathrm{St}_G(Y)$ on Y. As the latter action is minimal, the boundary ∂Y has to coincide with the empty set or Y. If $\partial Y = \emptyset$, then the set Y is clopen. If $\partial Y = Y$, then Y is nowhere dense in X. □

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LEMMA 6.8. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group. Suppose U is a non-empty clopen subset of X. Then, any orbit of the induced action of the stabilizer $\mathrm{St}_{\mathcal{G}}(U)$ on U is the intersection of an orbit of $\mathcal{G}$ with U. As a consequence, $\mathrm{St}_{\mathcal{G}}(U)$ acts minimally on U whenever $\mathcal{G}$ acts minimally on X.*

Proof. The induced action of the group $H = \mathrm{St}_{\mathcal{G}}(U)$ on U is the restriction of its action on X. Therefore, every orbit of the induced action is of the form $\mathrm{Orb}_H(x)$, where $x \in U$. We need to show that $\mathrm{Orb}_H(x) = \mathrm{Orb}_{\mathcal{G}}(x) \cap U$. The inclusion $\mathrm{Orb}_H(x) \subset \mathrm{Orb}_{\mathcal{G}}(x) \cap U$ is obvious. Conversely, take any point $y \in \mathrm{Orb}_{\mathcal{G}}(x) \cap U$ different from x. By Lemma 6.3, there exists a map $f \in \mathcal{G}$ such that $f(x) = y$ and f coincides with the identity map on the clopen set $X \setminus U$. Then, $f \in \mathrm{RiSt}_{\mathcal{G}}(U) \subset H$. Thus, $y \in \mathrm{Orb}_H(x)$.

Assuming the group $\mathcal{G}$ acts minimally on X, every orbit of $\mathcal{G}$ is dense in X. Since U is a clopen set, it follows that the intersection of any orbit of $\mathcal{G}$ with U is dense in U. By the above, this means that every orbit of the induced action of the group H on U is dense in U. Hence, the latter action is minimal. □

LEMMA 6.9. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group. Suppose Y is a closed subset of X such that the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ acts minimally when restricted to Y. Then, for any $g \in \mathcal{G} \setminus \mathrm{St}_{\mathcal{G}}(Y)$, the group $\langle \mathrm{St}_{\mathcal{G}}(Y) \cup \{g\} \rangle$ generated by $\mathrm{St}_{\mathcal{G}}(Y)$ and g has the same orbits as $\mathcal{G}$.*

Proof. Let $H = \mathrm{St}_{\mathcal{G}}(Y)$. For any $g \in \mathcal{G} \setminus \mathrm{St}_{\mathcal{G}}(Y)$, let $H_g = \langle \mathrm{St}_{\mathcal{G}}(Y) \cup \{g\} \rangle$. First, let us show that no orbit of the group H_g is contained in Y. Since g does not map the set Y onto itself, there exists a point $x_0 \in Y$ such that $g(x_0) \notin Y$ or $g^{-1}(x_0) \notin Y$. In either case, we have an element $g_0 \in H_g$ such that $g_0(x_0) \notin Y$. Note that $X \setminus Y$ is an open neighborhood of the point $g_0(x_0)$. Hence, the set $U = g_0^{-1}(X \setminus Y)$ is an open neighborhood of x_0. Take any $x \in Y$. Since the group H acts minimally when restricted to Y, the orbit $\mathrm{Orb}_H(x)$ is dense in Y. In particular, it has a point in $U \cap Y$. That is, $h(x) \in U$ for some $h \in H$. Then, $g_0 h \in H_g$ and $g_0 h(x) \in g_0(U) = X \setminus Y$. Therefore, the orbit $\mathrm{Orb}_{H_g}(x)$ has a point outside Y.

Since H_g is a subgroup of $\mathcal{G}$, for any $x \in X$, the orbit $\mathrm{Orb}_{H_g}(x)$ is contained in $\mathrm{Orb}_{\mathcal{G}}(x)$. We need to show that, conversely, $\mathrm{Orb}_{\mathcal{G}}(x) \subset \mathrm{Orb}_{H_g}(x)$. Take any point $y \in \mathrm{Orb}_{\mathcal{G}}(x)$. By the above, the orbits of the points x and y under the action of the group H_g are not contained in Y. Hence, there exist $h_1, h_2 \in H_g$ such that the points $x_1 = h_1(x)$ and $y_1 = h_2(y)$ are outside the set Y. If $x_1 = y_1$, then $y = h_2^{-1} h_1(x)$ so that $y \in \mathrm{Orb}_{H_g}(x)$. Now, assume $x_1 \neq y_1$. Both x_1 and y_1 do belong to $\mathrm{Orb}_{\mathcal{G}}(x)$. By Lemma 6.3, there exists a map $f \in \mathcal{G}$ such that $f(x_1) = y_1$ and $f(z) = z$ for all $z \in Y$. Then, $h_2^{-1} f h_1(x) = y$. Note that $f \in \mathrm{St}^{\star}_{\mathcal{G}}(Y) \subset H$, which implies that $h_2^{-1} f h_1 \in H_g$. Thus, $y \in \mathrm{Orb}_{H_g}(x)$. □

The simplest example of a closed subset of X is a finite set. To treat the stabilizers of finite sets, we do not even need to assume that the ample group acts minimally on X.

LEMMA 6.10. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group. Suppose $Y \subset X$ is a finite set contained in a single orbit of $\mathcal{G}$ and $Z \subset X$ is a closed (e.g., finite) set disjoint from Y. Then, for any permutation $\pi : Y \to Y$, there exists a map $f \in \mathcal{G}$ such that $f(x) = \pi(x)$ for all $x \in Y$ and $f(x) = x$ for all $x \in Z$.*

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Proof. First, we consider a particular case where the permutation π is a transposition. That is, $\pi = (x\ y)$, where x and y are distinct elements of Y. Let $Z_+ = (Y \cup Z) \setminus \{x, y\}$. Then, Z_+ is a closed set containing neither x nor y. By Lemma 6.3, there exists a map $f \in \mathcal{G}$ such that $f(x) = y$, $f(y) = x$, and $f(z) = z$ for all $z \in Z_+$. The map f coincides with π on the set Y and with the identity map on the set Z.

Now, consider the general case. If π is the identity map, we can take $f = \mathrm{id}_X$, which belongs to $\mathcal{G}$. Otherwise, the permutation π can be decomposed as a product, $\pi = \pi_1\pi_2 \cdots \pi_k$, where each π_i is a transposition exchanging two elements of Y. By the above, for any i, $1 \le i \le k$, there exists a map $f_i \in \mathcal{G}$ that coincides with π_i on the set Y and fixes all points of the set Z. Then, $f = f_1 f_2 \cdots f_k$ is an element of $\mathcal{G}$ that coincides with the permutation $\pi_1\pi_2 \cdots \pi_k = \pi$ on Y and fixes all points of Z. □

The next theorem is our main result on the stabilizers of finite sets.

THEOREM 6.11. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group. Suppose Y is a finite non-empty subset of X. Then, the following statements hold true.*

(i) *If Y is a proper subset of a single orbit of $\mathcal{G}$ and, moreover, Y does not contain exactly half of the points in the orbit, then the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ is a maximal subgroup of $\mathcal{G}$.*

(ii) *If Y is a subset of an orbit $\mathrm{Orb}_{\mathcal{G}}(x)$ that contains exactly half of its points, then $\mathrm{St}_{\mathcal{G}}(Y)$ is a subgroup of index 2 in the group $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$. The latter is a maximal subgroup of $\mathcal{G}$ unless Y consists of a single point, in which case, $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) = \mathcal{G}$.*

(iii) *If there are at least two orbits of $\mathcal{G}$ that intersect Y but are not contained in Y, then $\mathrm{St}_{\mathcal{G}}(Y)$ is a proper subgroup of $\mathcal{G}$ that is not maximal.*

(iv) *If Y is a union of orbits of $\mathcal{G}$, then $\mathrm{St}_{\mathcal{G}}(Y) = \mathcal{G}$.*

(v) *If $Y = Y_1 \sqcup Y_2$, where Y_2 is a union of orbits of $\mathcal{G}$, then $\mathrm{St}_{\mathcal{G}}(Y) = \mathrm{St}_{\mathcal{G}}(Y_1)$.*

Proof. Since $g(h(x)) = (gh)(x)$ for all maps $g, h \in \mathcal{G}$ and points $x \in X$, it follows that $g(\mathrm{Orb}_{\mathcal{G}}(x)) \subset \mathrm{Orb}_{\mathcal{G}}(x)$. For any $g \in \mathcal{G}$, the inverse map g^{-1} is also in $\mathcal{G}$. Hence, we also have $g^{-1}(\mathrm{Orb}_{\mathcal{G}}(x)) \subset \mathrm{Orb}_{\mathcal{G}}(x)$, which implies that, in fact, $g(\mathrm{Orb}_{\mathcal{G}}(x)) = \mathrm{Orb}_{\mathcal{G}}(x)$. It further follows that $g(Y) = Y$ for any set $Y \subset X$ that is a union of orbits of the group $\mathcal{G}$. Consequently, $\mathrm{St}_{\mathcal{G}}(Y) = \mathcal{G}$ for any such set Y, finite or not. Further, if a set $Y \subset X$ is represented as a disjoint union of two sets, $Y = Y_1 \sqcup Y_2$, then $g(Y) = g(Y_1) \sqcup g(Y_2)$ for any $g \in \mathcal{G}$. Assuming Y_2 is a union of orbits of $\mathcal{G}$, we have $g(Y_2) = Y_2$. Hence, $g(Y) = Y$ if and only if $g(Y_1) = Y_1$. As a result, $\mathrm{St}_{\mathcal{G}}(Y) = \mathrm{St}_{\mathcal{G}}(Y_1)$.

We have proved statements (iv) and (v). Next, we prove statement (iii). Suppose $\mathrm{Orb}_{\mathcal{G}}(x)$ and $\mathrm{Orb}_{\mathcal{G}}(y)$ are two distinct orbits of $\mathcal{G}$ that intersect a finite set $Y \subset X$, but are not contained in Y. Then, there exist points $x_1, x_2 \in \mathrm{Orb}_{\mathcal{G}}(x)$ and $y_1, y_2 \in \mathrm{Orb}_{\mathcal{G}}(y)$ such that x_1 and y_1 belong to Y, while x_2 and y_2 do not. By Lemma 6.10, the group $\mathcal{G}$ contains a map f that interchanges y_1 and y_2 while fixing all points of Y different from y_1. Note that $f(Y \cap \mathrm{Orb}_{\mathcal{G}}(x)) = Y \cap \mathrm{Orb}_{\mathcal{G}}(x)$, while $f(Y \cap \mathrm{Orb}_{\mathcal{G}}(y)) \neq Y \cap \mathrm{Orb}_{\mathcal{G}}(y)$. Likewise, there exists $h \in \mathcal{G}$ that interchanges x_1 and x_2, while fixing all points of Y different from x_1. Then, $h(Y \cap \mathrm{Orb}_{\mathcal{G}}(y)) = Y \cap \mathrm{Orb}_{\mathcal{G}}(y)$, while $h(Y \cap \mathrm{Orb}_{\mathcal{G}}(x)) \neq Y \cap \mathrm{Orb}_{\mathcal{G}}(x)$. It follows that neither of the stabilizers $\mathrm{St}_{\mathcal{G}}(Y \cap \mathrm{Orb}_{\mathcal{G}}(x))$ and $\mathrm{St}_{\mathcal{G}}(Y \cap \mathrm{Orb}_{\mathcal{G}}(y))$ contains

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the other. Therefore, both stabilizers are proper subgroups of $\mathcal{G}$ and their intersection is a proper subgroup of both of them. It remains to observe that the said intersection contains $\mathrm{St}_{\mathcal{G}}(Y)$. Indeed, any $g \in \mathcal{G}$ maps each orbit of $\mathcal{G}$ onto itself. Hence, $g(Y \cap \mathrm{Orb}_{\mathcal{G}}(x)) = g(Y) \cap \mathrm{Orb}_{\mathcal{G}}(x)$ and $g(Y \cap \mathrm{Orb}_{\mathcal{G}}(y)) = g(Y) \cap \mathrm{Orb}_{\mathcal{G}}(y)$. Consequently, the equality $g(Y) = Y$ implies that $g(Y \cap \mathrm{Orb}_{\mathcal{G}}(x)) = Y \cap \mathrm{Orb}_{\mathcal{G}}(x)$ and $g(Y \cap \mathrm{Orb}_{\mathcal{G}}(y)) = Y \cap \mathrm{Orb}_{\mathcal{G}}(y)$.

We proceed to the main part of the theorem, namely, statements (i) and (ii). Suppose that a finite non-empty set Y is a proper subset of a single orbit $\mathrm{Orb}_{\mathcal{G}}(x)$. Since $g(\mathrm{Orb}_{\mathcal{G}}(x)) = \mathrm{Orb}_{\mathcal{G}}(x)$ for any $g \in \mathcal{G}$, we have $g(Y) = Y$ if and only if $g(\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) = \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$. It follows that $\mathrm{St}_{\mathcal{G}}(Y)$ coincides with another set stabilizer $\mathrm{St}_{\mathcal{G}}(\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ as well as with the individual stabilizer $\mathrm{St}^{\bullet}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$. The collective stabilizer $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ acts on the set $\{Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y\}$ and $\mathrm{St}^{\bullet}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ is the kernel of this action. If the set Y does not contain exactly half of the points in $\mathrm{Orb}_{\mathcal{G}}(x)$, it cannot be mapped onto $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ by any invertible transformation of X. Then, the action is trivial so that $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) = \mathrm{St}^{\bullet}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) = \mathrm{St}_{\mathcal{G}}(Y)$. In the case where Y contains exactly half of the points in $\mathrm{Orb}_{\mathcal{G}}(x)$, the orbit is finite and the set $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ contains the same number of points as Y. Then, there exists a permutation π on $\mathrm{Orb}_{\mathcal{G}}(x)$ such that $\pi(Y) = \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ and $\pi(\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) = Y$. Lemma 6.10 implies that there is also a homeomorphism $f \in \mathcal{G}$ such that $f(Y) = \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ and $f(\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) = Y$. Hence, the action of the group $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ on the set $\{Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y\}$ is non-trivial, which implies that $\mathrm{St}_{\mathcal{G}}(Y) = \mathrm{St}^{\bullet}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ is a subgroup of index 2 in $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$.

