J. Plasma Phys. (2026), *vol.* 92, E6 © The Author(s), 2026. Published by Cambridge University Press. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution and reproduction, provided the original article is properly cited.
doi:10.1017/S0022377825101104

Conservative formulation of the drift-reduced fluid plasma model

Brenno De Lucca[1], Paolo Ricci[1], Micol Bassanini[1,2], Sergio García Herreros[1] and Zeno Tecchiolli[1]

[1]École Polytechnique Fédérale de Lausanne (EPFL), Swiss Plasma Center (SPC), CH-1015 Lausanne, Switzerland

[2]École Polytechnique Fédérale de Lausanne (EPFL), Institute of Mathematics, CH-1015 Lausanne, Switzerland

Corresponding author: Brenno De Lucca, brenno.delucca@epfl.ch

(Received 17 September 2025; revision received 13 November 2025; accepted 13 November 2025)

A conservative formulation of the drift-reduced fluid plasma model is constructed by analytically inverting the implicit relation defining the polarisation velocity as a function of the time derivative of the electric field. The obtained model satisfies exact conservation laws for energy, mass, charge and momentum, in arbitrary magnetic geometry, also when electromagnetic fluctuations are included.

Key words: plasma simulation, plasma nonlinear phenomena, fusion plasma

1. Introduction

In highly collisional regimes, magnetised plasma turbulence is often studied with drift-reduced fluid models (see e.g. Drake & Antonsen 1984; Zeiler, Drake & Rogers 1997; Scott 2003; Simakov & Catto 2003). These models are commonly employed to investigate the boundary region of fusion devices (Dudson *et al*. 2009; Zhu, Francisquez & Rogers 2018; Stegmeir *et al*. 2019; Giacomin *et al*. 2022*b*; Düll *et al*. 2024) and also find applications in basic plasma physics experiments (Riva *et al*. 2016).

The drift-reduced formulation of the fluid plasma is derived by perturbatively expanding the fluid moment equations in powers of the drift-expansion parameter $\epsilon \sim d_t/\Omega_c \ll 1$, which represents the ratio between the dynamical time scale, encoded in the material derivative d_t, and the cyclotron frequency $\Omega_c = qB/m$, expressed in terms of the charge q, the mass m and the magnetic field strength $B = |\boldsymbol{B}|$. In the limit $\epsilon \ll 1$, the projection of the fluid velocity in the plane orthogonal to the magnetic field, $\boldsymbol{v}_\perp$, decomposes in a set of drift contributions. Among these, the polarisation drift $\boldsymbol{v}_p \sim (B\Omega_c)^{-1} d_t \boldsymbol{E}$ is related to the time and spatial evolution of the electric field $\boldsymbol{E}$. This allows determining the electric field evolution through a vorticity equation. Indeed, by involving $\boldsymbol{v}_p$, the quasi-neutrality constraint $\nabla \cdot \boldsymbol{J} = 0$ results in an equation for the electrostatic potential. However, the drift-reduced models as currently expressed in the literature and implemented in the high-fidelity

fluid codes for tokamak and stellarator turbulence simulations (Dudson *et al*. 2009; Stegmeir *et al*. 2019; Giacomin *et al*. 2022*b*; Coelho *et al*. 2024; Düll *et al*. 2024) are known to lack exact conservative properties, with spurious source terms appearing at $O(\epsilon)$ (Reiser 2012; Halpern, Waltz & Bernard 2023).

To derive a conservative drift-reduced fluid model, it is necessary to invert an implicit equation to relate the polarisation velocity to the time derivative of the electric field, where formally small terms need to be kept to ensure conservation (Zeiler *et al*. 1997; Reiser 2012; Poulsen *et al*. 2020; Halpern *et al*. 2023). This was done in the geometry of a linear device for a cold-ion plasma in the electrostatic limit by Reiser (2012), by exploiting the cylindrical symmetry of the background magnetic field to derive an explicit time-evolution equation for the vorticity. In this work, we derive a closed-form, non-perturbative expression for the polarisation velocity $\boldsymbol{v}_p = \boldsymbol{v}_p(\partial_t \boldsymbol{E})$, in an arbitrary magnetic geometry and without imposing the electrostatic limit, and use it to build an exactly energy and momentum conserving drift-reduced fluid model. The construction holds for an arbitrary number of species and does not depend on the choice of fluid closure.

As drift-reduced fluid codes are becoming tools of reference in building predictive capabilities for the operation and design of fusion devices (Giacomin *et al*. 2022*a*; Oliveira *et al*. 2022; Bufferand *et al*. 2024; Zholobenko *et al*. 2024), the formulation of the drift-reduced model in a conservative form is relevant for several reasons. First, the existence of exact invariants is important when devising numerical schemes to increase code performance (LeVeque 1992). Second, it proves that the construction of a drift-reduced model obeying the same conservation properties as the non-drift-reduced system is possible in general, implying that this approximation scheme for fluid magnetised plasmas is well posed.

This paper is organised as follows. In § 2, we express the fluid equations for a multispecies quasi-neutral magnetised plasma and make explicit the corresponding energy and momentum conservation laws. In § 3, we state the drift-approximation ordering and explain why a perturbative expansion of the polarisation velocity as a function of leading-order quantities breaks the model's conservation properties. In § 4, the implicit and non-perturbative expression for the polarisation drift is inverted analytically, leading to an exactly conservative model. In § 5, the $O(\epsilon)$ drift-reduced model is given and shown to conserve the leading-order components of energy and momentum exactly (i.e. to all orders in ϵ) in § 6. Finally, the conclusions are drawn in § 7.

2. Fluid equations of a multispecies plasma

We consider a magnetised plasma containing an arbitrary number of distinct species, indexed by the label s. The fluid equations for a given species s, of mass m_s and electric charge q_s, are obtained by taking velocity moments of the Boltzmann equation (Braginskii 1965). The first three moment equations, evolving the density n_s, the fluid momentum density $\boldsymbol{\mathcal{M}}_s = m_s n_s \boldsymbol{V}_s$, with $\boldsymbol{V}_s$ the mean velocity of species s, and the scalar pressure $p_s = n_s T_s$, with T_s the temperature of species s, are given by

$$\frac{\partial n_s}{\partial t} + \boldsymbol{\nabla} \cdot (n_s \boldsymbol{V}_s) = \mathcal{S}_s, \tag{2.1}$$

$$\frac{\partial \boldsymbol{\mathcal{M}}_s}{\partial t} + \boldsymbol{\nabla} \cdot (\boldsymbol{V}_s \boldsymbol{\mathcal{M}}_s) + \boldsymbol{\nabla} \cdot \boldsymbol{P}_s - q_s n_s \left(\boldsymbol{E} + \boldsymbol{V}_s \times \boldsymbol{B}\right) = \boldsymbol{\mathcal{R}}_s, \tag{2.2}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2} p_s\right) + \boldsymbol{\nabla} \cdot \left(\frac{3}{2} p_s \boldsymbol{V}_s + \boldsymbol{q}_s\right) + \boldsymbol{P}_s : \boldsymbol{\nabla} \boldsymbol{V}_s = \mathcal{Q}_s + r_s \mathcal{K}_s, \tag{2.3}$$

https://doi.org/10.1017/S0022377825101104 Published online by Cambridge University Press

where $\boldsymbol{P}_s = p_s\mathbb{I} + \boldsymbol{\pi}_s$ is the pressure tensor of the species, $\boldsymbol{\pi}_s$ is the stress tensor, $\mathbb{I}$ is the identity matrix, $\boldsymbol{q}_s$ is the heat-flux density and $\mathcal{S}_s$, $\boldsymbol{\mathcal{R}}_s$, $\mathcal{Q}_s$ are source terms, arising from external drives and interactions between the species, e.g. via Coulomb collisions. In addition, we define $r_s \equiv \mathcal{S}_s/n_s$ as the rate at which particles of species s enter or leave the system and $\mathcal{K}_s \equiv m_s n_s V_s^2/2$ is the fluid kinetic energy density of species s. In (2.2) and (2.3), the notation $\boldsymbol{P} : \boldsymbol{\nabla V} \equiv P^{ij}\nabla_j V_i$ is introduced to represent the Frobenius product of the two tensors and the divergence of a tensor $\boldsymbol{\nabla} \cdot \boldsymbol{P}_s \equiv \nabla_i P^{ij}$ is defined as contracting over neighbouring indices. We express the application of a tensor on a vector with a similar abuse of notation and write $\boldsymbol{P} \cdot \boldsymbol{V} \equiv P_{ij}V^j$, with the tensorial property of the resulting quantity discernible from the context. For the sake of generality, we do not focus on the closure of the moment hierarchy in this work. We assume that an adequate closure is encoded in the heat-flux densities and stress tensors, as well as the contributions to the source terms arising from interparticle interactions. We note that the particle sources $\mathcal{S}_s$ cannot all be chosen independently, as charge neutrality imposes the constraint $\sum_s q_s\mathcal{S}_s = 0$.

Since the turbulent phenomena we wish to model tend to evolve on scales much larger than the Debye length, λ_D, and time scales much slower than the plasma frequency, ω_p, we assume

$$\lambda_D \boldsymbol{\nabla} \sim \frac{1}{\omega_p}\frac{\partial}{\partial t} \equiv \epsilon_D \to 0. \tag{2.4}$$

In this limit, the plasma is instantaneously locally quasi-neutral and the electric field $\boldsymbol{E}$ is not determined by the Gauss law but rather by the quasi-neutrality condition $\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$. Thus, Maxwell's equations in the limit $\epsilon_D \to 0$ reduce to

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} = \mu_0 \sum_s q_s n_s \boldsymbol{V}_s, \tag{2.5}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{2.6}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{2.7}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0. \tag{2.8}$$

Maxwell's equations, coupled to the fluid equations, (2.1)–(2.3), yield a closed model which conserves energy, momentum, mass and charge.

The energy density $\mathcal{H}$ can be expressed as

$$\mathcal{H} = \mathcal{H}_B + \sum_s \mathcal{H}_s = \frac{B^2}{2\mu_0} + \sum_s \left(\mathcal{K}_s + \frac{3}{2}p_s\right), \tag{2.9}$$

where $\mathcal{H}_s = \mathcal{K}_s + \mathcal{U}_s$ is the energy of each species, composed of the kinetic energy, $\mathcal{K}_s$, and the internal energy, $\mathcal{U}_s \equiv 3p_s/2$, while $\mathcal{H}_B \equiv B^2/2\mu_0$ is the field energy, which contains only the magnetic energy density contribution in the quasi-neutral limit. To demonstrate the conservation of $\mathcal{H}$, we take the scalar product of the momentum equation (2.2) with $\boldsymbol{V}_s$ and use the continuity equation (2.1). This leads to a transport equation for the fluid kinetic energy density $\mathcal{K}_s$:

$$\frac{\partial \mathcal{K}_s}{\partial t} + \boldsymbol{\nabla} \cdot (\mathcal{K}_s \boldsymbol{V}_s) = -\boldsymbol{V}_s \cdot \boldsymbol{\nabla} \cdot \boldsymbol{P}_s + q_s n_s \boldsymbol{V}_s \cdot \boldsymbol{E} + \boldsymbol{V}_s \cdot \boldsymbol{\mathcal{R}}_s - \mathcal{K}_s \frac{\mathcal{S}_s}{n_s}. \tag{2.10}$$

https://doi.org/10.1017/S0022377825101104 Published online by Cambridge University Press

Furthermore, we note that the electromagnetic field energy evolves according to Poynting's theorem and, in a quasi-neutral magnetised plasma, where $\omega/\omega_p \to 0$, the term due to the electric field energy, $\partial_t(\epsilon_0 E^2/2)$, is negligible. Indeed, the displacement current scales as $\partial_t \boldsymbol{E}/c^2 \sim (\omega\Omega_{ce}/\omega_{pe}^2)\mu_0 \boldsymbol{J}$, where the electron time scales are the dominant ones, so in the neutral limit $\omega/\omega_{pe} \to 0$, we have that $\partial_t \boldsymbol{E}/c^2 \to 0$ so long as Ω_{ce}/ω_{pe} does not diverge (which almost always holds in magnetised plasmas). In this limit, Poynting's theorem is given by

$$\frac{\partial \mathcal{H}_B}{\partial t} + \nabla \cdot \boldsymbol{S} + \boldsymbol{E} \cdot \boldsymbol{J} = 0, \tag{2.11}$$

with $\boldsymbol{S} \equiv \boldsymbol{E} \times \boldsymbol{B}/\mu_0$ the Poynting vector, as can be observed by taking the scalar product of Faraday's law with $\boldsymbol{B}$ and using the fact that $\boldsymbol{B} \cdot (\nabla \times \boldsymbol{E}) = \mu_0 \nabla \cdot \boldsymbol{S} + \boldsymbol{E} \cdot (\nabla \times \boldsymbol{B})$, leading to

