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λ-Operations in β-rings

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Abstract

A β-ring is supplied with operations β_H where H runs over the conjugacy classes of subgroups of the symmetric groups S_n. In an earlier paper we introduced a second set of operations λ_H and we show here that the two sets are related by the isomorphism

$$\beta_H(-X) \cong (-1)^n \lambda_H(X).$$

We then consider the operations β_H and λ_H as combinatorial species, in the sense of Joyal, and express their molecular decomposition as a finite sum of products of the exponential species with molecular species of degree at most n. We give combinatorial interpretations for β_{S_n}-structures and λ_{S_n}-structures and derive various species isomorphisms.

1. *Introduction*

The concept of a β-ring was introduced by Boorman [**2**], developed by Rymer [**10**] and Ochoa [**9**], given a sound basis by Joyal [**4**] and generalized to other representation functors by Vallejo [**11**]. In Joyal's formulation a β-ring is a commutative ring R supplied with an n-ary operation for each isomorphism class of bounded, virtual, n-ary species. Sum, product and substitution formulae for virtual species give rise to identities between these operations. In earlier versions of the theory only unary operations were defined, and in this paper we shall restrict our attention to these.

Denote by $R(S_n)$ and $A(S_n)$ the representation ring and the Burnside ring of the symmetric group S_n. The graded ring $A(S) = \oplus_{n \geqslant 0} A(S_n)$ is the free β-ring on one generator and plays the same fundamental role in the theory of β-rings that $R(S) = \oplus_{n \geqslant 0} R(S_n)$ does in the theory of λ-rings.

The Burnside rings $A(G)$ of finite groups G provide important examples of β-rings. Every transitive G-set is isomorphic to a set of left cosets (G/H) for some subgroup $H \leqslant G$. As a free Abelian group, $A(G)$ is generated by the set $\mathscr{T}_G$ of isomorphism classes $[G/H]$ of transitive G-sets. The assignment $[G/H] \mapsto [H]$ defines a bijection between $\mathscr{T}_G$ and the poset P_G of conjugacy classes of subgroups of G. When $G = S_n$ there is a second bijection Θ_n between $\mathscr{T}_{S_n}$ and the set $\mathscr{M}_n$ of isomorphism classes of molecular (or transitive) species of degree n.

The molecular species $[\![H]\!] = \Theta_n([S_n/H])$ is defined by

$$[\![H]\!][U] = \begin{cases} [S_U/H_U] & \text{when} \quad |U| = n, \\ \varnothing & \text{otherwise,} \end{cases}$$

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where S_U is the group of permutations of U and H_U is the image of H under the isomorphism between S_n and S_U determined by some chosen bijection between $\mathbb{N}_n = \{1, 2, \ldots, n\}$ and U. The bijection Θ_n between generating sets extends to a ring homomorphism between $A(S_n)$ and $\mathscr{VS}_n$, the ring of virtual species of degree n. See Joyal[**3**] and Yeh[**13**] for the basic properties of combinatorial species and J. Labelle & Yeh[**6**] for the connection with Burnside rings.

There are two alternative notations for molecular species. Recall that an arbitrary species $\mathbb{A}$ may be split into its kth degree components, $\mathbb{A} \cong \sum_{k \geqslant 0} \mathbb{A}_k$, and that the S_k-set $\mathbb{A}_k[\mathbb{N}_k]$ may be decomposed into transitive S_k-sets,

$$\mathbb{A}_k[\mathbb{N}_k] \cong \sum_{[J] \leqslant [S_k]} a_J(S_k/J),$$

from which we deduce the molecular decomposition

$$\mathbb{A} \cong \sum_{k \geqslant 0} \sum_{[J] \leqslant [S_k]} a_J[\![J]\!].$$

There is, in general, no simple formula for the a_J, but some standard examples are

$$\mathbb{X} \cong [\![I]\!], \quad \mathbb{E}_k \cong [\![S_k]\!], \quad \mathbb{S}(r)_k \cong [\![S_{k-r} \oplus S_r]\!], \quad \mathbb{C}_k \cong [\![C_k]\!], \quad \mathbb{L}_k \cong [\![I^k]\!],$$

where $\mathbb{E}$ is the exponential species, $\mathbb{X} \cong \mathbb{E}_1$, $\mathbb{S}(r)$ is the species of subsets of cardinality r, $\mathbb{C}$ is the species of circular permutations, $\mathbb{L}$ is the species of total orders, I^k is the identity subgroup of S_k, C_k is a subgroup generated by a k-cycle and $\oplus$ denotes the standard direct product of permutation groups.

The triangular table of marks for a group G is a valuable aid to calculations in $A(G)$. For $H, K \leqslant S_n$ the Cartesian product of species $[\![H]\!] \times [\![K]\!]$ is defined by

$$([\![H]\!] \times [\![K]\!])[\mathbb{N}_n] = ([\![H]\!][\mathbb{N}_n]) \times ([\![K]\!][\mathbb{N}_n]).$$

The molecular decomposition of this species may be calculated by taking the product of the supercharacters $\phi_{S_n/H}$ and $\phi_{S_n/K}$ and using the table of marks for S_n to split this product into transitive supercharacters. Thus for identities in β-rings and for operations on transitive species we require tables of marks for the symmetric groups. Tables for S_n up to $n = 9$ are available (see [**12**]). For further results on transitive species see J. Labelle[**5**] and Yeh[**14**].

For each $[H] \leqslant [S_n]$ the unary β-ring operation β_H is defined, for all $X \in A(G)$, by $\beta_H(X) = X^n \otimes_{S_n} (S_n/H)$. Efficient methods are required for performing calculations with virtual G-sets. One such is a combinatorial description of $\beta_H(-X)$ obtained by Joyal[**4**] using a 'rule of signs', and this is considered in Section 2.

In an earlier paper [**7**] we defined two further sets of unary operations, ψ_H and λ_H, in a β-ring. Recall that a λ-ring is a commutative ring with sets of operations λ_n, β_n, ψ_n $(n \in \mathbb{N})$. Every λ-ring is a β-ring with

$$\beta_{S_n} = \beta_n, \quad \psi_{S_n} = \psi_n \quad \text{and} \quad \lambda_{S_n} = \lambda_n.$$

Joyal suggested to us that the λ-ring identity

$$\beta_n(-X) = (-1)^n \lambda_n(X)$$

should extend to a β-ring identity

$$\beta_H(-X) \cong (-1)^n \lambda_H(X),$$

thus giving a more natural definition for λ_H. The first aim of this paper is to verify

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this identity; we accomplish it in Proposition 3·1. Then, using the β-ring axiom $\beta_H \circ \beta_K = \beta_{H[K]}$, where $H[K]$ is the standard permutation representation of the wreath product $K \int H$, formulae for $\beta_H \circ \lambda_K$ and $\lambda_H \circ \lambda_K$ are easily deduced.

In order to establish Proposition 3·1 we require results from [**8**] concerning functions $\tau = \gamma^{-1}\mu\Delta\zeta\gamma$ in the incidence algebra of the subgroup lattice of a finite group G.

The coefficients in the identities $\lambda_H = \sum \tau_{HK}\beta_K$ form an upper triangular, self-inverse matrix $M(\check{\tau}_-)$ with rows and columns indexed by the conjugacy classes of subgroups of S_n. We have shown [**7**, section 7] that the eigenvalues of $M(\check{\tau}_-)$ are all ± 1 and that $M(\check{\tau}_-)$ is transformed to diagonal form by the table of marks. In the case $n = 4$ the matrix $M(\check{\tau}_-)$ is given in [**7**] and when $n = 5$ the non-zero elements appear in table 7 of [**13**] as the coefficients in the expansion of $[\![H]\!].(-[\![I]\!])$. They also appear as diagonal blocks in Table 2 below. See [**12**] for $M(\check{\tau}_-)$ up to $n = 9$.

In section 4 we specialize to $A(S_n)$ and consider β_H as a β-species whose structures are selections with repetition, partitioned according to the orbits of H. It is not a simple matter to express an arbitrary species in terms of β-species or λ-species. In order to form such isomorphisms it is useful to have a supply of species with no structures of small degree. Accordingly we show that, for each $[H] \leqslant [S_n]$, the species $\mathbb{E}.[\![H]\!]$ is isomorphic to $\sum_{k \geqslant n} [\![S_{k-n} \oplus H]\!]$, where . denotes the usual product of species. It follows from a result of Vallejo [**11**] that the molecular decomposition of β_H may be expressed as

$$\beta_H \cong \sum_{r=1}^{n} \sum_{[J] \leqslant [S_r]} b_{H,J}(\mathbb{E}.[\![J]\!])$$

and that these β-species β_H provide an alternative additive basis for the species half-ring, as do the species $\mathbb{E}.[\![H]\!]$. The virtual λ-species $\lambda_H \cong \sum_{[H]} \tau_{KH}\beta_H$ form an additive basis for the Grothendieck ring of virtual species. We give combinatorial interpretations for the λ-species λ_{S_n} and for the β-species β_{S_π}, where S_π is a Young subgroup.

In Section 5 we express the exponential species as an alternating sum of λ-species and use this to express the species of subsets $\mathbb{S} \cong \mathbb{E} \circ (2\mathbb{X})$ as an alternating sum of the composite species $\lambda_{S_n} \circ (2\mathbb{X})$ and as a linear combination of products $\beta_{S_{n-2j}}\lambda_{S_j}$. In order to do this, we describe a reduction process which replaces an S_k-orbit of β_{S_n}-structures with an isomorphic S_k-orbit of $\beta_{S_{n-2j}}\lambda_{S_j}$ structures. We conclude by giving combinatorial interpretations for $\mathbb{S}[-X]$, $\mathbb{L}[-X]$ and $\mathbb{C}[-X]$.

