# NOTE ON THE ELECTRIFIED SPHERICAL CAP

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Copson (1) has shown that the problem of determining the surface density of electric charge induced on a thin circular disk kept at a constant potential in an external electrostatic field can be solved by two applications of the known solution of Abel's integral equation. This note shows that a similar result holds for a thin uniform spherical cap, and is a generalization of results obtained by the author using an alternative method (2).

We use spherical polar coordinates $(r, \theta, \phi)$ referred to the centre of the sphere $r = a$ as origin and the axis of the cap as polar axis, the cap being given by $r = a, 0 \leqslant \theta \leqslant \alpha$. If $V_1$ is the potential due to an external electrostatic field and $V$ the perturbation potential due to the presence of the cap in the field, we suppose that the total potential $V_1 + V$ has the constant value $V_0$ on the surface of the cap. The perturbation potential $V$ is $O(r^{-1})$ for large $r$ and has the value $V_0 - V_1$ on the surface of the cap. If it is assumed to be a single-valued continuous function of position on the cap, it can be expanded as a sum of terms of the type $f(\theta)\cos n(\phi+\beta)$, where $n$ is a positive integer or zero, $\beta$ a constant and $f(\theta)$ has the form $\sin^n\theta F(\cos\theta)$. We suppose that $V_0 - V_1$ reduces to a single term of this type, the general result being obtained by superposition.

We show that, if the potential at the point $(a, \theta, \phi)$ on the spherical cap $r = a$, $0 \leqslant \theta \leqslant \alpha$, is $f(\theta)\cos n(\phi+\beta)$, where $f(\theta)$ is continuously differentiable for $0 \leqslant \theta \leqslant \alpha$, the total surface-charge density $\sigma_1(\theta, \phi)$ on both faces of the cap at the point $(a, \theta, \phi)$ is $\sigma(\theta)\cos n(\phi+\beta)$, where

$$\sigma(\theta) = -\frac{1}{\pi}\tan^{n-1}\tfrac{1}{2}\theta\sec^2\tfrac{1}{2}\theta\frac{d}{d\theta}\int_\theta^\alpha \frac{\sin\psi\, S(\psi)\, d\psi}{(\cos\theta - \cos\psi)^{\frac{1}{2}}}, \tag{1}$$

$S(\psi)$ being given by

$$S(\psi) = \frac{1}{4\pi a}\cot^{2n}\tfrac{1}{2}\psi\frac{d}{d\psi}\int_0^\psi \frac{\sin\theta'\tan^n\tfrac{1}{2}\theta' f(\theta')\, d\theta'}{(\cos\theta' - \cos\psi)^{\frac{1}{2}}}. \tag{2}$$

The potential $V(r, \theta, \phi)$ at any point $(r, \theta, \phi)$ due to a distribution of charge of surface density $\sigma_1(\theta, \phi)$ on the cap is

$$V(r, \theta, \phi) = a^2\int_0^\alpha\int_0^{2\pi} \frac{\sigma_1(\theta', \phi')\sin\theta'\, d\phi'\, d\theta'}{(r^2 + a^2 - 2ar\cos\omega)^{\frac{1}{2}}},$$

where $\cos\omega = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$. We require $\sigma_1$ to be such that $V = f(\theta)\cos n(\phi+\beta)$ when $r = a$, and therefore choose

$$\sigma_1(\theta', \phi') = \sigma(\theta')\cos n(\phi' + \beta),$$

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where $\sigma(\theta')$ satisfies

$$a\int_0^{\alpha}\int_0^{2\pi}\frac{\sigma(\theta')\sin\theta'\cos n(\phi'+\beta)\,d\phi'\,d\theta'}{(2-2\cos\omega)^{\frac{1}{2}}}=f(\theta)\cos n(\phi+\beta) \tag{3}$$

for $0<\theta<\alpha$.