Both Y and $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ are non-empty sets. If each of them consists of a single point, then $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) = \mathrm{St}_{\mathcal{G}}(\mathrm{Orb}_{\mathcal{G}}(x)) = \mathcal{G}$. Otherwise, we can choose two distinct points x_1 and x_2 in one of the two sets and another point x_3 in the other set. By Lemma 6.10, the ample group $\mathcal{G}$ contains a map f such that $f(x_1) = x_1$, $f(x_2) = x_3$, and $f(x_3) = x_2$. Then, one of the points $f(x_1)$ and $f(x_2)$ is in Y, while the other one is in $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$. It follows that $f \notin \mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$. Hence, $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) \neq \mathcal{G}$.

It remains to prove that the stabilizer $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ is a maximal subgroup of $\mathcal{G}$ provided that at least one of the sets Y and $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ consists of more than one point. We have already shown that under this condition, $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ is a proper subgroup of $\mathcal{G}$. For convenience, let us assume that the number of points in Y is less than or equal to the number of points in $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ (note that the latter can be infinite). There is no loss of generality here as otherwise, we could replace Y by $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$. Given a map $g \in \mathcal{G}$ that does not belong to $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$, let H_g denote the subgroup of $\mathcal{G}$ generated by $\mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ and g. We need to show that $H_g = \mathcal{G}$. This requires some preparation.

Since $g(\mathrm{Orb}_{\mathcal{G}}(x)) = \mathrm{Orb}_{\mathcal{G}}(x)$, the condition $g \notin \mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ is equivalent to a pair of conditions $g(\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) \neq Y$ and $g(\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) \neq \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$. Note that the image $g(\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ cannot be a proper subset of Y as any proper subset of Y has strictly fewer points than $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$. In addition, $g(\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ cannot be a proper subset of $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ as that would imply that Y is a proper subset of $g(Y)$, which would contradict the fact that the set Y is finite. We conclude that $g(\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$ intersects both Y and $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$. Hence, we can choose points $x_+, x_- \in \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$

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such that $g(x_+) \in Y$, while $g(x_-) \in \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$. By Lemma 6.10, there exists a map $h \in \mathcal{G}$ such that the restriction of h to the set $Z = Y \cup g^{-1}(Y) \cup \{x_+, x_-\}$ coincides with the transposition $(x_+\ x_-)$. Then, the restriction of the map $\tilde{g} = ghg^{-1}$ to the set $g(Z) = Y \cup g(Y) \cup \{g(x_+), g(x_-)\}$ coincides with the transposition $(g(x_+)\ g(x_-))$. Since the map h fixes all points in Y, we have $h \in \mathrm{St}_{\mathcal{G}}(Y) \subset \mathrm{St}_{\mathcal{G}}(Y, \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$. It follows that $\tilde{g} \in H_g$. Let $y_+ = g(x_+)$ and $y_- = g(x_-)$. By construction, $y_+ \in Y$, $y_- \in \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$, $\tilde{g}(y_+) = y_-$, $\tilde{g}(y_-) = y_+$, and $\tilde{g}(y) = y$ for all points $y \in Y$ different from y_+.

Next, we prove that for any finite set $S \subset \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ containing the point y_-, there exists a map $h_S \in H_g$ such that the restriction of h_S to $Y \cup S$ coincides with the transposition $(y_+\ y_-)$. The proof is by induction on the number n of points in S. If $n = 1$, then $S = \{y_-\}$ and we can let $h_S = \tilde{g}$. In the case $n \geq 2$, we assume the claim holds for all sets of $n-1$ points. Take any point $z_0 \in S$ different from y_-. Then, the set $S_0 = S \setminus \{z_0\}$ consists of $n-1$ points including y_-. By the inductive assumption, there exists a map $h_{S_0} \in H_g$ such that the restriction of h_{S_0} to the set $Y \cup S_0$ coincides with the transposition $(y_+\ y_-)$. If the point $\tilde{z}_0 = h_{S_0}(z_0)$ coincides with z_0, then we can clearly let $h_S = h_{S_0}$. Otherwise, Lemma 6.10 implies the existence of a map $\phi \in \mathcal{G}$ such that its restriction to the set $Y \cup S \cup \{\tilde{z}_0\}$ coincides with the transposition $(z_0\ \tilde{z}_0)$. Since ϕ fixes all points in Y, it belongs to the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$, which is contained in H_g. Then, the map ϕh_{S_0} is in H_g as well. By construction, $\phi h_{S_0}(z_0) = z_0$. Note that $\tilde{z}_0 \notin Y \cup S_0$ as $Y \cup S_0 = h_{S_0}(Y \cup S_0)$. Hence, ϕh_{S_0} coincides with h_{S_0} on $Y \cup S_0$. Therefore, we can let $h_S = \phi h_{S_0}$ and complete the inductive step.

Next, we prove that for any pair of sets $S' \subset Y$ and $S'' \subset \mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$ of the same cardinality, there exists a map $h_{S',S''} \in H_g$ such that $h_{S',S''}(S') = S''$, $h^2_{S',S''}(y) = y$ for all $y \in S'$, and $h_{S',S''}(y) = y$ for all $y \in Y \setminus S'$. The proof is by induction on the number n of points in S'. If $n = 0$, then both sets are empty and we can let $h_{S',S''} = \mathrm{id}_X$. In the case $n \geq 1$, we assume the claim holds whenever the sets consist of $n-1$ points. Take any points $y_0 \in S'$ and $z_0 \in S''$. Then, the sets $S'_0 = S' \setminus \{y_0\}$ and $S''_0 = S'' \setminus \{z_0\}$ contain $n-1$ points each. By the inductive assumption, there exists a map $h_{S'_0,S''_0} \in H_g$ such that $h_{S'_0,S''_0}(S'_0) = S''_0$, $h^2_{S'_0,S''_0}(y) = y$ for all $y \in S'_0$, and $h_{S'_0,S''_0}(y) = y$ for all $y \in Y \setminus S'_0$. Note that $h_{S'_0,S''_0}(S''_0) = S'_0$ and hence, $h_{S'_0,S''_0}(Y \cup S''_0) = Y \cup S''_0$. As a consequence, the point $\tilde{z}_0 = h_{S'_0,S''_0}(z_0)$ is in $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$, but not in S''_0. Consider a set $Z = S''_0 \cup \{y_-\} \cup \{z_0\} \cup \{\tilde{z}_0\}$, which is a finite subset of $\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y$. By the above, there exists a map $h_Z \in H_g$ such that its restriction to $Y \cup Z$ coincides with the transposition $(y_+\ y_-)$. Next, by Lemma 6.10, for any permutation on the set $Y \cup Z$, there exists an element of the group $\mathcal{G}$ that coincides with this permutation on $Y \cup Z$. In particular, there exists a map $\phi \in \mathcal{G}$ such that $\phi(z_0) = \tilde{z}_0$, $\phi(\tilde{z}_0) = z_0$, and $\phi(y) = y$ for all other points $y \in Y \cup Z$. The restriction of ϕ to $Y \cup Z$ coincides with the transposition $(z_0\ \tilde{z}_0)$ if $z_0 \neq \tilde{z}_0$, and with the identity map if $z_0 = \tilde{z}_0$. Furthermore, there exists a map $\psi \in \mathcal{G}$ such that $\psi(y_+) = y_0$, $\psi(y_0) = y_+$, $\psi(y_-) = z_0$, $\psi(z_0) = y_-$, and $\psi(y) = y$ for all other points $y \in Y \cup Z$. The restriction of ψ to the set $Y \cup Z$ coincides with the permutation $(y_+\ y_0)(y_-\ z_0)$ if $y_+ \neq y_0$ and $y_- \neq z_0$, with the transposition $(y_+\ y_0)$ if $y_+ \neq y_0$ and $y_- = z_0$, with the transposition $(y_-\ z_0)$ if $y_+ = y_0$ and $y_- \neq z_0$, and

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with the identity map if $y_+ = y_0$ and $y_- = z_0$. It follows that the restriction of the map $\psi h_Z \psi^{-1}$ to $Y \cup Z$ coincides with the transposition $(\psi(y_-)\ \psi(y_+)) = (y_0\ z_0)$. Finally, let $h_{S',S''} = \psi h_Z \psi^{-1} \phi h_{S'_0,S''_0}$. By construction, $\phi(Y) = \psi(Y) = Y$ so that $\phi, \psi \in \mathrm{St}_{\mathcal{G}}(Y) \subset H_g$. Then, $h_{S',S''} \in H_g$ as well. The map $h_{S',S''}$ coincides with $h_{S'_0,S''_0}$ on the set $(Y \setminus \{y_0\}) \cup S''_0$. In addition, $h_{S',S''}(y_0) = \psi h_Z \psi^{-1} \phi(y_0) = \psi h_Z \psi^{-1}(y_0) = z_0$ and $h_{S',S''}(z_0) = \psi h_Z \psi^{-1} \phi(\tilde{z}_0) = \psi h_Z \psi^{-1}(z_0) = y_0$. The inductive step is complete.

Now, we are ready to show that $H_g = \mathcal{G}$. Take any map $f \in \mathcal{G}$. We define three sets $S_{++} = Y \cap f^{-1}(Y)$, $S_{+-} = Y \cap f^{-1}(\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y)$, and $S_{-+} = (\mathrm{Orb}_{\mathcal{G}}(x) \setminus Y) \cap f^{-1}(Y)$. Since $f(\mathrm{Orb}_{\mathcal{G}}(x)) = \mathrm{Orb}_{\mathcal{G}}(x)$, it follows that $Y = S_{++} \sqcup S_{+-} = f(S_{++}) \sqcup f(S_{-+})$. Since f is a one-to-one map, the set $f(S_{++})$ is of the same cardinality as S_{++}, while the set $f(S_{-+})$ is of the same cardinality as S_{-+}. Since the set Y is finite, it follows that S_{-+} is of the same cardinality as S_{+-}. By the above, there exists a map $h_f \in H_g$ such that $h_f(S_{+-}) = S_{-+}$ and $h_f(y) = y$ for all $y \in Y \setminus S_{+-} = S_{++}$. We obtain that $fh_f(S_{+-}) = f(h_f(S_{+-})) = f(S_{-+})$ and $fh_f(S_{++}) = f(h_f(S_{++})) = f(S_{++})$, which implies that $fh_f(Y) = Y$. Hence, the map fh_f belongs to the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$, which is contained in H_g. Then, $f = (fh_f)h_f^{-1}$ is in H_g as well. Thus, $H_g = \mathcal{G}$. □

Remark 6.12. Theorem 6.11 allows the determination, for any ample group $\mathcal{G} \subset \mathrm{Homeo}(X)$ and any finite set $Y \subset X$, whether the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ is a maximal subgroup of $\mathcal{G}$ and whether $\mathrm{St}_{\mathcal{G}}(Y) = \mathcal{G}$. Namely, $\mathrm{St}_{\mathcal{G}}(Y) = \mathcal{G}$ if and only if Y is a union of orbits of $\mathcal{G}$ (since the set Y is finite, this means the union of a finite number of finite orbits). The stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ is a maximal subgroup of $\mathcal{G}$ if and only if $Y = Y_1 \sqcup Y_2$, where Y_2 is a union of orbits of $\mathcal{G}$ (it may be empty) and Y_1 is a proper non-empty subset of a single orbit such that either Y does not contain exactly half of the points in the orbit or else Y is a one-point subset of a two-point orbit.

The following concise corollary of Theorem 6.11 is much less general, but it will be enough, e.g., when the topological space X is a Cantor set and the ample group $\mathcal{G}$ acts minimally on X.

THEOREM 6.13. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that has no finite orbits. Suppose Y is a finite non-empty subset of X. Then, the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ is a maximal subgroup of $\mathcal{G}$ if and only if Y is contained in a single orbit of $\mathcal{G}$.*

Proof. Since all orbits of the group $\mathcal{G}$ are infinite, the finite set Y cannot contain an entire orbit of $\mathcal{G}$ or exactly half of the elements in an orbit. If Y intersects at least two different orbits of $\mathcal{G}$, then the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ is not a maximal subgroup of $\mathcal{G}$ due to statement (iii) of Theorem 6.11. Otherwise, Y is contained in a single orbit of $\mathcal{G}$. Then, $\mathrm{St}_{\mathcal{G}}(Y)$ is a maximal subgroup of the group $\mathcal{G}$ due to statement (i) of Theorem 6.11. □

To treat the stabilizers of infinite closed sets, we need to develop a different approach. This approach will also apply to another class of subgroups that contains many maximal subgroups, namely, the stabilizers of partitions of X into clopen sets (see Definition 3.4).

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LEMMA 6.14. *Suppose $X = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k$ is a partition of X into clopen sets. Then, for any ample group $\mathcal{G} \subset \mathrm{Homeo}(X)$, the individual stabilizer $\mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2, \ldots, U_k)$ is the internal direct product of the local subgroups $\mathcal{G}_{U_i}$, $1 \le i \le k$.*

Proof. The group $H = \mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2, \ldots, U_k)$ is the intersection of the set stabilizers $\mathrm{St}_{\mathcal{G}}(U_i)$, $1 \le i \le k$. For any i, the local subgroup $\mathcal{G}_{U_i} = \mathrm{RiSt}_{\mathcal{G}}(U_i)$ is a subgroup of $\mathrm{St}_{\mathcal{G}}(U_i)$. In addition, $\mathcal{G}_{U_i} \subset \mathrm{St}^{\star}_{\mathcal{G}}(U_j) \subset \mathrm{St}_{\mathcal{G}}(U_j)$ whenever $i \neq j$ as then, $U_i \cap U_j = \emptyset$. Hence, $\mathcal{G}_{U_i} \subset H$ for all i.