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right) + \nabla \cdot \boldsymbol{S} + \frac{1}{\mu_0}\boldsymbol{E} \cdot (\nabla \times \boldsymbol{B}) = 0, \tag{2.12}$$

and then imposing $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$. Thus, summing the transport equations for the internal energy, (2.3), the fluid kinetic energy, (2.10), and the magnetic field energy, (2.11), results in an equation for the total energy density:

$$\frac{\partial \mathcal{H}}{\partial t} + \nabla \cdot \left[\boldsymbol{S} + \sum_s (\mathcal{H}_s \boldsymbol{V}_s + \boldsymbol{P}_s \cdot \boldsymbol{V}_s + \boldsymbol{q}_s)\right] = \sum_s (\mathcal{Q}_s + \boldsymbol{V}_s \cdot \boldsymbol{\mathcal{R}}_s), \tag{2.13}$$

where a further summation over all the species s has been performed and the symmetry property of the pressure tensor $\boldsymbol{P}_s = \boldsymbol{P}_s^T$ is used to write $\boldsymbol{P}_s \cdot \nabla \cdot \boldsymbol{P}_s + \boldsymbol{P}_s : \nabla \boldsymbol{V}_s = \nabla \cdot (\boldsymbol{P}_s \cdot \boldsymbol{V}_s)$ in (2.13). The energy content of the system changes due to heat sources, $\mathcal{Q}_s$, and forces, $\boldsymbol{V}_s \cdot \boldsymbol{\mathcal{R}}_s$. Defining the total power source $\mathcal{S}_\mathcal{H} = \sum_s (\mathcal{Q}_s + \boldsymbol{V}_s \cdot \boldsymbol{\mathcal{R}}_s)$ and the energy flux $\boldsymbol{\Gamma}_\mathcal{H} \equiv \boldsymbol{S} + \sum_s (\mathcal{H}_s \boldsymbol{V}_s + \boldsymbol{P}_s \cdot \boldsymbol{V}_s + \boldsymbol{q}_s)$, (2.13) can be written as a conservation law for $\mathcal{H}$:

$$\frac{\partial \mathcal{H}}{\partial t} + \nabla \cdot \boldsymbol{\Gamma}_\mathcal{H} = \mathcal{S}_\mathcal{H}. \tag{2.14}$$

The source term $\mathcal{S}_\mathcal{H}$ vanishes if the system is not externally driven and particle interactions are elastic, as is the case for the Coulomb interaction (Braginskii 1965). A transport equation can be similarly derived for the momentum density $\boldsymbol{\mathcal{M}} \equiv \sum_s \boldsymbol{\mathcal{M}}_s$:

$$\frac{\partial \boldsymbol{\mathcal{M}}}{\partial t} + \nabla \cdot \sum_s (\boldsymbol{V}_s \boldsymbol{\mathcal{M}}_s + \boldsymbol{P}_s) = \boldsymbol{J} \times \boldsymbol{B} + \sum_s \boldsymbol{\mathcal{R}}_s, \tag{2.15}$$

where, in the presence of elastic interactions and in the absence of external momentum input, the term $\sum_s \boldsymbol{\mathcal{R}}_s$ vanishes. Total mass is conserved by virtue of the continuity equation and, finally, charge is conserved by the requirement that $\nabla \cdot \boldsymbol{J} = 0$.

3. Drift approximation

We focus on low-frequency dynamics, therefore assuming that the physics of interest evolves on a scale of the order of the drift scale, $\partial_t \sim \boldsymbol{V}_s \cdot \nabla \sim \omega_{*s} \equiv \boldsymbol{k}_\perp \cdot \boldsymbol{v}_{*s}$,

https://doi.org/10.1017/S0022377825101104 Published online by Cambridge University Press

where $\boldsymbol{v}_{*s} = \boldsymbol{B} \times \nabla p_s/(q_s n_s B^2)$ is the diamagnetic drift velocity of species s and $\boldsymbol{k}_\perp$ is the wavenumber in the plane orthogonal to $\boldsymbol{B}$ (Zeiler *et al*. 1997). The drift scale is assumed much slower than the cyclotron frequency $\omega_{*s} \ll \Omega_{cs}$, thus $\partial_t/\Omega_{cs} \sim \boldsymbol{V}_s \cdot \nabla/\Omega_{cs} \equiv \epsilon$, with $\epsilon \ll 1$ a small parameter. This implies that the fluid velocities and length scales are ordered as $V_{\perp s} k_\perp \sim V_{\parallel s} k_\parallel \sim \epsilon \Omega_{cs}$, where $k_\perp = |\boldsymbol{k}_\perp|$ and $k_\parallel = \boldsymbol{k} \cdot \boldsymbol{b}$ denote the perpendicular and parallel wavenumbers, and $V_{\perp s}$ and $V_{\parallel s}$ represent the corresponding perpendicular and parallel components of the fluid velocity. Our ordering choice is consistent with both long-wavelength/large-flow drift-reduced assumptions, $k_\perp \rho_{Ls} \sim \sqrt{\epsilon}$, $V_{\perp s} \sim \sqrt{\epsilon} v_{Ts}$, and short-wavelength/small-flow gyrokinetics, $k_\perp \rho_{Ls} \sim 1$, $V_{\perp s} \sim \epsilon v_{Ts}$, where $v_{Ts} = \sqrt{T_s/m_s}$ is the thermal velocity of species s and $\rho_{Ls} = v_{Ts}/\Omega_{cs}$ its Larmor radius (Brizard & Hahm 2007). Indeed, here we adopt a more general ordering than the one typically assumed in drift-reduced models, in which only large-scale dynamics are retained $k_\perp \rho_{Ls} \sim \sqrt{\epsilon}$ (Drake & Antonsen 1984; Zeiler *et al*. 1997; Simakov & Catto 2003), as the large-scale assumption is not required to derive the results presented in this work.

In the $\epsilon = 0$ limit, key physical processes that give rise to electrostatic $E \times B$ turbulence are absent (Zeiler *et al*. 1997). In order to retain them in our model, we expand the fluid equations, (2.1)–(2.3), in powers of ϵ (Drake & Antonsen 1984). Decomposing the fluid velocity $\boldsymbol{V}_s$ of species s in terms of its parallel and perpendicular components with respect to the magnetic field $\boldsymbol{B}$,

$$\boldsymbol{V}_s = \boldsymbol{v}_{\parallel s} + \boldsymbol{v}_{\perp s}, \tag{3.1}$$

and taking the cross-product of the momentum equation (2.2) with $\boldsymbol{B}$, we find that $\boldsymbol{v}_{\perp s}$ decomposes into the sum of drift contributions (Drake & Antonsen 1984):

$$\boldsymbol{v}_{\perp s} = \boldsymbol{v}_E + \boldsymbol{v}_{*s} + \boldsymbol{v}_{\boldsymbol{\pi} s} + \boldsymbol{v}_{\boldsymbol{\mathcal{R}} s} + \boldsymbol{v}_{ps}, \tag{3.2}$$

that is, an $E \times B$ drift term, $\boldsymbol{v}_E = \boldsymbol{E} \times \boldsymbol{B}/B^2$, a diamagnetic drift related to isotropic pressure, $\boldsymbol{v}_{*s}$, a diamagnetic drift due to pressure anisotropies, $\boldsymbol{v}_{\boldsymbol{\pi} s} = \boldsymbol{B} \times \nabla \cdot \boldsymbol{\pi}_s/(q_s n_s B^2)$, a drift due to momentum source/sinks, $\boldsymbol{v}_{\boldsymbol{\mathcal{R}} s} = \boldsymbol{\mathcal{R}}_s \times \boldsymbol{B}/(q_s n_s B^2)$, and a drift related to the rate of change of the bulk flow, the polarisation velocity $\boldsymbol{v}_{ps}$, here defined as

$$\boldsymbol{v}_{ps} \equiv \frac{\boldsymbol{b}}{n_s \Omega_{cs}} \times \left[\frac{\partial (n_s \boldsymbol{V}_s)}{\partial t} + \nabla \cdot (n_s \boldsymbol{V}_s \boldsymbol{V}_s) \right], \tag{3.3}$$

where $\boldsymbol{b} = \boldsymbol{B}/B$ is the unit vector in the direction of $\boldsymbol{B}$. Using the continuity equation (2.1), the polarisation velocity can also be expressed as

$$\boldsymbol{v}_{ps} = \frac{\boldsymbol{b}}{\Omega_{cs}} \times \left(\frac{\mathrm{d}_s \boldsymbol{V}_s}{\mathrm{d}t} + r_s \boldsymbol{V}_s \right), \tag{3.4}$$

where $\mathrm{d}_s/\mathrm{d}t \equiv \partial_t + \boldsymbol{V}_s \cdot \nabla$ is the material derivative of species s. The polarisation drift results from the particle's finite inertia and is subleading in the imposed drift ordering, that is, $\boldsymbol{v}_{ps} \sim (d_t/\Omega_{cs}) \boldsymbol{V}_s = O(\epsilon \boldsymbol{V}_s)$, if the rate of particle injection is of the order of the drift frequency $r_s/\Omega_{cs} \sim \epsilon$, which we assume to hold. We can therefore write the total fluid velocity,

$$\boldsymbol{V}_s = \bar{\boldsymbol{v}}_s + \boldsymbol{v}_{ps} = (\boldsymbol{v}_{\parallel s} + \bar{\boldsymbol{v}}_{\perp s}) + \boldsymbol{v}_{ps}, \tag{3.5}$$

https://doi.org/10.1017/S0022377825101104 Published online by Cambridge University Press

as the sum of a leading-order flow, $\bar{\boldsymbol{v}}_s$, and the subleading correction, $\boldsymbol{v}_{ps}$, with $\boldsymbol{v}_{\parallel s}$, $\bar{\boldsymbol{v}}_s$ and $\bar{\boldsymbol{v}}_{\perp s}$ scaling as $O(V_s)$. Explicitly, the leading-order perpendicular flow is given by

$$\bar{\boldsymbol{v}}_{\perp s} = \boldsymbol{v}_E + \boldsymbol{v}_{*s} + \boldsymbol{v}_{\boldsymbol{\pi} s} + \boldsymbol{v}_{\boldsymbol{\mathcal{R}} s} \equiv \boldsymbol{v}_E + \boldsymbol{w}_{\perp s}, \tag{3.6}$$

where $\boldsymbol{w}_{\perp s}$ is the leading-order drift velocity of species s in the co-moving $E \times B$ frame. Separating the fluid velocity $\boldsymbol{V}_s$ appearing in (3.4) into its leading and subleading components, one obtains

$$\begin{aligned}\boldsymbol{v}_{ps} = \frac{\boldsymbol{b}}{\Omega_{cs}} \times \Bigg(& \frac{\partial \bar{\boldsymbol{v}}_s}{\partial t} + \bar{\boldsymbol{v}}_s \cdot \boldsymbol{\nabla} \bar{\boldsymbol{v}}_s + r_s \bar{\boldsymbol{v}}_s + \boldsymbol{v}_{ps} \cdot \boldsymbol{\nabla} \bar{\boldsymbol{v}}_s + \frac{\partial \boldsymbol{v}_{ps}}{\partial t} + \bar{\boldsymbol{v}}_s \cdot \boldsymbol{\nabla} \boldsymbol{v}_{ps} \\ & + r_s \boldsymbol{v}_{ps} + \boldsymbol{v}_{ps} \cdot \boldsymbol{\nabla} \boldsymbol{v}_{ps} \Bigg).\end{aligned} \tag{3.7}$$

Equation (3.7) yields an implicit equation for $\boldsymbol{v}_{ps}$, which is

$$\boldsymbol{v}_{ps} = F(\bar{\boldsymbol{v}}_s) + G(\bar{\boldsymbol{v}}_s, \boldsymbol{v}_{ps}) + K(\boldsymbol{v}_{ps}), \tag{3.8}$$

where

$$F(\bar{\boldsymbol{v}}_s) = \frac{\boldsymbol{b}}{\Omega_{cs}} \times \left(\frac{\partial \bar{\boldsymbol{v}}_s}{\partial t} + \bar{\boldsymbol{v}}_s \cdot \boldsymbol{\nabla} \bar{\boldsymbol{v}}_s + r_s \bar{\boldsymbol{v}}_s \right) = O(\epsilon \bar{\boldsymbol{v}}_s), \tag{3.9}$$

$$G(\bar{\boldsymbol{v}}_s, \boldsymbol{v}_{ps}) = \frac{\boldsymbol{b}}{\Omega_{cs}} \times \left(\boldsymbol{v}_{ps} \cdot \boldsymbol{\nabla} \bar{\boldsymbol{v}}_s + \frac{\partial \boldsymbol{v}_{ps}}{\partial t} + \bar{\boldsymbol{v}}_s \cdot \boldsymbol{\nabla} \boldsymbol{v}_{ps} + r_s \boldsymbol{v}_{ps} \right) = O(\epsilon \boldsymbol{v}_{ps} \bar{\boldsymbol{v}}_s), \tag{3.10}$$

$$K(\boldsymbol{v}_{ps}) = \frac{\boldsymbol{b}}{\Omega_{cs}} \times (\boldsymbol{v}_{ps} \cdot \boldsymbol{\nabla} \boldsymbol{v}_{ps}) = O(\epsilon \boldsymbol{v}_{ps} \boldsymbol{v}_{ps}). \tag{3.11}$$