Symmetric powers arise in many situations and it is usually possible to use these to define operations β_H and λ_H. In algebraic topology it is G-spaces rather than G-sets which are significant, and it is in this area that λ-rings have proved a useful tool. Symmetric powers also arise in the theory of monoidal categories. There are also generalizations of species where these results may be useful. For example, Bergeron [**1**] has developed a theory of permutation species $\mathbb{P}: \mathbb{S}\mathrm{et}P \to \mathbb{S}\mathrm{et}_f$ where $\mathbb{S}\mathrm{et}P$ is the category whose objects (U, α) comprise a set U and a permutation $\alpha \in S_U$. A morphism $\phi: (U, \alpha) \to (U', \alpha')$ is a bijection $\phi: U \to U'$ satisfying $\alpha' \circ \phi = \phi \circ \alpha$. A permutation species $\mathbb{P}$ is then a functor from $\mathbb{S}\mathrm{et}P$ to the category of sets and functions. It is easily verified that for each $[H] \leqslant [S_n]$ the β-operation $\beta_H: \mathbb{S}\mathrm{et}P \to \mathbb{S}\mathrm{et}P$ is a functor, where

$$\beta_H(U, \alpha) = (U^n \otimes_{S_n} (S_n/H), \beta_H(\alpha))$$

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and

$$\beta_H(\alpha)((u_1, u_2, \ldots, u_n) \otimes (\eta H)) = ((\alpha(u_1), \alpha(u_2), \ldots, \alpha(u_n)) \otimes (\eta H)).$$

Throughout this paper we shall use the following notation in order to distinguish between partitions (unordered) and compositions (ordered) of integers and of sets. The *multiplicity* of the partition $\pi = (\pi_1, \pi_2, \ldots, \pi_p) = (1^{p_1} 2^{p_2} \ldots n^{p_n}) \vdash n$ is the partition $m(\pi) = (p_1, p_2, \ldots, p_n) \vdash p$ where $p = l(\pi)$ is the *length* of π. Similarly, $\rho = (\rho_1, \rho_2, \ldots, \rho_r) = (1^{r_1} 2^{r_2} \ldots n^{r_n}) \vdash n$. Note that, on average, most of the p_i, r_i are zero and so are omitted from m(π) and $m(\rho)$. The *sign of* π is $(-1)^\pi = (-1)^{n-p}$. A *composition* κ of n is written $\kappa = (\kappa_1; \kappa_2; \ldots; \kappa_p) \vdash n$. Forgetting the order of the parts we obtain a partition $\pi_\kappa \vdash n$.

A partition $\sigma = /\sigma_1/\sigma_2/\ldots/\sigma_p/$ of $\mathbb{N}_n$ is said to be of *type* $\pi_\sigma = \pi \vdash n$ when $|\sigma_i| = \pi_i$ for $1 \leqslant i \leqslant p$. A *composition* $\omega = \backslash\omega_1\backslash\omega_2\backslash\ldots\backslash\omega_p\backslash$ of $\mathbb{N}_n$ is said to be of type $\kappa_\omega = \kappa$ when $|\omega_i| = \kappa_i$ for $1 \leqslant i \leqslant p$. Also, ω determines a partition $\sigma_\omega = /\omega_1/\omega_2/\ldots/\omega_p/$, by forgetting the order of the parts, and hence a partition $\pi_\omega = \pi_{(\sigma_\omega)} = \pi_{(\kappa_\omega)}$ of n. The *sign* and *length* of κ, σ, ω are defined in the obvious way. The *Young subgroup* S_ω of S_n is the direct product $S_{\omega_1} \oplus S_{\omega_2} \oplus \cdots \oplus S_{\omega_p}$ and belongs to the conjugacy class $[S_{\pi_\omega}]$ in S_n.

2. *The rule of signs*

Let Γ be a finite group, Y a right Γ-set and Z a left Γ-set. Let $\sim$ be the equivalence relation on $Y \times Z$ defined by $(y^\gamma, z) \sim (y, {}^\gamma z)$ for all $\gamma \in \Gamma$. Denote by $Y \otimes_\Gamma Z$ the set of equivalence classes $y \otimes z$ for this relation. This set has a right Γ-action given by $(y \otimes z)^\gamma = y^\gamma \otimes z$.

Given a left G-set X, the n-fold Cartesian product X^n has a left G-action ${}^g(x_1, x_2, \ldots, x_n) = ({}^g x_1, {}^g x_2, \ldots, {}^g x_n)$ and a right S_n-action $(x_1, x_2, \ldots, x_n)^\eta = (x_{\eta-1(1)}, x_{\eta-1(2)}, \ldots, x_{\eta-1}(n))$. For H a permutation group of degree n, the set of left cosets S_n/H is a left S_n-set with action ${}^\eta(\eta' H) = (\eta\eta')H$. The unary operation $\beta_H : A(G) \to A(G)$ is defined on X by

$$\beta_H(X) = X^n \otimes_{S_n} (S_n/H)$$

and $\beta_H(X)$ is a left G-set with action

$${}^g((x_1, x_2, \ldots, x_n) \otimes (\eta H)) = ({}^g x_1, {}^g x_2, \ldots, {}^g x_n) \otimes (\eta H).$$

The equivalent form $\beta_H(X) = X^n/H$ is more convenient for some calculations.

It follows from the Cauchy–Frobenius lemma that the cardinality of $\beta_H(X)$ is given by the cycle index of H:

$$|\beta_H(X)| = \frac{1}{|H|} \sum_{r=1}^{n} P(H, r) |X|^r,$$

where $P(H, r)$ is the number of permutations in H comprising r disjoint cycles. Since this polynomial has a factor $|X|$ we define $\beta_H(\varnothing) = \varnothing$ for all H.

Example 2·1. As a simple illustration of the calculation of $\beta_H(X)$ consider $\beta_{C_4}(\mathbb{N}_6)$. Using the notation $\mathbb{N}_6^4/C_4$, rather than $\mathbb{N}_6^4 \otimes (S_4/C_4)$, the $6^4 = 1296$ quadruples (a, b, c, d) in $\mathbb{N}_6^4$ form orbits

$$[abcd] = \{(a, b, c, d), (b, c, d, a), (c, d, a, b), (d, a, b, c)\}.$$

When a, b, c, d are not all different the orbit may contain just 1 or 2 quadruples. The

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number of orbits is obtained by substituting $k=6$ in the cycle index $\mathrm{Cyc}(C_4)=\frac{1}{4}(k^4+k^2+2k)$, namely 336. These orbits comprise:

6 of the form [1111], [2222], ..., [6666],
30 of the form [1112], [1113], ..., [11116], [22221], ..., [6665],
15 of the form [1122], ..., [5566], and of the form [1212], ..., [5656],
60 each of the forms [1123], ... and [1132], ... and [1213], ...,
90 of the form [1234], ..., [1432], ..., [3654].

To obtain the molecular decomposition of the degree 6 component of β_{C_4}, we let S_6 act on these orbits by permuting the elements in $\mathbb{N}_6$. Orbits [1111] and [1112] have stabilizers $S_{\{2,3,4,5,6\}}$ and $S_{\{3,4,5,6\}}$ respectively, while orbits [1122] and [1212] both have stabilizers $S_{\{1,2\}}\times S_{\{3,4,5,6\}}$. Since $[1123]\neq[1132]$ but $[1312]=[1213]$, $\mathrm{Stab}_{S_6}([1123])=S_{\{4,5,6\}}$ whereas $\mathrm{Stab}_{S_6}([1213])=S_{\{2,3\}}\times S_{\{4,5,6\}}$. Finally, $\mathrm{Stab}_{S_6}([1234])=\langle(1234),(56)\rangle\cong C_4\times C_2$ and so we obtain

$$(\beta_{C_4})_6\cong[\![S_5\oplus I]\!]+[\![S_4\oplus I^2]\!]+2[\![S_4\oplus C_2]\!]+[\![S_3\oplus I^3]\!]+[\![S_3\oplus C_2\oplus I]\!]+[\![C_4\oplus C_2]\!].$$

In [**4**] Joyal defined a virtual species ϵ, described below, and obtained a formula for $\beta_H(-X)$ using the structures of ϵ on $\mathbb{N}_n$. We shall show that $\epsilon[\mathbb{N}_n]$ is a unit in $A(S_n)$ and determine its transitive constituents.

Proposition 2·2 (Joyal).

$$\beta_H(-X)=X^n\otimes_{S_n}((S_n/H)\times\epsilon[\mathbb{N}_n]).$$

Let $\mathbb{E}$ be the exponential species with generating function e^x and structures $\mathbb{E}[U]=\{U\}$ for any set U. The virtual species ϵ is defined by $\epsilon[U]=\mathbb{E}[-U]$ and is the difference $(\epsilon_+-\epsilon_-)$ of two proper species. One combinatorial description of ϵ_+ and ϵ_- is

$$\epsilon_+[\varnothing]=\{\varnothing\},\quad \epsilon_-[\varnothing]=\varnothing,$$
$$\epsilon_+[U]=\{\omega\in\Omega(U)\colon \omega \text{ has an even number of parts}\},$$
$$\epsilon_-[U]=\{\omega\in\Omega(U)\colon \omega \text{ has an odd number of parts}\},$$

where $\Omega(U)$ is the poset of compositions $\backslash\omega_1\backslash\omega_2\backslash\ldots\backslash\omega_r\backslash$ of U ordered by refinement.

The generating functions for ϵ_+ and ϵ_- have the form

$$\epsilon(x)=e^{-X}=\epsilon_+(x)-\epsilon_-(x),$$

where

$$\epsilon_+(x)=1+Ox+2(x^2/2)+6(x^3/6)+38(x^4/24)+\ldots+\epsilon_{+n}(x^n/n!)+\ldots$$

and

$$\epsilon_-(x)=x+(x^2/2)+7(x^3/6)+37(x^4/24)+\ldots+\epsilon_{-n}(x^n/n!)+\ldots.$$

The coefficients of degree n in these series are the sums of the positive and negative terms in MacMahon's formula [**8**, section 4]:

$$1=\sum_{\rho\vdash n}\frac{(-1)^{n-l(\rho)}\,r!\,n!}{(1!)^{r_1}r_1!\,(2!)^{r_2}r_2!\ldots(n!)^{r_n}r_n!}. \tag{2·1}$$

The left S_U-action on $\Omega(U)$ is given, on a typical r-part composition, by

$$\eta(\backslash u_{11}u_{12}\ldots u_{1\rho_1}\backslash u_{21}u_{22}\ldots u_{2\rho_2}\backslash\ldots\backslash u_{r1}u_{r2}\ldots u_{r\rho_r}\backslash)$$
$$=\backslash v_{11}v_{12}\ldots v_{1\rho_1}\backslash v_{21}v_{22}\ldots v_{2\rho_2}\backslash\ldots\backslash v_{r1}v_{r2}\ldots v_{r\rho_r}\backslash,$$

where $v_{ij}=\eta(u_{ij})$ for all $1\leqslant i\leqslant r$ and $1\leqslant j\leqslant\rho_i$.