For each i, $1 \le i \le k$, choose a map $f_i \in \mathcal{G}_{U_i}$. Then, the maps $f_1, f_2, \ldots, f_k$ commute with one another as they have disjoint supports. In addition, the product $f_1 f_2 \cdots f_k$ coincides with f_i on U_i for all i, which implies that $f_1 f_2 \cdots f_k = \mathrm{id}_X$ if and only if each f_i is the identity map. It follows that the local subgroups $\mathcal{G}_{U_i}$, $1 \le i \le k$, form an internal direct product in H.

Now, take any $f \in H$. For each i, $1 \le i \le k$, consider a map $f_i : X \to X$ that coincides with f on U_i and with the identity map on $X \setminus U_i$. Since f is invertible and $f(U_i) = U_i$, it follows that the map f_i is invertible as well. By construction, f_i is piecewise an element of the ample group $\mathcal{G}$ and $\mathrm{supp}(f_i) \subset U_i$. Hence, $f_i \in \mathcal{G}_{U_i}$. Also, by construction, $f = f_1 f_2 \cdots f_k$. We conclude that the product of the local subgroups $\mathcal{G}_{U_1}, \mathcal{G}_{U_2}, \ldots, \mathcal{G}_{U_k}$ equals H. □

LEMMA 6.15. *Let $X = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k$ be a partition of X into clopen sets at most one of which consists of a single point. Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X. Suppose H is a subgroup of $\mathcal{G}$ such that $\mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2, \ldots, U_k) \subset H \subset \mathrm{St}_{\mathcal{G}}(U_1, U_2, \ldots, U_k)$. Then, H contains a local subgroup $\mathcal{G}_U$ if and only if $U \subset U_i$ for some i, $1 \le i \le k$.*

Proof. Lemma 6.14 implies that the individual stabilizer $\mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2, \ldots, U_k)$ contains local subgroups $\mathcal{G}_{U_1}, \mathcal{G}_{U_2}, \ldots, \mathcal{G}_{U_k}$. If a clopen set U is contained in U_i for some i, $1 \le i \le k$, then clearly, $\mathcal{G}_U \subset \mathcal{G}_{U_i}$. Consequently, $\mathcal{G}_U \subset H$. Now, assume that a clopen set U is not contained in a single element of the partition $X = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k$. Then, there are two different elements V_1 and V_2 of the partition such that the intersections $U \cap V_1$ and $U \cap V_2$ are not empty. By the assumption of the lemma, at least one of the sets V_1 and V_2 contains more than one point. We may assume without loss of generality that V_1 contains more than one point. Take any point $x \in U \cap V_1$ and let z be any point of V_1 different from x. Since the group $\mathcal{G}$ acts minimally on X, the orbit $\mathrm{Orb}_{\mathcal{G}}(x)$ is dense in X. In particular, this orbit has a point y in the non-empty clopen set $U \cap V_2$. By Lemma 6.3, there exists a map $f \in \mathcal{G}$ such that $f(x) = y$ and f coincides with the identity map on the closed set $(X \setminus U) \cup \{z\}$. Then, $f \in \mathcal{G}_U$. Since $f(x) = y$ and $f(z) = z$, the image $f(V_1)$ intersects both V_1 and V_2 so that it cannot be an element of the partition $X = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k$. As a consequence, f does not belong to the collective stabilizer $\mathrm{St}_{\mathcal{G}}(U_1, U_2, \ldots, U_k)$. In particular, $f \notin H$. Thus, the local subgroup $\mathcal{G}_U$ is not contained in H. □

Notice that the assumption of Lemma 6.15 that at most one element of the partition of X consists of a single point is essential. A one-point set $\{x\}$ is clopen if x is an isolated point of X. If $\{x\}$ and $\{y\}$ are two distinct elements of the partition, then the local subgroup $\mathcal{G}_{\{x,y\}}$ is contained in the stabilizer of the partition.

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LEMMA 6.16. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X. Suppose that $X = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k$ is a partition of X into non-empty clopen sets. Then, $\mathrm{St}_{\mathcal{G}}(U_1, U_2, \ldots, U_k) \neq \mathcal{G}$, unless this is the partition into points or the trivial partition consisting of only one set.*

Proof. The partition of X into points and the trivial partition are preserved by any invertible transformation of X. In particular, they are preserved by all elements of the group $\mathcal{G}$. Conversely, assume that $\mathrm{St}_{\mathcal{G}}(U_1, U_2, \ldots, U_k) = \mathcal{G}$. Then, the group $\mathcal{G}$ acts on the set $\{U_1, U_2, \ldots, U_k\}$. That action is transitive since the action of $\mathcal{G}$ on X is minimal. As a consequence, all elements of the partition are of the same cardinality. Hence, this is either the partition into points, or else a partition in which every element consists of more than one point. In the latter case, Lemma 6.15 implies that any local subgroup $\mathcal{G}_U$ is contained in the stabilizer $\mathrm{St}_{\mathcal{G}}(U_1, U_2, \ldots, U_k)$ if and only if the clopen set U is contained in a single element of the partition. Since $\mathcal{G}$ can be represented as the local subgroup $\mathcal{G}_X$, it follows that X is an element of the partition. As all elements of the partition are non-empty sets, we obtain that X is the only element so that the partition is trivial. □

PROPOSITION 6.17. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group with Property E. Suppose H is a subgroup of $\mathcal{G}$ that acts minimally on X and contains a local subgroup $\mathcal{G}_U$ for some clopen set U consisting of more than one point. Then, there exists a partition of X into clopen sets, $X = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k$, such that each U_i consists of more than one point and $\mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2, \ldots, U_k) \subset H \subset \mathrm{St}_{\mathcal{G}}(U_1, U_2, \ldots, U_k)$. Moreover, the partition is unique and the induced action of H on the set $\{U_1, U_2, \ldots, U_k\}$ is transitive.*

Proof. Let $\mathcal{L}_H$ be the set of all clopen sets $V \subset X$ such that the subgroup H contains the local subgroup $\mathcal{G}_V$. Note that $\mathcal{G}_{V_1} \subset \mathcal{G}_{V_2}$ whenever $V_1 \subset V_2$. Hence, any clopen set contained in an element of $\mathcal{L}_H$ is itself an element of $\mathcal{L}_H$. Since the group $\mathcal{G}$ has Property E, for any intersecting clopen sets V_1 and V_2, the local subgroup $\mathcal{G}_{V_1 \cup V_2}$ is generated by the union of $\mathcal{G}_{V_1}$ and $\mathcal{G}_{V_2}$. Therefore, $V_1 \cup V_2 \in \mathcal{L}_H$ whenever $V_1, V_2 \in \mathcal{L}_H$ and $V_1 \cap V_2 \neq \emptyset$. Furthermore, $\mathcal{G}_{g(V)} = g\mathcal{G}_V g^{-1}$ for any clopen set V and any $g \in \mathcal{G}$. This follows from the fact that $\mathrm{supp}(gfg^{-1}) = g(\mathrm{supp}(f))$ for any homeomorphism $f \in \mathrm{Homeo}(X)$. As a consequence, $h(V) \in \mathcal{L}_H$ for all $h \in H$ whenever $V \in \mathcal{L}_H$.

By assumption, some clopen set U consisting of more than one point is an element of $\mathcal{L}_H$. Minimality of the action of the group H on X implies that any orbit $\mathrm{Orb}_H(x)$ of H is dense in X. In particular, it has a point in U. That is, $h(x) \in U$ for some $h \in H$. Then, $h^{-1}(U)$ is an element of $\mathcal{L}_H$ containing the point x. We conclude that elements of $\mathcal{L}_H$ cover X. Since the topological space X is compact, there is a finite subcover. Let $\{U_1, U_2, \ldots, U_k\}$ be a finite subcover with the least possible number of elements. This choice implies that all elements of the subcover are non-empty sets and the union of any two or more of them is not an element of $\mathcal{L}_H$. Then, it follows from the above that any two elements of the subcover are disjoint. Hence, our subcover is actually a partition: $X = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k$. Observe that this is not the partition into points as otherwise, U would be the union of two or more elements of the partition. We claim that any set $V \in \mathcal{L}_H$ is contained in a single element of the partition. Assume the contrary, that is, V intersects U_i and U_j, where $i \neq j$. Then, the unions $V \cup U_i$ and $V \cup U_j$ are elements of $\mathcal{L}_H$. These

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unions are not disjoint as they both contain V. Therefore, their union $V \cup U_i \cup U_j$ belongs to $\mathcal{L}_H$ as well. Since $U_i \cup U_j$ is a clopen subset of $V \cup U_i \cup U_j$, we obtain that $U_i \cup U_j \in \mathcal{L}_H$, which yields a contradiction.

Take any $h \in H$. By the above, the sets $h(U_1), h(U_2), \dots, h(U_k)$ belong to $\mathcal{L}_H$. Therefore, each of them is contained in a single element of the partition $\{U_1, U_2, \dots, U_k\}$. Since h is an invertible map, the sets $h(U_1), h(U_2), \dots, h(U_k)$ themselves form a partition of X. Hence, we have a finite partition of X into non-empty sets, each element of which is contained in a single element of another partition of X into non-empty sets. Since both partitions consist of the same number of sets, this is possible only if both partitions are the same. In other words, h maps the sets $U_1, U_2, \dots, U_k$ onto one another. We conclude that the subgroup H is contained in $\mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$, the stabilizer of the partition. In addition, H contains the individual stabilizer $\mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$. Indeed, it follows from Lemma 6.14 that $\mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$ is generated by the local subgroups $\mathcal{G}_{U_i}$, $1 \le i \le k$, which are contained in H.

Since $H \subset \mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$, the group H acts on the set $\{U_1, U_2, \dots, U_k\}$. Take any orbit $\mathcal{O}$ of this action. Then, the union of all sets in $\mathcal{O}$ is invariant under the action of H on X. Note that this union is a non-empty clopen set. Minimality of the latter action implies that the union coincides with X. Hence, $\mathcal{O} = \{U_1, U_2, \dots, U_k\}$ so that the action of H on $\{U_1, U_2, \dots, U_k\}$ is transitive. In other words, all elements of the partition can be mapped onto one another by elements of H. As a consequence, all elements of the partition are of the same cardinality. Since $\{U_1, U_2, \dots, U_k\}$ is not the partition into points, it follows that each U_i consists of more than one point.

Let $\{V_1, V_2, \dots, V_\ell\}$ be an arbitrary partition of X into clopen sets such that each V_j consists of more than one point and $\mathrm{St}^{\bullet}_{\mathcal{G}}(V_1, V_2, \dots, V_\ell) \subset H \subset \mathrm{St}_{\mathcal{G}}(V_1, V_2, \dots, V_\ell)$. Lemma 6.14 implies that the sets $V_1, V_2, \dots, V_\ell$ belong to $\mathcal{L}_H$, just as the sets $U_1, U_2, \dots, U_k$ do. Note that the group $\mathcal{G}$ acts minimally on X since its subgroup H acts minimally. Hence, we can apply Lemma 6.15. It follows from the lemma that each element of either of the partitions $\{U_1, U_2, \dots, U_k\}$ and $\{V_1, V_2, \dots, V_\ell\}$ is contained in a single element of the other partition. Since both partitions consist of non-empty sets, this is possible only if both partitions are the same. □

Let us discuss the formulation of Proposition 6.17, specifically, why all clopen sets involved should consist of more than one point. If x is an isolated point of the topological space X, then $\{x\}$ is a non-empty clopen set, but the associated local subgroup $\mathcal{G}_{\{x\}}$ is trivial. Hence, the condition that a clopen set U contains more than one point is necessary for the local subgroup $\mathcal{G}_U$ to be non-trivial. In the case where the ample group $\mathcal{G}$ acts minimally on X, one can show that this condition is also sufficient. Further, if X is a finite set and $X = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k$ is the partition into points, then $\mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$ is a trivial group, while $\mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k) = \mathcal{G}$. The requirement that each U_i consists of more than one point rules out this degenerate case. Without it, we would not be able to show uniqueness of the partition. In addition, the requirement allows the use of Proposition 6.17 in conjunction with Lemma 6.15.

Proposition 6.17 is going to be crucial for the proofs of all subsequent results on maximal subgroups. The downside of this is that all those results will require the ample group to

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have Property E. We discuss this property and provide examples of ample groups with it in §8.

Our next result concerns the stabilizers of closed sets that are not open.

THEOREM 6.18. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X and has Property E. Suppose $Y \subset X$ is a closed set that is not open. Then, the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ is a maximal subgroup of $\mathcal{G}$ if and only if it acts minimally when restricted to Y, in which case, Y is necessarily nowhere dense in X.*

Proof. If the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ is a maximal subgroup of $\mathcal{G}$, then the induced action of $\mathrm{St}_{\mathcal{G}}(Y)$ on Y is minimal due to Theorem 6.6. Since the set Y is not clopen, Lemma 6.7 implies that it is nowhere dense in X.