Traditionally, (3.8) is treated with ordinary perturbation theory: $\boldsymbol{v}_{ps} = \boldsymbol{v}_{ps}^{(1)} + \boldsymbol{v}_{ps}^{(2)} + \cdots$, with $\boldsymbol{v}_{ps}^{(N)} \equiv O(\epsilon^N \bar{\boldsymbol{v}}_s)$. For example, solving (3.8) order by order, up to $O(\epsilon^3 \bar{\boldsymbol{v}}_s)$, yields

$$\boldsymbol{v}_{ps}^{(1)} = F(\bar{\boldsymbol{v}}_s) = \frac{\boldsymbol{b}}{\Omega_{cs}} \times \left(\frac{\partial \bar{\boldsymbol{v}}_s}{\partial t} + \bar{\boldsymbol{v}}_s \cdot \boldsymbol{\nabla} \bar{\boldsymbol{v}}_s + r_s \bar{\boldsymbol{v}}_s \right), \tag{3.12}$$

$$\boldsymbol{v}_{ps}^{(2)} = G(\bar{\boldsymbol{v}}_s, \boldsymbol{v}_{ps}^{(1)}) = \frac{\boldsymbol{b}}{\Omega_{cs}} \times \left(\boldsymbol{v}_{ps}^{(1)} \cdot \boldsymbol{\nabla} \bar{\boldsymbol{v}}_s + \frac{\partial \boldsymbol{v}_{ps}^{(1)}}{\partial t} + \bar{\boldsymbol{v}}_s \cdot \boldsymbol{\nabla} \boldsymbol{v}_{ps}^{(1)} + r_s \boldsymbol{v}_{ps}^{(1)} \right), \tag{3.13}$$

$$\boldsymbol{v}_{ps}^{(3)} = G\left(\bar{\boldsymbol{v}}_s, \boldsymbol{v}_{ps}^{(2)}\right) + K\left(\boldsymbol{v}_{ps}^{(1)}\right) = G\left(\bar{\boldsymbol{v}}_s, \boldsymbol{v}_{ps}^{(2)}\right) + \frac{\boldsymbol{b}}{\Omega_{cs}} \times \left(\boldsymbol{v}_{ps}^{(1)} \cdot \boldsymbol{\nabla} \boldsymbol{v}_{ps}^{(1)}\right). \tag{3.14}$$

The resulting drift-reduced model is presented in detail in Gath & Wiesenberger (2019). Truncating the expansion (3.12) to $O(\epsilon \bar{\boldsymbol{v}}_s)$ yields the models currently implemented in drift-reduced fluid codes (Dudson *et al*. 2015; Stegmeir *et al*. 2019; Giacomin *et al*. 2022*b*; Düll *et al*. 2024).

It is a known result that the models, based on the perturbative expansion in (3.12), are not conservative for both energy and momentum, with spurious source terms of $O(\epsilon)$ appearing in their time evolution (Reiser 2012; Halpern *et al*. 2023). The reason is that the expression for $\boldsymbol{v}_{ps}$ in (3.12) encodes the transport of the leading-order

https://doi.org/10.1017/S0022377825101104 Published online by Cambridge University Press

component of perpendicular momentum, which lacks advection by the polarisation drift (a subleading term; see (3.13)). On the other hand, the polarisation advection term is retained in other parts of the model, for instance in the continuity equation, which is expanded to $O(\epsilon)$. The resulting inconsistency between the transport of momentum and density breaks the energy and momentum conservation properties of the model (Reiser 2012). We further note that, given the recursive nature of the perturbative expansion method, exact conservation is not retrieved by expanding the model to higher order in ϵ. Indeed, for the drift-reduced model expanded to $O(\epsilon^N)$ with $N \geqslant 1$, energy and momentum conservation is broken by terms of the same order, namely $O(\epsilon^N)$.

The solution to the conservation problem of the drift approximation has been known since the original derivation of the model (Drake & Antonsen 1984; Zeiler *et al*. 1997; Scott 2003) and consists of retaining the polarisation advection term $\boldsymbol{b} \times (\boldsymbol{v}_{ps}^{(1)} \cdot \boldsymbol{\nabla} \bar{\boldsymbol{v}}_s)/\Omega_{cs}$ in the perpendicular momentum transport equation (3.12). More precisely, the polarisation drift $\boldsymbol{v}_{ps}^{(1)}$ should be defined implicitly as

$$\boldsymbol{v}_{ps}^{(1)} = \frac{\boldsymbol{b}}{\Omega_{cs}} \times \left(\frac{\partial \bar{\boldsymbol{v}}_s}{\partial t} + \bar{\boldsymbol{v}}_s \cdot \boldsymbol{\nabla} \bar{\boldsymbol{v}}_s + \boldsymbol{v}_{ps}^{(1)} \cdot \boldsymbol{\nabla} \bar{\boldsymbol{v}}_s + r_s \bar{\boldsymbol{v}}_s \right) \equiv \frac{\boldsymbol{b}}{\Omega_{cs}} \times \left(\frac{\mathrm{d}_s^{(1)} \bar{\boldsymbol{v}}_s}{\mathrm{d}t} + r_s \bar{\boldsymbol{v}}_s \right), \tag{3.15}$$

instead of the perturbative definition in (3.12). The expression for $\boldsymbol{v}_{ps}$ in (3.15) results in conservation laws for the leading-order contributions to energy and momentum, valid to all orders in ϵ. In the following section, we first invert (3.15) to express $\boldsymbol{v}_{ps}^{(1)}$ as a function of only the leading-order flow $\bar{\boldsymbol{v}}_s$ (§ 4). This was the main obstacle to overcome in obtaining an explicit and exactly conservative drift-reduced model. We then derive in § 5 a conservative drift-reduced model. Finally, mirroring previous work on the topic (Drake & Antonsen 1984; Zeiler *et al*. 1997; Reiser 2012), we demonstrate the conservation properties of the obtained drift-reduced system in § 6. Since we consider the expansion of the model only to first order in ϵ, we suppress henceforth the superscript in $\boldsymbol{v}_{ps}^{(1)}$ to lighten the notation in the rest of the paper.

4. Solution of the implicit equation for $\boldsymbol{v}_{ps}$

To the best of our knowledge, the expression of $\boldsymbol{v}_{ps}$, given by the solution of (3.15), has never been used in simulation codes, except for the case of an electrostatic cold-ion plasma in linear geometry (Reiser 2012). In that case, the symmetry properties of the magnetic field were exploited to obtain a vorticity equation and close the system equations. In general, however, the lack of an explicit form for the polarisation velocity as a function of the electric potential's rate of change precludes the direct use of (3.15). In fact, in the absence of an explicit relation $\boldsymbol{v}_{ps} = \boldsymbol{v}_{ps}(\partial_t \phi)$, imposing quasi-neutrality through $\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$ does not yield a scalar equation for ϕ. As a result, closing the drift-reduced fluid system requires explicit time evolution of the polarisation drift (on the cyclotron time scale), thereby defeating the purpose of the drift-reduced approximation (Halpern *et al*. 2023). In Scott (2003), the closure issue was avoided by assuming that the polarisation drift can be expressed in terms of the gradient of a scalar, but this relation does not hold in general.

In this section we seek to invert the expression for $\boldsymbol{v}_{ps}$, given by (3.15), as a function of the leading-order velocity $\bar{\boldsymbol{v}}_s$. Reordering terms in (3.15), we note that $\boldsymbol{v}_{ps}$ satisfies

$$\boldsymbol{v}_{ps} - \frac{\boldsymbol{b}}{\Omega_{cs}} \times (\boldsymbol{v}_{ps} \cdot \boldsymbol{\nabla}) \bar{\boldsymbol{v}}_s = \frac{\boldsymbol{b}}{\Omega_{cs}} \times \left(\frac{\partial \bar{\boldsymbol{v}}_s}{\partial t} + \bar{\boldsymbol{v}}_s \cdot \boldsymbol{\nabla} \bar{\boldsymbol{v}}_s + r_s \bar{\boldsymbol{v}}_s \right). \tag{4.1}$$

https://doi.org/10.1017/S0022377825101104 Published online by Cambridge University Press

We now show that the solution of (4.1) yields

$$\boldsymbol{v}_{ps} = \frac{1 + (\boldsymbol{b} \times \nabla \bar{\boldsymbol{v}}_s)/\Omega_{cs}}{1 + \boldsymbol{b} \cdot (\nabla \times \bar{\boldsymbol{v}}_s)/\Omega_{cs} + \det(\nabla \bar{\boldsymbol{v}}_s)_\perp/\Omega_{cs}^2} \cdot \left[\frac{\boldsymbol{b}}{\Omega_{cs}} \times \left(\frac{\partial \bar{\boldsymbol{v}}_s}{\partial t} + \bar{\boldsymbol{v}}_s \cdot \nabla \bar{\boldsymbol{v}}_s + r_s \bar{\boldsymbol{v}}_s \right) \right]. \tag{4.2}$$

For convenience, we define the inertial drift-velocity term, $\boldsymbol{U}_s$, to represent the right-hand side of (4.1), that is,

$$\boldsymbol{U}_s \equiv \frac{\boldsymbol{b}}{\Omega_{cs}} \times \left(\frac{\partial \bar{\boldsymbol{v}}_s}{\partial t} + \bar{\boldsymbol{v}}_s \cdot \nabla \bar{\boldsymbol{v}}_s + r_s \bar{\boldsymbol{v}}_s \right), \tag{4.3}$$

which contains only leading-order quantities related to the velocity $\bar{\boldsymbol{v}}_s$. We note that $\boldsymbol{U}_s \cdot \boldsymbol{b} = 0$. Equation (4.1) can be written in component form as

$$\left(\delta_l^i - \frac{1}{\Omega_{cs}} \epsilon^{ij}{}_k b_j \nabla_l \bar{v}_s^k \right) v_{ps}^l \equiv (Q_s^{-1})^i{}_l v_{ps}^l = U_s^i, \tag{4.4}$$

where v_{ps}^l are the contravariant components of $\boldsymbol{v}_{ps}$ in an arbitrary coordinate system, $\boldsymbol{Q}_s^{-1} \equiv (Q_s^{-1})^i{}_l$ is the linear operator we seek to invert, ϵ^{ijk} is the Levi-Civita tensor in $\mathbb{R}^3$ and $\delta_{ij} = \delta^{ij} = \delta^i_j$ is the Kronecker delta, used to raise/lower indices in $\mathbb{R}^3$. Einstein's convention of summation over repeated indices is used throughout the paper. We simplify the calculation by considering a basis composed of $\hat{\boldsymbol{w}}$, an arbitrary unit vector such that $\hat{\boldsymbol{w}} \cdot \boldsymbol{b} = 0$, $\boldsymbol{b} \times \hat{\boldsymbol{w}}$ and $\boldsymbol{b}$. To lighten the notation we define $\boldsymbol{b} \times \hat{\boldsymbol{w}} \equiv \hat{\boldsymbol{w}}^*$, and note that $\hat{\boldsymbol{w}} \cdot \hat{\boldsymbol{w}}^* = 0$, $\hat{w}^{*2} = \hat{w}^2 = 1$ and $(\hat{\boldsymbol{w}}^*)^* = \boldsymbol{b} \times (\boldsymbol{b} \times \hat{\boldsymbol{w}}) = -\hat{\boldsymbol{w}}$. Since the polarisation velocity lies in the plane orthogonal to $\boldsymbol{b}$, we have

$$\boldsymbol{v}_{ps} = \alpha_s \hat{\boldsymbol{w}} + \beta_s \hat{\boldsymbol{w}}^*, \tag{4.5}$$

where α_s and β_s are two scalar functions of $\boldsymbol{U}_s$ to be determined. Substituting this expression for $\boldsymbol{v}_p$ in (4.1), we have

$$\alpha_s \hat{\boldsymbol{w}} + \beta_s \hat{\boldsymbol{w}}^* - \alpha_s \frac{\boldsymbol{b}}{\Omega_{cs}} \times (\hat{\boldsymbol{w}} \cdot \nabla \bar{\boldsymbol{v}}_s) - \beta_s \frac{\boldsymbol{b}}{\Omega_{cs}} \times (\hat{\boldsymbol{w}}^* \cdot \nabla \bar{\boldsymbol{v}}_s) = \boldsymbol{U}_s. \tag{4.6}$$

Projecting (4.6) onto the basis elements $\hat{\boldsymbol{w}}$ and $\hat{\boldsymbol{w}}^*$, we obtain

$$\hat{\boldsymbol{w}} \cdot \boldsymbol{U}_s = \alpha_s \left(1 - \hat{\boldsymbol{w}} \cdot \left[\frac{\boldsymbol{b}}{\Omega_{cs}} \times (\hat{\boldsymbol{w}} \cdot \nabla \bar{\boldsymbol{v}}) \right] \right) - \beta_s \hat{\boldsymbol{w}} \cdot \left[\frac{\boldsymbol{b}}{\Omega_{cs}} \times \left(\hat{\boldsymbol{w}}^* \cdot \nabla \bar{\boldsymbol{v}}_s \right) \right], \tag{4.7}$$

$$\hat{\boldsymbol{w}}^* \cdot \boldsymbol{U}_s = \beta_s \left(1 - \hat{\boldsymbol{w}}^* \cdot \left[\frac{\boldsymbol{b}}{\Omega_{cs}} \times (\hat{\boldsymbol{w}}^* \cdot \nabla \bar{\boldsymbol{v}}_s) \right] \right) - \alpha_s \hat{\boldsymbol{w}}^* \cdot \left[\frac{\boldsymbol{b}}{\Omega_{cs}} \times (\hat{\boldsymbol{w}} \cdot \nabla \bar{\boldsymbol{v}}_s) \right]. \tag{4.8}$$