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The *period* $\sigma_H = /\sigma_1/\sigma_2/\ldots/\sigma_p/$ of H is the partition of $\mathbb{N}_n$ whose parts are the orbits of H, and the *orbit-type* of H is the partition $\pi_H = (|\sigma_1|, |\sigma_2|, \ldots, |\sigma_p|)$ of n. When H is a Young subgroup S_ω it is clear that $\sigma_{S_\omega} = \sigma_\omega$ and $\pi_{S_\omega} = \pi_\omega$. Denote by $[S_\rho]$ the conjugacy class of Young subgroups with orbit-type ρ, and by $\binom{l(\rho)}{m(\rho)}$ the multinomial coefficient

$$\binom{l(\rho)}{m(\rho)} = \frac{r!}{r_1!\, r_2! \ldots r_n!} = \frac{r!}{n!}|S_\rho|\,|[S_\rho]|.$$

PROPOSITION 2·3. *There is an isomorphism of S_n-sets*

$$\epsilon[\mathbb{N}_n] \cong \sum_{\rho \vdash n} (-1)^{l(\rho)} \binom{l(\rho)}{m(\rho)} (S_n/S_\rho).$$

Proof. The set $\Omega_\rho(\mathbb{N}_n)$ of compositions ω with $\pi_\omega = \rho$ is isomorphic as an S_n-set to $|\Omega_\rho(\mathbb{N}_n)|/[S_n : S_\rho]$ copies of (S_n/S_ρ), where

$$|\Omega_\rho(\mathbb{N}_n)| = \frac{r!\, n!}{r_1!(2!)^{r_2} r_2! \ldots (n!)^{r_n}} = \binom{l(\rho)}{m(\rho)} [S_n : S_\rho].$$

The result follows since $\Omega_\rho(\mathbb{N}_n)$ is a constituent of $\epsilon_+[\mathbb{N}_n]$ or $\epsilon_-[\mathbb{N}_n]$ according as $l(\rho)$ is even or odd. ∎

Recall that the *supercharacter* ϕ_T of a G-set T is a map from the poset P_G of conjugacy classes of subgroups of G to the integers. The values $\phi_T(J)$ of this map are the marks of G. The mark $\phi_{G/H}(J)$ for the class of transitive G-sets $[G/H]$ is the number of cosets in (G/H) fixed under left multiplication by J. For an arbitrary G-set T,

$$T \cong \sum_{[H]} c_H(G/H) \Rightarrow \phi_T = \sum_{[H]} c_H \phi_{G/H}.$$

In a recent paper [**8**] we have considered conjugates $\mu\Delta\zeta$ of a diagonal function Δ in the incidence algebra of the subgroup lattice L_G of G. For the supercharacter of $\epsilon[\mathbb{N}_n]$ we take $G = S_n$ and $\Delta(J) = (-1)^{l(J)}$, where $l(J)$ is the length of the period of J.

PROPOSITION 2·4. *The S_n-set $\epsilon[\mathbb{N}_n]$ is a unit in $A(S_n)$ with marks*

$$\phi_{\epsilon[\mathbb{N}_n]}(J) = (-1)^{l(J)}.$$

Proof. This is an immediate consequence of Proposition 2·3 and [**8**, proposition 4·2].

3. *The $\check{\tau}_-$-matrix for S_n*

We now recall from [**7**] the definition of the unary operations λ_H on a β-ring R together with some related notation. Let $\mathbb{Q}CA(L_n)$ be the algebra of rational conjugacy functions on the subgroup lattice $L_n = L_{S_n}$, let $\mathbb{Q}IA(P_n)$ be the rational incidence algebra on the poset $P_n = P_{S_n}$ of conjugacy classes of subgroups of S_n, and let $f \mapsto \check{f}$ be the algebra homomorphism from $\mathbb{Q}CA(L_n)$ to $\mathbb{Q}IA(P_n)$ defined by

$$\check{f}([J],[H]) = \sum_{H' \in [H]} f(J, H').$$

Consider $\Delta_- : L_n \to \mathbb{Z}$, mapping J to $(-1)^{n-l(J)}$, as a diagonal function in $\mathbb{Q}CA(L_n)$.

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Then set $\tau_- = \gamma^{-1}\mu\Delta_-\zeta\gamma$ where ζ, μ are the zeta and Möbius functions on L_n and γ is the diagonal mark function

$$\gamma(J) = \phi_{S_n/J}(J) = n!/(|J|\,|[J]|).$$

The λ-operations on a β-ring R are defined in terms of β-operations by

$$\lambda_H = \sum_{[J]} \check{\tau}_-([J],[H])\,\beta_J.$$

Proposition 3·1. *For every element X in a β-ring R and for every subgroup H of S_n, $\lambda_H(X) \cong (-1)^n \beta_H(-X)$.*

Proof. Since β-ring identities are determined by identities in $A(S)$ it is sufficient to verify the result for $X \in A(S_n)$. Let Y be a right G-set, Z_1, Z_2 two left G-sets and let $Z_1 \times Z_2$ have transitive decomposition $\sum_{[J] \leqslant [G]} n_J[G/J]$ in $A(G)$. Then

$$Y \otimes_G (Z_1 \times Z_2) \cong \sum_{[J] \leqslant [G]} n_J (Y \otimes_G (G/J)).$$

So, if we are given a decomposition

$$(-1)^n [S_n/H] \times \epsilon[\mathbb{N}_n] \cong \sum_{[J] \leqslant [S_n]} \tau_{JH}\,[S_n/J]$$

it follows from Proposition 2·2 that

$$(-1)^n \beta_H(-X) = (-1)^n X^n \otimes_{S_n} ((S_n/H) \times \epsilon[\mathbb{N}_n]) \cong \sum_{[J] \leqslant [S_n]} \tau_{JH}\,\beta_J(X)$$

in $A(S_n)$. Thus it is sufficient to verify that $\tau_{JH} = \check{\tau}_-([J],[H])$.

We have shown in [**8**, proposition 3·2] that, for each $T \in A(G)$, the multiplicity of $[G/J]$ in $T \times [G/H]$ is $\check{\tau}_T([J],[H])$ where $\tau_T = \gamma^{-1}\mu\Delta_T\,\zeta\gamma$ in $\mathbb{Q}CA(L_G)$ and $\Delta_T(J)$ is the mark $\phi_T(J)$. But, by Proposition 2·4, our chosen values $\Delta_-(J) = (-1)^{n-l(J)}$ are indeed the marks of $T_- = (-1)^n \epsilon[\mathbb{N}_n]$.

Corollary 3·2. $\lambda_H(-X) \cong (-1)^n \beta_H(X)$.

Restricting to Young subgroups $S_\rho \leqslant S_n$ we write $\tau_{\pi\rho}$ for $\tau_{S_\pi S_\rho}$ and $\tau(\rho)$ for $\tau_{S_\rho S_n}$. Recall that the coefficients in the identities $\lambda_\pi = \sum_{\rho \vdash n} \tau_{\pi\rho}\beta_\rho$ form a triangular submatrix of $M(\check{\tau}_-)$ and that this matrix is the transition matrix between the elementary and homogeneous symmetric functions e_π and h_π of degree n. It is well known that the number of compositions κ of n with $\pi_\kappa = \rho$ is $(-1)^\rho\ \tau(\rho)$ and that the lattice $\Omega(n)$ of compositions of n, ordered by refinement, is isomorphic to the lattice of vertices and edges of an $(n-1)$-dimensional cube. Thus from the Hasse diagram for $\Omega(5)$ in Fig. 1 we may read off the identity

$$\lambda_{S_5} = \beta_{I^5} - 4\beta_{C_2 I^3} + 3\beta_{S_3 I^2} + 3\beta_{KI} - 2\beta_{S_4 I} - 2\beta_{S_3 C_2} + \beta_{S_5}.$$

These two results enable us to obtain formulae for the composition of unary β-operations and λ-operations. Let $H[K]$ be the standard permutation representation of the wreath product $K \wr H$ with degree $(\deg H)(\deg K)$.

Proposition 3·3. *For each pair of permutation groups H and K,*

$$\lambda_H \circ \beta_K = \sum_{[J] \leqslant [H]} \tau_{JH}\,\beta_{J[K]}.$$

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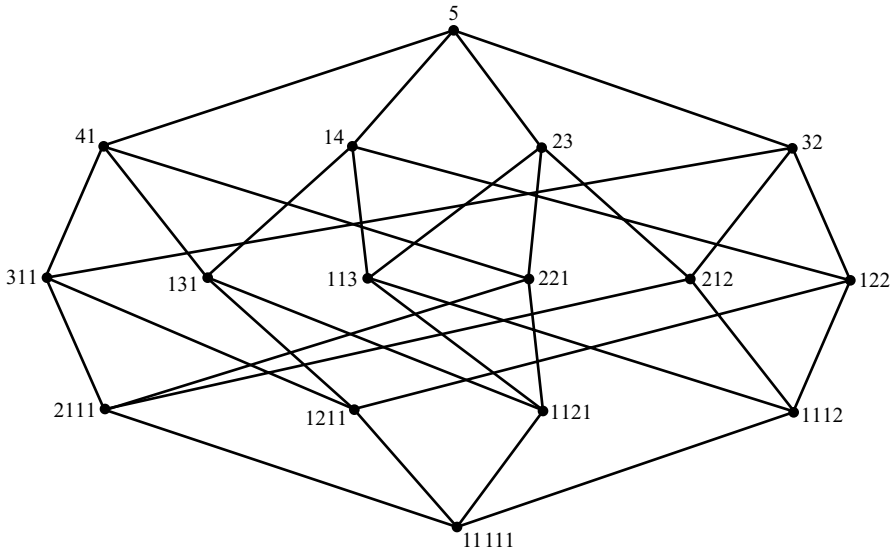


Fig. 1. Lattice $\Omega(5)$ of compositions of 5.

when K has even degree,

$$\beta_H \circ \lambda_K = \lambda_{H[K]} \quad \text{and} \quad \lambda_H \circ \lambda_K = \sum_{[J] \leqslant [H]} \tau_{JH} \lambda_{J[K]},$$

when K has odd degree,

$$\beta_H \circ \lambda_K = \sum_{[J] \leqslant [H]} \tau_{JH} \lambda_{J[K]} \quad \text{and} \quad \lambda_H \circ \lambda_K = \lambda_{H[K]}.$$

Proof. The formula for $\lambda_H \circ \beta_K$ follows immediately from the β-ring axiom $\beta_H \circ \beta_K = \beta_{H[K]}$.

Let H, K have degree h, k respectively. When k is even, applying Proposition 3·1 gives

$$\beta_H(\lambda_K(X)) = \beta_H(\beta_K(-X)) = \beta_{H[K]}(-X) = \lambda_{H[K]}(X).$$

The formula for $\lambda_H \circ \lambda_K$ follows from the definition of λ_H.