Now, assume that $\mathrm{St}_{\mathcal{G}}(Y)$ acts minimally when restricted to Y. Just as above, this implies that Y is nowhere dense in X. Also, since the closed set Y is not open, it is different from the empty set and X. Then, $\mathrm{St}_{\mathcal{G}}(Y) \neq \mathcal{G}$ due to Lemma 6.2. To prove that the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ is a maximal subgroup of $\mathcal{G}$, we are going to show that for any $g \in \mathcal{G} \setminus \mathrm{St}_{\mathcal{G}}(Y)$, the group $H_g = \langle \mathrm{St}_{\mathcal{G}}(Y) \cup \{g\} \rangle$ generated by $\mathrm{St}_{\mathcal{G}}(Y)$ and g coincides with $\mathcal{G}$. By Lemma 6.9, the group H_g has the same orbits as $\mathcal{G}$. As a consequence, H_g acts minimally on X. To apply Proposition 6.17 to the group H_g, we need it to contain a local subgroup $\mathcal{G}_U$ for some clopen set U containing more than one point. If $U \subset X \setminus Y$, then any map in $\mathcal{G}_U$ fixes all points of Y so that $\mathcal{G}_U \subset \mathrm{St}^{\star}_{\mathcal{G}}(Y) \subset \mathrm{St}_{\mathcal{G}}(Y) \subset H_g$. Note that the set $X \setminus Y$ is open, but not closed. Therefore, it is infinite. Any point $x \in X \setminus Y$ has a clopen neighborhood $V_x \subset X \setminus Y$. Take any distinct points $x, y \in X \setminus Y$. Then, $V_x \cup V_y$ is a clopen set consisting of more than one point. In addition, $V_x \cup V_y \subset X \setminus Y$ so that $\mathcal{G}_{V_x \cup V_y} \subset H_g$. Now, Proposition 6.17 implies that $\mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2, \ldots, U_k) \subset H_g \subset \mathrm{St}_{\mathcal{G}}(U_1, U_2, \ldots, U_k)$, where $\mathcal{P} = \{U_1, U_2, \ldots, U_k\}$ is a partition of X into clopen sets, each consisting of more than one point. Since the set Y is nowhere dense in X, its complement intersects any non-empty open set. In particular, for any i, $1 \leq i \leq k$, we can find a point $x_i \in U_i$ that does not belong to Y. Let $V = V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_k}$. Then, V is a clopen subset of $X \setminus Y$, which implies that $\mathcal{G}_V \subset H_g$. Lemma 6.15 further implies that V is contained in a single element of the partition $\mathcal{P}$. However, V was constructed so as to intersect every element of $\mathcal{P}$. We conclude that $\mathcal{P}$ consists of a single element, $\mathcal{P} = \{X\}$. Since $\mathrm{St}^{\bullet}_{\mathcal{G}}(X) = \mathrm{St}_{\mathcal{G}}(X) = \mathcal{G}$, we obtain that $H_g = \mathcal{G}$. □

If a closed set $Y \subset X$ is clopen, then for any ample group $\mathcal{G} \subset \mathrm{Homeo}(X)$ that acts minimally on X, the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ acts minimally when restricted to Y (due to Lemma 6.8). If Y is finite, then $\mathrm{St}_{\mathcal{G}}(Y)$ acts minimally on it if and only if Y is contained in a single orbit of $\mathcal{G}$. If the closed set Y is neither clopen nor finite, it is not clear whether minimality of the action of $\mathrm{St}_{\mathcal{G}}(Y)$ on Y is at all possible. In other words, it is not clear whether Theorem 6.18 yields any new examples of maximal subgroups compared with Theorem 6.11. We address this question in §7. In view of Lemma 5.2, the assumptions of Theorem 6.18 can be satisfied only in the case where X is a Cantor set. As it turns out, there are plenty of new examples in that case (see Proposition 7.2).

The next three theorems contain our main results on the stabilizers of partitions of X into clopen sets. Theorems 6.19 and 6.20 tell when such stabilizers are maximal subgroups

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of the corresponding ample groups (Theorem 6.19 for partitions into two sets, Theorem 6.20 for partitions into three or more sets). Theorem 6.21 provides a characterization of those maximal subgroups of ample groups that are the stabilizers of partitions into clopen sets (just as Theorem 6.6 provides a characterization of those maximal subgroups that are the stabilizers of closed sets). Theorem 6.19 also treats the stabilizers of clopen sets.

THEOREM 6.19. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X and has Property E. Suppose U is a clopen set different from the empty set and X. Then, $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$ is a maximal subgroup of $\mathcal{G}$ unless both U and $X \setminus U$ consist of a single point, in which case, $\mathrm{St}_{\mathcal{G}}(U, X \setminus U) = \mathcal{G}$. If U cannot be mapped onto $X \setminus U$ by an element of $\mathcal{G}$, then $\mathrm{St}_{\mathcal{G}}(U) = \mathrm{St}_{\mathcal{G}}(U, X \setminus U)$; otherwise, $\mathrm{St}_{\mathcal{G}}(U)$ is a subgroup of index 2 in $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$.*

Proof. For any invertible map $f : X \to X$, we have $f(U) = U$ if and only if $f(X \setminus U) = X \setminus U$. It follows that $\mathrm{St}_{\mathcal{G}}(U)$ coincides with another set stabilizer $\mathrm{St}_{\mathcal{G}}(X \setminus U)$ as well as with the individual stabilizer $\mathrm{St}^{\bullet}_{\mathcal{G}}(U, X \setminus U)$. The stabilizer $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$ of the partition $X = U \sqcup (X \setminus U)$ acts on the set $\{U, X \setminus U\}$ and $\mathrm{St}^{\bullet}_{\mathcal{G}}(U, X \setminus U)$ is the kernel of this action. If U cannot be mapped onto $X \setminus U$ by an element of $\mathcal{G}$, then the action is trivial so that $\mathrm{St}_{\mathcal{G}}(U, X \setminus U) = \mathrm{St}^{\bullet}_{\mathcal{G}}(U, X \setminus U) = \mathrm{St}_{\mathcal{G}}(U)$. Otherwise, $f(U) = X \setminus U$ for some $f \in \mathcal{G}$. Then, also $f(X \setminus U) = X \setminus f(U) = U$ so that $f \in \mathrm{St}_{\mathcal{G}}(U, X \setminus U)$. Hence, the action of the group $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$ on $\{U, X \setminus U\}$ is non-trivial, which implies that $\mathrm{St}_{\mathcal{G}}(U) = \mathrm{St}^{\bullet}_{\mathcal{G}}(U, X \setminus U)$ is a subgroup of index 2 in $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$.

If both U and $X \setminus U$ consist of a single point, then $X = U \sqcup (X \setminus U)$ is the partition into points, which implies that $\mathrm{St}_{\mathcal{G}}(U, X \setminus U) = \mathcal{G}$. Otherwise, $\mathrm{St}_{\mathcal{G}}(U, X \setminus U) \neq \mathcal{G}$ due to Lemma 6.16. To prove that $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$ is a maximal subgroup of $\mathcal{G}$ in the latter case, we are going to show that for any $g \in \mathcal{G} \setminus \mathrm{St}_{\mathcal{G}}(U, X \setminus U)$, the group $H_g = \langle \mathrm{St}_{\mathcal{G}}(U, X \setminus U) \cup \{g\} \rangle$ generated by $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$ and g coincides with $\mathcal{G}$. Since the group $\mathcal{G}$ acts minimally on X, it follows from Lemma 6.8 that the stabilizer $\mathrm{St}_{\mathcal{G}}(U)$ acts minimally when restricted to U. Then, Lemma 6.9 implies that the group $\langle \mathrm{St}_{\mathcal{G}}(U) \cup \{g\} \rangle$ generated by $\mathrm{St}_{\mathcal{G}}(U)$ and g has the same orbits as $\mathcal{G}$. Since the group H_g contains $\langle \mathrm{St}_{\mathcal{G}}(U) \cup \{g\} \rangle$, it also has the same orbits as $\mathcal{G}$. As a consequence, H_g acts minimally on X. Further, Lemma 6.14 implies that the stabilizer $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$ contains the local subgroups $\mathcal{G}_U$ and $\mathcal{G}_{X \setminus U}$. Hence, H_g contains them as well. Recall that we consider the case where at least one of the sets U and $X \setminus U$ consists of more than one point. Now, Proposition 6.17 implies that $\mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2, \ldots, U_k) \subset H_g \subset \mathrm{St}_{\mathcal{G}}(U_1, U_2, \ldots, U_k)$, where $\mathcal{P} = \{U_1, U_2, \ldots, U_k\}$ is a partition of X into clopen sets, each consisting of more than one point. By Lemma 6.15, any local subgroup $\mathcal{G}_V$ is a subgroup of H_g if and only if $V \subset U_i$ for some i, $1 \leq i \leq k$. In particular, each of the sets U and $X \setminus U$ is contained in a single element of the partition $\mathcal{P}$. This leaves only two possibilities, $\mathcal{P} = \{U, X \setminus U\}$ or $\mathcal{P} = \{X\}$. Since H_g is not contained in $\mathrm{St}_{\mathcal{G}}(U, X \setminus U)$, we conclude that $\mathcal{P} = \{X\}$. As $\mathrm{St}^{\bullet}_{\mathcal{G}}(X) = \mathrm{St}_{\mathcal{G}}(X) = \mathcal{G}$, we obtain that $H_g = \mathcal{G}$. □

THEOREM 6.20. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group that acts minimally on X and has Property E. Suppose $X = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k$ is a partition of X into at least three clopen sets, each consisting of more than one point. Then, $\mathrm{St}_{\mathcal{G}}(U_1, U_2, \ldots, U_k)$, the stabilizer*

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of the partition, is a maximal subgroup of $\mathcal{G}$ if and only if its induced action on the set $\{U_1, U_2, \dots, U_k\}$ is transitive.

Proof. Let $H = \mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$. Since the partition $\mathcal{P} = \{U_1, U_2, \dots, U_k\}$ is neither trivial nor the partition into points, Lemma 6.16 implies that $H \neq \mathcal{G}$. First, we consider the case where the induced action of the group H on $\mathcal{P}$ is not transitive. Take any orbit $\mathcal{O}$ of this action and let U be the union of all sets in $\mathcal{O}$. Then, U is a clopen set invariant under the action of H on X. It follows that $H \subset \mathrm{St}_{\mathcal{G}}(U)$. Note that the orbit $\mathcal{O}$ is different from the empty set and $\mathcal{P}$. Therefore, U is different from the empty set and X. Then, $\mathrm{St}_{\mathcal{G}}(U) \neq \mathcal{G}$ due to Lemma 6.2. Recall that the pointwise stabilizer $\mathrm{St}^{\star}_{\mathcal{G}}(U)$ and the rigid stabilizer $\mathrm{RiSt}_{\mathcal{G}}(U)$ are subgroups of $\mathrm{St}_{\mathcal{G}}(U)$. It is easy to see that $\mathrm{St}^{\star}_{\mathcal{G}}(U) = \mathcal{G}_{X\setminus U}$ and $\mathrm{RiSt}_{\mathcal{G}}(U) = \mathcal{G}_U$. However, the local subgroups $\mathcal{G}_U$ and $\mathcal{G}_{X\setminus U}$ cannot both be subgroups of H. Indeed otherwise, it would follow from Lemma 6.15 that each of the sets U and $X \setminus U$ is contained in a single element of the partition $\mathcal{P}$, which is not possible as $k \geq 3$. We conclude that $H \neq \mathrm{St}_{\mathcal{G}}(U)$. Hence, H is not a maximal subgroup of $\mathcal{G}$.

Now, consider the case where the induced action of H on the partition $\mathcal{P}$ is transitive. Let us show that in this case, the group H has the same orbits as $\mathcal{G}$. Any orbit of H has a point in every element of $\mathcal{P}$. Given $x \in X$, for any i, $1 \leq i \leq k$, choose a point $y_i \in \mathrm{Orb}_H(x) \cap U_i$. By Lemma 6.8, the intersection of the orbit $\mathrm{Orb}_{\mathcal{G}}(y_i) = \mathrm{Orb}_{\mathcal{G}}(x)$ with U_i coincides with the orbit of y_i under the action of the stabilizer $\mathrm{St}_{\mathcal{G}}(U_i)$. Note that $\mathrm{St}_{\mathcal{G}}(U_i) = \mathrm{St}_{\mathcal{G}}(X \setminus U_i) = \mathrm{St}^{\bullet}_{\mathcal{G}}(U_i, X \setminus U_i)$. By Lemma 6.14, the group $\mathrm{St}^{\bullet}_{\mathcal{G}}(U_i, X \setminus U_i)$ is the internal direct product of local subgroups $\mathcal{G}_{U_i}$ and $\mathcal{G}_{X\setminus U_i}$. Since $\mathcal{G}_{X\setminus U_i}$ acts trivially on U_i, it follows that the orbit of the point y_i under the action of $\mathrm{St}_{\mathcal{G}}(U_i)$ coincides with its orbit under the action of $\mathcal{G}_{U_i}$. As each $\mathcal{G}_{U_i}$ is obviously contained in H, we obtain that $\mathrm{Orb}_{\mathcal{G}}(x) \cap U_i \subset \mathrm{Orb}_H(y_i) = \mathrm{Orb}_H(x)$ for all i, which implies that $\mathrm{Orb}_H(x) = \mathrm{Orb}_{\mathcal{G}}(x)$.

