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Exponential sums over Möbius convolutions with applications to partitions

Debmalya Basak, Nicolas Robles, and Alexandru Zaharescu

Abstract. We establish bounds for exponential sums twisted by generalized Möbius functions and their convolutions. As an application, we prove asymptotic formulas for certain weighted chromatic partitions by using the Hardy–Littlewood circle method. Lastly, we provide an explicit formula relating the contributions from the major arcs with a sum over the zeros of the Riemann zeta-function.

1 Introduction

1.1 Motivation

A central problem in number theory is to estimate exponential sums of the form

$$\sum_{n\leqslant X} f(n)\,\mathrm{e}(n\alpha),$$

where $f(n)$ is an arithmetic function, $\mathrm{e}(x) = \exp(2\pi i x)$, and $\alpha \in \mathbb{R}$. Typically, such sums exhibit intricate oscillatory behavior that can be exploited to derive significant arithmetic results. A prominent example in this context is Vinogradov's famous estimate for exponential sums over primes. Let $\alpha \in \mathbb{R}$ and consider a reduced fraction a/q such that $|\alpha - a/q| \leqslant 1/q^2$. Expanding upon the work of Vinogradov [34], Vaughan [30] showed that for all $X \geqslant 2$, we have

$$\sum_{n\leqslant X} \Lambda(n)\,\mathrm{e}(n\alpha) \ll \left(X^{4/5} + \frac{X}{q^{1/2}} + X^{1/2}q^{1/2}\right)(\log X)^{5/2}, \tag{1.1}$$

where $\Lambda(n)$ is the Von-Mangoldt function. Indeed, the estimate (1.1) is crucial in bounding minor arc contributions in the Hardy–Littlewood circle method corresponding to the ternary Goldbach problem.

Our primary aim in this paper is to prove results on exponential sums involving generalized Möbius functions and their convolutions. As an application, we establish asymptotic formulas for certain weighted chromatic partitions by using the Hardy–Littlewood circle method.

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1.2 Main results

Consider the Möbius function denoted by μ, which is a multiplicative function defined in the following way: for each prime p,

$$\mu(p^k) := \begin{cases} -1, & k = 1 \\ 0, & k \geqslant 2. \end{cases}$$

By definition, $\mu^2(n) = 1$ if n is squarefree and $\mu^2(n) = 0$ otherwise. Let us write

$$S_\mu(X, \alpha) = \sum_{n \leqslant X} \mu(n)\, \mathrm{e}(n\alpha).$$

Then, we have the following result.

Theorem 1.1 *Let $\alpha \in \mathbb{R}$ and consider a reduced fraction a/q such that $|\alpha - a/q| \leqslant 1/q^2$. Then, for $X \geqslant 2$ and for any fixed $\varepsilon > 0$, we have*

$$S_\mu(X, \alpha) \ll_\varepsilon X^{4/5+\varepsilon} + \frac{X(\log X)^3}{q^{1/2}} + X^{1/2} q^{1/2} (\log X)^3.$$

More generally, for any positive integer k, we define the Popovici-Möbius function (see [28, §2.2] as well as [4] and [19, 26] for related applications) with parameter k by

$$\mu_k(n) := (\mu * \mu * \cdots * \mu)(n), \tag{1.2}$$

where the Dirichlet convolution is performed k times. The Dirichlet series for $\mu_k(n)$ is given by

$$\sum_{n=1}^{\infty} \frac{\mu_k(n)}{n^s} = \frac{1}{\zeta(s)^k},$$

for $\mathrm{Re}(s) > 1$. Let

$$S_{\mu_k}(X, \alpha) = \sum_{n \leqslant X} \mu_k(n)\, \mathrm{e}(n\alpha).$$

We prove the following theorem.

Theorem 1.2 *Let $\alpha \in \mathbb{R}$ and consider a reduced fraction a/q such that $|\alpha - a/q| \leqslant 1/q^2$. Consider the sequences $\{a_m\}, \{b_m\}$, and $\{c_m\}$ given by*

$$a_1 = \frac{4}{5} \quad \textit{and} \quad a_{m+1} = \frac{4 - a_m}{5 - 2a_m} \quad \textit{for} \quad m = 1, 2, \ldots,$$

$$b_1 = \frac{1}{2} \quad \textit{and} \quad b_{m+1} = \frac{b_m}{3} \quad \textit{for} \quad m = 1, 2, \ldots,$$

$$\textit{as well as} \quad c_1 = \frac{1}{2} \quad \textit{and} \quad c_{m+1} = \frac{4 - c_m}{5 - 2c_m} \quad \textit{for} \quad m = 1, 2, \ldots.$$

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Then, for $X \geqslant 2$, $k \geqslant 2$ and any fixed $\varepsilon > 0$, we have

$$S_{\mu_k}(X,\alpha) \ll_\varepsilon X^{a_k+\varepsilon} + \frac{X(\log X)^{k^2}}{q^{b_k}} + X^{c_k} q^{1-c_k} (\log X)^{k^2}.$$

We also establish results for exponential sums involving Dirichlet convolutions of the Möbius function with the characteristic function of squarefull and squarefree integers, respectively. To this end, we define the arithmetic function $\hat{\mu}$ by

$$\hat{\mu}(n) := (\mu * f)(n),$$

where f denotes the indicator function for the squarefull numbers. Writing

$$S_{\hat{\mu}}(X,\alpha) = \sum_{n \leqslant X} \hat{\mu}(n)\, \mathrm{e}(n\alpha),$$

we are able to prove the following estimate.

Theorem 1.3 *Let $\alpha \in \mathbb{R}$ and consider a reduced fraction a/q such that $|\alpha - a/q| \leqslant 1/q^2$. Then, for $X \geqslant 2$, we have*

$$S_{\hat{\mu}}(X,\alpha) \ll \left(X^{5/6} + q^{1/2} X^{1/2} + \frac{X}{q^{1/2}} \right) (\log X)^{5/2}.$$

Considering the convolution of the Möbius function with the characteristic function of squarefree integers, we define $\tilde{\mu}_k(n)$ in terms of coefficients of the following Dirichlet series. Let $k \geqslant 1$ and

$$\sum_{n=1}^{\infty} \frac{\tilde{\mu}_k(n)}{n^s} = \frac{1}{\zeta(ks)},$$

for $\mathrm{Re}(s) > 1/k$. As an example, for $k = 2$, one has $\tilde{\mu}_2(n) = (\mu * \mu^2)(n)$, where μ^2 is the indicator function for the squarefrees. Our next result provides an upper bound for the exponential sum

$$S_{\tilde{\mu}_2}(X,\alpha) = \sum_{n \leqslant X} \tilde{\mu}_2(n)\, \mathrm{e}(n\alpha).$$

Theorem 1.4 *Let $\alpha \in \mathbb{R}$ and consider a reduced fraction a/q such that $|\alpha - a/q| \leqslant 1/q^2$. Then, for $X \geqslant 2$, we have*

$$S_{\tilde{\mu}_2}(X,\alpha) \ll \left(X^{23/28} + \frac{X}{q^{1/4}} + X^{3/4} q^{1/4} \right) (\log X)^{10}.$$

Remark 1.1 One can apply the method from the proof of Theorem 1.2 to generalize the result in Theorem 1.4 to $\tilde{\mu}_k$ for any $k \geqslant 1$.

1.3 Applications to partitions

A partition of a positive integer n is a nondecreasing sequence of positive integers whose sum is equal to n. The partition function $p(n)$, which represents the number

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of partitions of n, was first studied by Hardy and Ramanujan [13] in 1918 by applying the Hardy–Littlewood circle method. Their result states that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \quad \text{as} \quad n \to \infty.$$

As an application of our results on exponential sums from Section 1.2, we consider a question about partitions of a positive integer that has a strong arithmetic flavor. To be precise, we look at the parts that appear in the partition, and more specifically, we consider the prime factorization of the parts. Let B (blue) and R (red) denote the set of squarefree numbers that have an even and, respectively, an odd number of prime factors. For the convenience of the reader, we use colors to show parity of these partitions. Thus, we write

$$B = \{1, 6, 10, 14, 15, 21, 22, 26, \ldots\},$$
$$R = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 30, \ldots\}.$$

We note in passing that the sets B and R are asymptotically just as numerous. Actually, it is expected that as x tends to infinity, the number of elements of B less than x differs from the number of elements of R less than x by at most $O_\varepsilon(x^{1/2+\varepsilon})$ for any $\varepsilon > 0$. As is well-known, this is equivalent to the Riemann hypothesis.

Definition 1.1 We call a partition of a positive integer n *admissible* if each part is either blue or red and each red part appears at most once. We say that an admissible partition is even (respectively odd) if the number of red parts used in the partitions is even (and respectively odd).

For example, there are exactly 15 admissible partitions in the case when $n = 10$. The partition

$$10 = 6 + 2 + 2$$

is not admissible because the red 2 appears twice. We see in Table 1.3 that there are 8 even admissible partitions and 7 odd admissible partitions in this situation. This naturally leads to the following question.

Question *Is the number of even admissible partitions of n asymptotic to the number of odd admissible partitions of n as n tends to infinity?*

We will address the above question as well as a more general version of this problem. To proceed, we start by considering the partition function $p_w(n)$ associated to a generating function of the form,

$$\Psi_w(z) = \sum_{n=1}^{\infty} p_w(n) z^n = \prod_{n \in \mathbb{N}} (1 - z^n)^{-w(n)}, \quad |z| < 1, \tag{1.3}$$

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Case for $n = 10$	Admissible partitions
Odd admissible partitions	$10 = 7+1+1+1$
	$10 = 6+3+1$
	$10 = 6+2+1+1$
	$10 = 5+3+2$
	$10 = 5+1+1+1+1+1$
	$10 = 3+1+1+1+1+1+1+1$
	$10 = 2+1+1+1+1+1+1+1+1$
Even admissible partitions	$10 = 1+1+1+1+1+1+1+1+1+1$
	$10 = 3+2+1+1+1+1+1$
	$10 = 5+2+1+1+1$
	$10 = 5+3+1+1$
	$10 = 6+1+1+1+1$
	$10 = 7+2+1$
	$10 = 7+3$
	$10 = 10$

where $w(n) \in \mathbb{Z}$ is the weight placed on each element $n \in \mathbb{N}$. We refer to such partitions as *weighted* partitions. Our cases of interest will have weights given by the Möbius function (i.e., $w(n) = \mu(n)$) or more generally by convolutions of the Möbius function (i.e., $w(n) = \mu_k(n)$, where $\mu_k(n)$ is defined by (1.2)).

In [32], Vaughan established an asymptotic formula for the number of partitions of a number n into primes when n is large by applying the Hardy-Littlewood circle method. In [33], Vaughan then extended this approach to study partition asymptotics regarding squares and other additive questions.

Since the publication of [32, 33], there has been considerable interest in the study of restricted or weighted partitions over arithmetically interesting sets. In [11], Gafni showed an asymptotic formula for the number of partitions of n into k-th powers, and in [12], Gafni proved results for partitions into prime powers. Berndt, Malik, and the third author [2] studied partitions into k-th powers in arithmetic progressions. Dunn and the second author [10] then considered the more general case where the parts are values of an arbitrary polynomial f of degree $d \geq 2$. More recently, Das, Berndt, Zeindler, and the second and third authors [3, 8, 27] studied partitions associated to generalized divisor functions, partitions into semiprimes and almost primes. General strategies for dealing with asymptotic weighted partitions have been established by Meinardus [21] and more recently by Debryune and Tenenbaum [9]. Notice that in the case of the above-mentioned results, the corresponding partitions have nonnegative weights, unlike our cases of interest where the weights take positive and negative values (see Figure 1.2). In our cases, one should be prepared in principle to have a main result which is an upper bound rather than an asymptotic formula. This indeed happens in our case, see Theorem 1.6 below.

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Very recently, in [5, 6], Daniels investigated partitions whose generating functions are given by

$$\Psi(z,f) = 1 + \sum_{n=1} p(n,f)z^n = \prod_{n=1}^{\infty}(1 - f(n)z^n)^{-1}, \quad |z| < 1. \tag{1.4}$$

Daniels refers to such partitions as *signed* partitions. In particular, he studied the case when $f(n) = \mu(n)$. Note the difference between placing $\mu(n)$ in the exponent of the factor $(1 - z^n)^{-\mu(n)}$ in (1.3) as opposed to the presence of $\mu(n)$ inside the factor $(1 - \mu(n)z^n)^{-1}$ in (1.4). This is a subtle difference, which surprisingly leads to the existence of main terms in Daniel's works and the non-existence of main terms in our context. For comparison, the reader is referred to Theorem 1.1 in [5] and Theorem 1.6 below. We remark that Theorem 1.6 below does lead to an asymptotic formula connecting the number of odd and even admissible partitions as in Definition 1.1.

It is straightforward to see that in the particular case when $w(n) = \mu(n)$, the partition function $p_\mu(n)$ is exactly the difference between the number of even admissible partitions and odd admissible partitions of n as in Definition 1.1. More precisely, suppose we denote by $\mathcal{A}(n)$ the number of admissible partitions of n. Additionally, let $\mathcal{E}(n)$ and $\mathcal{O}(n)$ be the number of even and odd admissible partitions of n, respectively. Then,

$$p_\mu(n) = \mathcal{E}(n) - \mathcal{O}(n), \tag{1.5}$$

$$\text{and} \quad \mathcal{A}(n) = \mathcal{E}(n) + \mathcal{O}(n). \tag{1.6}$$

Therefore, in order to show that the number of even admissible partitions is asymptotic to the number of odd admissible partitions as n tends to infinity, our goal in what follows is to provide an upper bound for their difference (i.e., for $p_\mu(n)$). We begin by answering the question raised previously in this section.