Equations (4.7) and (4.8) constitute a linear system for the unknowns α_s and β_s. Solving for α_s and β_s yields

$$\alpha_s = \frac{(1 - \tilde{\delta}_s) \hat{\boldsymbol{w}} \cdot \boldsymbol{U}_s + \gamma_s \hat{\boldsymbol{w}}^* \cdot \boldsymbol{U}_s}{\Delta_s}, \tag{4.9}$$

$$\beta_s = \frac{\tilde{\gamma}_s \hat{\boldsymbol{w}} \cdot \boldsymbol{U}_s + (1 - \delta_s) \hat{\boldsymbol{w}}^* \cdot \boldsymbol{U}_s}{\Delta_s}, \tag{4.10}$$

https://doi.org/10.1017/S0022377825101104 Published online by Cambridge University Press

where

$$\delta_s = \hat{\boldsymbol{w}} \cdot \left[\frac{\boldsymbol{b}}{\Omega_{cs}} \times (\hat{\boldsymbol{w}} \cdot \nabla \bar{\boldsymbol{v}}_s) \right], \tag{4.11}$$

$$\tilde{\delta}_s = \hat{\boldsymbol{w}}^* \cdot \left[\frac{\boldsymbol{b}}{\Omega_{cs}} \times (\hat{\boldsymbol{w}}^* \cdot \nabla \bar{\boldsymbol{v}}_s) \right], \tag{4.12}$$

$$\gamma_s = \hat{\boldsymbol{w}} \cdot \left[\frac{\boldsymbol{b}}{\Omega_{cs}} \times (\hat{\boldsymbol{w}}^* \cdot \nabla \bar{\boldsymbol{v}}_s) \right], \tag{4.13}$$

$$\tilde{\gamma}_s = \hat{\boldsymbol{w}}^* \cdot \left[\frac{\boldsymbol{b}}{\Omega_{cs}} \times (\hat{\boldsymbol{w}} \cdot \nabla \bar{\boldsymbol{v}}_s) \right], \tag{4.14}$$

and the determinant Δ_s is given by

$$\Delta_s = 1 - (\delta_s + \tilde{\delta}_s) + (\delta_s \tilde{\delta}_s - \gamma_s \tilde{\gamma}_s). \tag{4.15}$$

Using the cyclical property of the triple product $\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = \boldsymbol{a}_2 \cdot (\boldsymbol{a}_3 \times \boldsymbol{a}_1)$ and $(\hat{\boldsymbol{w}}^*)^* = -\hat{\boldsymbol{w}}$, the coefficients in (4.11)–(4.14) can be written in more transparent form as

$$\delta_s = -\hat{\boldsymbol{w}}^* \cdot \left(\frac{\boldsymbol{b} \times \nabla \bar{\boldsymbol{v}}_s}{\Omega_{cs}} \right) \cdot \hat{\boldsymbol{w}}^*, \tag{4.16}$$

$$\tilde{\delta}_s = -\hat{\boldsymbol{w}} \cdot \left(\frac{\boldsymbol{b} \times \nabla \bar{\boldsymbol{v}}_s}{\Omega_{cs}} \right) \cdot \hat{\boldsymbol{w}}, \tag{4.17}$$

$$\gamma_s = \hat{\boldsymbol{w}} \cdot \left(\frac{\boldsymbol{b} \times \nabla \bar{\boldsymbol{v}}_s}{\Omega_{cs}} \right) \cdot \hat{\boldsymbol{w}}^*, \tag{4.18}$$

$$\tilde{\gamma}_s = \hat{\boldsymbol{w}}^* \cdot \left(\frac{\boldsymbol{b} \times \nabla \bar{\boldsymbol{v}}_s}{\Omega_{cs}} \right) \cdot \hat{\boldsymbol{w}}. \tag{4.19}$$

Substituting the above expressions for the coefficients δ_s, $\tilde{\delta}_s$, γ_s and $\tilde{\gamma}_s$ into (4.9), (4.10) and (4.15), we obtain

$$\alpha_s = \frac{1}{\Delta_s} \left(\hat{\boldsymbol{w}} \cdot \boldsymbol{U}_s + \left(\hat{\boldsymbol{w}} \cdot \boldsymbol{T}_s \cdot \hat{\boldsymbol{w}} \right) \hat{\boldsymbol{w}} \cdot \boldsymbol{U}_s + \left(\hat{\boldsymbol{w}} \cdot \boldsymbol{T}_s \cdot \hat{\boldsymbol{w}}^* \right) \hat{\boldsymbol{w}}^* \cdot \boldsymbol{U}_s \right), \tag{4.20}$$

$$\beta_s = \frac{1}{\Delta_s} \left(\hat{\boldsymbol{w}}^* \cdot \boldsymbol{U}_s + \left(\hat{\boldsymbol{w}}^* \cdot \boldsymbol{T}_s \cdot \hat{\boldsymbol{w}}^* \right) \hat{\boldsymbol{w}}^* \cdot \boldsymbol{U}_s + \left(\hat{\boldsymbol{w}}^* \cdot \boldsymbol{T}_s \cdot \hat{\boldsymbol{w}} \right) \hat{\boldsymbol{w}} \cdot \boldsymbol{U}_s \right), \tag{4.21}$$

$$\begin{aligned} \Delta_s = 1 &+ (\hat{\boldsymbol{w}} \cdot \boldsymbol{T}_s \cdot \hat{\boldsymbol{w}} + \hat{\boldsymbol{w}}^* \cdot \boldsymbol{T}_s \cdot \hat{\boldsymbol{w}}^*) \\ &+ [(\hat{\boldsymbol{w}} \cdot \boldsymbol{T}_s \cdot \hat{\boldsymbol{w}})(\hat{\boldsymbol{w}}^* \cdot \boldsymbol{T}_s \cdot \hat{\boldsymbol{w}}^*) - (\hat{\boldsymbol{w}} \cdot \boldsymbol{T}_s \cdot \hat{\boldsymbol{w}}^*)(\hat{\boldsymbol{w}}^* \cdot \boldsymbol{T}_s \cdot \hat{\boldsymbol{w}})], \end{aligned} \tag{4.22}$$

with $\boldsymbol{T}_s \equiv (\boldsymbol{b}/\Omega_{cs} \times \nabla \bar{\boldsymbol{v}}_s)_\perp : W_\perp \to W_\perp$ the linear operator $\boldsymbol{b}/\Omega_{cs} \times \nabla \bar{\boldsymbol{v}}_s$ restricted to act on the subspace perpendicular to $\boldsymbol{b}(\boldsymbol{x}, t)$ at position $\boldsymbol{x}$ and time t, defined by $W_\perp \equiv \{\boldsymbol{y} \in \mathbb{R}^3 | \boldsymbol{b}(\boldsymbol{x}, t) \cdot \boldsymbol{y} = 0\}$. Substituting (4.20) and (4.21) into (4.5), and recalling

https://doi.org/10.1017/S0022377825101104 Published online by Cambridge University Press

that $\boldsymbol{U}_s = \hat{\boldsymbol{w}}(\hat{\boldsymbol{w}} \cdot \boldsymbol{U}_s) + \boldsymbol{w}^*(\hat{\boldsymbol{w}}^* \cdot \boldsymbol{U}_s)$, we obtain an expression for $\boldsymbol{v}_{ps}$:

$$\boldsymbol{v}_{ps} = \frac{1}{\Delta_s}(\boldsymbol{U}_s + [\boldsymbol{e}^i(\boldsymbol{e}_i \cdot \boldsymbol{T}_s \cdot \boldsymbol{e}^j)\boldsymbol{e}_j] \cdot \boldsymbol{U}_s), \tag{4.23}$$

with $\boldsymbol{e}^i = \{\hat{\boldsymbol{w}}, \hat{\boldsymbol{w}}^*\}$ the orthonormal basis elements of $W_\perp$. Since $\hat{\boldsymbol{w}}$ has arbitrary direction in the plane perpendicular to $\boldsymbol{b}$, we can express the result in a basis-independent form as

$$\boldsymbol{v}_{ps} = \frac{1 + \boldsymbol{T}_s}{\Delta_s} \cdot \boldsymbol{U}_s = \frac{1 + (\boldsymbol{b} \times \nabla \bar{\boldsymbol{v}}_s)_\perp / \Omega_{cs}}{\Delta_s} \cdot \boldsymbol{U}_s. \tag{4.24}$$

We note that $\boldsymbol{U}_s$ in (4.3) is an element of the orthogonal subspace $W_\perp$ and, therefore, $(\boldsymbol{b} \times \nabla \bar{\boldsymbol{v}}_s)_\perp \cdot \boldsymbol{U}_s = (\boldsymbol{b} \times \nabla \bar{\boldsymbol{v}}_s) \cdot \boldsymbol{U}_s$. The determinant Δ_s in basis-independent form is constructed from (4.22) and can be written as

$$\Delta_s = 1 + \mathrm{tr}(\boldsymbol{T}_s) + \det(\boldsymbol{T}_s), \tag{4.25}$$

or, explicitly, using the definition of $\boldsymbol{T}_s$, as

$$\Delta_s = 1 + \mathrm{tr}\left(\left[\frac{\boldsymbol{b}}{\Omega_{cs}} \times \nabla \bar{\boldsymbol{v}}_s\right]_\perp\right) + \det\left(\left[\frac{\boldsymbol{b}}{\Omega_{cs}} \times \nabla \bar{\boldsymbol{v}}_s\right]_\perp\right). \tag{4.26}$$

The trace term is given by

$$\mathrm{tr}\left(\left[\frac{\boldsymbol{b}}{\Omega_{cs}} \times \nabla \bar{\boldsymbol{v}}_s\right]_\perp\right) = \frac{1}{\Omega_{cs}} \epsilon_i{}^{jk} b_k \nabla_j \bar{v}_s^i = \frac{1}{\Omega_{cs}} \boldsymbol{b} \cdot (\nabla \times \bar{\boldsymbol{v}}_s). \tag{4.27}$$

Meanwhile, for any linear application $A : \mathbb{R}^3 \to \mathbb{R}^3$, the determinant of its restriction to the $W_\perp$ subspace can be expressed as $\det(A_\perp) = \epsilon_\perp^{kl} \epsilon_{\perp\, ij} A^i{}_k A^j{}_l / 2$, where $\epsilon_\perp^{ij} \equiv \epsilon^{ijk} b_k$ denotes the induced Levi-Civita symbol on $W_\perp$. The determinant term in (4.26) becomes

$$\begin{aligned} \det\left(\left[\frac{\boldsymbol{b}}{\Omega_{cs}} \times \nabla \bar{\boldsymbol{v}}_s\right]_\perp\right) &= \frac{1}{2} \epsilon_\perp^{kl} \epsilon_{\perp ij} \left(\frac{\boldsymbol{b}}{\Omega_{cs}} \times \nabla \bar{\boldsymbol{v}}_s\right)^i{}_k \left(\frac{\boldsymbol{b}}{\Omega_{cs}} \times \nabla \bar{\boldsymbol{v}}_s\right)^j{}_l \\ &= \frac{1}{2\Omega_{cs}^2} \epsilon_\perp^{kl} \epsilon_{\perp ij} (\nabla \bar{\boldsymbol{v}}_s)^i{}_k (\nabla \bar{\boldsymbol{v}}_s)^j{}_l = \frac{1}{\Omega_{cs}^2} \det(\nabla \bar{\boldsymbol{v}}_s)_\perp, \end{aligned} \tag{4.28}$$

where we used the fact that $(\boldsymbol{b} \times \nabla \bar{\boldsymbol{v}}_s)^i{}_k = \epsilon_\perp^{ij} \nabla_j \bar{v}_k$ and $\epsilon_\perp^{ki} \epsilon_{\perp kj} = (\Pi_\perp)^i{}_j$. Finally, the determinant Δ_s in (4.25) becomes

$$\Delta_s = 1 + \frac{\boldsymbol{b} \cdot (\nabla \times \bar{\boldsymbol{v}}_s)}{\Omega_{cs}} + \frac{\det(\nabla \bar{\boldsymbol{v}}_s)_\perp}{\Omega_{cs}^2}, \tag{4.29}$$

which contains a correction due to the parallel component of the vorticity $\boldsymbol{b} \cdot (\nabla \times \bar{\boldsymbol{v}}_s)/\Omega_{cs} \sim O(\epsilon)$ and a nonlinear correction related to flow shear $\det(\nabla \bar{\boldsymbol{v}}_s)_\perp / \Omega_{cs}^2 \sim O(\epsilon^2)$. For a linear geometry in the cold-ion approximation, and for the case where the leading-order ion perpendicular flow is purely the $E \times B$ drift $\bar{\boldsymbol{v}}_\perp = \boldsymbol{v}_E$, (4.29) reduces to equation (A6) in Reiser (2012). The final solution for the polarisation velocity is therefore given by (3.15), or, in more succinct form, by

$$\boldsymbol{v}_{ps} = \boldsymbol{Q}_s(\bar{\boldsymbol{v}}_s) \cdot \boldsymbol{U}_s, \tag{4.30}$$

https://doi.org/10.1017/S0022377825101104 Published online by Cambridge University Press

where

$$\boldsymbol{Q}_s(\bar{\boldsymbol{v}}_s) = \frac{1 + (\boldsymbol{b} \times \boldsymbol{\nabla}\bar{\boldsymbol{v}}_s)/\Omega_{cs}}{1 + \boldsymbol{b} \cdot (\boldsymbol{\nabla} \times \bar{\boldsymbol{v}}_s)/\Omega_{cs} + \det(\boldsymbol{\nabla}\bar{\boldsymbol{v}}_s)_\perp/\Omega_{cs}^2}. \tag{4.31}$$

Within the drift-ordering $\bar{\boldsymbol{v}}_s \cdot \boldsymbol{\nabla}/\Omega_{cs} \sim \epsilon$, the application $\boldsymbol{Q}_s^{-1}$ is invertible, as the determinant $\Delta_s = 1 + O(\epsilon)$ does not vanish. In Appendix A, we verify that the expression of $\boldsymbol{v}_{ps}$ given in (4.30) satisfies (4.1).