To prove that H is a maximal subgroup of $\mathcal{G}$, we need to show that for any $g \in \mathcal{G} \setminus H$, the group $H_g = \langle H \cup \{g\}\rangle$ generated by H and g coincides with $\mathcal{G}$. Since the group H has the same orbits as $\mathcal{G}$, the same is true for H_g. As a consequence, H_g acts minimally on X. Since the group H contains the local subgroups $\mathcal{G}_{U_i}$, $1 \leq i \leq k$, so does H_g. Now, Proposition 6.17 implies that $\mathrm{St}^{\bullet}_{\mathcal{G}}(V_1, V_2, \dots, V_\ell) \subset H_g \subset \mathrm{St}_{\mathcal{G}}(V_1, V_2, \dots, V_\ell)$, where $\mathcal{P}_g = \{V_1, V_2, \dots, V_\ell\}$ is a partition of X into clopen sets, each consisting of more than one point. By Lemma 6.15, any local subgroup $\mathcal{G}_U$ is contained in H_g if and only if $U \subset V_j$ for some j, $1 \leq j \leq \ell$. In particular, each element of the partition $\mathcal{P}$ is contained in a single element of the partition $\mathcal{P}_g$. It follows that each element of $\mathcal{P}_g$ is the union of one or more elements of $\mathcal{P}$. Note that $\mathcal{P}_g \neq \mathcal{P}$ since the group H_g is not contained in H. Hence, there exists j, $1 \leq j \leq \ell$, such that the set V_j contains sets U_{i_1} and U_{i_2}, where $i_1 \neq i_2$. Take any i, $1 \leq i \leq k$, different from i_1 and i_2 (as $k \geq 3$, at least one such i exists). Since the group H acts transitively on $\mathcal{P}$, we have $h(U_{i_2}) = U_i$ for some $h \in H$. Note that U_{i_2} and U_i are disjoint clopen sets. Hence, the generalized 2-cycle $f = \delta_{U_{i_2};h}$ is defined. Since $h \in \mathcal{G}$, it follows that $f \in \mathsf{F}(\mathcal{G}) = \mathcal{G}$. We have $f(U_{i_2}) = U_i$, $f(U_i) = U_{i_2}$, and $f(x) = x$ for any x not in U_{i_2} or U_i. Therefore, f preserves the partition $\mathcal{P}$ so that $f \in H$. Then, $f \in H_g$ so that f preserves the partition $\mathcal{P}_g$ as well. In particular, $f(V_j) \in \mathcal{P}_g$. Since $f(U_{i_1}) = U_{i_1}$ and $f(U_{i_2}) = U_i$, it follows that $f(V_j) = V_j$ and $U_i \subset V_j$. As i was chosen arbitrarily,

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we conclude that $V_j = X$. Thus, the partition $\mathcal{P}_g$ is trivial, $\mathcal{P}_g = \{X\}$. Since $\mathrm{St}^{\bullet}_{\mathcal{G}}(X) = \mathrm{St}_{\mathcal{G}}(X) = \mathcal{G}$, we obtain that $H_g = \mathcal{G}$. □

THEOREM 6.21. *Let $\mathcal{G} \subset \mathrm{Homeo}(X)$ be an ample group with Property E. Suppose H is a maximal subgroup of $\mathcal{G}$ that acts minimally on X and contains a local subgroup $\mathcal{G}_U$ for some clopen set U containing more than one point. Then, $H = \mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$, where $X = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k$ is a partition of X into clopen sets, each containing more than one point. Moreover, the partition is unique, it consists of at least two sets, and the induced action of H on the set $\{U_1, U_2, \dots, U_k\}$ is transitive.*

Proof. By Proposition 6.17, there exists a partition $X = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_k$ such that each U_i is a clopen set consisting of more than one point and $\mathrm{St}^{\bullet}_{\mathcal{G}}(U_1, U_2, \dots, U_k) \subset H \subset \mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$. Moreover, the partition is unique and the induced action of H on the set $\{U_1, U_2, \dots, U_k\}$ is transitive. The partition cannot be trivial (with $k = 1$ and $U_1 = X$) as $\mathrm{St}^{\bullet}_{\mathcal{G}}(X) = \mathrm{St}_{\mathcal{G}}(X) = \mathcal{G}$, while $H \neq \mathcal{G}$. Hence, $k \geq 2$. In addition, this is clearly not the partition into points. Then, $\mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k) \neq \mathcal{G}$ due to Lemma 6.16. Since $H \subset \mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$ and H is a maximal subgroup of $\mathcal{G}$, it follows that $H = \mathrm{St}_{\mathcal{G}}(U_1, U_2, \dots, U_k)$. □

7. *Nowhere dense closed sets*

In this section, we present a construction of a closed, nowhere dense subset Y of a topological space X and a group H of homeomorphisms of X that leave Y invariant. The closed set Y is a Cantor set and the group H acts minimally when restricted to Y. In the case where X is a Cantor set, the construction helps, given an ample group $\mathcal{G} \subset \mathrm{Homeo}(X)$ that acts minimally on X, to produce uncountably many infinite, nowhere dense closed sets Y such that the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ acts minimally on Y. By Theorem 6.18, all those stabilizers are maximal subgroups of $\mathcal{G}$ provided that the group $\mathcal{G}$ has Property E. In addition, it follows from Lemma 6.4 that all those stabilizers are different from one another and from the stabilizers of other closed sets.

The construction is general and we do not assume the topological space X to be a Cantor set. Suppose $U_0 \supset U_1 \supset U_2 \supset \cdots$ is a nested sequence of non-empty clopen subsets of X and x_0 is a common point of all these sets. Further, suppose that for any $n \geq 1$, we have a homeomorphism $g_n : X \to X$ such that the set $g_n(U_n)$ is disjoint from U_n and, moreover, the union $U_n \cup g_n(U_n)$ is a proper subset of U_{n-1}. Let H be a subgroup of $\mathrm{Homeo}(X)$ generated by the generalized 2-cycles $\delta_{U_n;g_n}$, $n = 1, 2, \dots$. Let $Y = \overline{\mathrm{Orb}_H(x_0)}$, the closure of the orbit of the point x_0 under the natural action of the group H on X. Then, Y is a closed set invariant under the action of H. Assuming all maps g_n, $n \geq 1$, are taken from a group $G \subset \mathrm{Homeo}(X)$, the group H is a subgroup of the ample group $\mathcal{G} = \mathsf{F}(G)$ so that $H \subset \mathrm{St}_{\mathcal{G}}(Y)$. For any $n \geq 1$, we let $f_n^{(0)} = \mathrm{id}_X$ and $f_n^{(1)} = \delta_{U_n;g_n}$. Note that $f_n^{(0)}(U_n) = U_n$ and $f_n^{(1)}(U_n) = g_n(U_n)$ are subsets of U_{n-1}. Hence, for any infinite string $\xi_1\xi_2\xi_3\cdots$ of 0s and 1s, we have a nested sequence of clopen sets $U_0 \supset f_1^{(\xi_1)}(U_1) \supset f_1^{(\xi_1)} f_2^{(\xi_2)}(U_2) \supset \cdots$. The intersection of these sets is non-empty.

LEMMA 7.1. *In terms of the above construction, suppose that for any infinite string $\xi_1\xi_2\xi_3\cdots$ of 0s and 1s, the intersection of the nested sets $U_0 \supset f_1^{(\xi_1)}(U_1) \supset$*

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$f_1^{(\xi_1)} f_2^{(\xi_2)}(U_2) \supset \cdots$ *consists of a single point. Then, Y is a Cantor set that is nowhere dense in X and the group H acts minimally when restricted to Y.*

Proof. We keep the same notation as used in the construction. First, let us describe an abstract model behind the construction. Let $\Xi = \{0, 1\}^{\mathbb{N}}$. We regard any element $\xi \in \Xi$ as an infinite string $\xi = \xi_1\xi_2\xi_3 \cdots$, where each $\xi_i \in \{0, 1\}$. We endow Ξ with the product topology (assuming the discrete topology on $\{0, 1\}$). Then, Ξ is a Cantor set. Let $\{0, 1\}^*$ be the set of all finite strings of 0s and 1s (including the empty string $\varnothing$). For any $w \in \{0, 1\}^*$, we denote by C_w the set of all infinite strings in Ξ that begin with w. Sets of the form C_w are called *cylinders*, they are clopen, and form a base of the topology on Ξ.

For any $n \geq 1$, let γ_n be a transformation of Ξ that changes only the nth character in any infinite string that begins with (at least) $n - 1$ consecutive zeros and fixes all the other elements of Ξ. The transformation γ_n is clearly an involution. In addition, it is continuous. Hence, $\gamma_n \in \mathrm{Homeo}(\Xi)$. Let Γ be the subgroup of $\mathrm{Homeo}(\Xi)$ generated by $\gamma_1, \gamma_2, \gamma_3, \ldots$.

For any $\xi = \xi_1\xi_2\xi_3 \cdots \in \Xi$ and integer $n \geq 0$, let $Z(\xi, n)$ denote a string $\zeta = \zeta_1\zeta_2\zeta_3 \cdots$ that begins with n zeros and coincides with the string ξ afterwards: $\zeta_k = 0$ if $k \leq n$ and $\zeta_k = \xi_k$ if $k > n$. Clearly, $Z(\xi, 0) = \xi$. For any $n \geq 1$, we have $Z(\xi, n) = Z(\xi, n - 1)$ if $\xi_n = 0$ and $Z(\xi, n) = \gamma_n(Z(\xi, n - 1))$ if $\xi_n = 1$. It follows that the strings $Z(\xi, 0), Z(\xi, 1), Z(\xi, 2), \ldots$ all belong to the orbit $\mathrm{Orb}_\Gamma(\xi)$. If two strings $\xi, \xi' \in \Xi$ coincide up to finitely many characters, then $Z(\xi, n) = Z(\xi', n)$ for all sufficiently large n, which implies that ξ and ξ' are in the same orbit of the group Γ. We conclude that any orbit of Γ has an element in every cylinder C_w, $w \in \{0, 1\}^*$. As a consequence, any orbit of Γ is dense in Ξ. Therefore, the group Γ acts minimally on Ξ.

Now, we are going to show that the actual construction of the set Y and the group H agrees with our abstract model (respectively Ξ and Γ). First, we define a map $\alpha : \Xi \to Y$. For any finite string $w = \xi_1\xi_2 \cdots \xi_k$ of 0s and 1s consisting of $k \geq 1$ characters, let $f^{(w)} = f_1^{(\xi_1)} f_2^{(\xi_2)} \cdots f_k^{(\xi_k)}$ and $V^{(w)} = f^{(w)}(U_k)$. Also, we let $f^{(\varnothing)} = \mathrm{id}_X$ and $V^{(\varnothing)} = U_0$. Given an infinite string $\xi = \xi_1\xi_2\xi_3 \cdots \in \Xi$, consider a nested sequence of clopen sets $V^{\varnothing} \supset V^{(\xi_1)} \supset V^{(\xi_1\xi_2)} \supset \cdots$. By assumption, the intersection of all these sets consists of a single point and we define $\alpha(\xi)$ to be that point. Let us show that $\alpha(\xi) \in Y$. Since the point x_0 belongs to each of the sets $U_0, U_1, U_2, \ldots$, it follows that $f^{(\xi_1\xi_2\cdots\xi_k)}(x_0) \in V^{(\xi_1\xi_2\cdots\xi_k)}$ for $k = 1, 2, \ldots$. This implies that every limit point of the sequence $x_0, f_1^{(\xi_1)}(x_0), f_2^{(\xi_1\xi_2)}(x_0), \ldots$ belongs to the intersection of the nested clopen sets $V^{\varnothing}, V^{(\xi_1)}, V^{(\xi_1\xi_2)}, \ldots$. As $\alpha(\xi)$ is the only point in the intersection, we conclude that the sequence converges to $\alpha(\xi)$. Since all points in the sequence belong to the orbit $\mathrm{Orb}_H(x_0)$, we obtain that $\alpha(\xi) \in \overline{\mathrm{Orb}_H(x_0)} = Y$.

By construction, $\alpha(C_w) \subset V^{(w)}$ for all $w \in \{0, 1\}^*$. For any strings $w, u \in \{0, 1\}^*$, let wu denote their concatenation. Notice that $C_{wu} \subset C_w$ and $V^{(wu)} \subset V^{(w)}$. Let k be the number of characters in the string w. Then, $V^{(w)} = f^{(w)}(U_k)$, $V^{(w0)} = f^{(w)}(U_{k+1})$, and $V^{(w1)} = f^{(w)}(f_{k+1}^{(1)}(U_{n+1}))$. By construction, U_{k+1} and $f_{k+1}^{(1)}(U_{k+1})$ are disjoint subsets of U_k and their union is different from U_k. It follows that $V^{(w)}$ is the disjoint union of three non-empty clopen sets $V^{(w0)}$, $V^{(w1)}$ and $V^{(w)} \setminus (V^{(w0)} \cup V^{(w1)})$. Note that any two strings in $\{0, 1\}^*$, neither of which is a beginning of the other, can be written as $w0u$ and $w1u'$,

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where $w, u, u' \in \{0,1\}^*$. Since $V^{(w0u)} \subset V^{(w0)}$ and $V^{(w1u')} \subset V^{(w1)}$, we obtain that the set $V^{(w0u)}$ is disjoint from $V^{(w1u')}$. In particular, the sets $V^{(w)}$ and $V^{(w')}$ are disjoint if w and w' are two different strings with the same number of characters.

Next, we prove that the map α is one-to-one. Given two infinite strings $\xi, \xi' \in \Xi$, let w be their longest common beginning. If $\xi \neq \xi'$, then w is a finite string. Moreover, one of the strings ξ and ξ' belongs to C_{w0}, while the other is in C_{w1}. Hence, one of the points $\alpha(\xi)$ and $\alpha(\xi')$ belongs to $V^{(w0)}$, while the other is in $V^{(w1)}$. By the above, the sets $V^{(w0)}$ and $V^{(w1)}$ are disjoint. As a consequence, $\alpha(\xi) \neq \alpha(\xi')$.

Given $\xi = \xi_1\xi_2\xi_3\cdots \in \Xi$, let us show that every open neighborhood of the point $\alpha(\xi)$ contains a clopen set of the form $V^{(w)}$, where the finite string $w \in \{0,1\}^*$ is a beginning of ξ. By construction, $V^{(\xi_1)} \supset V^{(\xi_1\xi_2)} \supset V^{(\xi_1\xi_2\xi_3)} \supset \cdots$ and the intersection of these clopen sets is $\{\alpha(\xi)\}$. Hence, $X \setminus V^{(\xi_1)} \subset X \setminus V^{(\xi_1\xi_2)} \subset X \setminus V^{(\xi_1\xi_2\xi_3)} \subset \cdots$ and the union of these complements is $X \setminus \{\alpha(\xi)\}$. Adding an arbitrary open neighborhood U of $\alpha(\xi)$ to the complements, we obtain an open cover of X. Since X is compact, there is a finite subcover. It further follows that X is covered by two sets U and $X \setminus V^{(\xi_1\xi_2\cdots\xi_n)}$ provided that n is large enough. Equivalently, $V^{(\xi_1\xi_2\cdots\xi_n)} \subset U$ if n is large enough.

Next, we prove that the map α is continuous. Let $\xi \in \Xi$ and U be an open neighborhood of the point $\alpha(\xi)$. We need to show that the preimage $\alpha^{-1}(U)$ contains an open neighborhood of ξ. By the above, U contains a set $V^{(w)}$ for some string w with which ξ begins. Then, the cylinder C_w is a clopen neighborhood of ξ. It is contained in $\alpha^{-1}(U)$ since $\alpha(C_w) \subset V^{(w)}$.