Theorem 1.5 *The number of odd admissible partitions of n is asymptotic to the number of even admissible partitions of n, as n tends to infinity.*

More generally, for any positive integer k, let $p_{\mu_k}(n)$ be the number of partitions of n weighted by $\mu_k(n)$; that is,

$$\sum_{n=1}^{\infty} p_{\mu_k}(n)z^n = \prod_{n=1}^{\infty}(1 - z^n)^{-\mu_k(n)},$$

where $\mu_k(n)$ is defined by (1.2). For simplicity, we will focus only on the case when $k = 2$. This is more involved than the case when $k = 1$ and already demonstrates all the ingredients necessary for a general k. For $k = 2$, we prove the following result.

Theorem 1.6 *For any fixed $B > 0$ and for all $n \geqslant 2$, we have*

$$\log p_{\mu*\mu}(n) = O_B\left(\frac{\sqrt{n}}{(\log n)^B}\right). \tag{1.7}$$

Remark 1.2 As we shall see in the proof of Theorem 1.6, the method can be generalized to any $k \geqslant 1$, with the same upper bound as in (1.7). In particular, when $k = 1$, we have

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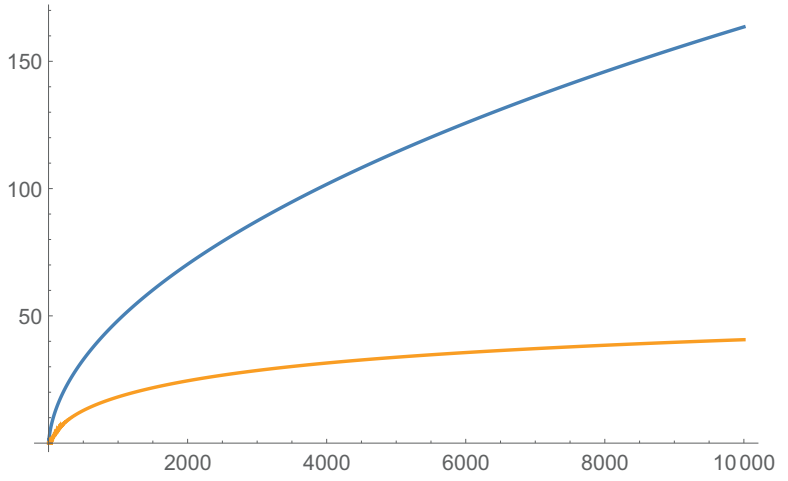


Figure 1.1: Plot of $\log \mathcal{A}(n)$ in blue and $\log p_\mu(n)$ in orange for $1 \leqslant n \leqslant 10000$.

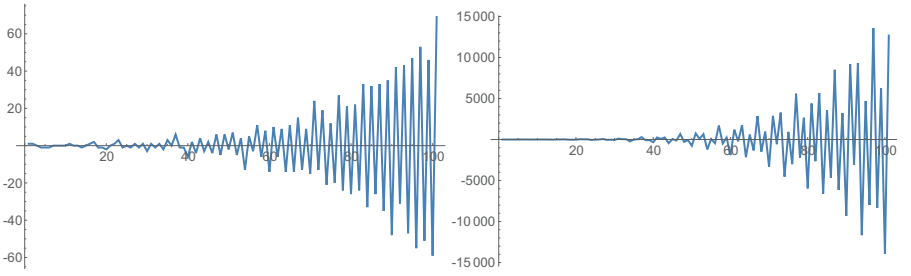


Figure 1.2: Left: Plot of partition function $p_w(n)$ with $w(n) = \mu(n)$, Right: Plot of partition function $p_w(n)$ with $w(n) = (\mu * \mu)(n)$.

$$\log p_\mu(n) = O_B\left(\frac{\sqrt{n}}{(\log n)^B}\right), \tag{1.8}$$

for any fixed $B > 0$ and all $n \geqslant 2$. Theorem 1.5 is an immediate consequence of (1.5) and (1.8). See Figure 1.1 for a plot of the logarithms of $\mathcal{A}(n)$ and $p_\mu(n)$.

Remark 1.3 It would be interesting to see what the analogues of Theorems 1.5 and 1.6 are if the Möbius function $\mu(n)$ is replaced by the Liouville function $\lambda(n)$, where

$$\lambda(n) = \prod_{i=1}^{k}(-1)^{a_i}, \quad \text{where} \quad n = \prod_{i=1}^{k} p_i^{a_i}.$$

In order to prove Theorem 1.6, we will employ the Hardy-Littlewood method. As we shall see in Section 8, the contribution from the major arcs is closely related to the

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zeros of the Riemann zeta-function $\zeta(s)$. To be precise, following (1.3), let us write

$$\Psi_{\mu*\mu}(z) = \sum_{n=1}^{\infty} p_{\mu*\mu}(n) z^n = \prod_{n=1}^{\infty} (1-z^n)^{-(\mu*\mu)(n)}, \quad |z| < 1. \tag{1.9}$$

Set

$$\Phi_{\mu*\mu}(z) := \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\mu*\mu)(n)}{j} z^{jn}, \quad |z| < 1. \tag{1.10}$$

Then, $\Psi_{\mu*\mu}(z) = \exp(\Phi_{\mu*\mu}(z))$. Let $X \geqslant 1$ and $\rho = \exp(-1/X)$. Choose $z = \rho\, \mathrm{e}(\theta)$ in (1.10) and let

$$\Delta := \frac{1}{(1+4\pi^2 X^2 \theta^2)^{1/2}}.$$

Then we have the following theorem.

Theorem 1.7 *Suppose that $\theta \in \mathbb{R}$ and $X \geqslant 1$. If $X\Delta^3 \geqslant 1$, then there exists a sequence T_v with $v \leqslant T_v \leqslant 2v$ such that*

$$\begin{aligned} \Phi_{\mu*\mu}(\rho\, \mathrm{e}(\theta)) &= 4 \log \frac{X}{1-2\pi i X\theta} - 8\log(2\pi) + 72 \frac{1-2\pi i X\theta}{X} \\ &\quad + \lim_{v\to\infty} \sum_{|\operatorname{Im}(\varpi)|<T_v} f(X,\theta,\varpi) + \sum_{n=1}^{\infty} g(X,\theta,n), \end{aligned} \tag{1.11}$$

where the functions f and g in (1.11) are given by

$$f(X,\theta,\varpi) := \left(\frac{X}{1-2\pi i X\theta}\right)^{\varpi} \frac{\zeta(1+\varpi)\Gamma(\varpi)}{\zeta'(\varpi)^2} \left(\log \frac{X}{1-2\pi i X\theta} + \psi(\varpi) - \frac{\zeta''}{\zeta'}(\varpi) + \frac{\zeta'}{\zeta}(1+\varpi) \right),$$

$$g(X,\theta,n) := \left(\frac{X}{1-2\pi i X\theta}\right)^{-2n} \left[\mathfrak{c}_1(n) \left(\log \frac{X}{1-2\pi i X\theta} \right)^2 + \mathfrak{c}_2(n) \log \frac{X}{1-2\pi i X\theta} + \mathfrak{c}_3(n) \right].$$

The coefficients $\mathfrak{c}_1(n)$, $\mathfrak{c}_2(n)$, and $\mathfrak{c}_3(n)$ are defined in (8.3), (8.4), and (8.5), respectively. The sum is taken over the nontrivial zeros ϖ of $\zeta(s)$ under the assumption, for notational ease, of simplicity.

Remark 1.4 As Titchmarsh puts it in [29, §14.27], "obvious modifications are required if" the zeros are not simple. Indeed, this assumption does not represent a technical difficulty, and it can be relaxed at the expense of cluttering the expression $f(X,\theta,\varpi)$ for the nontrivial zeros.

Remark 1.5 Interestingly, (1.11) is an instance of an explicit formula in the theory of partitions. It is reminiscent of the classical explicit formula

$$\psi_0(x) = \frac{1}{2} \lim_{\delta\to 0} (\psi(x+\delta) - \psi(x-\delta)) = x - \lim_{T\to\infty} \sum_{|\operatorname{Im}(\varpi)|<T} \frac{x^{\varpi}}{\varpi} - \log(2\pi) + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n}.$$

Here, ϖ runs over the nontrivial zeros of $\zeta(s)$ and

$$\psi(x) = \sum_{p^k \leqslant x} \log p, \tag{1.12}$$

where the sum in (1.12) is over prime powers.

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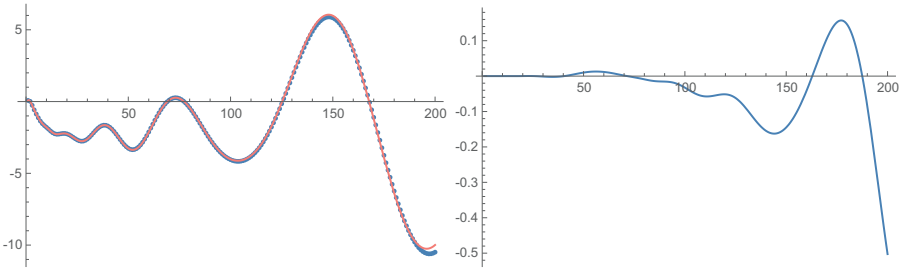


Figure 1.3: Left: Plot of $\mathrm{Re}(\Phi_1(X,\theta,120,800))$ in blue and $\mathrm{Re}(\Phi_2(X,\theta,20,10))$ in pink. Right: Plot of $\mathrm{Re}(\Phi_1(X,\theta,120,800)) - \mathrm{Re}(\Phi_2(X,\theta,20,10))$.

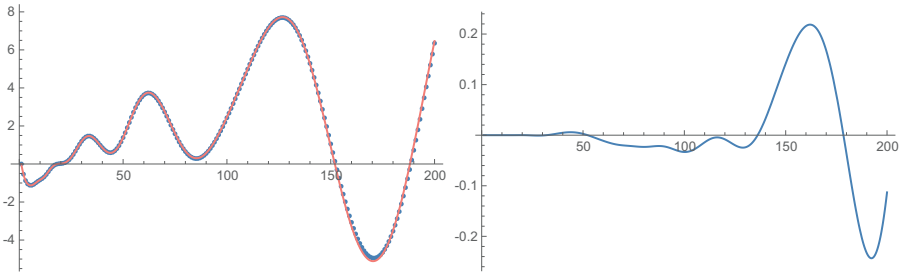


Figure 1.4: Left: Plot of $\mathrm{Im}(\Phi_1(X,\theta,120,800))$ in blue and $\mathrm{Im}(\Phi_2(X,\theta,20,10))$ in pink. Right: Plot of $\mathrm{Im}(\Phi_1(X,\theta,120,800)) - \mathrm{Im}(\Phi_2(X,\theta,20,10))$.

Remark 1.6 Consider the truncated arithmetic sum

$$\Phi_1 := \Phi_1(X,\theta,J,N) = \sum_{j=1}^{J}\sum_{n=1}^{N}\frac{(\mu*\mu)(n)}{j}\exp\left(-jn\left(\frac{1}{X}-2\pi i\theta\right)\right)$$

and its analytic truncated counterpart

$$\Phi_2 := \Phi_2(X,\theta,T,N) = 4\log\frac{X}{1-2\pi iX\theta} - 8\log(2\pi) + 72\frac{1-2\pi iX\theta}{X} + \sum_{|\mathrm{Im}(\omega)|<T} f(X,\theta,\omega) + \sum_{n=1}^{N} g(X,\theta,n).$$

We set $\theta = \theta(X)$ as

$$\theta(X) := \frac{1}{2\pi}\sqrt{\frac{1}{X^{4/3}} - \frac{1}{X^2}},$$

such that $X\Delta^3 = 1$. In Figures 1.3 and 1.4, we numerically illustrate our result from Theorem 1.7 in terms of the functions Φ_1 and Φ_2.

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1.4 Structure of the paper

The paper is organized as follows. In Section 2, we introduce some preliminary lemmas, such as Type I and Type II estimates on exponential sums involving Dirichlet convolutions, a result on the distribution of the Möbius function in arithmetic progressions, a divisor sum estimate, and an unconditional upper bound on $1/\zeta(s)$ in the critical strip along specific vertical line segments. Section 3 is devoted to the proofs of Theorems 1.1, 1.3, and 1.4. The proof of Theorem 1.2 is singled out in Section 4. In Section 5, we move on to the theory of partitions and set up the proof of Theorem 1.6. Section 6 and Section 7 will contain the details of the major and minor arc estimates, respectively. Finally, in 8, we present the proof of Theorem 1.7.

2 Preliminary Lemmas

In this section, we start by establishing some standard notations. Then, we collect some necessary results from the literature and proceed to prove some preliminary lemmas.

2.1 Notations

We employ some standard notation that will be used throughout the paper.