We note that (4.30) represents a non-perturbative result, containing terms of all orders in ϵ, as can be seen by Taylor-expanding the denominator in (4.31). All of the subleading terms contained in $\boldsymbol{Q}_s$ are required for the existence of exact conservation laws.

5. Conservative drift-reduced model

Based on (3.15) we construct now the drift-reduced model, valid to $O(\epsilon)$, which admits energy and momentum conservation laws satisfied exactly, i.e. to all orders in ϵ. The model evolves the density, parallel velocity and pressure of each species and is closed by a vorticity equation, an associated Poisson equation for ϕ and Ampère's equation for the vector potential $\boldsymbol{A}$.

Expressing the total fluid velocity as $\boldsymbol{V}_s = \bar{\boldsymbol{v}}_s + \boldsymbol{v}_{ps}$ in (2.1) and (2.3), we obtain

$$\frac{\partial n_s}{\partial t} + \boldsymbol{\nabla} \cdot [n_s(\bar{\boldsymbol{v}}_s + \boldsymbol{v}_{ps})] = \mathcal{S}_s, \tag{5.1}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_s\right) + \boldsymbol{\nabla} \cdot \left[\frac{3}{2}p_s(\bar{\boldsymbol{v}}_s + \boldsymbol{v}_{ps}) + \boldsymbol{q}_s\right] + \boldsymbol{P}_s : \boldsymbol{\nabla}(\bar{\boldsymbol{v}}_s + \boldsymbol{v}_{ps}) = \mathcal{Q}_s + r_s\overline{\mathcal{K}}_s, \tag{5.2}$$

where $\boldsymbol{v}_{ps}$ is given either implicitly by (3.15) or explicitly via (4.2), and we define $\overline{\mathcal{K}}_s \equiv m_s n_s \bar{v}_s^2/2$ to be the leading-order component of the fluid kinetic energy. The equation for the velocity component parallel to the magnetic field $v_{\parallel s} = \boldsymbol{V}_s \cdot \boldsymbol{b}$ of species s is obtained by projecting the $O(\epsilon)$ expanded momentum (2.2),

$$\frac{\partial(m_s n_s \bar{\boldsymbol{v}}_s)}{\partial t} + \boldsymbol{\nabla} \cdot (m_s n_s \boldsymbol{V}_s \bar{\boldsymbol{v}}_s) + \boldsymbol{\nabla} \cdot \boldsymbol{P}_s - q_s n_s(\boldsymbol{E} + \boldsymbol{V}_s \times \boldsymbol{B}) = \boldsymbol{\mathcal{R}}_s, \tag{5.3}$$

onto $\boldsymbol{b}$, obtaining

$$\frac{\partial(m_s n_s v_{\parallel s})}{\partial t} + \boldsymbol{\nabla} \cdot (m_s n_s v_{\parallel s} \boldsymbol{V}_s) - m_s n_s \bar{\boldsymbol{v}}_s \cdot \frac{\mathrm{d}_s \boldsymbol{b}}{\mathrm{d}t} + (\boldsymbol{\nabla} \cdot \boldsymbol{P}_s)_\parallel = \mathcal{R}_{\parallel s} + q_s n_s E_\parallel, \tag{5.4}$$

with $(\boldsymbol{\nabla} \cdot \boldsymbol{P}_s)_\parallel \equiv \boldsymbol{b} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{P}_s)$ and, similarly, for other quantities.

To derive the vorticity equation, we impose the quasi-neutrality constraint $\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$. We begin with the force-balance equation for the leading-order component of the total momentum density, $\overline{\boldsymbol{\mathcal{M}}} = \sum_s m_s n_s \bar{\boldsymbol{v}}_s$. As shown in § 6, this is given by

$$\frac{\partial \overline{\boldsymbol{\mathcal{M}}}}{\partial t} + \boldsymbol{\nabla} \cdot \sum_s (\boldsymbol{V}_s \overline{\boldsymbol{\mathcal{M}}}_s + \boldsymbol{P}_s) = \boldsymbol{J} \times \boldsymbol{B} + \sum_s \boldsymbol{\mathcal{R}}_s, \tag{5.5}$$

where the total current $\boldsymbol{J} = \boldsymbol{J}_\parallel + \bar{\boldsymbol{J}}_\perp + \boldsymbol{J}_p$ is the sum of the leading-order components, $\boldsymbol{J}_\parallel + \bar{\boldsymbol{J}}_\perp = \sum_s q_s n_s \boldsymbol{v}_{\parallel s} + \sum_s q_s n_s \bar{\boldsymbol{v}}_{\perp s}$, and the subleading polarisation term,

https://doi.org/10.1017/S0022377825101104 Published online by Cambridge University Press

$\boldsymbol{J}_p = \sum_s q_s n_s \boldsymbol{v}_{ps}$. We note the similarity between (5.5) and the non-drift-reduced form in (2.15). Taking the cross-product of (5.5) with $\boldsymbol{B}$ leads to

$$\boldsymbol{J} = \boldsymbol{J}_\parallel + \frac{\boldsymbol{b}}{B} \times \frac{\partial \overline{\boldsymbol{\mathcal{M}}}}{\partial t} + \frac{\boldsymbol{b}}{B} \times \left(\sum_s [\nabla \cdot (\boldsymbol{V}_s \overline{\boldsymbol{\mathcal{M}}}_s + \boldsymbol{P}_s) - \boldsymbol{\mathcal{R}}_s] \right), \tag{5.6}$$

which can be rewritten as

$$\boldsymbol{J} = \boldsymbol{J}_\parallel + \frac{\partial}{\partial t} \left(\frac{\boldsymbol{b} \times \overline{\boldsymbol{\mathcal{M}}}}{B} \right) + \frac{\boldsymbol{b}}{B} \times \left(\sum_s [\nabla \cdot (\boldsymbol{V}_s \overline{\boldsymbol{\mathcal{M}}}_s + \boldsymbol{P}_s) - \boldsymbol{\mathcal{R}}_s] \right) - \frac{\partial}{\partial t} \left(\frac{\boldsymbol{b}}{B} \right) \times \overline{\boldsymbol{\mathcal{M}}}. \tag{5.7}$$

Imposing the quasi-neutrality constraint $\nabla \cdot \boldsymbol{J} = 0$, we obtain the vorticity equation for $\varpi \equiv -\nabla \cdot (\boldsymbol{b} \times \overline{\boldsymbol{\mathcal{M}}}/B)$:

$$\begin{aligned} \frac{\partial \varpi}{\partial t} = {} & \nabla \cdot (\boldsymbol{J}_\parallel + \bar{\boldsymbol{J}}_\perp) + \nabla \cdot \left[\sum_s \frac{\boldsymbol{b}}{B} \times \nabla \cdot (\boldsymbol{V}_s \overline{\boldsymbol{\mathcal{M}}}_s) \right] \\ & + \nabla \cdot \left[\left(\frac{(\boldsymbol{\Pi}_\perp - \boldsymbol{bb}) \cdot \nabla \times \boldsymbol{E}}{B^2} \right) \times \overline{\boldsymbol{\mathcal{M}}} \right], \end{aligned} \tag{5.8}$$

where we use Faraday's law and, since $\boldsymbol{b} \cdot \partial_t \boldsymbol{b} = 0$, $\partial_t (\boldsymbol{b}/B) = 2\boldsymbol{bb} \cdot (\nabla \times \boldsymbol{E})/B^2 - \nabla \times \boldsymbol{E}/B^2 = (\boldsymbol{bb} - \boldsymbol{\Pi}_\perp) \cdot (\nabla \times \boldsymbol{E})/B^2$, with $\boldsymbol{\Pi}_\perp = \mathbb{I} - \boldsymbol{bb}$ the perpendicular projection operator. In (5.8), we also express the pressure and momentum drive terms as $\sum_s \boldsymbol{b} \times (\nabla \cdot \boldsymbol{P}_s - \boldsymbol{\mathcal{R}}_s)/B = \sum_s q_s n_s \bar{\boldsymbol{v}}_{\perp s} = \bar{\boldsymbol{J}}_\perp$. Using $\boldsymbol{V}_s = \bar{\boldsymbol{v}}_s + \boldsymbol{v}_{ps}$ in (5.8), we obtain

$$\begin{aligned} \frac{\partial \varpi}{\partial t} = {} & \nabla \cdot (\boldsymbol{J}_\parallel + \bar{\boldsymbol{J}}_\perp) + \nabla \cdot \left[\sum_s \frac{\boldsymbol{b}}{B} \times \nabla \cdot (\bar{\boldsymbol{v}}_s \overline{\boldsymbol{\mathcal{M}}}_s) \right] \\ & + \nabla \cdot \left[\left(\frac{(\boldsymbol{\Pi}_\perp - \boldsymbol{bb}) \cdot \nabla \times \boldsymbol{E}}{B^2} \right) \times \overline{\boldsymbol{\mathcal{M}}} \right] \\ & + \nabla \cdot \left[\sum_s \frac{\boldsymbol{b}}{B} \times \nabla \cdot (\boldsymbol{Q}_s(\bar{\boldsymbol{v}}_s) \cdot \boldsymbol{U}_s \overline{\boldsymbol{\mathcal{M}}}_s) \right], \end{aligned} \tag{5.9}$$

where $\boldsymbol{v}_{ps}$ is expressed in terms of leading-order quantities via (4.30), with $\boldsymbol{U}_s$ given in (4.3). The last term in (5.9) is the conservative correction to the usual vorticity equation found in the literature, accounting for polarisation advection of the leading-order momentum $\overline{\boldsymbol{\mathcal{M}}}$. Having evolved the scalar vorticity in time, the Poisson equation, defined via $\varpi = -\nabla \cdot (\boldsymbol{b} \times \overline{\boldsymbol{\mathcal{M}}}/B)$, takes its usual form (see e.g. Giacomin *et al*. 2022*b*):

$$\nabla \cdot \left(\frac{\rho}{B^2} \nabla_\perp \phi \right) = \varpi - \nabla \cdot \left(\sum_s \frac{(\nabla \cdot \boldsymbol{P}_s)_\perp - \boldsymbol{\mathcal{R}}_{\perp s}}{B \Omega_{cs}} \right) - \nabla \cdot \left(\frac{\rho}{B^2} \left. \frac{\partial \boldsymbol{A}}{\partial t} \right|_\perp \right), \tag{5.10}$$

where the terms due to $\boldsymbol{\mathcal{R}}_{\perp s}$ and $\partial_t \boldsymbol{A}|_\perp$ are usually neglected because of additional ordering assumptions on the size of perpendicular friction terms and strength of

https://doi.org/10.1017/S0022377825101104 Published online by Cambridge University Press

electromagnetic effects (Zeiler *et al*. 1997). In (5.10), $\rho \equiv \sum_s m_s n_s$ is the total mass density of the plasma and the perpendicular component of a given vector $\boldsymbol{W}$ is defined using the projection operator, $\boldsymbol{W}_\perp \equiv \boldsymbol{\Pi}_\perp \cdot \boldsymbol{W}$.