When k is odd,

$$\begin{aligned} \beta_H(\lambda_K(X)) &= \beta_H(-\beta_K(-X)) \\ &= (-1)^h \lambda_H(\beta_K(-X)) \\ &= (-1)^h \sum_{[J]} \tau_{JH} \beta_{J[K]}(-X) \\ &= (-1)^d \sum_{[J]} \tau_{JH} \lambda_{J[K]}(X), \end{aligned}$$

where $d = h + \deg(J[K]) = h(1+k)$ is even. Finally, with k odd,

$$\lambda_H(\lambda_K(X)) = \sum_{[J]} \tau_{JH} \beta_J(\lambda_K(X)) = \sum_{[L],\,[J]} \tau_{LJ} \tau_{JH} \lambda_{L[K]}(X)$$

which reduces to $\lambda_{H[K]}$ since $\tau_-^2 = \Delta_-^2$ is the identity function. ∎

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We complete this section by seeking formulae for evaluating λ_H on the Burnside ring $A(G)$. In particular we hope for an analogue of the λ-ring result that $\lambda_j(x) = 0$ when x has degree less than j.

PROPOSITION 3·4. *$\lambda_{S_j}(X) = 0$ when $|X| < j$.*

Proof. The operations $\lambda_j = \lambda_{S_j}$ define a pre-λ-ring structure on $A(G)$ (see [**7**, p. 266]). ∎

PROPOSITION 3·5. *Let X be the identity $[G/G]$ in $A(G)$. Then*

$$\lambda_H(X) = \begin{cases} X & \text{when} \quad H \leqslant A_n \\ 0 & \text{otherwise,} \end{cases}$$

where A_n is the alternating group of degree n.

Proof. Since $\beta_J(X) = X$ for all $J \leqslant S_n$, $\lambda_H(X) = (\sum_{[J]} \tau_{JH})X$. But, by [**8**, proposition 5·5], the entries in column $[H]$ of the matrix $M(\check{\tau}_-)$ sum to 1 or 0 according as H is, or is not, a subgroup of A_n. ∎

Calculations with groups of small degree suggest the following result.

CONJECTURE 3·6. *Let X be a G-set with fewer than j elements. Then $\lambda_H(X) = 0$ if H is a group of degree n having a symmetric Young subgroup in the class $[S_j \oplus I^{n-j}]$ with orbit-type $(1^{n-j}, j] \vdash n$.*

4. *β-species and λ-species*

We noted in the introduction that the theory of combinatorial species and the theory of β-rings of symmetric groups are essentially the same. In what follows we think of each β-operation β_H as a β-species and each λ_H as a λ-species. In this section we consider the molecular decompositions of β_H and λ_H and give combinatorial interpretations of these when $H = I^n$, S_n and S_π. Because $\xi \in \beta_H(X)$ involves $r \leqslant n$ elements in X, we expect $\beta_H(\mathbb{N}_k)$ to contain a term $[\![S_{k-r} \oplus J]\!]$, where J is some subgroup of S_r.

In his study of polynomial operations on Burnside rings, Vallejo [**11**, section 1·13] has defined, for each $[J] \leqslant [S_n]$, a *J-free power* operation F_n/J. He also showed [**11**, section 5·7] that these operations provide an alternative additive basis for the half ring of combinatorial species. A set X is mapped by F_n/J to the set of J-orbits of $\{(x_1, x_2, \ldots, x_n) \,|\, i \neq j \Rightarrow x_i \neq x_j\}$ and so, as a species, F_n/J is isomorphic to $\mathbb{E}.[\![J]\!]$. Rewriting [**11**, lemma 5·4] in our notation, we obtain

$$\mathbb{E}.[\![J]\!] \cong \sum_{k \geqslant r} [\![S_{k-r} \oplus J]\!].$$

For a group H of low degree it is easy to write down the molecular decomposition of β_H as a combinatorial species. In degree one, $[I] \cong \mathbb{X} \cong \mathbb{E}_1$ and $\beta_I[\mathbb{N}_k] = \mathbb{N}_k$ so $\beta_I \cong \sum_{k \geqslant 1} [\![S_{k-1} \oplus I]\!] \cong \mathbb{E}.\mathbb{X}$. In degree two, since $(S_k/S_{k-1})^2$ is isomorphic to $(S_k/S_{k-1}) + (S_k/S_{k-2})$ as an S_k-set, it follows that

$$\beta_{I^2} \cong \mathbb{E}.[\![I]\!] + \mathbb{E}.[\![I^2]\!].$$

Similarly,

$$\beta_{C_2} \cong \mathbb{E}.[\![I]\!] + \mathbb{E}.[\![C_2]\!]$$

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since

$$\beta_{C_2}(X) = \{[xx] : x \in X\} \cup \{[xy] : x, y \in X\}.$$

The next result shows that every β_H may be expressed as a sum of products $\mathbb{E}.[\![J]\!]$ with degree$(J) \leqslant$ degree(H).

PROPOSITION 4·1 (Vallejo[**11**]). *For H a permutation group of degree n, the molecular decomposition of β_H has the form*

$$\beta_H = \sum_{r=1}^{n} \sum_{[J] \leqslant [S_r]} b_{H,J}\, \mathbb{E}.[\![J]\!],$$

where the $b_{H,J}$ are non-negative integers. In particular, $b_{H,I} = 1$, $b_{H,H} = 1$ and $b_{H,J} = 0$ when $J \notin [H]$ has degree n.

COROLLARY 4·2. *For H a permutation group of degree n, the molecular decomposition of λ_H has the form*

$$\lambda_H = \sum_{r=1}^{n} \sum_{[J] \leqslant [S_r]} c_{H,J}\, \mathbb{E}.[\![J]\!],$$

where the $c_{H,J}$ *are integers. In particular, $c_{H,I} = 1$ when $H \leqslant A_n$ and 0 otherwise, $c_{H,H} = (-1)^H$ and, when J has degree n, $c_{H,J} = \tau_{JH} = \check{\tau}_{-}([J],[H])$.*

Proof. Substituting the molecular decomposition for β_K into the formula

$$\lambda_H = \sum \tau_{HK} \beta_k \quad \text{gives} \quad c_{H,J} = \sum_{[K] \leqslant [H]} \tau_{HK}\, b_{K,J}.$$

The special cases follow from those for the $b_{H,J}$ and Proposition 3·5. ∎

Tables 1 and 2 list the coefficients $b_{H,J}$ and $c_{H,J}$ in the expansions for β_H and λ_H up to degree 5. Notice the transposed $M(\check{\tau}_{-})$ forming diagonal blocks in Table 2.

We now present general formulae for the extreme cases $H = S_n$, $H = A_n$ and $H = I^n$. We may then interpret the β_{S_n}-structures on $\mathbb{N}_k$ as the unordered selections with repetition of n elements from $\mathbb{N}_k$. For each $\rho \vdash n$ the selections of type ρ form a single S_k-orbit and we may choose as orbit representative a selection with $\#1\text{'s} \geqslant \#2\text{'s} \geqslant \cdots \geqslant \#k\text{'s}$. The λ_{S_n}-structures on $\mathbb{N}_k$ are interpreted as *signed* compositions of n-element subsets of $\mathbb{N}_k$ forming $\tau(\rho)$ orbits for each $\rho \vdash n$ and having as orbit representative a composition of $\mathbb{N}_k$ with the elements in their natural order.

Example 4·3. Representatives of the S_k-orbits for $\beta_{S_4}(\mathbb{N}_k)$ and $\lambda_{S_4}(\mathbb{N}_k)$

ρ:	(1^4)	(21^2)	(31)	(2^2)	(4)
β_{S_4}:	$(1,2,3,4)$	$(1,1,2,3)$	$(1,1,1,2)$	$(1,1,2,2)$	$(1,1,1,1)$
λ_{S_4}:	$+\backslash 1\backslash 2\backslash 3\backslash 4\backslash$	$-\backslash 12\backslash 3\backslash 4\backslash$	$+\backslash 123\backslash 4\backslash$	$+\backslash 12\backslash 34\backslash$	$-\backslash 1234\backslash$
		$-\backslash 1\backslash 23\backslash 4\backslash$	$+\backslash 1\backslash 234\backslash$		
		$-\backslash 1\backslash 2\backslash 34\backslash$			

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Table 1. *Coefficients* $b_{H,J}$ *in* $\beta_H \cong \sum_{r=1}^{n} \sum_{[J]\leqslant[S_r]} b_{H,J} \sum_{k\geqslant r} [\![S_{k-r} \oplus J]\!]$

$H\backslash J$	I	I^2	C_2	I^3	C_2I	C_3	S_3	I^4	C_2I^2	C_3I	S_3I	C_2'	K	C_4	K'	D_4	A_4	S_4	
H has degree $\leqslant 4$																			
I	1																		
I^2	1	1																	
C_2	1		1																
I^3	1	3		1															
$C_2 \oplus I$	1	2			1														
C_3	1	1				1													
S_3	1	1					1												
I^4	1	7		6				1											
$C_2 \oplus I^2$	1	5		3	1				1										
$C_3 \oplus I$	1	3		2						1									
$S_3 \oplus I$	1	3		1	1						1								
C_2'	1	3	2	2	2							1							
K	1	3	1	1	2								1						
C_4	1	1	2	1	1									1					
K'	1	1	3		3										1				
D_4	1	1	2		2											1			
A_4	1	1	1		1												1		
S_4	1	1	1		1													1	
H, J of degree 5																			
I^5	1	15		25				10											$+\sum [\![S_{k-5} \oplus I^5]\!]$
$C_2 \oplus I^3$	1	11		15	1			4	3										$+\sum [\![S_{k-5} \oplus C_2' \oplus I^3]\!]$
$C_3 \oplus I^2$	1	7		9				3		1									$+\sum [\![S_{k-5} \oplus C_3 \oplus I^2]\!]$
$S_3 \oplus I^2$	1	7		7	1			1	2		1								$+\sum [\![S_{k-5} \oplus S_3 \oplus I^2]\!]$
$C_2' \oplus I$	1	9		11	4			4	2										$+\sum [\![S_{k-5} \oplus C_2' \oplus I]\!]$
$K \oplus I$	1	8		8	3			1	4										$+\sum [\![S_{k-5} \oplus K \oplus I]\!]$
$C_4 \oplus I$	1	5		5	3			2	1										$+\sum [\![S_{k-5} \oplus C_4 \oplus I]\!]$
$K' \oplus I$	1	6		4	6			1	3										$+\sum [\![S_{k-5} \oplus K' \oplus I]\!]$
$D_4 \oplus I$	1	5		3	4				3										$+\sum [\![S_{k-5} \oplus D_4 \oplus I]\!]$
$A_4 \oplus I$	1	4		2	2				1	1									$+\sum [\![S_{k-5} \oplus A_4 \oplus I]\!]$
$S_4 \oplus I$	1	4		2	2				1		1								$+\sum [\![S_{k-5} \oplus S_4 \oplus I]\!]$
C_6	1	5		5	1			1	1	1									$+\sum [\![S_{k-5} \oplus C_6]\!]$
S_3	1	5		3	4			1	1		1								$+\sum [\![S_{k-5} \oplus S_3]\!]$
D_6	1	5		3	3				2		1								$+\sum [\![S_{k-5} \oplus D_6]\!]$
C_5	1	3		5				2											$+\sum [\![S_{k-5} \oplus C_5]\!]$
D_5	1	3		1	4				1										$+\sum [\![S_{k-5} \oplus D_5]\!]$
Y_5	1	2			3				1										$+\sum [\![S_{k-5} \oplus Y_5]\!]$
A_5	1	2			2						1								$+\sum [\![S_{k-5} \oplus A_5]\!]$
S_5	1	2			2						1								$+\sum [\![S_{k-5} \oplus S_5]\!]$