- Throughout the paper, the expressions $f(X) = O(g(X))$, $f(X) \ll g(X)$, and $g(X) \gg f(X)$ are equivalent to the statement that $|f(X)| \leq C|g(X)|$ for all sufficiently large X, where $C > 0$ is an absolute constant. A subscript of the form $\ll_\alpha$ means the implied constant may depend on the parameter α. Dependence on several parameters is indicated in an analogous manner, as in $\ll_{\alpha,\lambda}$.
- The notation $f = o(g)$ as $x \to a$ means that $\lim_{x\to a} f(x)/g(x) = 0$ and $f \sim g$ as $x \to a$ denotes $\lim_{x\to a} f(x)/g(x) = 1$.
- We denote the set of real numbers by $\mathbb{R}$, the set of rational numbers by $\mathbb{Q}$, the set of integers by $\mathbb{Z}$, the set of natural numbers by $\mathbb{N}$, and the set of complex numbers by $\mathbb{C}$.
- For any set $\mathcal{A}$, $\mathrm{card}(\mathcal{A})$ denotes the cardinality of the set $\mathcal{A}$.
- The notation $\mathrm{e}(x)$ stands for $\exp(2\pi i x)$.
- Dyadic sums are represented by $\sum_{n\sim N} f(n) = \sum_{N<n\leqslant 2N} f(n)$.
- The divisor function is denoted by $\tau(n)$. The k-fold divisor function $\tau_k(n)$ is defined by the coefficients of the Dirichlet series

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{\tau_k(n)}{n^s}, \quad \text{for } \mathrm{Re}(s) > 1.$$

- The Möbius function is denoted by $\mu(n)$, whereas the Popovici-Möbius function will be denoted by $\mu_k(n) = (\mu * \mu * \cdots * \mu)(n)$, the convolution performed k times.
- We denote by $\hat{\mu}$ the convolution of the Möbius function with the characteristic function of the squarefull numbers. More precisely, we write

$$\hat{\mu}(n) := (\mu * f)(n),$$

where f denotes the indicator function for the squarefull numbers.

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- The arithmetic function $\tilde{\mu}_k(n)$ is defined by the coefficients of the Dirichlet series

$$\sum_{n\geqslant 1}\frac{\tilde{\mu}_k(n)}{n^s}=\frac{1}{\zeta(ks)}$$

for $\mathrm{Re}(s)>\frac{1}{k}$.

2.2 Exponential sums and Dirichlet convolutions

We first state the following two lemmas, due to Koukoulopoulos, that provide bounds on exponential sums involving Dirichlet convolutions.

Lemma 2.1 (Type I estimate) *Let $f:\mathbb{N}\to\mathbb{C}$ be supported on $[1,y]$, $v\geqslant 2$, $X\geqslant 2$, $\alpha\in\mathbb{R}$ and a/q be a reduced fraction such that $|\alpha-a/q|\leqslant 1/q^2$. Then,*

$$\sum_{n\leqslant X}(f*\log^v)(n)\,\mathrm{e}(n\alpha)\ll\left(y+\frac{X}{q}+q\right)(\log X)^{v+1}\|f\|_\infty.$$

Proof See [16, Theorem 23.5]. ∎

Lemma 2.2 (Type II estimate) *Let $f,g:\mathbb{N}\to\mathbb{C}$ be two arithmetic functions supported on $[1,y]$ and $[1,z]$, respectively. Let $X\geqslant 2$, $\alpha\in\mathbb{R}$ and consider a reduced fraction a/q such that $|\alpha-a/q|\leqslant 1/q^2$. Then,*

$$\sum_{n\leqslant X}(f*g)(n)\,\mathrm{e}(n\alpha)\ll\left(q+y+z+\frac{yz}{q}\right)^{1/2}\sqrt{\log 2q}\cdot\|f\|_2\|g\|_2.$$

Proof See [16, Theorem 23.6]. ∎

We would like to establish more general versions of Lemmas 2.1 and 2.2, in the case when $\alpha\in\mathbb{R}$ and there exists some reduced fraction a/q such that $|\alpha-a/q|\leqslant\gamma/q^2$, for some $\gamma\geqslant 1$. In this regard, we first prove the following proposition.

Proposition 2.1 *Let $\alpha\in\mathbb{R}$ and consider a reduced fraction a/q such that $|\alpha-a/q|\leqslant\gamma/q^2$ for some $\gamma\geqslant 1$. Then, for each $m_1\in\{1,2,\dots,q\}$, there exists at most 14γ choices of $m_2\in\{1,2,\dots,q\}$ such that*

$$\left|\|m_1\alpha\|-\|m_2\alpha\|\right|<\frac{1}{q},\tag{2.1}$$

where $\|x\|$ denotes the smallest distance of x to an integer.

Proof Fix $m_1\in\{1,2,\dots,q\}$ and suppose $m_2\in\{1,2\dots,q\}$ such that (2.1) is satisfied. Writing $\alpha=a/q+\beta$ with $|\beta|\leqslant\gamma/q^2$, we have

$$m_1\alpha=\frac{m_1a}{q}+m_1\beta\quad\text{and}\quad m_2\alpha=\frac{m_2a}{q}+m_2\beta.$$

Therefore, we obtain

$$\left|\|m_1\alpha\|-\left\|\frac{m_1a}{q}\right\|\right|\leqslant|m_1\beta|\leqslant\frac{\gamma}{q}\quad\text{and}\quad\left|\|m_2\alpha\|-\left\|\frac{m_2a}{q}\right\|\right|\leqslant|m_2\beta|\leqslant\frac{\gamma}{q}.\tag{2.2}$$

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From (2.1) and (2.2), it follows that

$$\left|\left\|\frac{m_1 a}{q}\right\| - \left\|\frac{m_2 a}{q}\right\|\right| \leqslant \frac{2y+1}{q} \leqslant \frac{3y}{q}. \tag{2.3}$$

Next, we write

$$\left\|\frac{m_1 a}{q}\right\| = \frac{\ell_1}{q} \quad \text{and} \quad \left\|\frac{m_2 a}{q}\right\| = \frac{\ell_2}{q},$$

where $\ell_1, \ell_2 \in \{0, 1, \dots, \lfloor q/2 \rfloor\}$. Note that when m_1 is fixed, ℓ_1 is fixed. Also, for each choice of ℓ_2, there are at most two different choices of m_2. Hence, in order to satisfy (2.3), there are at most $7y$ choices of ℓ_2. This implies there are at most $14y$ choices of m_2 which satisfy (2.1). ∎

With Proposition 2.1 in hand, we have the following two lemmas.

Lemma 2.3 *Let $f: \mathbb{N} \to \mathbb{C}$ be supported on $[1, y]$, $\nu \geqslant 2$, $X \geqslant 2$, $\alpha \in \mathbb{R}$ and a/q be a reduced fraction such that $|\alpha - a/q| \leqslant \gamma/q^2$ for some $\gamma \geqslant 1$. Then,*

$$\sum_{n \leqslant X} (f * \log^{\nu})(n)\, \mathrm{e}(n\alpha) \ll \gamma\left(y + \frac{X}{q} + q\right)(\log x)^{\nu+1}\|f\|_{\infty}, \tag{2.4}$$

where the implied constant in (2.4) is absolute.

Lemma 2.4 *Let $f, g: \mathbb{N} \to \mathbb{C}$ be two arithmetic functions supported on $[1, y]$ and $[1, z]$, respectively. Let $X \geqslant 2$, $\alpha \in \mathbb{R}$ and consider a reduced fraction a/q such that $|\alpha - a/q| \leqslant \gamma/q^2$ for some $\gamma \geqslant 1$. Then,*

$$\sum_{n \leqslant X} (f * g)(n)\, \mathrm{e}(n\alpha) \ll \gamma\left(q + y + z + \frac{yz}{q}\right)^{1/2} \sqrt{\log 2q} \cdot \|f\|_2 \|g\|_2, \tag{2.5}$$

where the implied constant in (2.5) is absolute.

We only prove Lemma 2.3. The proof of Lemma 2.4 is similar.

Proof of Lemma 2.3 Following the proof of Theorem 23.5 in [16], it suffices to bound the sum

$$\sum_{k \leqslant y} \min\left\{\frac{X}{k}, \frac{1}{\|k\alpha\|}\right\}.$$

To do so, we divide the interval $[1, y]$ into subintervals of length q and write

$$S_m := \sum_{mq < k \leqslant (m+1)q} \min\left\{\frac{X}{k}, \frac{1}{\|k\alpha\|}\right\}.$$

Note that

$$\sum_{k \leqslant y} \min\left\{\frac{X}{k}, \frac{1}{\|k\alpha\|}\right\} \leqslant \sum_{m=0}^{\lfloor y/q \rfloor} S_m.$$

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We write $\alpha = a/q + \beta$ with $|\beta| \leqslant \gamma/q^2$ and consider two cases.

Case 1: $m \geqslant 1$. Writing $k = mq + r$ where $1 \leqslant r \leqslant q$, we have $\|k\alpha\| = \|r\alpha + mq\beta\|$. Define the intervals

$$I_j = \left[\frac{j-1}{2q}, \frac{j}{2q}\right), \quad j \in \{1, 2, \ldots, \lceil q \rceil\}.$$

Fix some $k_1 = mq + r_1$, such that $\|k_1\alpha\| \in I_j$ for some j. Since I_j is an interval of length $1/2q$, it follows that $\|r_1\alpha\|$ is contained in the union of at most two intervals, say T_1 and T_2, each of length at most $1/2q$. Suppose for some $k_2 = mq + r_2$, $\|k_2\alpha\| \in I_j$ as well. Then, $\|r_2\alpha\|$ is also contained in $T_1 \cup T_2$. But using Proposition 2.1, each of T_1 or T_2 can contain at most 14γ choices of r_2. Hence, there can be at most 28γ different k_2 such that both $\|k_1\alpha\|$ and $\|k_2\alpha\|$ lie in I_ℓ. Therefore, the contribution from those k's for which $\|k\alpha\| \in I_j$ to the sum S_m is

$$\leqslant \frac{60\gamma q}{j-1},$$

when $j > 1$. For the specific case when $j = 1$, we trivially bound the contribution by $30\gamma/mq$. Therefore, for $m \geqslant 1$, we obtain

$$S_m \leqslant 30\gamma\left(\frac{X}{mq} + \sum_{2 \leqslant j \leqslant \lceil q \rceil} \frac{2q}{j-1}\right) \ll \gamma\left(\frac{X}{mq} + q\log q\right).$$

Case 2: $m = 0$. For all $r \geqslant q/2\gamma$, an argument similar to the previous case shows that the contribution from such terms is $\ll \gamma(X/q + q\log q)$. Finally, we are left with the cases when $r < q/2\gamma$. Then, $|r\beta| \leqslant 1/2q$, which implies $\|r\alpha\| \geqslant 1/2q$. Therefore, the total contribution from the cases when $r < q/2\gamma$ is

$$\ll \gamma(2q + q\log q) \ll \gamma q \log q.$$

The rest of the proof follows along the lines of the proof of [16, Theorem 23.5], and noting that there are $\ll \gamma/q$ choices of m. ∎

2.3 Exponential sums and the divisor function

We record the following result achieved by Mikawa that provides bounds on exponential sums involving the three-fold divisor function.

Lemma 2.5 *Let $\alpha \in \mathbb{R}$ and consider a reduced fraction a/q such that $|\alpha - a/q| \leqslant 1/q^2$. Then, for $0 < M, J \leqslant X$, we have*

$$\sum_{m \sim M} \sum_{j \sim J} \tau_3(j) \min\left(\frac{X}{m^2 j}, \frac{1}{\|\alpha m^2 j\|}\right) \ll MJ(\log X)^3 + \frac{X^{3/4}}{M}\left(Xq^{-1} + XM^{-1} + q\right)^{1/4} (\log X)^8,$$

where τ_3 denotes the three-fold divisor function.

Proof See [22]. ∎

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2.4 The Möbius function in arithmetic progressions

We collect below a result on the Möbius function in arithmetic progressions, due to Davenport, which is an application of the Siegel-Walfisz theorem with the standard zero-free region for Dirichlet L-functions.

Lemma 2.6 *Let $X \geqslant 2$ and $B > 0$. Let $r, q \geqslant 1$ be two integers with $q \leqslant (\log X)^B$. Then, there exists a constant $c_B > 0$ depending only on B such that*

$$\sum_{\substack{n \leqslant X \\ n \equiv r \bmod q}} \mu(n) \ll_B X \exp(-c_B \sqrt{\log X}).$$

Proof See [7, Lemma 6]. ∎

2.5 A divisor sum estimate

Let $k \in \mathbb{N}$ and let $\tau_k(n)$ be the k-fold divisor function. Norton established the following result.

Lemma 2.7 *For $X \geqslant 1$, and $k, r \geqslant 2$, we have*

$$\sum_{n \leqslant X} (\tau_k(n))^r \ll X(\log X)^{k^r - 1}.$$

Proof See [24, Eq. 1.10]. ∎

2.6 Bounding $1/\zeta(s)$ in the critical strip.

In order to prove the explicit formula in Section 8, we will require a contour integration involving the term $1/\zeta^2(s)$. To prove our results unconditionally, we shall use the following result due to works of Ramachandra and Sankaranarayanan, and Inoue, which provide unconditional upper bounds for $1/\zeta(s)$ inside the critical strip along a specific vertical line segment.

Lemma 2.8 *Let $T > 0$ be a sufficiently large positive number and $H = T^{1/3}$. Then, one has*

$$\min_{T \leqslant t \leqslant T+H} \max_{\frac{1}{2} \leqslant \sigma \leqslant 2} |\zeta(\sigma + it)|^{-1} \leqslant \exp(C(\log\log T)^2)$$

with an absolute constant $C > 0$. In particular, there exists a real $T_* \in [T, T + T^{1/3}]$ such that

$$\frac{1}{\zeta(\sigma + iT_*)} \leqslant T_*^{\varepsilon} \quad \text{where} \quad \frac{1}{2} \leqslant \sigma \leqslant 2,$$

for any $\varepsilon > 0$.

Proof See [25, Theorem 2] and [14, Lemma 1]. ∎

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3 Proofs of Theorems 1.1, 1.3, and 1.4

In this section, we present the proofs of Theorems 1.1, 1.3, and 1.4.