Next, we prove that $f_n^{(1)}\alpha = \alpha\gamma_n$ on Ξ for all $n \geq 1$. Take any $\xi = \xi_1\xi_2\xi_3\cdots \in \Xi$ and let $\gamma_n(\xi) = \eta_1\eta_2\eta_3\cdots$. By the above, $f^{(\xi_1\xi_2\cdots\xi_k)}(x_0) \to \alpha(\xi)$ and $f^{(\eta_1\eta_2\cdots\eta_k)}(x_0) \to \alpha(\gamma_n(\xi))$ as $k \to \infty$. Then, $f_n^{(1)} f^{(\xi_1\xi_2\cdots\xi_k)}(x_0) \to f_n^{(1)}(\alpha(\xi))$ as $k \to \infty$. We are going to show that $f_n^{(1)} f^{(\xi_1\xi_2\cdots\xi_k)}(x_0) = f^{(\eta_1\eta_2\cdots\eta_k)}(x_0)$ for $k \geq n$. This will imply that $f_n^{(1)}(\alpha(\xi)) = \alpha(\gamma_n(\xi))$. First, consider the case where $\xi_i = 0$ for all $i < n$. Then, $\eta_n \neq \xi_n$ and $\eta_i = \xi_i$ for all $i \neq n$. Since $f_n^{(1)}$ is an involution and $f_n^{(0)} = \mathrm{id}_X$, we obtain that $f_n^{(1)} f_n^{(\xi_n)} = f_n^{(\eta_n)}$. As $f_i^{(\xi_i)} = \mathrm{id}_X$ for $i < n$, it follows that $f_n^{(1)} f^{(\xi_1\xi_2\cdots\xi_k)} = f^{(\eta_1\eta_2\cdots\eta_k)}$ for all $k \geq n$. Now, consider the case where $\xi_i = 1$ for some $i < n$. In this case, $\gamma_n(\xi) = \xi$. Since $x_0 \in U_k$ for any $k \geq n$, it follows that the point $y = f_{i+1}^{(\xi_{i+1})} f_{i+2}^{(\xi_{i+2})} \cdots f_k^{(\xi_k)}(x_0)$ belongs to U_i. Then, the point $f_i^{(1)}(y)$ does not belong to U_i and neither does $f^{(\xi_1\xi_2\cdots\xi_k)}(x_0)$. Since $\mathrm{supp}(f_n^{(1)}) \subset U_{n-1} \subset U_i$, we conclude that $f_n^{(1)} f^{(\xi_1\xi_2\cdots\xi_k)}(x_0) = f^{(\xi_1\xi_2\cdots\xi_k)}(x_0) = f^{(\eta_1\eta_2\cdots\eta_k)}(x_0)$.

Next, we prove that the map α is onto. For any $n \geq 1$, the equality $f_n^{(1)}\alpha = \alpha\gamma_n$ implies that $f_n^{(1)}(\alpha(\Xi)) = \alpha(\gamma_n(\Xi)) = \alpha(\Xi)$. Since the group H is generated by the maps $f_1^{(1)}, f_2^{(1)}, f_3^{(1)}, \ldots$, it follows that $h(\alpha(\Xi)) = \alpha(\Xi)$ for all $h \in H$. Notice that $x_0 \in \alpha(\Xi)$, namely, x_0 is the image of the infinite string of all 0s. Since the set $\alpha(\Xi)$ is invariant under the action of the group H, we obtain that $\mathrm{Orb}_H(x_0) \subset \alpha(\Xi)$. Further, the set $\alpha(\Xi)$ is compact as Ξ is compact and the map α is continuous. As a consequence, $\alpha(\Xi)$ is a closed subset of X. Since $\mathrm{Orb}_H(x_0) \subset \alpha(\Xi) \subset Y$ and $Y = \overline{\mathrm{Orb}_H(x_0)}$, we conclude that $\alpha(\Xi) = Y$.

Since the map α is one-to-one and onto, it is invertible. The inverse map $\alpha^{-1} : Y \to \Xi$ is continuous since α is continuous and Ξ is compact. Hence, α is a homeomorphism. As a consequence, Y is a Cantor set.

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Now, we derive minimality of the action of the group H on Y from minimality of the action of the group Γ on Ξ. For any closed set $Y_0 \subset Y$, the set $\alpha^{-1}(Y_0)$ is a closed subset of Ξ. Assuming Y_0 is invariant under the action of H, we have $f_n^{(1)}(Y_0) = Y_0$ for all $n \geq 1$. By the above, $f_n^{(1)}\alpha = \alpha\gamma_n$ on Ξ. Therefore, $\gamma_n = \alpha^{-1} f_n^{(1)}\alpha$ on Ξ. Then, $\gamma_n(\alpha^{-1}(Y_0)) = \alpha^{-1} f_n^{(1)}(Y_0) = \alpha^{-1}(Y_0)$. Since the group Γ is generated by maps $\gamma_1, \gamma_2, \gamma_3, \ldots$, it follows that $\gamma(\alpha^{-1}(Y_0)) = \alpha^{-1}(Y_0)$ for all $\gamma \in \Gamma$, that is, the set $\alpha^{-1}(Y_0)$ is invariant under the action of the group Γ on Ξ. Minimality of the latter action implies that $\alpha^{-1}(Y_0)$ is either Ξ or the empty set. Then, Y_0 is either Y or the empty set.

It remains to prove that the set Y is nowhere dense in X. Since Y is closed, it is enough to show that it has no interior points. Let $y \in Y$ and U be an open neighborhood of the point y. We have $y = \alpha(\xi)$ for some $\xi \in \Xi$. By the above, U contains a set $V^{(w)}$ for some string $w \in \{0, 1\}^*$ with which ξ begins. In addition, by the above, $V^{(w)}$ is the disjoint union of three non-empty clopen sets $V^{(w0)}$, $V^{(w1)}$, and $V^{(w)} \setminus (V^{(w0)} \cup V^{(w1)})$. Take any string $\xi' \in \Xi$ and let w' be the beginning of ξ' that has the same number of characters as w. If $w' \neq w$, then the set $V^{(w')}$, which contains the point $\alpha(\xi')$, is disjoint from $V^{(w)}$ so that $\alpha(\xi') \notin V^{(w)}$. If $w' = w$, then $\alpha(\xi') \in V^{(w0)} \cup V^{(w1)}$. We conclude that the non-empty set $V^{(w)} \setminus (V^{(w0)} \cup V^{(w1)})$ is disjoint from $\alpha(\Xi) = Y$. Therefore, the open set U is not fully contained in Y. □

One can show that the group H considered in Lemma 7.1 is locally finite, that is, any finitely generated subgroup of H is finite.

PROPOSITION 7.2. *Suppose X is a Cantor set and $\mathcal{G} \subset \mathrm{Homeo}(X)$ is an ample group that acts minimally on X. Then, there are uncountably many infinite, nowhere dense closed sets $Y \subset X$ such that the stabilizer $\mathrm{St}_{\mathcal{G}}(Y)$ acts minimally when restricted to Y.*

Proof. Let $\rho : X \times X \to \mathbb{R}$ be any distance function on the Cantor set X compatible with its topology. For any $x \in X$ and $r > 0$, denote by $B(x, r)$ the ball of radius r centered at the point x: $B(x, r) = \{y \in X \mid \rho(y, x) < r\}$. The ball $B(x, r)$ is an open neighborhood of x of diameter at most $2r$.

We are going to construct a nested sequence $U_0 \supset U_1 \supset U_2 \supset \cdots$ of non-empty clopen subsets of X and two sequences, $g_1, g_2, g_3, \ldots$ and $h_1, h_2, h_3, \ldots$, of elements of the group $\mathcal{G}$. The elements of $\mathcal{G}$ will be chosen so that for any $n \geq 1$, the sets U_n, $g_n(U_n)$, and $h_n(U_n)$ are disjoint subsets of U_{n-1}. Assuming this, generalized 2-cycles $f_n^{(1)} = \delta_{U_n;g_n}$ and $f_n^{(2)} = \delta_{U_n;h_n}$ are defined. Since $g_n, h_n \in \mathcal{G}$, the maps $f_n^{(1)}$ and $f_n^{(2)}$ belong to the group $\mathsf{F}(\mathcal{G}) = \mathcal{G}$. We also let $f_n^{(0)} = \mathrm{id}_X$. The clopen sets $U_0, U_1, U_2, \ldots$ will be chosen so that for any string $\xi_1\xi_2 \cdots \xi_n$ of 0s, 1s, and 2s, the set $f_1^{(\xi_1)} f_2^{(\xi_2)} \cdots f_n^{(\xi_n)}(U_n)$ is of diameter at most 2^{-n}.

The construction is done inductively. First, we let $U_0 = X$. Now, assume that for some $n \geq 1$, the set U_{n-1} is already chosen, and so are the maps g_k and h_k, $1 \leq k \leq n-1$. Since X is a Cantor set, the non-empty clopen set U_{n-1} is infinite. Therefore, we can find three disjoint non-empty clopen sets $V, V', V'' \subset U_{n-1}$. Take any point $y \in V$. Since the group $\mathcal{G}$ acts minimally on X, the orbit $\mathrm{Orb}_{\mathcal{G}}(y)$ is dense in X. In particular, this orbit has a point in V' and in V''. Hence, we can choose $g_n, h_n \in \mathcal{G}$ such that $g_n(y) \in V'$ and $h_n(y) \in V''$. Then, $W = V \cap g_n^{-1}(V') \cap h_n^{-1}(V'')$ is a clopen neighborhood of the point y. By construction, $W \subset V$, $g_n(W) \subset V'$, and $h_n(W) \subset V''$, which implies that W,

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$g_n(W)$, and $h_n(W)$ are disjoint subsets of U_{n-1}. Now that the maps g_k and h_k are chosen for any k, $1 \le k \le n$, we can define the maps $f_k^{(i)}$, $i \in \{0, 1, 2\}$, $1 \le k \le n$, as described above. For any string $w = \xi_1\xi_2 \cdots \xi_n$ of 0s, 1s, and 2s that has exactly n characters, we let $f^{(w)} = f_1^{(\xi_1)} f_2^{(\xi_2)} \cdots f_n^{(\xi_n)}$ and $y_w = f^{(w)}(y)$. Further, let $\widetilde{W}$ be the intersection of W and all sets of the form $(f^{(w)})^{-1}(B(y_w, 2^{-n-1}))$. Then, $\widetilde{W}$ is an open neighborhood of the point y. As X is a Cantor set, the set $\widetilde{W}$ contains a clopen neighborhood of y. We choose the latter as U_n. The clopen set U_n is not empty since $y \in U_n$. Since $U_n \subset \widetilde{W} \subset W$, the sets U_n, $g_n(U_n)$, and $h_n(U_n)$ are disjoint subsets of U_{n-1}. By construction, $f^{(w)}(U_n) \subset B(y_w, 2^{-n-1})$ for any string $w = \xi_1\xi_2 \cdots \xi_n$ of 0s, 1s, and 2s, which implies that the diameter of the set $f^{(w)}(U_n)$ is at most 2^{-n}. The inductive step of the construction is complete.

Let $\Xi = \{0, 1, 2\}^{\mathbb{N}}$. We regard any element $\xi \in \Xi$ as an infinite string $\xi = \xi_1\xi_2\xi_3 \cdots$, where each $\xi_i \in \{0, 1, 2\}$. Note that for any $n \ge 1$, the sets $f_n^{(0)}(U_n) = U_n$, $f_n^{(1)}(U_n) = g_n(U_n)$, and $f_n^{(2)}(U_n) = h_n(U_n)$ are subsets of U_{n-1}. Therefore, for any infinite string $\xi = \xi_1\xi_2\xi_3 \cdots$ of 0s, 1s, and 2s, we have a nested sequence of clopen sets $U_0 \supset f_1^{(\xi_1)}(U_1) \supset f_1^{(\xi_1)} f_2^{(\xi_2)}(U_2) \supset \cdots$. By construction, the diameters of those sets tend to 0. Hence, their intersection consists of a single point, which we denote $\beta(\xi)$. Now, we have a map $\beta : \Xi \to X$. Let $x_0 = \beta(000 \cdots)$. The point x_0 is in the intersection of sets $U_0, U_1, U_2, \ldots$.

Consider the set $\Omega = \{1, 2\}^{\mathbb{N}}$, which is a subset of Ξ. For any infinite string $\omega = \omega_1\omega_2\omega_3 \cdots \in \Omega$, let H_ω denote a subgroup of $\mathcal{G}$ generated by elements $f_n^{(\omega_n)}$, $n \ge 1$. Further, let $Y_\omega = \overline{\mathrm{Orb}_{H_\omega}(x_0)}$. Then, Y_ω is a closed subset of X invariant under the action of the group H_ω. Since for any $n \ge 1$, the sets U_n, $g_n(U_n)$, and $h_n(U_n)$ are disjoint subsets of U_{n-1}, it follows that the union of U_n and $f_n^{(\omega_n)}(U_n)$ is a proper subset of U_{n-1}. Now, Lemma 7.1 implies that Y_ω is a Cantor set which is nowhere dense in X, and the group H_ω acts minimally when restricted to Y_ω. Since $H_\omega \subset \mathrm{St}_{\mathcal{G}}(Y_\omega)$, the stabilizer $\mathrm{St}_{\mathcal{G}}(Y_\omega)$ also acts minimally when restricted to Y_ω.

As the set Ω is uncountable, it remains to demonstrate that $Y_\omega \ne Y_{\omega'}$ whenever ω and ω' are different elements of Ω. Just as in the proof of Lemma 7.1 we showed that the map α is one-to-one, we can show that the map β is one-to-one. Just as in the proof of Lemma 7.1 we showed that the map α is onto, we can show that for any $\omega = \omega_1\omega_2\omega_3 \cdots \in \Omega$, the set Y_ω consists of all points of the form $\beta(\xi_1\xi_2\xi_3 \cdots)$, where $\xi_n \in \{0, \omega_n\}$ for each $n \ge 1$. Since the map β is one-to-one, it follows that $Y_\omega \cap \beta(\Omega) = \{\beta(\omega)\}$. As a consequence, $Y_\omega \ne Y_{\omega'}$ whenever $\omega \ne \omega'$. □

8. *Property E*

The majority of our results on maximal subgroups of ample groups obtained in §6 require that the ample group has Property E (to be precise, these are Theorems 6.18, 6.19, 6.20, and 6.21). Hence, they have no value until we provide examples of ample groups with this property. We should note that Property E is complex and its verification is not going to be easy. We begin with establishing a weakened version of Property E that involves the generalized symmetric groups.