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Table 2. *Coefficients* $c_{H,J}$ *in* $\lambda_H \cong \sum_{r=1}^{n} \sum_{[J] \leqslant [S_r]} c_{H,J} \sum_{k \geqslant r} [\![S_{k-r} \oplus J]\!]$

(a) $H \backslash J$	I	I^2	C_2	I^3	$C_2 I$	C_3	S_3	I^4	$C_2 I^2$	$C_3 I$	$S_3 I$	C_2'	K	C_4	K'	D_4	A_4	S_4
(a)																		
H has degree $\leqslant 4$																		
I	1																	
I^2	1	1																
C_2	0	1	−1															
I^3	1	3		1														
$C_2 \oplus I$	0	1		1	−1													
C_3	1	1				1												
S_3	0	0		1	−2		1											
I^4	1	7		6				1										
$C_2 \oplus I^2$	0	2		3	−1			1	−1									
$C_3 \oplus I$	1	3		2						1								
$S_3 \oplus I$	0	0		1	−1			1	−2		1							
C_2'	1	3	2	2	2							1						
K	0	0	1	1	0			1	−2				1					
C_4	0	2	0	1	1							1		−1				
K'	1	1	3		3			−1				3			−1			
D_4	0	0	1		1				−1			1	1			−1		
A_4	1	1	1		1			−1		2		1					−1	
S_4	0	0	0		0			1	−3		2		1					−1
H of degree 5 and J of degree < 5																		
I^5	1	15		25				10										
$C_2 \oplus I^3$	0	4		10	−1			6	−3									
$C_3 \oplus I^2$	1	7		9				3		1								
$S_3 \oplus I^2$	0	0		2	−1			3	−4		1							
$C_2' \oplus I$	1	9		11	4			4	2									
$K \oplus I$	0	1		3	1			3	−2									
$C_4 \oplus I$	0	4		6	1			2	1									
$K' \oplus I$	1	6		4	6			1	3									
$D_4 \oplus I$	0	1		1	2			1										
$A_4 \oplus I$	1	4		2	2				1	1								
$S_4 \oplus I$	0	0		0	0			1	−2		1							

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Table 2. (*cont.*)

(*a*) $H\backslash J$	I	I^2	C_2	I^3	C_2I	C_3	S_3	I^4	C_2I^2	C_3I	S_3I	C_2'	K	C_4	K'	D_4	A_4	S_4
C_6	0	2		4	−1			2	−1									
S_3'	1	5		3	4			0	3	1	−1							
D_6	0	0		0	1			1	−1									
C_5	1	3		5				2										
D_5	1	3		1	4				1									
Y_5	0	1		1	1				1									
A_5	1	2			2			−1	2	1	−1							
S_5	0	0			0						0							

(*b*) $H\backslash J$	I^5	C_2I^3	C_3I^2	S_3I^2	$C_2'I$	KI	C_4I	$K'I$	D_4I	A_4I	S_4I	C_6	S_3'	D_6	C_5	D_5	Y_5	A_5	S_5
I^5	1																		
$C_2 \oplus I^3$	1	−1																	
$C_3 \oplus I^2$			1																
$S_3 \oplus I^2$	1	−2		1															
$C_2' \oplus I$					1														
$K \oplus I$	1	−2				1													
$C_4 \oplus I$					1		−1												
$K' \oplus I$	−1				3			−1											
$D_4 \oplus I$		−1			1	1			−1										
$A_4 \oplus I$	−1		2		1					−1									
$S_4 \oplus I$	1	−3		2		1					−1								
C_6			1									−1							
S_3'	−1		1		2								−1						
D_6	1	−3		1		2								−1					
C_5															1				
D_5																1			
Y_5					1		−2										1		
A_5	−2		3		3					−2			−2					1	
S_5	1	−4		3		3					−2			−2					1

PROPOSITION 4·4.

(i) $\beta_{S_n} \cong \sum_{\rho \vdash n} \mathbb{E}.[\![S_m(\rho)]\!]$,

(ii) $\lambda_{S_n} \cong \sum_{\rho \vdash n} (-1)^\rho \tau(\rho)\, \mathbb{E}.[\![S_\rho]\!]$,

(iii) $\beta_{A_n} \cong \beta_{S_n} - \mathbb{E}.[\![S_n]\!] + \mathbb{E}.[\![A_n]\!]$,

(iv) $\beta_{I^n} = \lambda_{I^n} \cong \sum_{r=1}^{n} S(n,r)\, \mathbb{E}.[\![I^r]\!]$,

where $S(n,r)$ is the Stirling number of the second kind, counting partitions of $\mathbb{N}_n$ with r parts.

Proof. (i) Let X have cardinality k. For each n-tuple $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in X^n$ we require the S_k-stabilizer $\mathrm{Stab}(\xi)$ of $\xi = \mathbf{y} \otimes (eS_n)$. Suppose that the y_i comprise r distinct elements $z_1, \ldots, z_r$ with z_j occurring ρ_j times and that $\rho_1 \leqslant \rho_2 \leqslant \ldots \leqslant \rho_r$, forming a partition ρ. For each $\theta \in S_n$, $\mathbf{y}^\theta \otimes (eS_n) = \xi$ so we may assume that $\mathbf{y} = (z_1, \ldots, z_1, z_2, \ldots, z_2, \ldots, z_r)$.

The S_k-action on ξ is such that ${}^g\xi = {}^g\mathbf{y}^\theta \otimes \theta^{-1}(eS_n)$ and g fixes ξ provided there exists $\theta \in S_n$ such that ${}^g\mathbf{y}^\theta = \mathbf{y}$. If $\rho_i = \rho_j$, for some $i \neq j$, let g act as the transposition $(z_i z_j)$ on X. Then there exists $\theta \in S_n$ interchanging the positions of the z_i's and the z_j's so that ${}^g\mathbf{y}^\theta = \mathbf{y}$ and $g \in \mathrm{Stab}(\xi)$. Thus ξ is also stabilized by $S_{m(\rho)} = S_{r_1} \oplus S_{r_2} \oplus \ldots \oplus S_{r_n}$, with the qth summand S_{r_q} permuting those z_i which occur q times in ξ. When $r < k$, $\mathrm{Stab}(\xi)$ also has a direct summand S_{k-r} permuting the elements in $X \backslash \{z_1, \ldots, z_r\}$.

(ii) When $|X| < n$, $\lambda_{S_n}[X] = 0$ by Proposition 3·4. By Corollary 4·2

$$\lambda_{S_n} \cong \sum_{[J] \leqslant [S_n]} c_{S_n, J}\, \mathbb{E}.[\![J]\!],$$

and by [**7**, proposition 7·8], $c_{S_n, J} = 0$ when J is not a Young subgroup and $c_{S_n, S_p} = (-1)^\rho \tau(\rho)$.

(iii) If $x_i = x_j$ in $\underline{\mathbf{x}} = (x_1, x_2, \ldots, x_n)$ then $\underline{\mathbf{x}}^{(ij)} = \mathbf{x}$ and the A_n-orbit of $\underline{\mathbf{x}}$ is equal to the S_n-orbit. When the x_i are all distinct the S_n-orbit splits into two A_n-orbits, one containing all the even permutations of $\underline{\mathbf{x}}$ and the other all the odd permutations. These two elements of $\beta_{A_n}(X)$ belong to the same S_k-orbit having stabilizer class $[S_{k-n} \oplus A_n]$.

(iv) This is a result of Vallejo's [**11**, corollary 5·3. (1)]. ∎

Formulae for the series expansion of a general Young subgroup species β_{S_π} are essentially λ-ring identities and so may be expressed in terms of partitions or compositions.

Definition 4·5. Let $\pi = (\pi_1, \pi_2, \ldots, \pi_p)$ and $\rho = (1^{r_1} 2^{r_2} \ldots n^{r_n})$ be partitions of n where exactly b of the multiplicities r_i are non-zero. Let $\mathscr{A}_{\pi\rho}$ be the set of $p \times b$ arrays where

(i) rows are labelled $1 \ldots p$ and columns are labelled by terms (i^{r_i}) with $r_i > 0$,

(ii) the (i,j)th entry is a vector of length r_j whose elements are non-negative integers,

(iii) the vector sum of the entries in column j is $(j, j, \ldots, j)$,

(iv) the sum of all the elements in the entries in row i is π_i.

The Young subgroup $S_{m(\rho)} = S_{r_1} \oplus S_{r_2} \oplus \cdots \oplus S_{r_n}$ acts on each $A \in \mathscr{A}_{\pi\rho}$ with S_{r_j} permuting the elements of each vector in column j. Let S_A denote the Young

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subgroup $\mathrm{Stab}_{S_{m(\rho)}}(A)$ and let $\bar{\mathscr{I}}_{\pi\rho}$ be a set of representatives for the orbits under this action.