3.1 Proof of Theorem 1.1

We can assume $q \leqslant X$; otherwise, the conclusion follows trivially. Let $U, V \geqslant 2$ be parameters to be chosen later on. Applications of the Möbius inversion formula yield

$$\begin{aligned}\mu(n) &= \mu_{\leqslant U}(n) + \mu_{\leqslant V}(n) + (\mu_{>U} * \mu_{>V} * 1)(n) - (\mu_{\leqslant U} * \mu_{\leqslant V} * 1)(n)\\ &= \mu^{(1)} + \mu^{(2)} + \mu^{(3)} - \mu^{(4)}, \end{aligned} \tag{3.1}$$

say. Consider $\mu^{(1)}$ and $\mu^{(2)}$ first. Since our choice of parameters U and V will be small compared to X, bounding trivially, we arrive at

$$\sum_{n\leqslant X} \mu^{(1)}(n)\,\mathrm{e}(n\alpha) \ll U \tag{3.2}$$

$$\sum_{n\leqslant X} \mu^{(2)}(n)\,\mathrm{e}(n\alpha) \ll V. \tag{3.3}$$

For $\mu^{(3)}(n)$, we will use Lemma 2.2. However, applying the lemma directly does not suffice our purpose. We consider breaking our range into dyadic intervals. By definition, we have

$$\mu^{(3)}(n) = \sum_{\substack{k\ell=n\\ k>U,\ell>V}} \mu_{>U}(k)(\mu_{>V} * 1)(\ell).$$

To obtain better control of the support of the variables k and ℓ, we break the interval $(U, X/V]$ into dyadic intervals $\left(2^{j-1}, 2^j\right]$, where $2^j \in (U, 2X/V]$. If $k \in \left(2^{j-1}, 2^j\right]$, then $\ell = n/k \leqslant X/2^{j-1}$. With this more accurate decomposition, we have

$$\mu^{(3)}(n) = \sum_{U<2^j\leqslant 2X/V} \left(f_j * g_j\right)(n), \qquad \text{for } n \leqslant X,$$

where $f_j(k) = \mu_{>U}(k)\cdot 1_{2^{j-1}<k\leqslant 2^j}$ and $g_j(\ell) = (\mu_{>V} * 1)(\ell)\cdot 1_{V<\ell\leqslant X/2^{j-1}}$. Then, applying Lemma 2.2, we obtain

$$\begin{aligned}\sum_{n\leqslant X} \mu^{(3)}(n)\,\mathrm{e}(n\alpha) &= \sum_{U<2^j\leqslant 2X/V} \sum_{n\leqslant X} \left(f_j * g_j\right)(n)\,\mathrm{e}(n\alpha)\\ &\ll \sum_{U<2^j\leqslant 2X/V} \left(q^{1/2} + \frac{X^{1/2}}{U^{1/2}} + \frac{X^{1/2}}{V^{1/2}} + \frac{X^{1/2}}{q^{1/2}}\right)\sqrt{\log 2q}\cdot \|f_j\|_2\|g_j\|_2\\ &\ll \left(X^{1/2}q^{1/2} + \frac{X}{U^{1/2}} + \frac{X}{V^{1/2}} + \frac{X}{q^{1/2}}\right)(\log X)^3. \end{aligned} \tag{3.4}$$

Note. In the rest of this section, we will always apply the above more accurate decomposition technique before applying Lemma 2.2.

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Finally, we focus on $\mu^{(4)}(n)$. We apply Lemma 2.1 with $y = UV$, $v = 0$, and $f = \mu_{\leqslant U} * \mu_{\leqslant V}$. Since $\|f\|_\infty \ll_\varepsilon (UV)^\varepsilon$ for any fixed $\varepsilon > 0$, it follows that

$$\sum_{n\leqslant X} \mu^{(4)}(n)\,\mathrm{e}(n\alpha) \ll_\varepsilon (UV)^\varepsilon\left(UV + \frac{X}{q} + q\right)(\log X). \tag{3.5}$$

We choose $U = V = \min\{X^{2/5}, q, X/q\}$ and combine the estimates (3.2), (3.3), (3.4), and (3.5) to obtain

$$\sum_{n\leqslant X} \mu(n)\,\mathrm{e}(n\alpha) \ll_\varepsilon X^{4/5+\varepsilon} + \frac{X(\log X)^3}{q^{1/2}} + X^{1/2}q^{1/2}(\log X)^3.$$

This ends the proof.

3.2 Proof of Theorem 1.3

The proof follows along the lines of the proof of Theorem 1.1. Again, we can assume $q \leqslant X$. Let $U, V \geqslant 2$ be parameters to be chosen later on. Employing the Möbius inversion formula, we obtain

$$\hat{\mu}(n) = \hat{\mu}_{\leqslant U}(n) + (\hat{\mu}_{>U} * \mu_{>V} * 1)(n) + (f * \mu_{\leqslant V})(n) - (\hat{\mu}_{\leqslant U} * \mu_{\leqslant V} * 1)(n).$$

Trivially, we have

$$\sum_{n\leqslant X} \hat{\mu}_{\leqslant U}(n)\,\mathrm{e}(n\alpha) \ll U\log U. \tag{3.6}$$

Next, by Lemma 2.2, similar to how we obtained (3.4), we have

$$\begin{aligned}\sum_{n\leqslant X} (\hat{\mu}_{>U} * \mu_{>V} * 1)(n)\,\mathrm{e}(n\alpha) &\ll X^{1/2}\left(\frac{X}{U} + \frac{X}{V} + q + \frac{X}{q}\right)^{1/2}(\log X)^{5/2}\\ &\ll \left(X^{1/2}q^{1/2} + \frac{X}{U^{1/2}} + \frac{X}{V^{1/2}} + \frac{X}{q^{1/2}}\right)(\log X)^{5/2}. \end{aligned}\tag{3.7}$$

Observing that $\|f\cdot 1_{n\leqslant X}\|_2 \ll X^{1/4}$, again, an application of Lemma 2.2 shows that

$$\begin{aligned}\sum_{n\leqslant X} (f * \mu_{\leqslant V})(n)\,\mathrm{e}(n\alpha) &\ll \left(q + X + V + \frac{X}{q}\right)^{1/2} X^{1/4}V^{1/4}(\log X)^{3/2}\\ &\ll \left(X^{1/4}V^{1/4}q^{1/2} + X^{3/4}V^{1/4}\right)(\log X)^{3/2}. \end{aligned}\tag{3.8}$$

Arguing as in the proof of (3.5) for the case of the Möbius function, we have

$$\sum_{n\leqslant X} (\hat{\mu}_{\leqslant U} * \mu_{\leqslant V} * 1)(n)\,\mathrm{e}(n\alpha) \ll_\varepsilon (UV)^\varepsilon\left(UV + \frac{X}{q} + q\right)(\log X), \tag{3.9}$$

for any fixed $\varepsilon > 0$. Finally, we choose $U = V = \min\{X^{1/3}, q, X/q\}$ and combine (3.6), (3.7), (3.8), and (3.9) to obtain

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$$\sum_{n\leqslant X}\hat{\mu}(n)\,\mathrm{e}(n\alpha)\ll\left(X^{5/6}+q^{1/2}X^{1/2}+\frac{X}{q^{1/2}}\right)(\log X)^{5/2},$$

which is the desired result.

3.3 Proof of Theorem 1.4

Similar to the previous proofs, we can assume $q\leqslant X$. Let $U, V\geqslant 2$ be parameters to be chosen later on. We start with the following identity:

$$\begin{aligned}\tilde{\mu}_2(n)=\tilde{\mu}_2(n)\cdot 1_{n\leqslant U}+(\tilde{\mu}_2\cdot 1_{n>U}*\mu_{>V}*1)(n)\\+(\mu^2*\mu_{\leqslant V})(n)-(\tilde{\mu}_2\cdot 1_{n\leqslant U}*\mu_{\leqslant V}*1)(n).\end{aligned}$$

Arguing as in the previous proofs, we can show that

$$\sum_{n\leqslant X}\tilde{\mu}_2(n)\cdot 1_{n\leqslant U}\,\mathrm{e}(n\alpha)\ll U\log U, \tag{3.10}$$

$$\sum_{n\leqslant X}(\tilde{\mu}_2\cdot 1_{n>U}*\mu_{>V}*1)(n)\,\mathrm{e}(n\alpha)\ll\left(X^{1/2}q^{1/2}+\frac{X}{U^{1/2}}+\frac{X}{V^{1/2}}+\frac{X}{q^{1/2}}\right)(\log X)^3, \tag{3.11}$$

$$\text{and}\quad\sum_{n\leqslant X}(\mu_{\leqslant V}*\tilde{\mu}_2\cdot 1_{n\leqslant U}*1)(n)\,\mathrm{e}(n\alpha)\ll_\varepsilon (UV)^\varepsilon\left(UV+\frac{X}{q}+q\right)(\log X), \tag{3.12}$$

for any fixed $\varepsilon>0$. It remains to estimate the sum

$$\sum_{n\leqslant X}(\mu^2*\mu_{\leqslant V})(n)\,\mathrm{e}(n\alpha).$$

A treatment like (3.8) is not enough in this case because $\|\mu^2\cdot 1_{n\leqslant X}\|_2\ll X^{1/2}$. Therefore, we proceed in a different way. Using the relation $\mu^2(n)=\sum_{ab^2=n}\mu(b)$, we can write

$$\begin{aligned}\sum_{n\leqslant X}(\mu^2*\mu_{\leqslant V})(n)\,\mathrm{e}(n\alpha)&=\sum_{\substack{mb^2\leqslant X\\ m\leqslant V}}\mu(b)\mu(m)\sum_{a\leqslant X/(mb^2)}\mathrm{e}(amb^2\alpha)\\&\ll\sum_{b\leqslant X^{1/2}}\sum_{m\leqslant V}\min\left\{\frac{X}{b^2m},\frac{1}{\|b^2m\alpha\|}\right\}.\end{aligned} \tag{3.13}$$

We need to break the range of b in (3.13) into two parts: for large b's, we will use the bound from Lemma 2.5, and for small b's, we will use the Cauchy-Schwarz inequality along with Lemma 2.1. Indeed, we write

$$\sum_{b\leqslant X^{1/2}}\sum_{m\leqslant V}\min\left\{\frac{X}{b^2m},\frac{1}{\|b^2m\alpha\|}\right\}=\left(\sum_{b\leqslant B}\sum_{m\leqslant V}+\sum_{B<b\leqslant X^{1/2}}\sum_{m\leqslant V}\right)\min\left\{\frac{X}{b^2m},\frac{1}{\|b^2m\alpha\|}\right\}.$$

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Then, by applying the Cauchy-Schwarz inequality,

$$\begin{aligned}\sum_{b\leqslant B}\sum_{m\leqslant V}\min\left\{\frac{X}{b^2m},\frac{1}{\|b^2m\alpha\|}\right\} &\leqslant \sum_{m\leqslant B^2V}\tau_3(m)\min\left\{\frac{X}{m},\frac{1}{\|m\alpha\|}\right\}\\ &\ll X^{1/2}\left(B^2V+q+\frac{X}{q}\right)^{1/2}(\log X)^6\\ &\ll \left(X^{1/2}BV^{1/2}+X^{1/2}q^{1/2}+\frac{X}{q^{1/2}}\right)(\log X)^6. \qquad (3.14)\end{aligned}$$

Next, by Lemma 2.5 dyadically, we have

$$\sum_{B<b\leqslant X^{1/2}}\sum_{m\leqslant V}\min\left\{\frac{X}{b^2m},\frac{1}{\|b^2m\alpha\|}\right\}\ll\left(X^{1/2}V+\frac{X}{q^{1/4}B}+\frac{X}{B^{5/4}}+\frac{X^{3/4}q^{1/4}}{B}\right)(\log X)^{10}. \qquad (3.15)$$

Choose $B=(X/V)^{2/9}$ and combining (3.14) and (3.15), we obtain

$$\sum_{b\leqslant X^{1/2}}\sum_{m\leqslant V}\min\left\{\frac{X}{b^2m},\frac{1}{\|b^2m\alpha\|}\right\}\ll\left(X^{13/18}V^{5/18}+X^{1/2}V+\frac{X}{q^{1/4}}+X^{3/4}q^{1/4}\right)(\log X)^{10}. \qquad (3.16)$$

Finally, we choose $U=V=\min\{X^{5/14},q,X/q\}$ and combine (3.10), (3.11), (3.12), and (3.16) to obtain

$$\sum_{n\leqslant X}\tilde{\mu}_2(n)\,\mathrm{e}(n\alpha)\ll\left(X^{23/28}+\frac{X}{q^{1/4}}+X^{3/4}q^{1/4}\right)(\log X)^{10},$$

as it was to be shown.

4 Proof of Theorem 1.2

In order to prove Theorem 1.2, we will require the following lemma.

Lemma 4.1 *Let $\alpha\in\mathbb{R}$ and consider a reduced fraction a/q such that $|\alpha-a/q|\leqslant\gamma/q^2$ for some $\gamma\geqslant 1$. Then, for $X\geqslant 2$ and for any fixed $\varepsilon>0$, we have*

$$S_\mu(X,\alpha)\ll_\varepsilon\gamma\left(X^{4/5+\varepsilon}+\frac{X(\log X)^3}{q^{1/2}}+X^{1/2}q^{1/2}(\log X)^3\right), \qquad (4.1)$$

where the implied constant in (4.1) depends at most on ε.

Proof The proof follows exactly along the lines of the proof of Theorem 1.1, except that we apply Lemmas 2.3 and 2.4 instead of Lemmas 2.1 and 2.2, respectively. ∎

We explicitly show the proof of Theorem 1.2 in the case when $k=2$. The argument then follows similarly for higher values of k using induction. We can assume $q\leqslant X$; otherwise, the conclusion is trivial. We break our proof into several steps.

