INITIAL STRESS AND ELASTIC INSTABILITY

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By a well-known theorem of Kirchhoff the strain energy of an elastic solid is less in the equilibrium position than in any other position satisfying the same boundary conditions, and under the same body forces. The theorem contradicts the fact that elastic instability can occur, since two or more positions of equilibrium can then exist, and both cannot have the smaller strain energy. Numerous writers (Bryan (1), Southwell (2), Dean (3)) have explained the apparent discrepancy as due to the neglect of second-order terms in the elastic equations. This is correct so far as it goes, but it does not explain why the usual discussions of elastic instability give the right answers. The elastic constants vary somewhat with stress and in any case will be different according as a stressed or an unstressed state is taken as the standard. If any second-order terms should be included, we might expect that this variation would make a contribution comparable with those hitherto considered. Further, several theories of elasticity now exist that differ in the higher terms (Dean (3), Seth (8), Murnaghan (6)), and it may be asked whether they should lead to the same estimates of the critical loads.

Let us consider the stress in a body initially stressed, but consider only displacements from this stressed state whose squares can be neglected. The stress p_{ik} at a given point x_i after displacement will differ from that before displacement in three ways. We notice that to any order of accuracy a rigid body displacement requires no change of stress with respect to axes defined by the same particles. The matter near x_i after displacement may be regarded as having reached that region by (1) a translation u_i equal to the displacement of the particle that reaches x_i, (2) a rotation specified by the antisymmetrical tensor ξ_{ik}, (3) distortions expressed by the strain tensor e_{ik} (notation as in Jeffreys (5), chapter 8). We may consider these as applied in turn. The first two simply transport the initial stress, the third requires additional stresses. Before discussing the first two it is convenient to consider the following question: if the stress distribution at x_i is given by p_{ik}, what is the stress at $x_i + u_i$ with reference to coordinate planes rotated according to the tensor ξ_{ik}? The components with respect to axes parallel to the original ones are

$$p'_{ik} = p_{ik} + u_m \frac{\partial p_{ik}}{\partial x_m}, \tag{1}$$

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If the suffixes j, l refer to the rotated planes, the direction cosines of the further transformation are $\delta_{ij}-\xi_{ij}$, and the required components are

$$p''_{jl} = (\delta_{ij}-\xi_{ij})(\delta_{kl}-\xi_{kl})\,p'_{ik} = p'_{jl}-\xi_{ij}p'_{il}-\xi_{kl}p'_{jk}$$
$$= p_{jl}+u_m\frac{\partial p_{jl}}{\partial x_m}-\xi_{ij}p_{il}-\xi_{kl}p_{jk}, \tag{2}$$

to the first order in the new displacements.

The stress transported from x_i-u_i will be obtained by giving p_{jl} the initial values denoted by p^0_{jl} and reversing u and ξ. Hence it is

$$p^0_{jl}-u_m\frac{\partial p^0_{jl}}{\partial x_m}+\xi_{ij}p^0_{il}+\xi_{k.}\,p^0_{jk}. \tag{3}$$

The stress associated with the new strains will be, as usual,

$$p_{jl} = \lambda\frac{\partial u_m}{\partial x_m}\delta_{jl}+2\mu e_{jl} \tag{4}$$

if the material is isotropic. This, however, cannot be exact. Murnaghan (6) has given a full discussion of finite strain and shown incidentally (p. 251) that 'the assumption of isotropy forces the stress in the initial position to be of the nature of a hydrostatic pressure'. In other words, if a bar is initially compressed in the x_1 direction, it will not appear to have the same rigidity for additional stresses p_{12} and p_{23}. We therefore need, strictly, to take the general set of relations

$$p_{jl} = c_{jl,rs}e_{rs}. \tag{5}$$

But the stress needed to change the elastic constants by a large fraction of themselves is of the order of the elastic constants, and if the material is isotropic when unstressed we can write

$$p_{jl} = \lambda\frac{\partial u_m}{\partial x_m}\delta_{jl}+2\mu e_{jl}+O(p^0_{jl}e_{rs}). \tag{6}$$

In all, the stresses are, with a rearrangement of the suffixes,

$$p_{ik} = p^0_{ik}-u_m\frac{\partial p^0_{ik}}{\partial x_m}-\xi_{im}p^0_{mk}-\xi_{km}p^0_{im}+\lambda\frac{\partial u_m}{\partial x_m}\delta_{ik}+2\mu e_{ik}+O(p^0_{ik}e_{rs}). \tag{7}$$