PROPOSITION 8.1. *Suppose a group $G \subset \mathrm{Homeo}(X)$ acts minimally on X. Then, for any clopen sets U_1 and U_2 that intersect, the generalized symmetric group over the local*

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subgroup $\mathsf{F}_{U_1\cup U_2}(G)$ *is generated by the union of generalized symmetric groups over* $\mathsf{F}_{U_1}(G)$ *and* $\mathsf{F}_{U_2}(G)$*:* $\mathsf{S}(\mathsf{F}_{U_1\cup U_2}(G)) = \langle \mathsf{S}(\mathsf{F}_{U_1}(G)) \cup \mathsf{S}(\mathsf{F}_{U_2}(G))\rangle$.

Proof. For any clopen set $W \subset X$, let D_W denote the set of all generalized 2-cycles in the local subgroup $\mathsf{F}_W(G)$. Since the group $\mathsf{F}_W(G)$ is ample, it follows from Lemma 4.7 that the generalized symmetric group $\mathsf{S}(\mathsf{F}_W(G))$ is generated by D_W. As a consequence, for any clopen sets $U_1, U_2 \subset X$, the group $\mathsf{S}(\mathsf{F}_{U_1\cup U_2}(G))$ is generated by $D_{U_1\cup U_2}$, while the group $\langle \mathsf{S}(\mathsf{F}_{U_1}(G)) \cup \mathsf{S}(\mathsf{F}_{U_2}(G))\rangle$ is generated by $D_{U_1} \cup D_{U_2}$. Note that the sets D_{U_1} and D_{U_2} are contained in $D_{U_1\cup U_2}$ since the local subgroups $\mathsf{F}_{U_1}(G)$ and $\mathsf{F}_{U_2}(G)$ are contained in $\mathsf{F}_{U_1\cup U_2}(G)$. To prove the proposition, it is enough to show that, assuming the intersection $U_1 \cap U_2$ is non-empty, every element of $D_{U_1\cup U_2}$ belongs to the group $\langle D_{U_1} \cup D_{U_2}\rangle$. This is trivial when one of the sets U_1 and U_2 contains the other. Hence, we may further assume that each of the sets $P_1 = U_1 \setminus U_2$, $P_2 = U_2 \setminus U_1$, and $P_3 = U_1 \cap U_2$ is non-empty.

Given $f \in D_{U_1\cup U_2}$, we have $f = \delta_{U;f'}$ for some clopen set $U \subset X$ and homeomorphism $f' : X \to X$. Observe that the generalized 2-cycle $\delta_{U;f}$ is defined and $f = \delta_{U;f}$. First, we consider a special case where one of the sets U and $f(U)$ is contained in P_1 and the other is contained in P_2. Note that $f^{-1} = f$. Therefore, the generalized 2-cycle $\delta_{f(U);f}$ is defined and $\delta_{f(U);f} = \delta_{U;f} = f$. Hence, we may assume without loss of generality that $U \subset P_1$ and $f(U) \subset P_2$. Since the group G acts minimally on X, Lemma 5.3 implies that for some $k \geq 1$, there exist disjoint clopen sets $V_1, V_2, \ldots, V_k$, and maps $g_1, g_2, \ldots, g_k \in G$ such that $U = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$ and $g_i(V_i) \subset P_3$ for all i. As the set $f(U)$ is disjoint from U, it follows that the sets $V_1, V_2, \ldots, V_k, f(V_1), f(V_2), \ldots, f(V_k)$ are disjoint from one another. In particular, the generalized 2-cycle $\delta_{V_i;f}$ is defined for each i, $1 \leq i \leq k$. Since the map f belongs to the ample group $\mathsf{F}(G)$, so does $\delta_{V_i;f}$. Since $\mathrm{supp}(\delta_{V_i;f}) = V_i \cup f(V_i)$, the maps $\delta_{V_1;f}, \delta_{V_2;f}, \ldots, \delta_{V_k;f}$ have pairwise disjoint supports. Therefore, these maps commute with one another. Each $\delta_{V_i;f}$ coincides with $\delta_{U;f}$ on $V_i \cup f(V_i)$ and with the identity map everywhere else. Since sets $V_i \cup f(V_i)$, $1 \leq i \leq k$ form a partition of the set $U \cup f(U)$, it follows that $f = \delta_{V_1;f}\, \delta_{V_2;f} \cdots \delta_{V_k;f}$. We are going to show that each $\delta_{V_i;f}$ belongs to the group $\langle D_{U_1} \cup D_{U_2}\rangle$. Then, $f \in \langle D_{U_1} \cup D_{U_2}\rangle$ as well.

For any i, $1 \leq i \leq k$, the sets V_i, $f(V_i)$, and $g_i(V_i)$ are disjoint since $V_1 \subset P_1$, $f(V_i) \subset P_2$, and $g_i(V_i) \subset P_3$. Therefore, the generalized permutation $h_\pi = \mu[V_i; \mathrm{id}_X, f, g_i; \pi]$ is defined for any permutation $\pi \in \mathsf{S}_3$. Moreover, $h_\pi \in \mathsf{F}(G)$. Since $(1\,2) = (1\,3)(2\,3)(1\,3)$, it follows from Lemma 4.2 that $h_{(1\,2)} = h_{(1\,3)}h_{(2\,3)}h_{(1\,3)}$. Observe that $h_{(1\,2)} = \delta_{V_i;f}$, $h_{(1\,3)} = \delta_{V_i;g_i}$ and $h_{(2\,3)} = \delta_{f(V_i);g_i f}$. Then, $\mathrm{supp}(h_{(1\,3)}) = V_i \cup g_i(V_i) \subset U_1$ and $\mathrm{supp}(h_{(2\,3)}) = f(V_i) \cup g_i(V_i) \subset U_2$, which means that $h_{(1\,3)} \in \mathsf{F}_{U_1}(G)$ and $h_{(2\,3)} \in \mathsf{F}_{U_2}(G)$. Hence, $h_{(1\,3)} \in D_{U_1}$ and $h_{(2\,3)} \in D_{U_2}$. Consequently, the map $\delta_{V_i;f} = h_{(1\,3)}h_{(2\,3)}h_{(1\,3)}$ belongs to the group $\langle D_{U_1} \cup D_{U_2}\rangle$.

Now, consider an arbitrary generalized 2-cycle $f = \delta_{U;f}$ in $D_{U_1\cup U_2}$. Note that the support $\mathrm{supp}(f) = U \cup f(U)$ is contained in $U_1 \cup U_2$. For any $i, j \in \{1, 2, 3\}$, let $Q_{ij} = P_i \cap f(P_j)$. Since $U_1 \cup U_2 = P_1 \sqcup P_2 \sqcup P_3$, $f(U_1 \cup U_2) = U_1 \cup U_2$, and $f^{-1} = f$, it follows that $U_1 \cup U_2$ is the disjoint union of 9 clopen sets Q_{ij}, $i, j \in \{1, 2, 3\}$. Consequently, U is the disjoint union of 9 clopen sets $U \cap Q_{ij}$, $i, j \in \{1, 2, 3\}$. Just as above, we obtain that generalized 2-cycles $f_{ij} = \delta_{U\cap Q_{ij};f}$, $i, j \in \{1, 2, 3\}$ are defined, they commute

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with one another, and f is their product. All 9 maps belong to $D_{U_1 \cup U_2}$. By construction, $U \cap Q_{ij} \subset P_i$ and $f_{ij}(U \cap Q_{ij}) = f(U \cap Q_{ij}) \subset P_j$. Hence, the maps f_{12} and f_{21} have been covered by the special case. By the above, both maps belong to $\langle D_{U_1} \cup D_{U_2} \rangle$. As for the other 7 maps, the supports of f_{11}, f_{13}, f_{31}, and f_{33} are contained in U_1, while the supports of f_{22}, f_{23}, and f_{32} are contained in U_2. Hence, $f_{11}, f_{13}, f_{31}, f_{33} \in D_{U_1}$ and $f_{22}, f_{23}, f_{32} \in D_{U_2}$. We conclude that all 9 maps f_{ij}, $i, j \in \{1, 2, 3\}$ belong to the group $\langle D_{U_1} \cup D_{U_2} \rangle$. Then, $f \in \langle D_{U_1} \cup D_{U_2} \rangle$ as well. □

Remark 8.2. If X is a finite set with the discrete topology, then all subsets of X are clopen. Hence, the group $\mathrm{Homeo}(X)$ coincides with the symmetric group S_X of all permutations on X. Any permutation on X is also a generalized permutation (in the sense of Definition 4.1). Therefore, for any group $G \subset \mathsf{S}_X$, the ample group $\mathsf{F}(G)$ coincides with the generalized symmetric group $\mathsf{S}(G)$. Moreover, for any subset $U \subset X$, the local subgroup $\mathsf{F}_U(G)$ coincides with $\mathsf{S}(\mathsf{F}_U(G))$. In this case, the conclusion of Proposition 8.1 means that the group G has Property E. The hypothesis of Proposition 8.1 is that G acts minimally on X. For a finite X, this means transitivity, that is, the entire set X forms a single orbit. Then, every permutation on X is pointwise (and hence piecewise) an element of G, which implies that $\mathsf{F}(G) = \mathsf{S}_X$. Hence, Theorems 6.19, 6.20, and 6.21 apply to the symmetric group S_X (Theorem 6.18 is vacuous if X is finite). One of the hypotheses of Theorem 6.21 is that the subgroup H of the ample group $\mathcal{G}$ contains a local subgroup $\mathcal{G}_U$ for some clopen set U containing more than one point. In the case where $\mathcal{G} = \mathsf{S}_X$, this is equivalent to the condition that H contains a transposition. Thus, as a byproduct of our proceedings, we obtain the following classical result.

PROPOSITION 8.3. (Jordan [**36**]) *Any proper primitive subgroup of a finite symmetric group S_n contains no transposition.*

Proposition 8.3 is a direct corollary of the statement proved in the book [**36**, Note C] (for an explicit formulation, see [**15**, Theorem 3.3A]).

Proof of Proposition 8.3. Suppose H is a maximal subgroup of S_n that contains a transposition $(x\ y)$. Then, H contains the local subgroup $\mathsf{F}_{\{x,y\}}(\mathsf{S}_n) = \{\mathrm{id}_X, (x\ y)\}$. The symmetric group S_n has Property E (as explained in Remark 8.2). In the case where the subgroup H acts transitively on the set X, it follows from Theorem 6.21 that H is the stabilizer of a partition of X into at least two sets, each consisting of at least two points. Hence, H is not primitive. In the case where the action of H on X is not transitive, it is not primitive either.

Now, assume H_0 is a proper primitive subgroup of S_n. The subgroup H_0 can be extended to a maximal subgroup H. Since H_0 is primitive, so is H. Then, it follows from the above that the group H contains no transposition. The same is true for its subgroup H_0. □

Remark 8.4. Another case where $\mathsf{F}_U(G) = \mathsf{S}(\mathsf{F}_U(G))$ for any clopen set $U \subset X$ so that Proposition 8.1 yields Property E is when every non-trivial local subgroup $\mathsf{F}_U(G)$ is simple. Indeed, $\mathsf{S}(\mathsf{F}_U(G))$ is always a normal subgroup of $\mathsf{F}_U(G)$ (due to Lemma 4.4). One example of such a group is Thompson's group V. The group V, which is simple, can be represented as an ample group acting minimally on the Cantor set $\{0, 1\}^{\mathbb{N}}$ (see, e.g., [**51**,

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Example 2.3.2.1] or [**11**, Definition 1.1]). It is not hard to show that for every non-empty clopen set $U \subset \{0,1\}^{\mathbb{N}}$, there exists a homeomorphism $\phi : \{0,1\}^{\mathbb{N}} \to U$ that is piecewise an element of the group V. This homeomorphism conjugates the ample group V with its local subgroup V_U. Hence, the subgroup V_U is isomorphic to the entire group V. In particular, V_U is simple.

In general, Proposition 8.1 does not yield Property E, but it helps in deriving this property from simpler properties introduced in §5.

LEMMA 8.5. *For any ample group acting minimally, Property CI implies Property E.*

Proof. Suppose $\mathcal{G} \subset \mathrm{Homeo}(X)$ is an ample group that acts minimally on X and has Property CI. For any clopen sets $U_1, U_2 \subset X$, the local subgroups $\mathcal{G}_{U_1}$ and $\mathcal{G}_{U_2}$ are contained in the local subgroup $\mathcal{G}_{U_1 \cup U_2}$. Assuming the intersection $U_1 \cap U_2$ is non-empty, we need to show that, conversely, every map $f \in \mathcal{G}_{U_1 \cup U_2}$ belongs to the group $\langle \mathcal{G}_{U_1} \cup \mathcal{G}_{U_2} \rangle$. This is trivial when one of the sets U_1 and U_2 contains the other. Hence, we may further assume that each of the sets $U_1 \setminus U_2$, $U_2 \setminus U_1$, and $U_1 \cap U_2$ is non-empty.

Since the group $\mathcal{G}$ acts minimally and $U_1 \cap U_2 \neq \emptyset$, Proposition 8.1 implies that the generalized symmetric group $\mathsf{S}(\mathcal{G}_{U_1 \cup U_2})$ is generated by its subgroups $\mathsf{S}(\mathcal{G}_{U_1})$ and $\mathsf{S}(\mathcal{G}_{U_2})$. Since $\mathsf{S}(\mathcal{G}_{U_1}) \subset \mathcal{G}_{U_1}$ and $\mathsf{S}(\mathcal{G}_{U_2}) \subset \mathcal{G}_{U_2}$, we conclude that $\mathsf{S}(\mathcal{G}_{U_1 \cup U_2}) \subset \langle \mathcal{G}_{U_1} \cup \mathcal{G}_{U_2} \rangle$.