To summarise, the drift-reduced fluid model we propose, obeying exact conservation laws for the leading-order energy $\overline{\mathcal{H}}$, leading-order momentum $\overline{\boldsymbol{\mathcal{M}}}$, mass and charge (as shown in § 6), is given by the moment equations (5.1), (5.2) and (5.4), the vorticity equation (5.9) for ϖ, the Poisson equation (5.10) for ϕ, the polarisation drift expression (4.2) for $\boldsymbol{v}_{ps}$ and the Ampère equation for $\boldsymbol{A}$, which in the Coulomb gauge, $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$, is given by

$$\boldsymbol{\nabla}^2 \boldsymbol{A} = -\mu_0 (\boldsymbol{J}_\parallel + \bar{\boldsymbol{J}}_\perp + \boldsymbol{J}_p). \tag{5.11}$$

Computing the polarisation velocity requires evaluating $\partial_t \bar{\boldsymbol{v}}_s$ in (4.2), which is itself a function of $\boldsymbol{v}_{ps}$. Solution of this coupled system can be implemented either implicitly via an iterative procedure or explicitly by using $\boldsymbol{v}_{ps}$ from the previous time step to first evolve the dynamical fields, then calculating $\partial_t \bar{\boldsymbol{v}}_s$ to update $\boldsymbol{v}_{ps}$. Devising a numerical implementation which preserves the conservation properties of the system is, however, left for future work. The commonly used non-conservative drift-reduced vorticity equation is obtained from (5.9) by keeping only the leading-order terms. This amounts to neglecting the last term in (5.9), yielding

$$\frac{\partial \varpi}{\partial t} \simeq \boldsymbol{\nabla} \cdot (\boldsymbol{J}_\parallel + \bar{\boldsymbol{J}}_\perp) + \boldsymbol{\nabla} \cdot \left[\sum_s \frac{\boldsymbol{b}}{B} \times \boldsymbol{\nabla} \cdot (\bar{\boldsymbol{v}}_s \overline{\boldsymbol{\mathcal{M}}}_s) \right] + \boldsymbol{\nabla} \cdot \left[\left(\frac{(\boldsymbol{\Pi}_\perp - \boldsymbol{bb}) \cdot \boldsymbol{\nabla} \times \boldsymbol{E}}{B^2} \right) \times \overline{\boldsymbol{\mathcal{M}}} \right], \tag{5.12}$$

which is the form found in the literature and implemented in drift-reduced turbulence codes (Gath & Wiesenberger 2019; Stegmeir *et al*. 2019; Giacomin *et al*. 2022*b*; Düll *et al*. 2024). For the case considered in Reiser (2012), the use of a conservative formulation had a small influence on the overall turbulent dynamics. In fact, the corrections needed to ensure energy–momentum consistency are subdominant in the drift-ordering expansion. However, in regimes where flow shear becomes large, and the tensor $\boldsymbol{Q}_s$ in (4.31) deviates sufficiently from unity, a self-consistent treatment of the vorticity equation, as given in (5.9), could become necessary to accurately describe the nonlinear dynamics.

6. Conservation properties

To prove that the system of equations (5.1), (5.2), (5.4), (5.9) and (5.10) conserves both energy and momentum, we follow the procedure in Halpern *et al*. (2023). To obtain the transport equation for the parallel kinetic energy density $\mathcal{K}_{\parallel s} \equiv m_s n_s v_{\parallel s}^2/2$ of species s, we multiply (5.4) by $v_{\parallel s}$. After straightforward algebra, this leads to

$$\frac{\partial \mathcal{K}_{\parallel s}}{\partial t} + \boldsymbol{\nabla} \cdot \left(\mathcal{K}_{\parallel s} \boldsymbol{V}_s \right) - m_s n_s v_{\parallel s} \bar{\boldsymbol{v}}_s \cdot \frac{\mathrm{d}_s \boldsymbol{b}}{\mathrm{d}t} + \boldsymbol{v}_{\parallel s} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{P}_s) = \boldsymbol{v}_{\parallel s} \cdot \boldsymbol{\mathcal{R}}_s + q_s n_s \boldsymbol{v}_{\parallel s} \cdot \boldsymbol{E} - r_s \mathcal{K}_{\parallel s}. \tag{6.1}$$

https://doi.org/10.1017/S0022377825101104 Published online by Cambridge University Press

We note that $\boldsymbol{b}\cdot \mathrm{d}_s\boldsymbol{b}/\mathrm{d}t = \mathrm{d}_s(b^2/2)/\mathrm{d}t = 0$ and, as a consequence, $\bar{\boldsymbol{v}}\cdot d_s\boldsymbol{b}/dt = \bar{\boldsymbol{v}}_{\perp s}\cdot d_s\boldsymbol{b}/dt$. Adding the internal energy $\mathcal{U}_s = 3p_s/2$ contribution in (5.2)–(6.1), we have

$$\frac{\partial(\mathcal{K}_{\parallel s}+\mathcal{U}_s)}{\partial t} + \nabla\cdot\left[(\mathcal{K}_{\parallel s}+\mathcal{U}_s)\boldsymbol{V}_s + \boldsymbol{q}_s\right] - m_s n_s v_{\parallel s}\bar{\boldsymbol{v}}_{\perp s}\cdot\frac{\mathrm{d}_s\boldsymbol{b}}{\mathrm{d}t} + \boldsymbol{P}_s : \nabla\boldsymbol{V}_s + \boldsymbol{v}_{\parallel s}\cdot(\nabla\cdot\boldsymbol{P}_s) = \mathcal{Q}_s + \boldsymbol{v}_{\parallel s}\cdot\boldsymbol{\mathcal{R}}_s + q_s n_s \boldsymbol{v}_{\parallel s}\cdot\boldsymbol{E} + r_s(\overline{\mathcal{K}}_s - \mathcal{K}_{\parallel s}). \tag{6.2}$$

To find the transport equation for the leading-order perpendicular kinetic energy $\overline{\mathcal{K}}_{\perp s} \equiv m_s n_s \bar{v}_{\perp s}^2/2 = \overline{\mathcal{K}}_s - \mathcal{K}_{\parallel s}$ and the magnetic field energy $\mathcal{H}_B$, we multiply the quasi-neutrality equation $\nabla\cdot\boldsymbol{J} = 0$ by ϕ and express this as

$$0 = \nabla\cdot(\phi\boldsymbol{J}) - (\boldsymbol{J}_\parallel + \bar{\boldsymbol{J}}_\perp + \boldsymbol{J}_p)\cdot\nabla\phi, \tag{6.3}$$

where we recall $\boldsymbol{J} \equiv \sum_s q_s n_s \boldsymbol{V}_s = \boldsymbol{J}_\parallel + \bar{\boldsymbol{J}}_\perp + \boldsymbol{J}_p$. The total polarisation current can be expressed as $\boldsymbol{J}_p \equiv \sum_s \boldsymbol{J}_{ps} = \sum_s q_s n_s \boldsymbol{v}_{ps}$, with $\boldsymbol{v}_{ps}$ given by (3.15). The polarisation current density of species s, $\boldsymbol{J}_{ps}$, is therefore

$$\boldsymbol{J}_{ps} = \frac{m_s n_s}{B}\boldsymbol{b}\times\left(\frac{\mathrm{d}_s\bar{\boldsymbol{v}}_s}{\mathrm{d}t} + r_s\bar{\boldsymbol{v}}_s\right). \tag{6.4}$$

As a first step, we note that, since the polarisation current is orthogonal to $\boldsymbol{b}$, we can express

$$\boldsymbol{J}_p\cdot\nabla\phi = \boldsymbol{J}_p\cdot\nabla_\perp\phi = -\boldsymbol{J}_p\cdot[\boldsymbol{b}\times(\boldsymbol{b}\times\nabla\phi)], \tag{6.5}$$

and, given that $\nabla\phi = -\boldsymbol{E} - \partial_t\boldsymbol{A}$, we can rewrite (6.5) as

$$\boldsymbol{J}_p\cdot\nabla\phi = -\boldsymbol{J}_p\cdot(\boldsymbol{B}\times\boldsymbol{v}_E) - \boldsymbol{J}_p\cdot\partial_t\boldsymbol{A}. \tag{6.6}$$

Moreover, since $\boldsymbol{v}_E$ is species-independent, we obtain

$$\boldsymbol{J}_p\cdot\nabla\phi = -\sum_s \boldsymbol{J}_{ps}\cdot(\boldsymbol{B}\times\boldsymbol{v}_E) - \boldsymbol{J}_p\cdot\partial_t\boldsymbol{A}. \tag{6.7}$$

Expressing the $E\times B$ drift in (6.7) as $\boldsymbol{v}_E = \bar{\boldsymbol{v}}_{\perp s} - \boldsymbol{w}_{\perp s}$, with $\boldsymbol{w}_{\perp s}$ the drift velocity of species s in the frame co-moving with the $E\times B$ velocity (cf. (3.6)), we have

$$\sum_s \boldsymbol{J}_{ps}\cdot(\boldsymbol{B}\times\boldsymbol{v}_E) = \sum_s\left[m_s n_s\left(\frac{\mathrm{d}_s\bar{\boldsymbol{v}}_s}{\mathrm{d}t} + r_s\bar{\boldsymbol{v}}_s\right)\cdot\bar{\boldsymbol{v}}_{\perp s}\right] - \sum_s \boldsymbol{J}_{ps}\cdot(\boldsymbol{B}\times\boldsymbol{w}_{\perp s}). \tag{6.8}$$

Decomposing $\bar{\boldsymbol{v}}_s = v_{\parallel s}\boldsymbol{b} + \bar{\boldsymbol{v}}_{\perp s}$ into its parallel and perpendicular components, the first term on the right-hand side of (6.8) becomes

$$\sum_s\left[m_s n_s\left(\frac{\mathrm{d}_s\bar{\boldsymbol{v}}_s}{\mathrm{d}t} + r_s\bar{\boldsymbol{v}}_s\right)\cdot\bar{\boldsymbol{v}}_{\perp s}\right] = \sum_s m_s n_s\left(\bar{\boldsymbol{v}}_{\perp s}\cdot\frac{\mathrm{d}_s\bar{\boldsymbol{v}}_{\perp s}}{\mathrm{d}t} + r_s\bar{v}_{\perp s}^2\right) + \sum_s m_s n_s v_{\parallel s}\bar{\boldsymbol{v}}_{\perp s}\cdot\frac{\mathrm{d}_s\boldsymbol{b}}{\mathrm{d}t}. \tag{6.9}$$

https://doi.org/10.1017/S0022377825101104 Published online by Cambridge University Press

The first term on the right-hand side of (6.9) leads to a transport law for the leading-order perpendicular kinetic energy of species s, $\overline{\mathcal{K}}_{\perp s}$. Indeed, we have

$$m_s n_s \bar{\boldsymbol{v}}_{\perp s} \cdot \left(\frac{\mathrm{d}_s \bar{\boldsymbol{v}}_{\perp s}}{\mathrm{d}t} + r_s \bar{\boldsymbol{v}}_{\perp s} \right) = \frac{\partial \overline{\mathcal{K}}_{\perp s}}{\partial t} + \boldsymbol{\nabla} \cdot (\overline{\mathcal{K}}_{\perp s} \boldsymbol{V}_s) + r_s \overline{\mathcal{K}}_{\perp s}, \tag{6.10}$$

where the continuity equation $\mathrm{d}_s n_s / \mathrm{d}t = r_s n_s - n_s \boldsymbol{\nabla} \cdot \boldsymbol{V}_s$ is used. Using (6.7)–(6.10), we find

$$\begin{aligned} \boldsymbol{J}_p \cdot \boldsymbol{\nabla} \phi = & - \sum_s \left[\frac{\partial \overline{\mathcal{K}}_{\perp s}}{\partial t} + \boldsymbol{\nabla} \cdot (\overline{\mathcal{K}}_{\perp s} \boldsymbol{V}_s) + r_s \overline{\mathcal{K}}_{\perp s} \right] \\ & - \sum_s m_s n_s v_{\parallel s} \bar{\boldsymbol{v}}_{\perp s} \cdot \frac{\mathrm{d}_s \boldsymbol{b}}{\mathrm{d}t} + \sum_s \boldsymbol{J}_{ps} \cdot (\boldsymbol{B} \times \boldsymbol{w}_{\perp s}) - \boldsymbol{J}_p \cdot \partial_t \boldsymbol{A}. \end{aligned} \tag{6.11}$$

Given (6.11), we can rewrite (6.3) as

$$\begin{aligned} & \boldsymbol{\nabla} \cdot (\phi \boldsymbol{J}) - (\boldsymbol{J}_\parallel + \bar{\boldsymbol{J}}_\perp) \cdot \boldsymbol{\nabla} \phi - \sum_s \boldsymbol{J}_{ps} \cdot (\boldsymbol{B} \times \boldsymbol{w}_{\perp s}) + \boldsymbol{J}_p \cdot \partial_t \boldsymbol{A} \\ & \quad + \sum_s \left[\frac{\partial \overline{\mathcal{K}}_{\perp s}}{\partial t} + \boldsymbol{\nabla} \cdot (\overline{\mathcal{K}}_{\perp s} \boldsymbol{V}_s) + r_s \overline{\mathcal{K}}_{\perp s} \right] + \sum_s m_s n_s v_{\parallel s} \bar{\boldsymbol{v}}_{\perp s} \cdot \frac{\mathrm{d}_s \boldsymbol{b}}{\mathrm{d}t} = 0. \end{aligned} \tag{6.12}$$