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4.1 Initial step

Let

$$S_{\mu_2}(X,\alpha) := \sum_{n\leqslant X}(\mu*\mu)(n)\,\mathrm{e}(n\alpha).$$

Then, for some parameters $2 \leqslant M, N \leqslant X$ to be chosen later, we can write

$$\begin{aligned} S_{\mu_2}(X,\alpha) &= \sum_{\substack{mn\leqslant X\\ m>M,n>N}}\mu(m)\mu(n)\,\mathrm{e}(mn\alpha) + \sum_{m\leqslant M}\mu(m)\sum_{n\leqslant X/m}\mu(n)\,\mathrm{e}(mn\alpha)\\ &\quad + \sum_{n\leqslant N}\mu(n)\sum_{m\leqslant X/n}\mu(m)\,\mathrm{e}(mn\alpha) - \sum_{\substack{m\leqslant M\\ n\leqslant N}}\mu(m)\mu(n)\,\mathrm{e}(mn\alpha)\\ &= S^{(1)}_{\mu_2}(X,\alpha) + S^{(2)}_{\mu_2}(X,\alpha) + S^{(3)}_{\mu_2}(X,\alpha) - S^{(4)}_{\mu_2}(X,\alpha), \end{aligned} \tag{4.2}$$

say. We now turn our attention to each of these terms.

4.2 Estimation of $S^{(1)}_{\mu_2}(X,\alpha)$ and $S^{(4)}_{\mu_2}(X,\alpha)$.

Applying Lemma 2.2 after the dyadic decomposition following the proof of Theorem 1.1, we deduce that

$$\begin{aligned} S^{(1)}_{\mu_2}(X,\alpha) &\ll X^{1/2}\left(q+\frac{X}{M}+\frac{X}{N}+\frac{X}{q}\right)^{1/2}(\log X)^2\\ &\ll \left(X^{1/2}q^{1/2}+\frac{X}{M^{1/2}}+\frac{X}{N^{1/2}}+\frac{X}{q^{1/2}}\right)(\log X)^2. \end{aligned} \tag{4.3}$$

Trivially, we have

$$S^{(4)}_{\mu_2}(X,\alpha) \ll MN. \tag{4.4}$$

4.3 Estimation of $S^{(2)}_{\mu_2}(X,\alpha)$ and $S^{(3)}_{\mu_2}(X,\alpha)$.

Fix $\varepsilon > 0$. Writing $\alpha = a/q + \beta$, one has

$$m\alpha = \frac{ma}{q} + m\beta = \frac{ma/(m,q)}{q/(m,q)} + m\beta \quad \text{and} \quad |m\beta| \leqslant \frac{m/(m,q)^2}{q^2/(m,q)^2} \leqslant \frac{m/(m,q)}{q^2/(m,q)^2}.$$

Therefore, using Lemma (4.1), we obtain

$$\begin{aligned} S^{(2)}_{\mu_2}(X,\alpha) &\ll_\varepsilon \sum_{m\leqslant M}\frac{m}{(m,q)}\left\{\left(\frac{X}{m}\right)^{4/5+\varepsilon} + \frac{X(m,q)^{1/2}(\log X)^3}{mq^{1/2}} + \left(\frac{X}{m}\right)^{1/2}\frac{q^{1/2}(\log X)^3}{(m,q)^{1/2}}\right\}\\ &\ll_\varepsilon X^{4/5+\varepsilon}M^{6/5-\varepsilon} + \frac{XM(\log X)^3}{q^{1/2}} + X^{1/2}q^{1/2}M^{3/2}(\log X)^3. \end{aligned} \tag{4.5}$$

Similarly, we can show that

$$S^{(3)}_{\mu_2}(X,\alpha) \ll_\varepsilon X^{4/5+\varepsilon}N^{6/5-\varepsilon} + \frac{XN(\log X)^3}{q^{1/2}} + X^{1/2}q^{1/2}N^{3/2}(\log X)^3. \tag{4.6}$$

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4.4 Parameter optimization

Substituting the estimates (4.3), (4.5), (4.6), and (4.4) in (4.2) and choosing $M = N$, we deduce that

(4.7)
$$S_{\mu_2}(X,\alpha) \ll_\varepsilon X^{4/5+\varepsilon}M^{6/5-\varepsilon} + \left(\frac{X}{M^{1/2}} + X^{1/2}q^{1/2} + \frac{XM}{q^{1/2}} + X^{1/2}q^{1/2}M^{3/2}\right)(\log X)^3.$$

We are now left with the crucial step: to choose our parameter M optimally. Since we will choose $M \geqslant 1$, we may omit the term $X^{1/2}q^{1/2}$ in (4.7). Let $M = y^5$, $A = \sqrt{q}$, $B = X^{1/5}$, and $C = \sqrt{X/q}$. To optimally choose M, it suffices to minimize the expression

$$H(y) := \max\left\{\frac{1}{y^{5/2}}, \frac{y^5}{A}, \frac{y^6}{B}, \frac{y^{15/2}}{C}\right\}. \tag{4.8}$$

Note that as a function of y, only $1/y^{5/2}$ is decreasing. Hence, the minimum value in (4.8) is attained when $1/y^{5/2}$ is equal to one of the other three functions. We consider the following cases.

Case 1. Suppose we have

$$\frac{y^5}{A} \geqslant \max\left\{\frac{y^6}{B}, \frac{y^{15/2}}{C}\right\}. \tag{4.9}$$

Then the minimum value in (4.8) is attained when

$$\frac{1}{y^{5/2}} = \frac{y^5}{A},$$

which implies that $y = A^{2/15}$. With this particular choice of y, we see that

$$\frac{y^5}{A} \geqslant \frac{y^6}{B}$$

is equivalent to $A^{-1/3} \geqslant A^{4/5}B^{-1}$ which occurs when $1 \leqslant q \leqslant X^{6/17}$. However, the inequality

$$\frac{y^5}{A} \geqslant \frac{y^{15/2}}{C}$$

is equivalent to $A^{-1/3} \geqslant AC^{-1}$ which occurs when $1 \leqslant q \leqslant X^{6/14}$. Combining the two ranges, we see that (4.9) holds when $1 \leqslant q \leqslant X^{6/17}$. In this range, we have $H(y) = A^{-1/3} = q^{-1/6}$, and thus,

$$S_{\mu_2}(X,\alpha) \ll \frac{X(\log X)^3}{q^{1/6}}. \tag{4.10}$$

Case 2. Now let's assume the scenario

$$\frac{y^6}{B} \geqslant \max\left\{\frac{y^5}{A}, \frac{y^{15/2}}{C}\right\}. \tag{4.11}$$

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Then the minimum value in (4.8) is attained when

$$\frac{1}{y^{5/2}} = \frac{y^6}{B},$$

which implies that $y = B^{2/17}$. Therefore, the inequality

$$\frac{y^6}{B} \geqslant \frac{y^5}{A}$$

holds with $y = B^{2/17}$ when $A \geqslant B^{15/17}$, which implies that $q \geqslant X^{6/17}$. However, the inequality

$$\frac{y^6}{B} \geqslant \frac{y^{15/2}}{C}$$

holds with $y = B^{2/17}$ when $B \geqslant C^{17/20}$, which implies that $q \leqslant X^{9/17}$. Putting together the two ranges, we see that (4.11) holds when $X^{6/17} \leqslant q \leqslant X^{9/17}$. In this range, $H(y) = B^{-5/17} = X^{-1/17}$, and we have

$$S_{\mu_2}(X, \alpha) \ll_\varepsilon X^{16/17+\varepsilon}. \tag{4.12}$$

Case 3. Arguments similar to Cases 1 and 2 show that when $X^{9/17} \leqslant q \leqslant X$, then the minimum value in (4.8) is $H(y) = C^{-1/4} = q^{1/8}X^{-1/8}$. Therefore in this range, we obtain

$$S_{\mu_2}(X, \alpha) \ll X^{7/8}q^{1/8}(\log X)^3. \tag{4.13}$$

Combining the estimates (4.10), (4.12), and (4.13), we conclude that

$$\sum_{n\leqslant X}(\mu * \mu)(n)\,\mathrm{e}(n\alpha) \ll_\varepsilon X^{16/17+\varepsilon} + \left(\frac{X}{q^{1/6}} + X^{7/8}q^{1/8}\right)(\log X)^3.$$

This completes the proof for the case when $k = 2$.

4.5 Induction step

Fix $\varepsilon > 0$. Assume that for $X \geqslant 2$,

$$\sum_{n\leqslant X}\mu_k(n)\,\mathrm{e}(n\alpha) \ll_\varepsilon X^{a_k+\varepsilon} + \frac{X(\log X)^{k^2}}{q^{b_k}} + X^{c_k}q^{1-c_k}(\log X)^{k^2},$$

for some $k = \ell \geqslant 2$, and we want to prove our desired result when $k = \ell + 1$. Similar to Lemma 4.1, one can show that if $\alpha \in \mathbb{R}$ such that $|\alpha - a/q| \leqslant \gamma/q^2$ for some $(a, q) = 1$ and $\gamma \geqslant 1$, then

$$\sum_{n\leqslant X}\mu_\ell(n)\,\mathrm{e}(n\alpha) \ll_\varepsilon \gamma\left(X^{a_\ell+\varepsilon} + \frac{X(\log X)^{\ell^2}}{q^{b_\ell}} + X^{c_\ell}q^{1-c_\ell}(\log X)^{\ell^2}\right), \tag{4.14}$$

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where the implied constant in (4.14) depends at most on ε. Equipped with (4.14), for some parameters $2 \leqslant M, N \leqslant X$ to be chosen later, we can write

$$\begin{aligned} S_{\mu_{\ell+1}}(X,\alpha) = &\sum_{\substack{mn\leqslant X\\ m>M,n>N}} \mu(m)\mu_\ell(n)\,\mathrm{e}(mn\alpha) + \sum_{m\leqslant M}\mu(m)\sum_{n\leqslant X/m}\mu_\ell(n)\,\mathrm{e}(mn\alpha)\\ &+\sum_{n\leqslant N}\mu_\ell(n)\sum_{m\leqslant X/n}\mu(m)\,\mathrm{e}(mn\alpha) - \sum_{\substack{m\leqslant M\\ n\leqslant N}}\mu(m)\mu_\ell(n)\,\mathrm{e}(mn\alpha)\\ &= S^{(1)}_{\mu_{\ell+1}}(X,\alpha) + S^{(2)}_{\mu_{\ell+1}}(X,\alpha) + S^{(3)}_{\mu_{\ell+1}}(X,\alpha) - S^{(4)}_{\mu_{\ell+1}}(X,\alpha). \end{aligned} \tag{4.15}$$

Again, applying Lemma 2.2, we obtain

$$S^{(1)}_{\mu_{\ell+1}}(X,\alpha) \ll \left(X^{1/2}q^{1/2} + \frac{X}{M^{1/2}} + \frac{X}{N^{1/2}} + \frac{X}{q^{1/2}}\right)(\log X)^{(\ell+1)^2}. \tag{4.16}$$

Trivially, we have

$$S^{(4)}_{\mu_{\ell+1}}(X,\alpha) \ll MN(\log N)^{\ell-1}. \tag{4.17}$$

Choosing $\gamma = m/(m,q)$ and applying (4.14), we arrive at

$$S^{(2)}_{\mu_{\ell+1}}(X,\alpha) \ll_\varepsilon X^{a_\ell+\varepsilon}M^{2-a_\ell-\varepsilon} + \frac{XM(\log X)^{(\ell+1)^2}}{q^{b_\ell}} + X^{c_\ell}q^{1-c_\ell}M^{2-c_\ell}(\log X)^{(\ell+1)^2}. \tag{4.18}$$

Note that Lemma 2.7 shows that

$$\sum_{n\leqslant N}\tau_\ell^2(n) \ll N(\log N)^{\ell^2-1}.$$

Hence, applying Lemma (4.1) and Cauchy-Schwartz inequality in conjunction, we can write

$$\begin{aligned} S^{(3)}_{\mu_{\ell+1}}(X,\alpha) &\ll_\varepsilon \sum_{n\leqslant N}\tau_\ell(n)\cdot\frac{n}{(n,q)}\left\{\left(\frac{X}{n}\right)^{4/5+\varepsilon} + \frac{X(n,q)^{1/2}(\log X)^3}{nq^{1/2}} + \left(\frac{X}{n}\right)^{1/2}\frac{q^{1/2}(\log X)^3}{(n,q)^{1/2}}\right\}\\ &\ll_\varepsilon X^{4/5+\varepsilon}N^{6/5-\varepsilon} + \frac{XN(\log X)^{(\ell+1)^2}}{q^{1/2}} + X^{1/2}q^{1/2}N^{3/2}(\log X)^{(\ell+1)^2}. \end{aligned} \tag{4.19}$$

Substituting the estimates (4.16), (4.17), (4.18), and (4.19) in (4.15) and choosing $M = N$, we have

$$S_{\mu_{\ell+1}}(X,\alpha) \ll_\varepsilon X^{a_\ell+\varepsilon}M^{2-a_\ell-\varepsilon} + \left(\frac{X}{M^{1/2}} + \frac{XM}{q^{b_\ell}} + X^{c_\ell}q^{1-c_\ell}M^{2-c_\ell} + X^{1/2}q^{1/2}M^{3/2}\right)(\log X)^{(\ell+1)^2}.$$

An optimization procedure similar to that for the case $k = 2$ then shows that

$$S_{\mu_{\ell+1}}(X,\alpha) \ll_\varepsilon X^{a_{\ell+1}+\varepsilon} + \frac{X(\log X)^{(\ell+1)^2}}{q^{b_{\ell+1}}} + X^{c_{\ell+1}}q^{1-c_{\ell+1}}(\log X)^{(\ell+1)^2},$$

thereby completing the proof.