Now suppose that p_{ik} and p^0_{ik} both satisfy the equations of equilibrium with the same body force; then at all points $\partial p'_{ik}/\partial x_k = 0$, where now $p'_{ik} = p_{ik}-p^0_{ik}$. Hence

$$0 = \int u_i\frac{\partial p'_{ik}}{\partial x_k}\,d\tau = \int l_k u_i p'_{ik}\,dS-\int p'_{ik}\frac{\partial u_i}{\partial x_k}\,d\tau \tag{8}$$

taken through the solid. The surface integral vanishes if either the displacements of the boundary or the stresses across it are given to be the same in both positions.

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By the usual method the terms in λ and μ give $-2V$, where V is the strain energy of the displacements u_i. If e is a typical strain and v is the volume, it is of order $\mu e^2 v$. The term in u_m gives

$$\int u_m e_{ik} \frac{\partial p^0_{ik}}{\partial x_m} d\tau = \int l_m u_m e_{ik} p^0_{ik} \, dS - \int p^0_{ik} \frac{\partial c}{\partial x_m} (u_m e_{ik}) \, d\tau. \tag{9}$$

For parts of the boundary where the displacement is prescribed, $u_m = 0$, and for free parts $p^0_{ik} = 0$. Hence in most cases the surface integral vanishes, and the volume integral is of order $p^0 e \xi v$ or $p^0 e^2 v$, whichever is the larger. If ξ is a typical rotation, the rotation terms are of order $\xi p^0 e v$. The remainder due to departure from isotropy is of order $p^0 e^2 v$. Then since with ordinary substances the initial stress is small compared with the elastic moduli, the equality (8) as between the two suggested positions of equilibrium can be satisfied only if e/ξ is of order p^0/μ, and therefore is small for ordinary solids. The other terms are all small compared with V. In discussion of elastic instability the rotation terms are the only part of the allowance for the initial stress that need consideration; and it can arise only if the form of the body is such that in some possible displacements the rotations are much larger than the strains. This is, I think, a little clearer than, for instance, Love's statement ((7), p. 30) that 'large *displacements* must be accompanied by very small strains'. It is satisfied, for instance, in the simple problem of the buckling of a beam under thrust at the ends. If the curvature of the median section is y'', the length $2l$ and the thickness $2h$, the strains are of order hy'' and the rotations of order ly'', which is much larger for a thin beam.

Love ((7), p. 108) quotes a form of allowance for initial stress, due to Cauchy, which in the present notation would be

$$p^0_{ik}\left(1 - \frac{\partial u_m}{\partial x_m}\right) + p^0_{im} \frac{\partial u_k}{\partial x_m} + p^0_{km} \frac{\partial u_i}{\partial x_m}. \tag{10}$$

Several derivations of this result have been given and severely criticized. We may notice that it does not contain the term $-u_m \partial p^0_{ik}/\partial x_m$ and therefore, as Love remarks, seems to be limited to the case of initial homogeneous stress. Apart from this, on account of the relations

$$\frac{\partial u_i}{\partial x_m} = -\xi_{im} + e_{im}, \tag{11}$$

the difference between Cauchy's formula and the corresponding terms in (7) is

$$-p^0_{ik} \frac{\partial u_m}{\partial x_m} + p^0_{im} e_{km} + p^0_{km} e_{im}, \tag{12}$$

which is of the order of magnitude of the remainder term. Hence the difference between the two formulae would make no practical difference. It does illustrate

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Murnaghan's result, since (12) cannot in general be put in the form (4), and it gives one possible form of the allowance for the variation of the elastic properties with the initial stress, since it is a tensor. But the general argument shows that within the accuracy of experiment and, usually, of the approximations made in the calculation, elastic instability depends only on the elastic constants for small strains, their variation with the strain not needing to be considered.

The form (7) without the remainder term may be useful in application to special problems because it avoids the need to use the second order theory every time. In problems where the initial stress is large and nearly hydrostatic, and the form is such that the rotations must be comparable with the strains, as in many geophysical problems, we see that the initial stress can be taken as hydrostatic without serious error. Equations for this case have already been given (Jeffreys (4)).

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