The term $\boldsymbol{\nabla} \cdot (\phi \boldsymbol{J})$ can be expressed in terms of the Poynting flux as

$$\boldsymbol{\nabla} \cdot (\phi \boldsymbol{J}) = \boldsymbol{\nabla} \cdot \boldsymbol{S} + \boldsymbol{\nabla} \cdot \left(\frac{\partial_t \boldsymbol{A} \times \boldsymbol{B}}{\mu_0} \right), \tag{6.13}$$

where $\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$ is used. Writing the term $\boldsymbol{\nabla} \cdot (\partial_t \boldsymbol{A} \times \boldsymbol{B} / \mu_0)$ as

$$\boldsymbol{\nabla} \cdot \left(\frac{\partial_t \boldsymbol{A} \times \boldsymbol{B}}{\mu_0} \right) = \frac{\boldsymbol{B}}{\mu_0} \cdot \partial_t \boldsymbol{\nabla} \times \boldsymbol{A} - \frac{\boldsymbol{\nabla} \times \boldsymbol{B}}{\mu_0} \cdot \partial_t \boldsymbol{A} = \frac{\partial \mathcal{H}_B}{\partial t} - \boldsymbol{J} \cdot \partial_t \boldsymbol{A}, \tag{6.14}$$

where we recall that $\mathcal{H}_B = B^2/(2\mu_0)$ is the magnetic energy density, we obtain

$$\boldsymbol{\nabla} \cdot (\phi \boldsymbol{J}) = \frac{\partial \mathcal{H}_B}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{S} - \boldsymbol{J} \cdot \partial_t \boldsymbol{A}. \tag{6.15}$$

Substituting (6.15) into (6.12), we find the transport equation for the sum of the field energy, $\mathcal{H}_B$, and the total leading-order perpendicular kinetic energy, $\overline{\mathcal{K}}_\perp = \sum_s \overline{\mathcal{K}}_{\perp s}$, given by

$$\begin{aligned} & \frac{\partial (\mathcal{H}_B + \overline{\mathcal{K}}_\perp)}{\partial t} + \boldsymbol{\nabla} \cdot \left(\boldsymbol{S} + \sum_s \overline{\mathcal{K}}_{\perp s} \boldsymbol{V}_s \right) + (\boldsymbol{J}_\parallel + \bar{\boldsymbol{J}}_\perp) \cdot \boldsymbol{E} \\ & + \sum_s r_s \overline{\mathcal{K}}_{\perp s} + \sum_s m_s n_s v_{\parallel s} \bar{\boldsymbol{v}}_{\perp s} \cdot \frac{\mathrm{d}_s \boldsymbol{b}}{\mathrm{d}t} - \sum_s \boldsymbol{J}_{ps} \cdot (\boldsymbol{B} \times \boldsymbol{w}_{\perp s}) = 0. \end{aligned} \tag{6.16}$$

We now add the contribution from the parallel and internal energy densities in (6.2), having summed it over the species indices s, to (6.16). We thus obtain that the

https://doi.org/10.1017/S0022377825101104 Published online by Cambridge University Press

total leading-order energy density, $\overline{\mathcal{H}} \equiv \mathcal{H}_B + \sum_s (\overline{\mathcal{K}}_s + \mathcal{U}_s)$, evolves according to

$$\frac{\partial \overline{\mathcal{H}}}{\partial t} + \nabla \cdot \left(\boldsymbol{S} + \sum_s [\overline{\mathcal{H}}_s \boldsymbol{V}_s + \boldsymbol{q}_s] \right) + \sum_s \boldsymbol{P}_s : \nabla \boldsymbol{V}_s + \sum_s \boldsymbol{v}_{\parallel s} \cdot (\nabla \cdot \boldsymbol{P}_s) + (\boldsymbol{J}_\parallel + \bar{\boldsymbol{J}}_\perp) \cdot \boldsymbol{E} - \sum_s q_s n_s \boldsymbol{v}_{\parallel s} \cdot \boldsymbol{E} - \sum_s \boldsymbol{J}_{ps} \cdot (\boldsymbol{B} \times \boldsymbol{w}_{\perp s}) = \sum_s (\mathcal{Q}_s + \boldsymbol{v}_{\parallel s} \cdot \boldsymbol{\mathcal{R}}_s), \tag{6.17}$$

where we introduce the species-specific energy density, $\overline{\mathcal{H}}_s \equiv \overline{\mathcal{K}}_s + \mathcal{U}_s$. Finally, we rewrite the term $\sum_s \boldsymbol{J}_{ps} \cdot (\boldsymbol{B} \times \boldsymbol{w}_{\perp s})$ by decomposing $\boldsymbol{v}_{ps} = \boldsymbol{v}_{\perp s} - \bar{\boldsymbol{v}}_{\perp s} = \boldsymbol{v}_{\perp s} - \boldsymbol{v}_E - \boldsymbol{w}_{\perp s}$:

$$\boldsymbol{J}_{ps} \cdot (\boldsymbol{B} \times \boldsymbol{w}_{\perp s}) = q_s n_s (\boldsymbol{v}_{\perp s} - \boldsymbol{v}_E - \boldsymbol{w}_{\perp s}) \cdot (\boldsymbol{B} \times \boldsymbol{w}_{\perp s}) = q_s n_s (\boldsymbol{v}_{\perp s} - \boldsymbol{v}_E) \cdot (\boldsymbol{B} \times \boldsymbol{w}_{\perp s}). \tag{6.18}$$

Given that $\boldsymbol{v}_E = \boldsymbol{E} \times \boldsymbol{B}/B^2$, one deduces that the term $qn\boldsymbol{v}_E \cdot (\boldsymbol{B} \times \boldsymbol{w}_{\perp s}) = -q_s n_s \boldsymbol{w}_{\perp s} \cdot \boldsymbol{E}$. Furthermore, we have

$$q_s n_s (\boldsymbol{B} \times \boldsymbol{w}_{\perp s}) = \boldsymbol{B} \times (\boldsymbol{v}_{*s} + \boldsymbol{v}_{\boldsymbol{\pi} s} + \boldsymbol{v}_{\boldsymbol{\mathcal{R}} s}) = -(\nabla \cdot \boldsymbol{P}_s)_\perp + \boldsymbol{\mathcal{R}}_{\perp s}. \tag{6.19}$$

Substituting the expression (6.19) into (6.18) we obtain

$$\boldsymbol{J}_{ps} \cdot (\boldsymbol{B} \times \boldsymbol{w}_{\perp s}) = -\boldsymbol{v}_{\perp s} \cdot (\nabla \cdot \boldsymbol{P}_s) + \boldsymbol{v}_{\perp s} \cdot \boldsymbol{\mathcal{R}}_s + q_s n_s \boldsymbol{w}_{\perp s} \cdot \boldsymbol{E}, \tag{6.20}$$

which, substituted into (6.17), yields

$$\frac{\partial \overline{\mathcal{H}}}{\partial t} + \nabla \cdot \left(\boldsymbol{S} + \sum_s [\overline{\mathcal{H}}_s \boldsymbol{V}_s + \boldsymbol{P}_s \cdot \boldsymbol{V}_s + \boldsymbol{q}_s] \right) + (\boldsymbol{J}_\parallel + \bar{\boldsymbol{J}}_\perp) \cdot \boldsymbol{E} - \sum_s q_s n_s (\boldsymbol{v}_{\parallel s} + \boldsymbol{w}_{\perp s}) \cdot \boldsymbol{E} = \sum_s (\mathcal{Q}_s + \boldsymbol{V}_s \cdot \boldsymbol{\mathcal{R}}_s). \tag{6.21}$$

Recalling that

$$\boldsymbol{J}_\parallel + \bar{\boldsymbol{J}}_\perp = \sum_s q_s n_s (\boldsymbol{v}_{\parallel s} + \boldsymbol{w}_{\perp s}), \tag{6.22}$$

we find that the ohmic heating terms in (6.21) cancel, and the leading-order energy transport equation is given by

$$\frac{\partial \overline{\mathcal{H}}}{\partial t} + \nabla \cdot \left(\boldsymbol{S} + \sum_s [\overline{\mathcal{H}}_s \boldsymbol{V}_s + \boldsymbol{P}_s \cdot \boldsymbol{V}_s + \boldsymbol{q}_s] \right) = \sum_s (\mathcal{Q}_s + \boldsymbol{V}_s \cdot \boldsymbol{\mathcal{R}}_s). \tag{6.23}$$

Comparing this result with the time evolution of the total energy density $\mathcal{H}$, given by (2.13), we find that they have identical form. The model given by (5.1), (5.2) and (5.4), with $\boldsymbol{v}_{ps}$ given by (3.15), and coupled to the quasi-neutral Maxwell equations (2.5)–(2.8), therefore conserves the leading-order component $\overline{\mathcal{H}}$ of the total energy $\mathcal{H} = \overline{\mathcal{H}}[1 + O(\epsilon)]$ exactly, that is, to all orders in ϵ. This result can be extended to plasmas with finite Debye length, though for simplicity we have assumed $\partial_t/\omega_p = 0$.

https://doi.org/10.1017/S0022377825101104 Published online by Cambridge University Press

The leading-order component of the total fluid momentum density $\overline{\mathcal{M}} \equiv \sum_s m_s n_s \bar{\boldsymbol{v}}_s = \mathcal{M}_\parallel + \overline{\mathcal{M}}_\perp$ is also a conserved quantity. The equation for the parallel momentum of species s, $\mathcal{M}_{\parallel s} \equiv m_s n_s \boldsymbol{v}_{\parallel s}$, is obtained by multiplying (5.4) by $\boldsymbol{b}$:

$$\frac{\partial \mathcal{M}_{\parallel s}}{\partial t} + \boldsymbol{\nabla}\cdot(\boldsymbol{V}_s \mathcal{M}_{\parallel s}) - \mathcal{M}_{\parallel s}\frac{\mathrm{d}_s \boldsymbol{b}}{\mathrm{d}t} - \overline{\mathcal{M}}_{\perp s}\cdot\frac{\mathrm{d}_s \boldsymbol{b}}{\mathrm{d}t}\boldsymbol{b} + (\boldsymbol{\nabla}\cdot\boldsymbol{P}_s)_\parallel \boldsymbol{b} - q_s n_s \boldsymbol{E}_\parallel = \mathcal{R}_{\parallel s}, \tag{6.24}$$

where $\overline{\mathcal{M}}_{\perp s} \equiv m_s n_s \bar{\boldsymbol{v}}_{\perp s}$ is defined. The evolution of the perpendicular momentum $\overline{\mathcal{M}}_{\perp s}$ is obtained from the force balance equation:

$$m_s n_s \left.\frac{\mathrm{d}_s \bar{\boldsymbol{v}}_s}{\mathrm{d}t}\right|_\perp + r_s m_s n_s \bar{\boldsymbol{v}}_{\perp s} - q_s n_s \boldsymbol{B}\times\bar{\boldsymbol{v}}_{\perp s} = q_s n_s \boldsymbol{V}_s \times \boldsymbol{B}, \tag{6.25}$$

which is a rewriting of the polarisation velocity definition (3.15). Using the continuity equation (5.1), (6.25) can be recast as

$$\left.\frac{\mathrm{d}_s \overline{\mathcal{M}}_s}{\mathrm{d}t}\right|_\perp + \overline{\mathcal{M}}_{\perp s}\boldsymbol{\nabla}\cdot\boldsymbol{V}_s - q_s n_s \boldsymbol{B}\times\bar{\boldsymbol{v}}_{\perp s} = q_s n_s \boldsymbol{V}_s\times\boldsymbol{B}. \tag{6.26}$$

Using the fact that

$$\left.\frac{\mathrm{d}_s \overline{\mathcal{M}}_s}{\mathrm{d}t}\right|_\perp = \frac{\mathrm{d}_s \overline{\mathcal{M}}_{\perp s}}{\mathrm{d}t} + \overline{\mathcal{M}}_{\perp s}\cdot\frac{\mathrm{d}_s \boldsymbol{b}}{\mathrm{d}t}\boldsymbol{b} + \mathcal{M}_{\parallel s}\frac{\mathrm{d}_s \boldsymbol{b}}{\mathrm{d}t}, \tag{6.27}$$

we have, upon summing together (6.24) and (6.26), that

$$\frac{\partial \overline{\mathcal{M}}_s}{\partial t} + \boldsymbol{\nabla}\cdot(\boldsymbol{V}_s\overline{\mathcal{M}}_s) + (\boldsymbol{\nabla}\cdot\boldsymbol{P}_s)_\parallel\boldsymbol{b} - q_s n_s \boldsymbol{B}\times\bar{\boldsymbol{v}}_{\perp s} = \mathcal{R}_{\parallel s} + q_s n_s(\boldsymbol{E}_\parallel + \boldsymbol{V}_s\times\boldsymbol{B}). \tag{6.28}$$

Finally, given that $q_s n_s \boldsymbol{B}\times\bar{\boldsymbol{v}}_{\perp s} = q_s n_s \boldsymbol{E}_\perp - (\boldsymbol{\nabla}\cdot\boldsymbol{P}_s)_\perp + \mathcal{R}_{\perp s}$, we have

$$\frac{\partial \overline{\mathcal{M}}_s}{\partial t} + \boldsymbol{\nabla}\cdot(\boldsymbol{V}_s\overline{\mathcal{M}}_s + \boldsymbol{P}_s) = q_s n_s(\boldsymbol{E} + \boldsymbol{V}_s\times\boldsymbol{B}) + \mathcal{R}_s, \tag{6.29}$$

and, upon summing over all species s, we obtain the analogue of the non-drift-reduced result, (2.15), that is,

$$\frac{\partial \overline{\mathcal{M}}}{\partial t} + \boldsymbol{\nabla}\cdot\sum_s(\boldsymbol{V}_s\overline{\mathcal{M}}_s + \boldsymbol{P}_s) = \boldsymbol{J}\times\boldsymbol{B} + \sum_s \mathcal{R}_s, \tag{6.30}$$

showing that the leading-order contribution to the total fluid momentum is conserved.