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5 Proof of Theorem 1.6

We now set up the structure of the proof of Theorem 1.6. Recall from (1.9) and (1.10) that

$$\Psi_{\mu*\mu}(z) = \sum_{n=1}^{\infty} p_{\mu*\mu}(n) z^n = \prod_{n=1}^{\infty} (1-z^n)^{-(\mu*\mu)(n)}, \quad |z|<1,$$
$$\Phi_{\mu*\mu}(z) = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \frac{(\mu*\mu)(n)}{j} z^{jn}, \quad |z|<1,$$

and $\Psi_{\mu*\mu}(z) = \exp(\Phi_{\mu*\mu}(z))$. The strategy to prove Theorem 1.6 is to employ the Hardy-Littlewood circle method. An application of Cauchy's theorem yields

$$p_{\mu*\mu}(n) = \rho^{-n} \int_0^1 \exp(\Phi_{\mu*\mu}(\rho\, \mathrm{e}(\theta)) - 2\pi i n\theta) d\theta, \tag{5.1}$$

where $0<\rho<1$. We write $\rho = \exp(-1/X)$, for some parameter $X>0$. As we shall see, for each value of n, we will choose $X = X(n)$, and therefore, $\rho = \rho(n)$ in (5.1) to be such that as $n\to\infty$, $X\to\infty$, and $\rho\to 1$. Therefore, in what follows, we can assume X to be sufficiently large. Invoking the periodicity of the integrand in (5.1), we can replace the unit interval by the interval $\mathcal{U} = [-X^{-1}(\log X)^A, 1-X^{-1}(\log X)^A)$, where $A>0$ is a sufficiently large constant. This choice $\mathcal{U}$ keeps the region around the origin from being split into two regions.

Now we are ready to set up the major and minor arcs. Let $a\in\mathbb{N}\cup\{0\}$ and $q\in\mathbb{N}$ such that $0\leqslant a<q$ and $(a,q)=1$. Choose

$$\delta_q = q^{-1}X^{-1}(\log X)^A \quad \text{and} \quad Q = (\log X)^A, \tag{5.2}$$

and define

$$\mathfrak{M}(q,a) = \left\{\theta\in\mathcal{U} : \left|\theta - \frac{a}{q}\right| \leqslant \delta_q\right\}. \tag{5.3}$$

The major arcs $\mathfrak{M}$ and the minor arcs $\mathfrak{m}$ are

$$\mathfrak{M} = \bigcup_{\substack{1\leqslant a\leqslant q\leqslant Q\\(a,q)=1}} \mathfrak{M}(q,a) \quad \text{and} \quad \mathfrak{m} = \mathcal{U}\backslash\mathfrak{M}. \tag{5.4}$$

We will show that if $\theta\in\mathfrak{M}$, then

$$\Phi_{\mu*\mu}(\rho\, \mathrm{e}(\theta)) \ll_A \frac{X}{(\log X)^A}. \tag{5.5}$$

However, if $\theta\in\mathfrak{m}$, then one has

$$\Phi_{\mu*\mu}(\rho\, \mathrm{e}(\theta)) \ll_A \frac{X}{(\log X)^{A/9}}. \tag{5.6}$$

Assuming for now the validity of (5.5) and (5.6), we proceed to prove Theorem 1.6.

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Proof of Theorem 1.6 We rewrite (5.1) as

$$p_{\mu*\mu}(n) = \rho^{-n}\left(\int_{\mathfrak{M}} + \int_{\mathfrak{m}}\right)\exp(\Phi_{\mu*\mu}(\rho\,\mathrm{e}(\theta)) - 2\pi i n\theta)d\theta. \tag{5.7}$$

First, we consider the integral over the minor arcs. Let $X = \sqrt{n}(\log n)^{A/18}$. Since $\rho = \exp(-1/X)$, using (5.6), we can write

$$\rho^{-n}\int_{\mathfrak{m}} \exp(\Phi_{\mu*\mu}(\rho\,\mathrm{e}(\theta)) - 2\pi i n\theta)d\theta \ll \exp\left(\frac{n}{X} + \frac{X}{(\log X)^{A/9}}\right) \ll \exp\left(\frac{\sqrt{n}}{(\log n)^{A/20}}\right). \tag{5.8}$$

A concomitant argument holds for the major arcs. Therefore, choosing A sufficiently large, we obtain

$$\log p_{\mu*\mu}(n) \ll_B \frac{\sqrt{n}}{(\log n)^B},$$

for any fixed $B > 0$. ∎

Therefore, it suffices to establish (5.5) and (5.6), which we do in Sections 6 and 7, respectively.

6 Major arcs

We begin our discussion on the major arcs by estimating Möbius convolutions in arithmetic progressions. The key ingredient required for the following result is Lemma 2.6.

Lemma 6.1 *Let $X \geqslant 2$. Suppose $B > 0$ is fixed. Let $r, q \geqslant 1$ be positive integers with $q \leqslant (\log X)^B$. Then there exists a constant $c_B > 0$ depending only on B such that*

$$S(X) := \sum_{\substack{n\leqslant X\\ n\equiv r \bmod q}} (\mu*\mu)(n) \ll_B X\exp(-c_B\sqrt{\log X}).$$

Proof By Dirichlet's hyperbola method, we break $S(X)$ into three sub-sums; that is, $S(X) = S_1(X) + S_2(X) - S_3(X)$, where

$$S_1(X) = \sum_{n_1\leqslant\sqrt{X}}\mu(n_1)\sum_{\substack{n_2\leqslant X/n_1\\ n_1n_2\equiv r \bmod q}}\mu(n_2),$$

$$S_2(X) = \sum_{n_2\leqslant\sqrt{X}}\mu(n_2)\sum_{\substack{n_1\leqslant X/n_2\\ n_1n_2\equiv r \bmod q}}\mu(n_2),$$

$$S_3(X) = \sum_{n_1\leqslant\sqrt{X}}\mu(n_1)\sum_{\substack{n_2\leqslant\sqrt{X}\\ n_1n_2\equiv r \bmod q}}\mu(n_2).$$

We treat $S_3(X)$ first. Consider the inner sum over n_2. Note that if $n_1n_2 \equiv r \bmod q$, then $n_1(n_2+q) \equiv r \bmod q$. With this in mind, we write

$$\mathcal{B} := \{x \bmod q : n_1x \equiv r \bmod q\}. \tag{6.1}$$

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Then using Lemma 2.6, it follows that

$$\sum_{\substack{n_2\leqslant\sqrt{X}\\ n_1n_2\equiv r \bmod q}}\mu(n_2)=\sum_{b\in\mathcal{B}}\sum_{\substack{n_2\equiv b \bmod q\\ n_2\leqslant\sqrt{X}}}\mu(n_2)\ll_B \operatorname{card}(\mathcal{B})\sqrt{X}\exp(-c_1\sqrt{\log X}), \tag{6.2}$$

for some constant $c_1=c_1(B)>0$. The bound in (6.2) holds for all $q\leqslant(\log X)^B$. Therefore, by trivially bounding the sum over n_1, we obtain

$$S_3(X)\ll_B\Bigg(\sum_{n_1\leqslant\sqrt{X}}|\mu(n_1)|\Bigg)\operatorname{card}(\mathcal{B})\sqrt{X}\exp(-c_1\sqrt{\log X})\ll_B X\exp(-c_2\sqrt{\log X}), \tag{6.3}$$

for some constant $c_2=c_2(B)>0$.

Note that $S_1(X)=S_2(X)$ by symmetry. Therefore, it is enough to estimate the sum $S_1(X)$. We analyze the inner sum over n_2 first. Again applying Lemma 2.6, we obtain

$$\sum_{\substack{n_2\leqslant X/n_1\\ n_1n_2\equiv r \bmod q}}\mu(n_2)=\sum_{b\in\mathcal{B}}\sum_{\substack{n_2\equiv b \bmod q\\ n_2\leqslant X/n_1}}\mu(n_2)\ll_B \operatorname{card}(\mathcal{B})\frac{X}{n_1}\exp\Bigg(-c_3\sqrt{\log\frac{X}{n_1}}\Bigg),$$

for some constant $c_3=c_3(B)>0$, where $\mathcal{B}$ is as in (6.1). This leaves us with

$$S_1(X)\ll_B\sum_{n_1\leqslant\sqrt{X}}\operatorname{card}(\mathcal{B})\frac{X}{n_1}\exp\Bigg(-c_3\sqrt{\log\frac{X}{n_1}}\Bigg)\ll_B X\exp\big(-c_4\sqrt{\log X}\big), \tag{6.4}$$

for some constant $c_4=c_4(B)>0$. Combining (6.3) and (6.4), we arrive at the conclusion. ∎

Applying Lemma 6.1, one can quickly conclude the following.

Lemma 6.2 *Let $X\geqslant 2$ and suppose $B>0$ is fixed. Let $\theta\in\mathbb{R}$ such that $\theta=a/q+\beta$, where $a,q\in\mathbb{Z}$ with $0\leqslant a<q\leqslant(\log X)^B$, $(a,q)=1$ and $|\beta|\leqslant X^{-1}(\log X)^B$. Then there exists a constant $c_B>0$ depending only on B such that*

$$\sum_{n\leqslant X}(\mu*\mu)(n)\,\mathrm{e}(n\theta)\ll_B X\exp(-c_B\sqrt{\log X}).$$

Proof By partial summation, we can write

$$\sum_{n\leqslant X}(\mu*\mu)(n)\,\mathrm{e}(n\theta)\ll_B(1+|\beta|X)\max_{1\leqslant y\leqslant X}\Bigg|\sum_{n\leqslant y}(\mu*\mu)(n)\,\mathrm{e}(na/q)\Bigg|. \tag{6.5}$$

Consequently, it is enough to estimate the sum

$$\sum_{n\leqslant y}(\mu*\mu)(n)\,\mathrm{e}(na/q)=\sum_{b=1}^{q}\mathrm{e}\Big(\frac{ab}{q}\Big)\sum_{\substack{n\leqslant y\\ n\equiv b \bmod q}}(\mu*\mu)(n). \tag{6.6}$$

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An immediate application of Lemma 6.1 shows that for some constant $c_1 = c_1(B) > 0$ depending only on B, one has

$$\sum_{\substack{n\leqslant y\\ n\equiv b \bmod q}} (\mu*\mu)(n) \ll_B y\exp(-c_1\sqrt{\log y}). \tag{6.7}$$

Combining (6.5), (6.6), and (6.7), we conclude that

$$\sum_{n\leqslant X}(\mu*\mu)(n)\,\mathrm{e}(n\theta) \ll_B (1+|\beta|X)qX\exp(-c_1\sqrt{\log X}) \ll_B X\exp(-c_B\sqrt{\log X}),$$

for some constant $c_B > 0$ depending only on B. ∎

Recall from (5.4) that if $\theta \in \mathfrak{M}$, we can write $\theta = a/q + \beta$ with $q \leqslant (\log X)^A$ and $|\beta| \leqslant \delta_q \leqslant X^{-1}(\log X)^A$, where $A > 0$ is sufficiently large. Using Lemmas 6.1 and 6.2, we can now prove (5.5) for $\theta \in \mathfrak{M}$.

Lemma 6.3 *Let $\theta \in \mathfrak{M}$ as defined in (5.4). Then, for any fixed $A > 0$,*

$$\Phi_{\mu*\mu}(\rho\,\mathrm{e}(\theta)) \ll_A \frac{X}{(\log X)^A}.$$

Proof Since $\rho = \exp(-1/X)$, it follows from (1.10) that

$$\Phi_{\mu*\mu}(\rho\,\mathrm{e}(\theta)) = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{j}(\mu*\mu)(n)e^{-nj/X}\,\mathrm{e}(jn\theta).$$

We replace the exponentially damped term above by the integral

$$e^{-nj/X} = \int_n^{\infty} jX^{-1}e^{-tj/X}\,dt,$$

leaving us with

$$\Phi_{\mu*\mu}(\rho\,\mathrm{e}(\theta)) = \sum_{j=1}^{\infty}\frac{1}{j}\int_1^{\infty} jX^{-1}e^{-tj/X}\sum_{n\leqslant t}(\mu*\mu)(n)\,\mathrm{e}(jn\theta)\,dt.$$

Suppose $J = (\log X)^{2A}$. Trivially, we obtain

$$\begin{aligned}\sum_{j>J}\frac{1}{j}\int_1^{\infty} jX^{-1}e^{-tj/X}\sum_{n\leqslant t}(\mu*\mu)(n)\,\mathrm{e}(jn\theta)\,dt &\\ \ll \sum_{j>J}\frac{1}{j}\int_1^{\infty} jX^{-1}e^{-tj/X}t\log t\,dt &\\ \ll \sum_{j>J}\frac{X\log X}{j^2}\int_{j/X}^{\infty} ue^{-u}\log u\,du \ll_A \frac{X}{(\log X)^A}.&\end{aligned} \tag{6.8}$$

Hence, in what follows, it suffices to work with the sum

$$\sum_{j=1}^{J}\frac{1}{j}\int_1^{\infty} jX^{-1}e^{-tj/X}\sum_{n\leqslant t}(\mu*\mu)(n)\,\mathrm{e}(jn\theta)\,dt. \tag{6.9}$$