Example 4·6. We consider the case $n = 7$, $\pi = (3,2,2)$, $\rho = (1^1 2^3)$. The arrays listed below are representatives for the 11 orbits in $\bar{\mathscr{I}}_{\pi\rho}$.

$$A_1 = \begin{pmatrix} (1) & (2,0,0) \\ (0) & (0,2,0) \\ (0) & (0,0,2) \end{pmatrix}, \quad A_2 = \begin{pmatrix} (1) & (2,0,0) \\ (0) & (0,1,1) \\ (0) & (0,1,1) \end{pmatrix}, \quad A_3 = \begin{pmatrix} (1) & (0,1,1) \\ (0) & (2,0,0) \\ (0) & (0,1,1) \end{pmatrix},$$

$$A_4 = \begin{pmatrix} (1) & (0,1,1) \\ (0) & (0,1,1) \\ (0) & (2,0,0) \end{pmatrix}, \quad A_5 = \begin{pmatrix} (1) & (0,1,1) \\ (0) & (1,0,1) \\ (0) & (1,1,0) \end{pmatrix}, \quad A_6 = \begin{pmatrix} (0) & (2,1,0) \\ (1) & (0,1,0) \\ (0) & (0,0,2) \end{pmatrix},$$

$$A_7 = \begin{pmatrix} (0) & (2,1,0) \\ (1) & (0,0,1) \\ (0) & (0,1,1) \end{pmatrix}, \quad A_8 = \begin{pmatrix} (0) & (1,1,1) \\ (1) & (1,0,0) \\ (0) & (0,1,1) \end{pmatrix}, \quad A_9 = \begin{pmatrix} (0) & (2,1,0) \\ (0) & (0,0,2) \\ (1) & (0,1,0) \end{pmatrix},$$

$$A_{10} = \begin{pmatrix} (0) & (2,1,0) \\ (0) & (0,1,1) \\ (1) & (0,0,1) \end{pmatrix}, \quad A_{11} = \begin{pmatrix} (0) & (1,1,1) \\ (0) & (0,1,1) \\ (1) & (1,0,0) \end{pmatrix}.$$

Arrays A_2, A_3, A_4, A_8 and A_{11} have stabilizer $I^2 \oplus C_2$, the rest have stabilizer I^4. Note that each second column sums to $(2,2,2)$ and defines a refinement vector of compositions, for example $((2,0,0), (1,0,1), (0,1,1))$ for A_7. The array A_5 exhibits the need for the vector entries. If these entries were partitions rather than compositions, we should have

$$A_5 = \begin{pmatrix} (1) & (1^2) \\ (0) & (1^2) \\ (0) & (1^2) \end{pmatrix}$$

and the second column might suggest a non-trivial stabilizer.

Proposition 4·7. *With the notation of Definition* 4·5,

$$\beta_{S_\pi} \cong \sum_{\rho \vdash n} \sum_{A \in \bar{\mathscr{I}}_{\pi\rho}} \sum_{k \geqslant r} [\![S_{k-r} \oplus S_A]\!].$$

Proof. Let $\xi \in X^n/S_\pi$ and let ξ contain l_j copies of x_j for each $1 \leqslant j \leqslant r$, where the x_j are all distinct elements of X. Then the array A encodes information about the orbit of ξ as follows. Let π_i be the qth of those parts of π which are equal to j. The qth element of the vector in row i and column j is taken to be the multiplicity of x_j in the set of elements permuted by the summand S_{π_i} of S_π.

5. *Subset species*

In this section we consider the species $\mathbb{E}.[\![H]\!]$ and in particular $\mathbb{E}.[\![S_n]\!]$ which is isomorphic to the species of n-element subsets. The well-known identity $\mathbb{S} \cong \mathbb{E} \circ (2\mathbb{X})$ for the species of subsets causes us to investigate the composite species $\beta_{S_n} \circ (2\mathbb{X})$ and $\lambda_{S_n} \circ (2\mathbb{X})$.

It is not a simple matter to express a typical combinatorial species, such as endofunctions or labelled trees, in terms of β- or λ-species. However, there is a remarkably simple formula for the exponential species.

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PROPOSITION 5·1. $\mathbb{E} \cong \mathbb{E}_0 + \sum_{k \geqslant 1} [\![S_k]\!] \cong \mathbb{E}_0 + \sum_{k \geqslant 1} (-1)^{k+1} \lambda_{S_k}$.

Proof. The molecular decomposition is well known. The expansion as a combination of λ-species is essentially an application of Newton's formula for homogeneous and elementary symmetric functions, since the identity $\lambda_{S_n} = \sum_{\pi \vdash n} \tau(\pi) \beta_{S_\pi}$ corresponds directly to the symmetric function formula $e_n = \sum_{\pi \vdash n} \tau(\pi) h_\pi$ ([**7**, p. 255]).

Let $\Lambda = \lambda_I - \lambda_{C_2} + \lambda_{S_3} - \lambda_{S_4} + \ldots + (-1)^{n+1} \lambda_{S_n} + \ldots$. We calculate, for each $\rho = \vdash n$ except $\rho = (n)$, the coefficient of $[\![S_\rho]\!]$ in the molecular decomposition of Λ. From Proposition 4·4(ii) we know that terms λ_{S_k} with $k > n$ make no contribution. For each $1 \leqslant j \leqslant n$, if $r_j \neq 0$ and $\rho \backslash (j)$ is the partition of $(n-j)$ obtained by deleting a j from ρ, then $S_j \oplus S_{\rho \backslash (j)} \cong S_\rho$ and $\lambda_{S_{n-j}}$ contributes a term $\tau_{S_{\rho \backslash (j)} S_{n-j}} [\![S_\rho]\!]$ to Λ. Thus the coefficient of $[\![S\rho]\!]$ in Λ *is*

$$\sum_{r_j \neq 0} (-1)^{n-j+1} \tau_{S_{\rho \backslash (j)} S_{n-j}} + (-1)^{n+1} \tau(\rho).$$

It is easily checked that the last term in this expression has sign $(-1)^{l(\rho)+1}$ while the other terms all have sign $(-1)^{l(\rho)}$. This expression is precisely the coefficient of h_ρ obtained by substituting $\sum_{\pi \vdash j} \tau(\pi) h_\pi$ for each e_j in Newton's formula $h_n = h_{n-1} e_1 - h_{n-2} e_2 + \cdots + (-1)^{n+1} e_n$, and so is zero when $\rho \neq (n)$. ∎

Formulae expressing $\mathbb{E}.[\![H]\!]$ in terms of β-species may be obtained by back substitution on the rows of Table 1. Note that these formulae involve β-species with decreasing degree. Vallejo [**11**, corollary 5·3. (2)] has shown that

$$\mathbb{E}.[\![I^n]\!] \cong \sum_{r=1}^{n} (-1)^{n-r} s(n, r) \beta_{I^r},$$

where $s(n, r)$ is the Stirling number of the first kind, counting permutations of $\mathbb{N}_n$ comprising r disjoint cycles. We shall define a reduction process which enables a formula to be determined for $\mathbb{E}.[\![S_n]\!]$.

Clearly $\mathbb{E}.[\![S_n]\!]$ is isomorphic to the species $\mathbb{S}(n)$ of n-element subsets. The identities up to degree four are:

$$\mathbb{S}(1) \cong \beta_I \quad \mathbb{S}(2) \cong \beta_{C_2} - \beta_I \quad \mathbb{S}(3) \cong \beta_{S_3} - \beta_{I^2}$$

and

$$\mathbb{S}(4) \cong \beta_{S_4} - \beta_{C_2 \oplus I} + (\beta_{I^2} - \beta_{C_2}) \cong \beta_{S_4} - \beta_{C_2} \lambda_I + \lambda_{C_2}.$$

Before proving a general formula we describe a construction which, given a molecular β_{S_n}-structure B on $\mathbb{N}_k$, not isomorphic to $[\![S_n \oplus S_{k-n}]\!]$, produces a variety of $\beta_{S_{n-2i}} \lambda_{S_i}$-structures, all isomorphic to B as S_k-sets.

Consider B as a set of selections of n elements from $\mathbb{N}_k$ of type $\rho = (1^q h_1^{q_1} h_2^{q_2} \ldots h_b^{q_b}) \vdash n$, with $q \geqslant 0$ and $q_i \geqslant 1$, and multiplicity $m(\rho) = (q, q_1, \ldots, q_b) \vdash r$ where r is the number of distinct elements in B. Choose a representative selection containing q non-repeated elements $a_{01}, \ldots, a_{0q}$ and elements a_{ij} repeated i times with $1 \leqslant i \leqslant b$ and $1 \leqslant j \leqslant q_i$. Choose i from $\{1, \ldots, b\}$. Removing two copies of $a_{h_i 1}, a_{h_i 2}, \ldots, a_{h_i r_i}$ from the selection produces a $\beta_{S_{n-2q_i}}$-structure B_i'. Then $B_i = (B_i', \backslash a_{h_i 1} a_{h_i 2} \ldots a_{h_i q_i} \backslash)$ is a $(-1)^{q_i} \beta_{S_{n-2q_i}} \lambda_{S_{q_i}}$-structure on $\mathbb{N}_k$. Note that both B and B_i have S_k-stabilizer class $[S_{m(\rho)} \oplus S_{k-r}]$ although B_i' may have a larger stabilizer. We call this the *reduction* construction since the degree of the operation is reduced by q_i.

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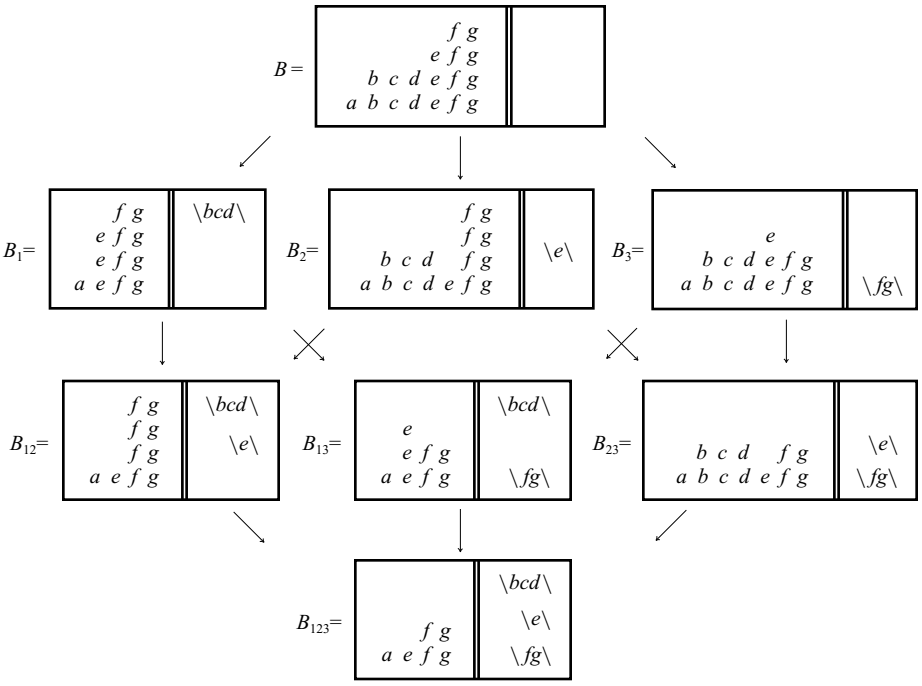


Fig. 2. Reduced structures for $\rho = (12^3 34^2)$.