Given a map $f \in \mathcal{G}_{U_1 \cup U_2}$, let $V_+ = \{x \in U_1 \mid f(x) \notin U_1\}$ and $V_- = \{x \in U_1 \mid f^{-1}(x) \notin U_1\}$. The sets V_+ and V_- are clopen as $V_+ = U_1 \cap f^{-1}(X \setminus U_1)$ and $V_- = U_1 \cap f(X \setminus U_1)$. First, we consider an easy case where $V_+ = V_- = \emptyset$. An equivalent condition is that $f(U_1) = U_1$. Since $\mathrm{supp}(f) \subset U_1 \cup U_2$, we also have $f(U_2 \setminus U_1) = U_2 \setminus U_1$. Let f_1 be a map on X that coincides with f on U_1 and with the identity map everywhere else. Let f_2 be another map on X that coincides with f on $U_2 \setminus U_1$ and with the identity map everywhere else. Since $f(U_1) = U_1$ and $f(U_2 \setminus U_1) = U_2 \setminus U_1$, it follows that f_1 and f_2 are homeomorphisms. By construction, both f_1 and f_2 are piecewise elements of the group $\mathcal{G}$. Since $\mathcal{G}$ is ample, it contains them. Also, by construction, $\mathrm{supp}(f_1) \subset U_1$, $\mathrm{supp}(f_2) \subset U_2 \setminus U_1$ and $f = f_1 f_2$. Therefore, $f_1 \in \mathcal{G}_{U_1}$, $f_2 \in \mathcal{G}_{U_2}$ and $f \in \langle \mathcal{G}_{U_1} \cup \mathcal{G}_{U_2} \rangle$.

Next, consider the case where one of the sets V_+ and V_- coincides with U_1. Then, the other set coincides with U_1 as well. In this case, the image $f(U_1)$ is disjoint from U_1. Hence, the generalized 2-cycle $g = \delta_{U_1; f}$ is defined. Moreover, $g \in \mathsf{F}(\mathcal{G}) = \mathcal{G}$. Note that $\mathrm{supp}(g) \subset \mathrm{supp}(f) \subset U_1 \cup U_2$. Therefore, $g \in \mathcal{G}_{U_1 \cup U_2}$. Then, the map $h = gf$ is also in $\mathcal{G}_{U_1 \cup U_2}$. Since g coincides with f^{-1} on the set $f(U_1)$, we obtain that h coincides with the identity map on U_1. It follows that $h \in \mathcal{G}_{U_2 \setminus U_1} \subset \mathcal{G}_{U_2}$. In addition, $g \in \mathsf{S}(\mathcal{G}_{U_1 \cup U_2}) \subset \langle \mathcal{G}_{U_1} \cup \mathcal{G}_{U_2} \rangle$. As a consequence, the map $f = g^{-1}h$ belongs to $\langle \mathcal{G}_{U_1} \cup \mathcal{G}_{U_2} \rangle$.

Now, consider the case where V_+ and V_- are both different from the empty set and U_1. In this case, the unions $V_+ \cup (X \setminus U_1)$ and $V_- \cup (X \setminus U_1)$ are both different from X. It is easy to see that f maps the first of these unions onto the second. Since the set $X \setminus U_1$ is disjoint from V_+ and V_-, Property CI of the group $\mathcal{G}$ implies that $f_0(V_+) = V_-$ for some $f_0 \in \mathcal{G}$. Note that the set $f^{-1}(V_-)$ is disjoint from U_1. It follows that the generalized 2-cycle $g = \delta_{V_+; f^{-1} f_0}$ is defined and belongs to $\mathsf{F}(\mathcal{G}) = \mathcal{G}$. Since $\mathrm{supp}(f) \subset U_1 \cup U_2$, we obtain that $f^{-1}(V_-) \subset U_2 \setminus U_1$. Then, the support $\mathrm{supp}(g) = V_+ \cup f^{-1}(V_-)$ is contained

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in $U_1 \cup U_2$ as well. Therefore, the map g belongs to $\mathcal{G}_{U_1 \cup U_2}$ and so does the map $h = fg$. By construction, $h(V_+) = V_-$ and $h(U_1 \setminus V_+) = f(U_1 \setminus V_+)$, which implies that $h(U_1) = U_1$. Hence, the map h has been covered by the first case. By the above, $h \in \langle \mathcal{G}_{U_1} \cup \mathcal{G}_{U_2} \rangle$. Since $g \in \mathsf{S}(\mathcal{G}_{U_1 \cup U_2})$, we conclude that the map $f = hg^{-1}$ belongs to $\langle \mathcal{G}_{U_1} \cup \mathcal{G}_{U_2} \rangle$ as well.

It remains to consider the case where one of the sets V_+ and V_- is empty while the other is not. This case will be reduced to the previous one. Without loss of generality, we may assume that $V_+ = \emptyset$ and $V_- \neq \emptyset$ (otherwise, we could replace f with f^{-1}). Then, $f(U_1)$ is a proper subset of U_1. It follows that for any $j \in \mathbb{Z}$, the set $f^{j+1}(U_1)$ is a proper subset of $f^j(U_1)$. Note that $V_- = U_1 \setminus f(U_1)$. Since $f^j(V_-) = f^j(U_1) \setminus f^{j+1}(U_1)$ for all $j \in \mathbb{Z}$, we obtain that the sets $f^j(V_-)$, $j \in \mathbb{Z}$ are disjoint from one another. The set $f^j(V_-)$ is contained in U_1 for $j \geq 0$ and disjoint from U_1 for $j < 0$. Since $\mathrm{supp}(f) \in U_1 \cup U_2$, we have $f^j(V_-) \subset U_2 \setminus U_1$ for $j < 0$. Consider a generalized 2-cycle $g = \delta_{f(V_-);f^{-2}} = \mu[V_-; f, f^{-1}; (1\,2)]$. The map g is well defined and belongs to the ample group $\mathcal{G}$. Since $\mathrm{supp}(g) = f(V_-) \cup f^{-1}(V_-)$, this map is in $\mathcal{G}_{U_1 \cup U_2}$ and so is the map $h = gf$. By construction, $h(V_-) = f^{-1}(V_-)$ and $h = f$ on the set $U_1 \setminus V_-$. Furthermore, $h^{-1}(f(V_-)) = f^{-2}(V_-)$ and $h^{-1} = f^{-1}$ on $U_1 \setminus f(V_-)$. It follows that $\{x \in U_1 \mid h(x) \notin U_1\} = V_-$ and $\{x \in U_1 \mid h^{-1}(x) \notin U_1\} = V_- \cup f(V_-)$. Observe that both of the latter sets are different from $\emptyset$ and U_1. Therefore, the map h has been covered by the previous case. By the above, $h \in \langle \mathcal{G}_{U_1} \cup \mathcal{G}_{U_2} \rangle$. Since $g \in \mathsf{S}(\mathcal{G}_{U_1 \cup U_2})$, we obtain that the map $f = g^{-1}h$ belongs to $\langle \mathcal{G}_{U_1} \cup \mathcal{G}_{U_2} \rangle$ as well. □

PROPOSITION 8.6. *For an ample group acting minimally on a Cantor set, any of Properties UR, NC, SC, UC, CI, and HS implies Property E.*

Proof. By Lemma 5.4, Property UR is equivalent to Property NC. It follows from Lemma 5.6 that for an ample group acting minimally on a Cantor set, Property SC is equivalent to Property UC. For an ample group, either of Properties NC and UC implies Property CI due to Lemmas 5.10 and 5.11. By Lemma 5.16, Property HS implies Property CI as well. Finally, by Lemma 8.5, Property CI implies Property E for any ample group acting minimally. □

Now, equipped with Proposition 8.6, we are going to provide more examples of ample groups with Property E.

Definition 8.7. A homeomorphism $f : X \to X$ of a Cantor set X is called *minimal* if there is no closed set $Y \subset X$ different from the empty set and X that is invariant under f: $f(Y) \subset Y$.

An equivalent condition is that for any point $x \in X$, the *forward orbit* $x, f(x), f^2(x), \ldots$ is dense in X. It is easy to show that the homeomorphism f is minimal if and only if the cyclic group $\langle f \rangle$ generated by f acts minimally on X (in the sense of Definition 5.1).

There are many known examples of minimal homeomorphisms of Cantor sets (they are referred to as *Cantor minimal systems*).

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THEOREM 8.8. *For any minimal homeomorphism f of a Cantor set, the ample group $\mathsf{F}(\langle f\rangle)$ has Property E.*

It is not hard to show that the cyclic group $\langle f\rangle$ has Property UR (and the equivalent Property NC). However, it is not clear whether the ample group $\mathsf{F}(\langle f\rangle)$ has these properties. In general, it is not clear whether Property NC survives when the group is amplified.

To prove Theorem 8.8, we are going to use some advanced results of Matui. In [**46**], Matui introduced the notion of an *almost finite* groupoid (see [**46**, Definition 6.2]). Later, Kerr [**38**] defined almost finiteness for group actions (see [**38**, Definition 8.2]). Any continuous action of a countable group on a Cantor set gives rise to a groupoid. In this particular case, both notions are equivalent (see [**38**, Example 8.5]).

THEOREM 8.9. (Matui [**46**]) *Let G be a group of homeomorphisms of a Cantor set X. If the natural action of the group G on X is minimal and almost finite, then the ample group $\mathsf{F}(G)$ has Property HS.*

Note that Matui's result [**46**, Theorem 6.12] is formulated in terms of groupoids. Theorem 8.9 is a reformulation of it in a particular case where the groupoid is associated to a group action.

PROPOSITION 8.10. (Matui [**46**]) *Any free action of the group $\mathbb{Z}^n$ on a Cantor set is almost finite.*

Proposition 8.10 is a less general version of [**46**, Lemma 6.3].

Proof of Theorem 8.8. Suppose f is a minimal homeomorphism of a Cantor set X. Minimality implies that f has no periodic points. Hence, the natural action of the cyclic group $\langle f\rangle$ on X is free. Since $\langle f\rangle$ is isomorphic to $\mathbb{Z}$, that action is almost finite due to Proposition 8.10. In addition, the action of $\langle f\rangle$ is minimal. Then, it follows from Theorem 8.9 that the ample group $\mathsf{F}(\langle f\rangle)$ has Property HS. Since the group $\langle f\rangle$ acts minimally on X, so does its amplification. It remains to apply Proposition 8.6. □

The notion of almost finiteness has been extensively studied. It is expected that any free minimal action of a countable amenable group on the Cantor set is almost finite (see [**38**, Problem 8.9]). Let us mention a recent result of Kerr and Naryshkin [**39**] in that direction.

THEOREM 8.11. (Kerr, Naryshkin [**39**]) *Every free continuous action of a countable elementary amenable group on a finite-dimensional compact metrizable space is almost finite.*

In view of Theorem 8.11 [**39**, Theorem A], the cyclic group $\langle f\rangle$ in Theorem 8.8 can be replaced by any countable elementary amenable group $G \subset \mathrm{Homeo}(X)$ such that the natural action of G on the Cantor set X is minimal and free.

There is an alternative way to prove Theorem 8.8 using more elementary means. Let us outline it. Suppose f is a minimal homeomorphism of a Cantor set X. For any point $x \in X$, consider its forward orbit $\{f^k(x) \mid k \geq 0\}$ with respect to f and the stabilizer of this orbit

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under the natural action of the ample group $\mathcal{G} = \mathsf{F}(\langle f \rangle)$. By Lemma 3.3, the stabilizer $\mathrm{St}_{\mathcal{G}}(\{f^k(x) \mid k \geq 0\})$ is also an ample group.

PROPOSITION 8.12. (Putnam [54]) *The stabilizer* $\mathrm{St}_{\mathcal{G}}(\{f^k(x) \mid k \geq 0\})$ *is a locally finite group.*

Proposition 8.12 can be extracted from the results of [54, §5] (as asserted in [23]). Note that the group $\mathsf{F}(\langle f \rangle)$ appears only implicitly in [54] as everything is formulated in terms of a C^*-algebra associated with the minimal homeomorphism f. A more general fact is explicitly formulated and proved by Juschenko and Monod (see [37, Lemmas 4.1 and 4.2]).

PROPOSITION 8.13. *The action of the subgroup* $\mathrm{St}_{\mathcal{G}}(\{f^k(x) \mid k \geq 0\})$ *on clopen subsets of the Cantor set X has the same orbits as the action of the entire group* $\mathcal{G}$.

Assuming Proposition 8.13 is proved, the proof of Theorem 8.8 goes as follows. Since the stabilizer $\mathrm{St}_{\mathcal{G}}(\{f^k(x) \mid k \geq 0\})$ is locally finite, every element has finite order. Hence, this group obviously has Property UR. Also, it has Property NC (this is easy to check directly or we can use Lemma 5.4). As the stabilizer is an ample group, Lemma 5.10 implies that it has Property CI. Then, it follows from Proposition 8.13 that the group $\mathcal{G}$ has Property CI as well. It remains to apply Proposition 8.6.

We do not include the proof of Proposition 8.13 as it is quite lengthy. It will be published elsewhere.

There are also completely different kinds of ample groups with Property E where that property holds since the groups allow strong contraction of clopen sets (Property SC). First of all, let us mention Thompson's group V (already considered above) as well as its various generalizations including the Higman–Thompson groups $V_{n,r}$, also denoted $G_{n,r}$ (see, e.g., [12]), the Brin–Thompson groups nV (see [14]), and others. Each of those groups can be realized as a group of homeomorphisms of a Cantor set that is easily shown to be ample. Some of the groups are not simple so that the argument in Remark 8.4 may not apply to them. However, it is not hard to prove that any of the groups $V_{n,r}$ and nV acts transitively on the set of all non-trivial clopen subsets of the respective Cantor set. This immediately implies minimality and Properties SC, UC, CI, and HS.

Another class of ample groups with Property E is the (topological) *full groups* of minimal, purely infinite étale groupoids considered by Matui in [47, 48]. The full group is ample by construction. Minimality of the groupoid translates into minimality of the natural action of the group, while pure infiniteness translates into Property SC. For example, a purely infinite groupoid can be associated to any one-sided subshift of finite type over a finite alphabet (see [47, §6]). Note that the ample group obtained from the groupoid associated to the full shift on n symbols is isomorphic to the Higman–Thompson group $V_{n,1}$.

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