7. Conclusion

The drift-reduced fluid model is a widely used tool for simulating plasma turbulence in the collisional regime. Here, we derive a conservative drift-reduced system that holds in arbitrary magnetic geometry without enforcing the electrostatic limit, a crucial property for interpreting long-time dynamics and for constructing robust

https://doi.org/10.1017/S0022377825101104 Published online by Cambridge University Press

numerical solvers. The central step is a non-perturbative, analytic inversion of the defining relation for the polarisation velocity as a function of $\partial_t \boldsymbol{E}$, which yields a consistent transport equation for the leading-order perpendicular momentum and a closed set of conservative fluid equations. The derivation presented here is independent of the closure and thus applies, for example, to Braginskii's two-fluid closure as well as to multispecies closures (Braginskii 1965; Zhdanov 2002; Poulsen *et al*. 2020; Raghunathan *et al*. 2022). Unlike in variational approaches based on expanding the guiding-centre Lagrangian (Brizard 2005; Jorge, Ricci & Loureiro 2017; Mencke & Ricci 2025), we performed the drift reduction directly at the fluid level, starting from the equations of motion and requiring that conservation laws should be satisfied exactly. The implications of exact conservation properties on turbulence in drift-reduced fluid plasmas will be assessed in future work.

Acknowledgements

The authors are grateful to D. Mancini, J. Mencke, L. Stenger and G. Van Parys for valuable discussions concerning drift-reduced models and guiding-centre Lagrangian approaches, and thank the anonymous referees for their help in improving the manuscript.

Editor Thierry Passot thanks the referees for their advice in evaluating this article.

Funding

This work has been carried out within the framework of the EUROfusion Consortium, via the Euratom Research and Training Programme (grant agreement no. 101052200 – EUROfusion) and funded by the Swiss State Secretariat for Education, Research and Innovation (SERI). Views and opinions expressed are, however, those of the authors only and do not necessarily reflect those of the European Union, the European Commission, or SERI. Neither the European Union nor the European Commission nor SERI can be held responsible for them. This work was supported in part by the Swiss National Science Foundation.

Declaration of interests

The authors report no conflict of interest.

Appendix A. Proof of $\boldsymbol{v}_{ps}$ exact inversion

We prove that the expression of $\boldsymbol{v}_{ps}$ in (4.30) solves (4.1). Since the computation is identical for all species, we omit the species index s to avoid clutter. Substituting (4.30) into (4.1), we have

$$(\Delta - 1)\boldsymbol{U} = \frac{1}{\Omega_c}(\boldsymbol{b} \times \nabla \bar{\boldsymbol{v}})\boldsymbol{U} - \frac{1}{\Omega_c}\boldsymbol{b} \times (\boldsymbol{U} \cdot \nabla)\bar{\boldsymbol{v}} - \frac{1}{\Omega_c^2}\boldsymbol{b} \times [((\boldsymbol{b} \times \nabla \bar{\boldsymbol{v}})\boldsymbol{U}) \cdot \nabla \bar{\boldsymbol{v}}]. \quad \text{(A1)}$$

Expressing the above in component form:

$$(\Delta - 1)U^k = \frac{1}{\Omega_c}\epsilon^{kmn} b_m \nabla_n v_s U^s - \frac{1}{\Omega_c}\epsilon^{kmn} b_m \nabla_s v_n U^s - \frac{1}{\Omega_c^2}\epsilon^{kmn} b_m \epsilon^{pqr} b_q \nabla_p v_n \nabla_r v_s U^s. \quad \text{(A2)}$$

https://doi.org/10.1017/S0022377825101104 Published online by Cambridge University Press

Progress can be made by factorising one of the Levi-Civita terms and introducing again the tensor $\epsilon_\perp^{ij}$. We have

$$(\Delta - 1)U^k = -\frac{1}{\Omega_c}\epsilon_\perp^{kn}\left[\nabla_n v_s - \nabla_s v_n + \frac{1}{\Omega_c}\epsilon_\perp^{pr}\nabla_p v_n \nabla_r v_s\right] U^s \equiv -\frac{1}{\Omega_c}\epsilon_\perp^{kn} L_{ns} U^s. \tag{A3}$$

We note that the tensor

$$L_{ns} = \nabla_n v_s - \nabla_s v_n + \frac{1}{\Omega_c}\epsilon_\perp^{pr}\nabla_p v_n \nabla_r v_s \tag{A4}$$

is antisymmetric, $L_{ns} = -L_{sn}$. Indeed,

$$L_{sn} = \nabla_n v_s - \nabla_s v_n + \frac{1}{\Omega_c}\epsilon_\perp^{pr}\nabla_p v_s \nabla_r v_n = \nabla_s v_n - \nabla_n v_s - \frac{1}{\Omega_c}\epsilon_\perp^{rp}\nabla_r v_n \nabla_p v_s = -L_{ns}. \tag{A5}$$

On $W_\perp$, any antisymmetric tensor takes the form $L_{ns} = \eta\epsilon_{\perp ns}$, for some scalar η. To compute η, we contract (A4) with $\epsilon_\perp^{ns}$. Recalling that $\epsilon_{\perp ns}\epsilon_\perp^{ns} = 2$, we obtain

$$\eta = \frac{1}{2}L_{ns}\epsilon_\perp^{ns} = \frac{1}{2}(\nabla_n v_s - \nabla_s v_n)\epsilon_\perp^{ns} + \frac{1}{\Omega_c}\epsilon_\perp^{pr}\epsilon_\perp^{ns}\nabla_p v_n \nabla_r v_s = \epsilon_\perp^{ns}\nabla_n v_s + \frac{1}{\Omega_c}\epsilon_\perp^{pr}\epsilon_\perp^{ns}\nabla_p v_n \nabla_r v_s. \tag{A6}$$

Expressing this result in terms of the trace of $\boldsymbol{b} \times \nabla\bar{\boldsymbol{v}}$ and the determinant of $(\nabla\bar{\boldsymbol{v}})_\perp$ as was done in (4.27) and (4.28), we have

$$\eta = (\boldsymbol{b} \times \nabla) \cdot \bar{\boldsymbol{v}} + \frac{1}{\Omega_c}\det(\nabla\bar{\boldsymbol{v}})_\perp = \Omega_c(\Delta - 1). \tag{A7}$$

Substituting the expression $L_{ns} = \eta\epsilon_{\perp ns}$, with $\eta = \Omega_c(\Delta - 1)$, into (A3), we find

$$U^k = -\epsilon_\perp^{kn}\epsilon_{\perp ns}U^s = (\Pi_\perp)^k{}_s U^s = U^k, \tag{A8}$$

completing the proof.

REFERENCES

BRAGINSKII, S.I. 1965 Transport processes in a plasma. *Rev. Plasma Phys.* **1**, 205.

BRIZARD, A.J. 2005 Energy-conserving finite-beta electromagnetic drift-fluid equations. *Phys. Plasmas* **12**, 092302.

BRIZARD, A.J. & HAHM, T.S. 2007 Foundations of nonlinear gyrokinetic theory. *Rev. Mod. Phys.* **79**, 421–468.

BUFFERAND, H. *et al.* 2024 Global 3d full-scale turbulence simulations of tcv-x21 experiments with soledge3x. *Nucl. Mater. Energy* **41**, 101824.

COELHO, A.J., LOIZU, J., RICCI, P. & TECCHIOLLI, Z. 2024 Global fluid simulation of plasma turbulence in stellarators with the gbs code. *Nucl. Fusion* **64**, 076057.

DRAKE, J.F. & ANTONSEN, T.M., Jr 1984 Nonlinear reduced fluid equations for toroidal plasmas. *Phys. Fluids* **27**, 898–908.

DUDSON, B.D. *et al.* 2015 Bout++: Recent and current developments. *J. Plasma Phys.* **81**, 365810104.

https://doi.org/10.1017/S0022377825101104 Published online by Cambridge University Press

DUDSON, B.D., UMANSKY, M.V., XU, X.Q., SNYDER, P.B. & WILSON, H.R. 2009 Bout++: a framework for parallel plasma fluid simulations. *Comput. Phys. Commun.* **180**, 1467–1480.

DÜLL, R., BUFFERAND, H., SERRE, E., CIRAOLO, G., QUADRI, V., RIVALS, N. & TAMAIN, P. 2024 Introducing electromagnetic effects in soledge3x. *Contrib. Plasm. Phys.* **64**, e202300147.

GATH, J. & WIESENBERGER, M. 2019 Consistency in drift-ordered fluid equations. *Phys. Plasmas* **26**, 032304.

GIACOMIN, M., PAU, A., RICCI, P., SAUTER, O., EICH, T., THE ASDEX UPGRADE TEAM, JET CONTRIBUTORS, THE TCV TEAM 2022*a* First-principles density limit scaling in tokamaks based on edge turbulent transport and implications for iter. *Phys. Rev. Lett.* **128**, 185003.

GIACOMIN, M., RICCI, P., COROADO, A., FOURESTEY, G., GALASSI, D., LANTI, E., MANCINI, D., RICHART, N., STENGER, L.N. & VARINI, N. 2022*b* The g*b*s code for the self-consistent simulation of plasma turbulence and kinetic neutral dynamics in the tokamak boundary. *J. Comput. Phys.* **463**, 111294.

HALPERN, F.D., WALTZ, R.E. & BERNARD, T.N. 2023 Drift-ordered fluid vorticity equation with energy consistency. *Phys. Plasmas* **30**, 032302.

JORGE, R., RICCI, P. & LOUREIRO, N.F. 2017 A drift-kinetic analytical model for scrape-off layer plasma dynamics at arbitrary collisionality. *J. Plasma Phys.* **83**, 905830606.

LEVEQUE, R.J. 1992 *Numerical Methods for Conservation Laws*. Lectures in Mathematics, 2nd edn. ETH Zürich.

MENCKE, J.E. & RICCI, P. 2025 Extended drift-kinetic full-f turbulent simulation of a linear plasma device using the gyro-moment approach. *Plasma Phys. Control. Fusion* **67**, 035029.

OLIVEIRA, D.S. *et al.* 2022 Validation of edge turbulence codes against the tcv-x21 diverted l-mode reference case. *Nucl. Fusion* **62**, 096001.

POULSEN, A., RASMUSSEN, J.J., WIESENBERGER, M. & NAULIN, V. 2020 Collisional multispecies drift fluid model. *Phys. Plasmas* **27**, 032305.

RAGHUNATHAN, M., MARANDET, Y., BUFFERAND, H., CIRAOLO, G., GHENDRIH, P., TAMAIN, P. & SERRE, E. 2022 Multi-temperature generalized zhdanov closure for scrape-off layer/edge applications. *Plasma Phys. Control. Fusion* **64**, 045005.

REISER, D. 2012 Revisited global drift fluid model for linear devices. *Phys. Plasmas* **19**, 072317.

RIVA, F. *et al.* 2016 Blob dynamics in the torpex experiment: a multi-code validation. *Plasma Phys. Control. Fusion* **58**, 044005.

SCOTT, B. 2003 The character of transport caused by E × B drift turbulence. *Phys. Plasmas* **10**, 963–976.

SIMAKOV, A.N. & CATTO, P.J. 2003 Drift-ordered fluid equations for field-aligned modes in low-beta collisional plasma with equilibrium pressure pedestals. *Phys. Plasmas* **10**, 4744–4757.

STEGMEIR, A. *et al.* 2019 Global turbulence simulations of the tokamak edge region with grillix. *Phys. Plasmas* **26**, 052517.

ZEILER, A., DRAKE, J.F. & ROGERS, B. 1997 Nonlinear reduced braginskii equations with ion thermal dynamics in toroidal plasma. *Phys. Plasmas* **4**, 2134–2138.

ZHDANOV, V.M. 2002 Transport processes in multicomponent plasma. *Plasma Phys. Control. Fusion* **44**, 2283.

ZHOLOBENKO, W. *et al.* 2024 Tokamak edge-sol turbulence in h-mode conditions simulated with a global, electromagnetic, transcollisional drift-fluid model. *Nucl. Fusion* **64**, 106066.

ZHU, B., FRANCISQUEZ, M., ROGERS, B.N. 2018 Gdb: a global 3d two-fluid model of plasma turbulence and transport in the tokamak edge. *Comput. Phys. Commun.* **232**, 46–58.

https://doi.org/10.1017/S0022377825101104 Published online by Cambridge University Press