When $b \geqslant 2$ this construction may be applied to produce b reduced structures B_i and then composed in all possible ways to produce $(2^b - 1)$ structures from the original B. We illustrate this in Fig. 2 for the case $\rho = (12^3 34^2) \vdash 18$ where all the structures belong to orbits with stabilizer class $[I \oplus S_3 \oplus I \oplus C_2 \oplus S_{k-7}]$. From the original $\beta_{S_{18}}$-structure we have produced structures for $\beta_{S_{12}}\lambda_{S_3}$, $\beta_{S_{16}}\lambda_I$, $\beta_{S_{14}}\lambda_{C_2}$, $\beta_{S_{10}}\lambda_{S_4}$, $\beta_{S_8}\lambda_{S_5}$, $\beta_{S_{12}}\lambda_{S_3}$ and $\beta_{S_6}\lambda_{S_6}$ respectively.

Note that when the construction has been composed $c \geqslant 2$ times there are in fact $c!$ different structures in which the c parts $\backslash a_{h_i 1} a_{h_i 2} \dots a_{h_i r_i} \backslash$ are ordered in $c!$ ways.

Proposition 5·2.

$$\mathbb{S}(n) \cong \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \beta_{S_{n-2j}} \lambda_{S_j}.$$

Proof. Elements of type (1^n) in $\beta_{S_n}[\mathbb{N}_k]$ form a single S_k-orbit, isomorphic to $\mathbb{S}(n)[\mathbb{N}_k]$, with stabilizer class $[S_n \oplus S_{k-n}]$. We have to show that the S_k-orbit of selections of type $\rho \neq (1^n)$ is cancelled by the S_k-orbits of $\beta_{S_{n-2j}}\lambda_{S_j}$-structures:

$$\beta_{S_n}[\mathbb{N}_k] \cong (\mathbb{S}(n) + \beta_{S_{n-2}}\lambda_I - \beta_{S_{n-4}}\lambda_{C_2} + \dots + (-1)^{i-1}\beta_{S_{n-2j}}\lambda_{S_j} + \dots)[\mathbb{N}_k].$$

Suppose $\rho = (1^r h_1^{r_1} h_2^{r_2} \dots h_b^{r_b})$ has b distinct parts greater than one. For each $1 \leqslant i \leqslant b$ the reduction construction applied to a selection B of type ρ produces a $(-1)^{r_i - 1}\beta_{S_{n-2r_i}}\lambda_{S_{r_i}}$-structure isomorphic to B as an S_k-set. Applying the construction to both the $h_i^{r_i}$ and $h_j^{r_j}$ terms produces *three* $\beta_{S_{n-2r_i-2r_j}}\lambda_{S_{r_i+r_j}}$-structures of the form

$$\begin{aligned}(-1)^{r_i+r_j}\{ &+ (B'_{ij}, \backslash(a_{h_i 1} a_{h_i 2} \dots a_{h_i r_i} \backslash a_{h_j 1} a_{h_j 2} \dots a_{h_j r_j} \backslash),\\ &+ (B'_{ij}, \backslash a_{h_j 1} a_{h_j 2} \dots a_{h_j r_j} \backslash a_{h_i 1} a_{h_i 2} \dots a_{h_i r_i} \backslash),\\ &- (B'_{ij}, \backslash a_{h_i 1} a_{h_i 2} \dots a_{h_i r_i} a_{h_j 1} a_{h_j 2} \dots a_{h_j r_j} \backslash)\}.\end{aligned}$$

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Since all three have the same stabilizer as B, these structures are isomorphic as S_k-sets and cancel to give just one structure. Similarly, when t different reductions are composed, cancellation of S_k-orbits leaves, by (2·1), just one $\beta_{S_{n-2(r_{i_1}+\cdots+r_{i_t})}}\lambda_{S_{(r_{i_1}+\cdots+r_{i_t})}}$-structure. Since the terms in the formula alternate in sign, the 2^b orbits cancel exactly (as in the expansion of $(1-1)^b = 0$).

We have shown that all but the first term in $\beta_{S_n}[\mathbb{N}_k]$ cancels, but we must also consider extra structures introduced by a $\beta_{S_{n-2j}}\lambda_{S_j}$-term which cannot be obtained from any $\beta_{S_{n-2i}}\lambda_{S_i}$-structure with $i < j$. The simplest of these is the $\beta_{C_2}\lambda_I$-structure $(aa, \backslash b\backslash)$ which occurs as an extra structure in the formula for $\beta_{S_4}(\mathbb{N}_2)$ because $(aabb)$ gets mapped to $(\varnothing, \backslash ab\backslash)$. Every structure, maximal with respect to the reduction, must have a repeated element in the β-component. If there are d parts in the λ-component, we obtain 2^{b-d} reduced structures which cancel as before.

COROLLARY 5·3. *The species of non-empty subsets with at most n elements may be expressed as*

$$\mathbb{S}(1 \dots n) \cong \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{i} (-1)^j \beta_{S_{n-i-j}} \lambda_{S_j}$$

with no terms of degree less than $\lfloor n/2 \rfloor$.

Proof. Collecting together the terms of degree d in the expansion of $\sum_{i=0}^{n} \mathbb{S}(i)$ we obtain $\sum (-1)^j \beta_{S_{d-j}} \lambda_{S_j}$, where j takes values from 0 to $\min\{d, n-d\}$. When $d \leqslant \lfloor n/2 \rfloor$ this sum is zero by Newton's identity. ∎

Note that the reduction construction may be generalised to β_{S_π}-structures. To illustrate this we reduce array A_3 in Example 4·6:

$$A_3 = \begin{bmatrix} (1)\ (0,1,1) \\ (0)\ (2,0,0) \\ (0)\ (0,1,1) \end{bmatrix} : \begin{array}{|ccc|c|} \hline a & c & d & \\ \hline b & b & & \\ \hline c & d & & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \hline a & \backslash b\backslash \\ \hline & \backslash cd\backslash \\ \hline \end{array}$$

and so the formula for $\mathbb{E}.[\![S_{(32^2)}]\!]$ will contain a term $-\beta_I \lambda_{C_2 I} = -\lambda_{C_2 I^2}$.

Now let $2\mathbb{X} \cong 2[\![I]\!]$ be the species with no structures when $k \neq 1$ and just two when $k = 1$, usually taken to be $\{0, 1\}$ or $\{+, -\}$ or {True, False}. One of the standard examples of composition of species is $\mathbb{S} \cong \mathbb{E} \circ (2\mathbb{X})$ [**3**, p. 13]. Applying Proposition 5·1 we obtain

$$\mathbb{S} \cong \mathbb{S}_0 + \sum_{n \geqslant 1} (-1)^{n+1} \lambda_{S_n} \circ (2\mathbb{X}),$$

so we wish to describe the $\beta_{S_n} \circ (2\mathbb{X})$- and $\lambda_{S_n} \circ (2\mathbb{X})$-structures.

For an arbitrary species $\mathbb{A}$ the $\mathbb{A} \circ (2\mathbb{X})$-structures on $\mathbb{N}_k$ may be considered as $\mathbb{A}$-structures on $\mathbb{N}_k$ with the elements $1, 2, \dots, k$ labelled $+$ or $-$. When $\mathbb{A}$ has exponential series $\mathbb{A}(x) = \sum a_n x^n/n!$ then $\mathbb{A} \circ 2\mathbb{X}(x) = \sum (2^n a_n) x^n/n!$. So if B is a selection of n elements from $\mathbb{N}_k$ of type ρ the corresponding $\beta_{S_n} \circ (2\mathbb{X})$-structures have the form $((B^+ | B^-), (T^+, T^-))$ where T is a subset of $\mathbb{N}_k$, B^+ and B^- are selections comprising those elements which are or are not in T, T^+ is the subset of T with elements not selected for B and T^- is the subset of $\mathbb{N}_k$ with elements neither in T nor B. Suppose, as usual, that $\rho = (1^{r_1} 2^{r_2} \dots n^{r_n})$ has multiplicity $m(\rho) \vdash r$. Then the subset T will contain $j_i \in \{0, 1, \dots, r_i\}$ of the elements repeated i times in B and

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$j_0 \in \{0, 1, \ldots, k-r\}$ of the elements not in B. Thus the total number of S_k-orbits of these structures is $(k-r+1)\prod_{i=1}^{n}(r_i+1)$. A typical orbit may be specified by an $(n+1)$-tuple $(j_0, j_1, \ldots, j_n)$ where $0 \leqslant j_0 \leqslant k-r$ and $0 \leqslant j_i \leqslant r_i$. When B^+, B^- are selections of type ρ^+, ρ^- respectively, the S_k-stabilizer of $((B^+|B^-), (T^+, T^-))$ is isomorphic to $S_{\rho^+} \oplus S_{\rho^-} \oplus S_{|T^+|} \oplus S_{|T^-|}$. Thus

$$(\beta_{S_n} \circ (2\mathbb{X}))_k \cong \sum_{\rho \vdash n} \sum_{(j_0, j_1, \ldots, j_n)} \left[\!\!\left[S_{j_0} \oplus S_{(k-r-j_0)} \bigoplus_{i=1}^{n} (S_{j_i} \oplus S_{r_i - j_i}) \right]\!\!\right]$$

and, summing over all k, we obtain the following result.

Proposition 5·4.

$$\beta_{S_n} \circ (2\mathbb{X}) \cong \sum_{\rho \vdash n} \sum_{(j_0, j_1, \ldots, j_n)} \mathbb{E}.\left[\!\!\left[S_{j_0} \bigoplus_{i=1}^{n} (S_{j_i} \oplus S_{r_i - j_i}) \right]\!\!\right].$$

A description of the $\beta_{S_n} \circ (2\mathbb{X})$-structures may be obtained in a similar manner. For each $T \subseteq \mathbb{N}_k$ and for each array $A \in \mathscr{A}_{\pi\rho}$ of Definition 4·5 there is a structure $((A^+|A^-), (T^+, T^-))$.