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Writing $\theta = a/q + \beta$ with $q \leqslant (\log X)^A$ and $|\beta| \leqslant \delta_q \leqslant X^{-1}(\log X)^A$, one has

$$\theta_j := j\theta = \frac{aj/(j,q)}{q/(j,q)} + j\beta = \frac{a_j}{q_j} + \beta_j,$$

where $(a_j, q_j) = 1$, $1 \leqslant q_j \leqslant q \leqslant (\log X)^A$ and $\beta_j \leqslant X^{-1}(\log X)^{3A}$. We can now break the inner integral in (6.9) as

$$\begin{aligned} I &= \int_1^\infty jX^{-1}e^{-tj/X} \sum_{n\leqslant t} (\mu * \mu)(n)\, \mathrm{e}(jn\theta)\, dt \\ &= \left(\int_1^{X\log X} + \int_{X\log X}^\infty \right) jX^{-1}e^{-tj/X} \sum_{n\leqslant t} (\mu * \mu)(n)\, \mathrm{e}(jn\theta)\, dt = I_1 + I_2, \end{aligned}$$

say. For I_2, one has

(6.10)
$$I_2 \ll \int_{X\log X}^\infty jX^{-1}e^{-tj/X} t \log t\, dt \ll \frac{X\log X}{j} \int_{\log X}^\infty ue^{-u} \log u\, du \ll_A \frac{X}{(\log X)^{3A}}.$$

For I_1, an application of Lemma 6.2 shows that

$$\begin{aligned} I_1 &\ll_A X\log X \exp(-C_A\sqrt{\log X}) \int_1^{X\log X} jX^{-1}e^{-tj/X}\, dt \\ &\ll_A X\log X \exp(-C_A\sqrt{\log X}) \int_1^{X\log X} e^{-u}\, du \ll_A \frac{X}{(\log X)^{3A}}, \end{aligned} \tag{6.11}$$

where $C_A > 0$ is a constant depending only on A. Putting together (6.10) and (6.11), we obtain

$$\sum_{j=1}^J \frac{1}{j} \int_1^\infty jX^{-1}e^{-tj/X} \sum_{n\leqslant t} (\mu * \mu)(n)\, \mathrm{e}(jn\theta)\, dt \ll_A \frac{X}{(\log X)^A}.$$

Combining this with (6.8), we arrive at our desired conclusion. ∎

Remark 6.1 The above arguments and the statements of Lemmas 6.1, 6.2, and 6.3 hold for a general μ_k with appropriate changes.

7 Minor Arcs

In general, being able to show that the contribution of the minor arcs is of lower order than the principal arcs is the most challenging aspect of successfully deploying the circle method. In situations such as [2, 11], which involved partitioning with respect to powers or powers in arithmetic progressions, one requires Weyl's inequality [31, Lemma 2.4] in conjunction with Hua's inequality [31, Theorem 4.1]. Adapting Weyl's inequality was also a key aspect for partitioning with respect to polynomials in [10]. Restricted partitions with respect to primes [32] necessitated Vinogradov's inequality [31, Chapter 3]. However, Vinogradov's inequality was not sufficient for functions of primes. In [12], Gafni employed a bound by Kawada and Wooley [15], and in [8], a new bound altogether for exponential sums dealing with semiprimes was introduced and proved. This was also the case in [3, §6] where a bound for exponential

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sums twisted by generalized divisor functions had to be supplied. In our current situation, the exponential sums discussed in Section 1 and proved in Sections 3 and 4 will serve as the main technology to bound the contributions from the minor arcs. It is worth remarking that interesting exponential sums for minor arcs can also be found in [20].

We begin by recording the immediate corollary from Theorem 1.2 for the case when $k = 2$.

Corollary 7.1 *Let $\alpha \in \mathbb{R}$ and consider a reduced fraction a/q such that $|\alpha - a/q| \leqslant 1/q^2$. Then, for $X \geqslant 2$ and any fixed $\varepsilon > 0$, we have*

$$\sum_{n \leqslant X} (\mu * \mu)(n)\, \mathrm{e}(n\alpha) \ll_{\varepsilon} X^{16/17+\varepsilon} + \left(\frac{X}{q^{1/6}} + X^{7/8} q^{1/8}\right)(\log X)^3.$$

Now we are ready to prove (5.6) for $\theta \in \mathfrak{m}$.

Lemma 7.2 *Let $\theta \in \mathfrak{m}$ as defined in (5.4). Then, for any fixed $A > 0$,*

$$\Phi_{\mu*\mu}(\rho\, \mathrm{e}(\theta)) \ll_A \frac{X}{(\log X)^{A/9}}.$$

Proof As in the proof of Lemma 6.3, it suffices to only consider the sum

$$\sum_{j=1}^{J} \frac{1}{j} \int_1^{\infty} jX^{-1} e^{-tj/X} \sum_{n \leqslant t} (\mu * \mu)(n)\, \mathrm{e}(jn\theta)\, dt,$$

where $J = (\log X)^{2A}$. For each $j \leqslant J$, we employ Dirichlet's theorem (see [31, Lemma 2.1]) to choose $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$ such that

$$\left| j\theta - \frac{a}{q} \right| \leqslant q^{-1} X^{-1} (\log X)^A \quad \text{and} \quad q < X(\log X)^{-A}.$$

We now set $a_j := a/(a, j)$ and $q_j := jq/(a, j)$. The definition of δ_q in (5.2) implies that

$$\left| \alpha - \frac{a_j}{q_j} \right| \leqslant \delta_{q_j}.$$

Since $\alpha \in \mathfrak{m}$, it follows that $q_j > Q$, where $Q = (\log X)^A$. Now, applying Corollary 7.1, we obtain

$$\sum_{n \leqslant t} (\mu * \mu)(n)\, \mathrm{e}(jn\theta) \ll_{\varepsilon} t^{16/17+\varepsilon} + \frac{t(\log t)^3}{q^{1/6}} + t^{7/8} q^{1/8} (\log t)^3. \tag{7.1}$$

Integrating each of the three terms on the right-hand side of (7.1) yields

$$E_1(X, j) = \int_1^{\infty} t^{16/17+\varepsilon} jX^{-1} e^{-tj/X}\, dt \ll_{\varepsilon} \left(\frac{X}{j}\right)^{16/17+\varepsilon},$$

$$E_2(X, j) = \int_1^{\infty} \frac{t}{q^{1/6}} (\log t)^3 jX^{-1} e^{-tj/X}\, dt \ll \frac{X}{jq^{1/6}} (\log X)^3,$$

$$E_3(X, j) = \int_1^{\infty} t^{\frac{7}{8}} q^{\frac{1}{8}} (\log t)^3 jX^{-1} e^{-tj/X}\, dt \ll \left(\frac{X}{j}\right)^{\frac{7}{8}} q^{1/8} (\log X)^3.$$

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Note that $jq \geqslant q_j > (\log X)^A$, and by our choice, $q < X(\log X)^{-A}$. Therefore, we conclude that

$$\sum_{j=1}^{J} \frac{1}{j} \int_1^{\infty} jX^{-1} e^{-tj/X} \sum_{n \leqslant t} (\mu * \mu)(n)\, \mathrm{e}(jn\theta)\, dt \ll \sum_{j=1}^{J} \frac{1}{j}(E_1 + E_2 + E_3) \ll \frac{X}{(\log X)^{A/9}},$$

which is the bound that we wanted to show. ∎

Remark 7.1 The statement of Lemma 7.2 holds more generally for μ_k by applying Theorem 1.2 appropriately in place of Corollary 7.1.

8 Proof of Theorem 1.7

In order to prove Theorem 1.7, we need to investigate how $\Phi_{\mu*\mu}(z)$ behaves in the major arc and examine the effect that the nontrivial zeros of the Riemann zeta-function have on its behavior. We break our proof into several steps.

Initial setup

Recall that

$$\Phi_{\mu*\mu}(\rho\, \mathrm{e}(\theta)) = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\mu * \mu)(n)}{j} \exp\left(- jn\left(\frac{1}{X} - 2\pi i\theta \right) \right), \tag{8.1}$$

where $\rho = \exp(-1/X)$. We first apply the Cahen-Mellin formula and rewrite (8.1) to obtain

$$\Phi_{\mu*\mu}(\rho\, \mathrm{e}(\theta)) = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\mu * \mu)(n)}{j} \frac{1}{2\pi i} \int_{(c)} \Gamma(s) j^{-s} n^{-s} \left(\frac{X}{1 - 2\pi i X\theta} \right)^s ds,$$

for $c > 0$. Interchanging the summations and the integration, for any real $c > 1$, we have

$$\begin{aligned} \Phi_{\mu*\mu}(\rho\, \mathrm{e}(\theta)) &= \frac{1}{2\pi i} \int_{(c)} \Gamma(s) \left(\sum_{j=1}^{\infty} \frac{1}{j^{s+1}} \right) \left(\sum_{n=1}^{\infty} \frac{(\mu * \mu)(n)}{n^s} \right) \left(\frac{X}{1 - 2\pi i X\theta} \right)^s ds \\ &= \frac{1}{2\pi i} \int_{(c)} \Gamma(s) \zeta(s+1) \frac{1}{\zeta^2(s)} \left(\frac{X}{1 - 2\pi i X\theta} \right)^s ds. \end{aligned} \tag{8.2}$$

Contour shifting.

The path of integration in 8.2 is a vertical line with $\mathrm{Re}(s) = c > 1$. We move the portion $|t| \leqslant T$ of this path to the left, replacing it by a rectangular path joining the points $c \pm iT$ and $-u \pm iT$, where $u = 2N + 1$, for some large $N \in \mathbb{N}$. Thus, the new path of integration is of the form $L = \bigcup_{i=1}^{5} L_i$, with $L_1 = (c - i\infty, c - iT]$, $L_2 = [c - iT, -u - iT]$, $L_3 = [-u - iT, -u + iT]$, $L_4 = [-u + iT, c + iT]$ and $L_5 = [c + iT, c + i\infty)$. With this change of the path of integration, an application of the Cauchy residue theorem shows that

$$\Phi_{\mu*\mu}(\rho\, \mathrm{e}(\theta)) = M + \frac{1}{2\pi i} \sum_{j=1}^{5} I_j,$$

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where M denotes the contribution of the residues at singularities of the integrand in the region enclosed by the two paths and I_j denotes the integral over the path L_j. Our method will require an analysis of the I_j's. Equipped with Lemma 2.8, we will establish our results unconditionally. For similar techniques, the reader is referred to the works of Bartz [1], Ramachandra and Sankaranarayanan [25], and Inoue [14].

Residue computations

Set

$$y := \frac{X}{1-2\pi i X\theta}.$$

A singularity analysis regarding the integrand in (8.2) shows the following. We have a double pole at $s=0$ coming from $\Gamma(s)\zeta(s+1)$ for which

$$\operatorname*{res}_{s=0}\Gamma(s)\zeta(s+1)\frac{1}{\zeta^2(s)}y^s = 4\log y - 8\log(2\pi).$$

The simple pole at $s=-1$ yields

$$\operatorname*{res}_{s=-1}\Gamma(s)\zeta(s+1)\frac{1}{\zeta^2(s)}y^s = -\frac{\zeta(0)}{\zeta^2(-1)y} = \frac{72}{y}.$$

Unlike previous cases in the literature [2, 3, 8, 10, 11, 12], we now capture residues from the trivial and nontrivial zeros of the Riemann zeta-function. We start with the nontrivial zeros. Those take place at $s=\varpi$ and yield double poles for which

$$\operatorname*{res}_{s=\varpi}\Gamma(s)\zeta(s+1)\frac{1}{\zeta^2(s)}y^s = y^{\varpi}\Gamma(\varpi)\frac{\zeta(1+\varpi)}{\zeta'(\varpi)^2}\left(\log y + \psi(\varpi) - \frac{\zeta''}{\zeta'}(\varpi) + \frac{\zeta'}{\zeta}(1+\varpi)\right),$$

where we have used the simplicity of the zeros to alleviate the notation. For the trivial zeros, we have triple poles at $s=-2n$, for $n=1,2,3,\dots$, coming from $\Gamma(s)/\zeta^2(s)$. These residues can be written as

$$\operatorname*{res}_{s=-2n}\Gamma(s)\zeta(s+1)\frac{1}{\zeta^2(s)}y^s = y^{-2n}(\mathfrak{c}_1(n)(\log y)^2 + \mathfrak{c}_2(n)\log y + \mathfrak{c}_3(n)),$$

where the coefficients are given by

$$\mathfrak{c}_1(n) := -\frac{B_{2n}}{8n^2\Gamma(2n)\zeta'(-2n)^2}, \tag{8.3}$$

$$\mathfrak{c}_2(n) := \frac{B_{2n}\zeta''(-2n) + \zeta'(-2n)\left(2n\zeta'(1-2n) - B_{2n}\psi(2n+1)\right)}{2n(2n)!\zeta'(-2n)^3}, \tag{8.4}$$

and

$$\begin{aligned}\mathfrak{c}_3(n) := {}& \frac{1}{48n^2\Gamma(2n)\zeta'(-2n)^4}\\ &\times\Big[B_{2n}\Big\{-9\zeta''(-2n)^2 - 2\left(3\psi(2n+1)^2 - 3\psi'(2n+1) + \pi^2\right)\zeta'(-2n)^2\\ &\quad + 4\zeta'(-2n)(\zeta'''(-2n) + 3\psi(2n+1)\zeta''(-2n))\Big\}\\ &\quad + 12n\zeta'(-2n)\left(\zeta'(-2n)\zeta''(1-2n) + 2\zeta'(1-2n)\left(\psi(2n+1)\zeta'(-2n) - \zeta''(-2n)\right)\right)\Big].\end{aligned} \tag{8.5}$$

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Here, $\psi(x) = \Gamma'/\Gamma(x)$ denotes the digamma function and B_n denotes the Bernoulli numbers.