The $\lambda_{S_n} \circ (2\mathbb{X})$-structures on $\mathbb{N}_k$ involve only selections from $\mathbb{N}_k$ without repetition. For each $T \subseteq \mathbb{N}_k$ and for each λ_{S_n}-structure $\backslash\omega_1\backslash\omega_2\backslash\ldots\backslash\omega_p\backslash$ there are structures $(\backslash\omega_1^+|\omega_1^-\backslash\omega_2^+|\omega_2^-\backslash\ldots\backslash\omega_p^+|\omega_p^-\backslash, (T^+, T^-))$ where $\omega_i^+ = \omega_i \cap T$, $\omega_i^- = \omega_i \backslash \omega_i^+$ and T^+, T^- are as above. The stabilizer of such a structure is

$$S_{T^+} \oplus S_{T^-} \bigoplus_{i=1}^{n} (S_{\omega_i^+} \oplus S_{\omega_i^-}).$$

We thus obtain a double series expansion for the species $\lambda_{S_n} \circ (2\mathbb{X})$. For $n \leqslant 4$ we may calculate

$$\begin{aligned}
\lambda_I \circ (2\mathbb{X}) &\cong 2\mathbb{E}.[\![I]\!], \\
\lambda_{C_2} \circ (2\mathbb{X}) &\cong 3\mathbb{E}.[\![I^2]\!] - 2\mathbb{E}.[\![C_2]\!], \\
\lambda_{S_3} \circ (2\mathbb{X}) &\cong 4\mathbb{E}.[\![I^3]\!] - 6\mathbb{E}.[\![C_2 I]\!] + 2\mathbb{E}.[\![S_3]\!], \\
\lambda_{S_4} \circ (2\mathbb{X}) &\cong 5\mathbb{E}.[\![I^4]\!] - 12\mathbb{E}.[\![C_2 I^2]\!] + 6\mathbb{E}.[\![S_3 I]\!] + 3\mathbb{E}.[\![C_2{}^2]\!] - 2\mathbb{E}.[\![S_4]\!].
\end{aligned}$$

Since the coefficient of S_ρ in the molecular expansion for λ_{S_n} depends on $m(\rho)$, it comes as no surprise that the same is true for these expansions.

Before stating the general formula, we investigate which compositions ω of $\mathbb{N}_n$ with $\pi_\omega = \pi$ can split to form a composition ω' with $\pi_{\omega'} = \rho$. A *j-splitting* of a composition κ of n is a refinement of κ obtained by choosing j of the parts κ_i in κ and splitting each into two non-zero parts $\kappa_i \mapsto \kappa_i^+|\kappa_i^-$. For example, $\kappa' = (3;2|3;1;1|2)$ is a 2-splitting of $\kappa = (3;5;1;3)$. It is convenient to use '|' rather than ';' as the separator to make clear which parts have been split. We write $\kappa'|\kappa$ when κ' is a j-splitting of κ for some $j \geqslant 0$. This defines a *non-transitive* relation | on $\Omega(n)$, compatible with refinement.

A *j-splitting* of a composition $\omega = \backslash\omega_1\backslash\omega_2\backslash\ldots\backslash\omega_p\backslash$ of $\mathbb{N}_n$ is a refinement ω' obtained by splitting each part, $\omega_i \mapsto \omega_i^+|\omega_i^-$, where $\omega_i^+, \omega_i^- \neq \varnothing$ for j of the parts and the remaining $p-j$ parts are rewritten as $\omega_i|\varnothing$ or $\varnothing|\omega_i$. Clearly the type κ' or ω' is a j-splitting of the type κ of ω. Let $\Omega_\kappa(\mathbb{N}_k)$ be the transitive S_k-set, isomorphic to $(S_k/(S_{\pi_\kappa} \oplus S_{k-n}))$, of compositions of n-element subsets of $\mathbb{N}_k$ of type κ. Let κ' be a j-

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splitting of κ where parts $\kappa_{i_1}, \kappa_{i_2}, \ldots, \kappa_{i_j}$ are split. The number of j-splittings of elements of $\Omega_\kappa(\mathbb{N}_k)$ having type κ' is

$$\binom{\kappa_{i_1}}{\kappa_{i_1}^+}\binom{\kappa_{i_2}}{\kappa_{i_2}^+}\cdots\binom{\kappa_{i_j}}{\kappa_{i_j}^+}2^{p-j},$$

where the factor 2^{p-j} comes from the choices of position for the empty parts. These compositions have S_k-stabilizer class $[S_{\pi_{\kappa'}} \oplus S_{k-n}]$ and form 2^{p-j} S_k-orbits. For example $\Omega_{(3;2;2)}(\mathbb{N}_k)$, with representative $\backslash 123\backslash 45\backslash 67\backslash$, has two orbits of 2-splittings of type $(2\,|\,1\,;1\,|\,1\,;2)$ with representatives $\backslash 12\,|\,3\backslash 4\,|\,5\backslash 67\,|\,\varnothing\backslash$ and $\backslash 12\,|\,3\backslash 4\,|\,5\backslash\varnothing\,|\,67\backslash$.

PROPOSITION 5·5.

$$\lambda_{S_n} \circ (2\mathbb{X}) \cong \sum_{\rho \vdash n} (-1)^\rho \tau(\rho)\,(l(\rho)+1) \sum_{m \geqslant 0} \mathbb{E}.[\![S_\rho \oplus S_m]\!].$$

Proof. We have seen above that $\lambda_{S_n} \circ (2\mathbb{X})[\mathbb{N}_k] = \varnothing$ when $k < n$ so we first consider the case $k = n$ and verify the coefficient of $[\![S_\rho]\!]$. For each $\rho \vdash n$ we denote by $Q(\rho)$ the poset whose elements are the compositions κ of n which have, for some $0 \leqslant j \leqslant \lfloor r/2 \rfloor$, a j-splitting κ' with $\pi_{\kappa'} = \rho$. Then $Q(\rho)$ is a subposet of the lattice $\Omega(n)$ but is not, in general, a fully subposet. Let $\Omega_\rho(n)$ be the set of all compositions κ with $\pi_\kappa = \rho$. Then $Q(\rho)$ has $\Omega_\rho(n)$ as the set of least elements and the remaining elements in $Q(\rho)$ are those compositions which may be obtained by combining one or more pairs of adjacent parts in some $\kappa \in \Omega(\rho)$. Every least element in $Q(\rho)$ has r parts and is less than exactly $\binom{r-j}{j}$ elements with $r-j$ parts. If $\kappa \in Q(\rho)$ and $\pi_\kappa = \pi$ then $Q(\rho)$ contains all $\tau(\pi)$ compositions κ' with $\pi_{\kappa'} = \pi$.

A structure $(\backslash\omega_1^+\,|\,\omega_1^-\backslash\omega_2^+|\omega_2^-\backslash\ldots\backslash\omega_p^+\,|\,\omega_p^-\backslash,\ (\varnothing, \varnothing))$ with stabilizer class $[S_\rho]$ in $\lambda_{S_n} \circ (2\mathbb{X})$ must occur as a j-splitting for some ω of type $\kappa_\omega \in Q(\rho)$. The S_k-orbits of compositions ω with $\pi_\omega = \rho$ correspond to the $\kappa' \in \Omega_\rho(n)$. We have seen that there are $\binom{r-j}{j}$ elements $\kappa \in Q(\rho)$ which have κ' as a j-splitting. Each κ corresponds to the S_n-orbit $(-1)^\kappa [\![S_{\pi_\kappa}]\!]$ is $\lambda_{S_n}[\mathbb{N}_n]$ and gives rise to $2^{l(\kappa)-j} = 2^{r-2j}$ S_n-orbits. Thus the coefficient of $[\![\mathbb{S}_\rho]\!]$ is

$$\tau(\rho) \sum_{j=0}^{\lfloor r/2 \rfloor} (-1)^{n-(r-j)} \binom{r-j}{j} 2^{r-2j}$$

and using induction we find that this simplifies to $(-1)^\rho \tau(\rho)\,(r+1)$.

Finally, when $k > n$, we count S_k-orbits of structures $(\backslash\omega_1^+\,|\,\omega_1^-\backslash\omega_2^+\,|\,\omega_2^-\backslash\ldots\backslash\omega_p^+\,|\,\omega_p^-\backslash,$ $(T^+, T^-))$. Two such structures with identical first component are in the same S_k-orbit provided $|T_1^+| = |T_2^+| = m$ and $|T_1^-| = |T_2^-| = k-n-m$ for some $0 \leqslant m \leqslant k-n$. Hence

$$(\lambda_{S_n} \circ (2\mathbb{X}))_k \cong \sum_{\rho \vdash n} (-1)^\rho \tau(\rho)\,(r+1) \sum_{m=0}^{n-k} [\![S_\rho \oplus S_m \oplus S_{k-n-m}]\!]. \quad \blacksquare$$

We conclude by observing that, when we have expressed a species $\mathbb{A}$ as a combination of β- and λ-species, we may use Proposition 3·1 and Corollary 3·2 to write down an expression of $\mathbb{A}[-X]$. We consider, in particular, the species $\mathbb{S}$, $\mathbb{L}$ and $\mathbb{C}$.

For the subset species we obtain $\mathbb{S}[-X] \cong \epsilon[2X]$ and we may interpret the structures as compositions with the elements labelled $+$ or $-$.

Since $\mathbb{L}[-X] \cong (-1)^{|X|}\mathbb{L}[X]$, the structures are simply signed total orders.

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Finally, consider the species $\mathbb{C}$ of circular permutations. Since $\mathbb{C}_k \cong [\![C_k]\!]$ we may write

$$\mathbb{C} \cong \sum_{k=1}^{\infty} (\beta_{C_k})_k$$

and so

$$\mathbb{C}[-\mathbb{N}_n] \cong \sum_{k=1}^{\infty} (-1)^k (\lambda_{C_k})_k [\mathbb{N}_n] \cong (-1)^n \lambda_{C_n}[\mathbb{N}_n].$$

Now $\lambda_{C_n} = \beta_{C_n}$, when n is odd, whereas $\lambda_{C_{2n}} = \beta_{C'_n} - \beta_{C_{2n}}$, where $[C'_n]$ is the class of cyclic subgroups generated by squares of circular permutations of length $2n$. Thus

$$\mathbb{C}[-\mathbb{N}_{2n}] \cong (S_{2n}/C'_n) - (S_{2n}/C_{2n}).$$

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