Estimating the vertical integral I_3

Using Stirling's formula, for $\mathrm{Re}(s) \leqslant -3/2$,

$$\Gamma(s)\zeta(s+1)\zeta(s) \ll (2\pi)^{2\,\mathrm{Re}(s)}|s|^{-\mathrm{Re}(s)-1/2}e^{-\pi|t|/2}. \tag{8.6}$$

Therefore, we have

$$\begin{aligned} I_3 = \int_{-u-iT}^{-u+iT} \Gamma(s)\zeta(s+1)\frac{1}{\zeta^2(s)}y^s\,ds &\ll \int_{-u-iT}^{-u+iT} \frac{(2\pi)^{-2u}|s|^{u-1/2}e^{-\pi|t|/2}}{\zeta^3(s)}y^s\,ds \\ &\ll \frac{1}{(2\pi)^{2u}}\int_{-u-iT}^{-u+iT} \frac{(u^2+|t|^2)^{u/2-1/4}e^{-\pi|t|/2}}{\zeta^3(s)}y^s\,ds. \end{aligned} \tag{8.7}$$

We apply the functional equation for $\zeta(s)$ in (8.7) alongside the observation that $\zeta^{-1}(1-s) \ll 1$. Also, following [33] (See Lemma 2.8),

$$|y|^s \leqslant (X\Delta)^{-u}e^{-|t|(\Delta-\pi/2)}.$$

Hence, we obtain

$$I_3 \ll \frac{2^{3u}\pi^{3(u-1)}}{(2\pi)^{2u}(X\Delta)^u}\int_{-u-iT}^{-u+iT} \frac{(u^2+|t|^2)^{u/2-1/4}e^{-|t|\Delta}}{\sin^3(\pi s/2)\Gamma^3(1-s)}\,ds. \tag{8.8}$$

Next, we note that in our range of integration,

$$\frac{1}{\Gamma(1-s)} \ll e^{(u+1)-(u+\frac{1}{2})\log(u+1)+\frac{1}{2}\pi|t|} \quad \text{and} \quad \frac{1}{\sin(\pi s/2)} \ll e^{-\frac{1}{2}\pi|t|}.$$

Inserting these two bounds in (8.8), we arrive at

$$\begin{aligned} I_3 &\ll \frac{(2\pi)^u e^{3(u+1)-3(u+\frac{1}{2})\log(u+1)}}{(X\Delta)^u}\int_{-T}^{T}(u^2+|t|^2)^{u/2-1/4}e^{-|t|\Delta}\,dt, \\ &\ll \frac{(2\pi)^u e^{3(u+1)-3(u+\frac{1}{2})\log(u+1)}}{(X\Delta)^u}\int_{0}^{T}(u^2+t^2)^{u/2-1/4}e^{-t\Delta}\,dt. \end{aligned} \tag{8.9}$$

We consider the inside integral in (8.9). We divide this into two cases. When $t \leqslant u$, we have

$$\int_0^T (u^2+t^2)^{u/2-1/4}e^{-t\Delta}\,dt \ll 2^{u/2}u^{u-1/2}\int_0^\infty e^{-t\Delta}\,dt \ll 2^{u/2}u^{u-1/2}\Delta^{-1}. \tag{8.10}$$

However, when $t > u$,

$$\int_0^T (u^2+t^2)^{u/2-1/4}e^{-t\Delta}\,dt \ll 2^{u/2}\int_0^\infty t^{u-1/2}e^{-t\Delta}\,dt \ll 2^{u/2}\Gamma(u+1/2)\Delta^{-u-1/2}. \tag{8.11}$$

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Therefore, applying (8.10) and (8.11) in (8.9) and using Stirling's approximation for $\Gamma(u+1/2)$, we obtain

$$I_3 \ll \frac{(2\sqrt{2}\pi)^u e^{3(u+1)-3(u+\frac{1}{2})\log(u+1)}}{(X\Delta)^u}(u^{u-1/2}\Delta^{-1}+\Gamma(u+1/2)\Delta^{-u-1/2})$$

$$\ll \left(\frac{2\sqrt{2}\pi e^2}{X\Delta^2}\right)^u (u^{-2u-3/2}\Delta^{-1/2}). \tag{8.12}$$

Estimating the horizontal integrals I_2 and I_4.

We consider I_4 only. The treatment for I_2 is similar. Dividing the contour into two parts, we write

$$I_4 = \int_{-u+iT}^{-3/2+iT} + \int_{-3/2+iT}^{c+iT} = I_{4,1} + I_{4,2},$$

say. We work with $I_{4,1}$ first. Applying (8.6), we have

$$I_{4,1} = \int_{-u+iT}^{-3/2+iT} \Gamma(s)\zeta(s+1)\frac{1}{\zeta^2(s)} y^s\, ds \ll \int_{-u+iT}^{-3/2+iT} \frac{(2\pi)^{2\sigma}|s|^{-\sigma-1/2}e^{-\pi T/2}}{\zeta^3(s)} y^s\, ds$$

$$\ll e^{-\pi T/2}\int_{-u+iT}^{-3/2+iT} \frac{(2\pi)^{2\sigma}(\sigma^2+T^2)^{-\sigma/2-1/4}}{\zeta^3(s)} y^s\, ds. \tag{8.13}$$

Proceeding as in the case of I_3, we apply the functional equation for $\zeta(s)$ in (8.13) to obtain

$$I_{4,1} \ll e^{-T\Delta}\int_{-u+iT}^{-3/2+iT} \frac{(2\pi)^{2\sigma}(X\Delta)^{\sigma}(\sigma^2+T^2)^{-\sigma/2-1/4}}{2^{3\sigma}\pi^{3(\sigma-1)}\sin^3(\pi s/2)\Gamma^3(1-s)}\, ds.$$

In our range of integration,

$$\frac{1}{\Gamma(1-s)} \ll e^{1-\sigma-(\frac{1}{2}-\sigma)\log(1-\sigma)+\frac{1}{2}\pi T} \quad \text{as well as} \quad \frac{1}{\sin(\pi s/2)} \ll e^{-\frac{1}{2}\pi T}.$$

Therefore, we arrive at

$$I_{4,1} \ll e^{-T\Delta}\int_{-u}^{-3/2}(2\pi)^{-\sigma}(X\Delta)^{\sigma}(\sigma^2+T^2)^{-\sigma/2-1/4}e^{3(1-\sigma)-3(\frac{1}{2}-\sigma)\log(1-\sigma)}\, d\sigma$$

$$\ll e^{-T\Delta}\int_{3/2}^{u}\left(\frac{2\pi e^3}{X\Delta}\right)^{\sigma}\frac{(\sigma^2+T^2)^{\sigma/2-1/4}}{(\sigma+1)^{3(\sigma+\frac{1}{2})}}\, d\sigma. \tag{8.14}$$

We write (8.14) as

$$e^{-T\Delta}\int_{3/2}^{u}\left(\frac{2\pi e^3}{X\Delta}\right)^{\sigma}\frac{(\sigma^2+T^2)^{\sigma/2-1/4}}{(\sigma+1)^{3(\sigma+\frac{1}{2})}}\, d\sigma = e^{-T\Delta}\int_{3/2}^{T} + e^{-T\Delta}\int_{T}^{u} = J_1 + J_2,$$

say. Consider J_1 first. We have

$$J_1 \ll e^{-T\Delta}\int_{3/2}^{T}\left(\frac{2\sqrt{2}\pi e^3 T}{\sigma^3}\right)^{\sigma} d\sigma.$$

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Let f be defined as

$$f(\sigma) = \left(\frac{2\sqrt{2}\pi e^3 T}{\sigma^3}\right)^{\sigma},$$

and suppose $g(\sigma) = \log f(\sigma)$. The maximum of $g(\sigma)$ is attained at $\sigma_0 = (2\sqrt{2}\pi T)^{1/3}$. Hence, the maximum value attained by f in our range of integration is $\exp(3(2\sqrt{2}\pi T)^{1/3})$. Therefore, we have

$$J_1 \ll e^{-T\Delta} e^{3(2\sqrt{2}\pi T)^{1/3}} T \ll e^{-T\Delta/2}. \tag{8.15}$$

For J_2, we obtain

$$e^{-T\Delta} \int_T^u \left(\frac{2\pi e^3}{X\Delta}\right)^{\sigma} \frac{(\sigma^2+T^2)^{\sigma/2-1/4}}{(\sigma+1)^{3(\sigma+\frac{1}{2})}}\, d\sigma \ll e^{-T\Delta} \int_T^u \left(\frac{2\sqrt{2}\pi e^3}{\sigma^2}\right)^{\sigma} d\sigma \ll e^{-T\Delta/2}. \tag{8.16}$$

Now, we consider $I_{4,2}$, which is given by

$$I_{4,2} = \int_{-3/2+iT}^{c+iT} \Gamma(s)\zeta(s+1)\frac{1}{\zeta^2(s)} y^s\, ds.$$

In our range of integration,

$$\Gamma(s) \ll T^{\sigma-1/2} e^{-\pi T/2}, \quad \text{and} \quad \zeta(s+1) \ll T^{\eta}, \tag{8.17}$$

for some fixed constant $\eta > 0$. We write

$$I_{4,2} = \left(\int_{-3/2+iT}^{-1+iT} + \int_{-1+iT}^{c+iT}\right)\Gamma(s)\zeta(s+1)\frac{1}{\zeta^2(s)} y^s\, ds.$$

For the first part, we apply the functional equation for $\zeta(s)$ to obtain

$$\begin{aligned}\int_{-3/2+iT}^{-1+iT} \Gamma(s)\zeta(s+1)\frac{1}{\zeta^2(s)} y^s\, ds &\ll \frac{T^{\eta-1/2}}{e^{T\Delta}} \int_{-3/2}^{-1} \left(\frac{X\Delta T}{2\pi e}\right)^{\sigma} \frac{1}{(1-\sigma)^{1/2-\sigma}}\, d\sigma \\ &\ll \frac{T^{\eta-1/2}}{e^{T\Delta}} \int_1^{3/2} \left(\frac{2\pi e}{X\Delta T}\right)^{\sigma} \frac{1}{(1+\sigma)^{1/2+\sigma}}\, d\sigma \ll e^{-T\Delta/2}.\end{aligned} \tag{8.18}$$

For the second part, using Lemma 2.8, for $-1 \leqslant \mathrm{Re}(s) = \sigma \leqslant 2$ and any $\varepsilon > 0$, for sufficiently large $V > 0$, there exists some $V \leqslant T_* \leqslant 2V$ such that

$$\frac{1}{\zeta^2(\sigma + iT_*)} \ll T_*^{\varepsilon}. \tag{8.19}$$

Therefore, using the bounds (8.17) and (8.19) and choosing $T = T_*$, we arrive at

$$\int_{-1+iT}^{c+iT} \Gamma(s)\zeta(s+1)\frac{1}{\zeta^2(s)} y^s\, ds \ll \frac{T^{-1/2+\eta+\varepsilon}}{e^{T\Delta}} \int_{-1}^{c} (X\Delta T)^{\sigma}\, d\sigma \ll e^{-T\Delta/2}. \tag{8.20}$$

Finally, combining the estimates (8.15), (8.16), (8.18), and (8.20), we conclude that $I_4 \ll e^{-T\Delta/2}$.

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Estimating the vertical integrals I_1 and I_5.

Writing

$$I_1 = \int_{c+iT}^{c+i\infty} \Gamma(s)\zeta(s+1)\frac{1}{\zeta^2(s)} y^s \, ds,$$

we note that in our range of integration,

$$\Gamma(s) \ll e^{-\pi T/2}, \quad \zeta(s+1) \ll 1, \quad \zeta^{-2}(s) \ll 1, \quad \text{and} \quad |y|^s \leqslant (X\Delta)^\sigma e^{-T(\Delta-\pi/2)}.$$

Combining these estimates, we obtain $I_1 \ll e^{-T\Delta/2}$. A similar argument holds true for I_5.

Final steps

Combining the estimates from the integrals and letting $N \to \infty$, we obtain

$$\begin{aligned}\Phi_{\mu*\mu}(\rho\, \mathrm{e}(\theta)) &= 4\log\frac{X}{1-2\pi iX\theta} - 8\log(2\pi) + 72\frac{1-2\pi iX\theta}{X} \\ &\quad + \sum_{|\operatorname{Im}(\varpi)|<T_\nu} f(X,\theta,\varpi) + \sum_{n=1}^{\infty} g(X,\theta,n) + O(e^{-T_\nu\Delta/2}),\end{aligned} \tag{8.21}$$

where $\nu \leqslant T_\nu \leqslant 2\nu$, $\nu \in \mathbb{N}$, the sum over ϖ runs along all the nontrivial zeros of ζ, and f, g are as defined in the statement of the theorem. Finally, we let $\nu \to \infty$ and choose our sequence of T_ν appropriately such that (8.19) is satisfied. We conclude that

$$\begin{aligned}\Phi_{\mu*\mu}(\rho\, \mathrm{e}(\theta)) &= 4\log\frac{X}{1-2\pi iX\theta} - 8\log(2\pi) + 72\frac{1-2\pi iX\theta}{X} \\ &\quad + \lim_{\nu\to\infty} \sum_{|\operatorname{Im}(\varpi)|<T_\nu} f(X,\theta,\varpi) + \sum_{n=1}^{\infty} g(X,\theta,n),\end{aligned}$$

thereby finishing the last step of the proof.

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