Using the first of Copson's lemmas (1) and the identity

$$\begin{aligned}2-2\cos\omega &= 4(\sin^2\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\theta'+\sin^2\tfrac{1}{2}\theta'\cos^2\tfrac{1}{2}\theta)-8\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta'\sin\tfrac{1}{2}\theta'\cos\tfrac{1}{2}\theta\cos(\phi-\phi')\\ &= c^2+d^2-2cd\cos(\phi-\phi'),\end{aligned}$$

where $c=2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta'$, $d=2\sin\frac{1}{2}\theta'\cos\frac{1}{2}\theta$ and both $c$ and $d$ are positive for all $\theta$ and $\theta'$ since $0\leqslant\theta,\theta'\leqslant\alpha$, we find that

$$\begin{aligned}\int_0^{2\pi}\frac{\cos n(\phi'+\beta)\,d\phi'}{(2-2\cos\omega)^{\frac{1}{2}}} &= \int_0^{2\pi}\frac{\cos n(\phi'+\beta)\,d\phi'}{(c^2+d^2-2cd\cos(\phi-\phi'))^{\frac{1}{2}}}\\ &= \frac{4\cos n(\phi+\beta)}{(cd)^n}\int_0^{\mathrm{Min}(c,d)}\frac{s^{2n}\,ds}{[(c^2-s^2)(d^2-s^2)]^{\frac{1}{2}}}.\end{aligned}$$

Since $c<d$ when $\theta<\theta'$ and $c>d$ when $\theta>\theta'$, equation (3) can be written

$$\begin{aligned}\frac{4a}{\sin^n\theta}\Bigg\{&\int_0^{\theta}\sigma(\theta')\sin^{1-n}\theta'\int_0^{d}\frac{s^{2n}\,ds}{[(c^2-s^2)(d^2-s^2)]^{\frac{1}{2}}}\,d\theta'\\ &+\int_{\theta}^{\alpha}\sigma(\theta')\sin^{1-n}\theta'\int_0^{c}\frac{s^{2n}\,ds}{[(c^2-s^2)(d^2-s^2)]^{\frac{1}{2}}}\,d\theta'\Bigg\}=f(\theta).\end{aligned}$$

On making the change of variable

$$s=2\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta'\tan\tfrac{1}{2}\psi,$$

we obtain

$$\begin{aligned}\frac{4a}{\tan^n\frac{1}{2}\theta}\Bigg\{&\int_0^{\theta}\sigma(\theta')\cot^{n-1}\tfrac{1}{2}\theta'\cos^2\tfrac{1}{2}\theta'\int_0^{\theta'}\frac{\tan^{2n}\frac{1}{2}\psi\,d\psi}{[(\cos\psi-\cos\theta)(\cos\psi-\cos\theta')]^{\frac{1}{2}}}\,d\theta'\\ &+\int_{\theta}^{\alpha}\sigma(\theta')\cot^{n-1}\tfrac{1}{2}\theta'\cos^2\tfrac{1}{2}\theta'\int_0^{\theta}\frac{\tan^{2n}\frac{1}{2}\psi\,d\psi}{[(\cos\psi-\cos\theta)(\cos\psi-\cos\theta')]^{\frac{1}{2}}}\,d\theta'\Bigg\}=f(\theta).\end{aligned}$$

Inverting the order of integration, we have

$$\int_0^{\theta}\frac{\tan^{2n}\frac{1}{2}\psi}{(\cos\psi-\cos\theta)^{\frac{1}{2}}}\int_{\psi}^{\alpha}\frac{\sigma(\theta')\cot^{n-1}\frac{1}{2}\theta'\cos^2\frac{1}{2}\theta'\,d\theta'\,d\psi}{(\cos\psi-\cos\theta')^{\frac{1}{2}}}=\frac{1}{4a}\tan^n\tfrac{1}{2}\theta f(\theta)\quad(0<\theta<\alpha).$$

To solve this repeated integral equation we write

$$S(\psi)=\int_{\psi}^{\alpha}\frac{\sigma(\theta')\cot^{n-1}\frac{1}{2}\theta'\cos^2\frac{1}{2}\theta'\,d\theta'}{(\cos\psi-\cos\theta')^{\frac{1}{2}}}\quad(0<\psi<\alpha), \tag{4}$$

and obtain

$$\int_0^{\theta}\frac{\tan^{2n}\frac{1}{2}\psi\,S(\psi)\,d\psi}{(\cos\psi-\cos\theta)^{\frac{1}{2}}}=\frac{1}{4a}\tan^n\tfrac{1}{2}\theta f(\theta)\quad(0<\theta<\alpha). \tag{5}$$

Equations (4) and (5) are similar to Abel's equation. Provided $f(\theta)$ is continuously differentiable for $0\leqslant\theta\leqslant\alpha$, the solution of (5) can be shown to be

$$S(\psi)=\frac{\cot^{2n}\frac{1}{2}\psi}{4\pi a}\frac{d}{d\psi}\int_0^{\psi}\frac{\sin\theta\tan^n\frac{1}{2}\theta f(\theta)\,d\theta}{(\cos\theta-\cos\psi)^{\frac{1}{2}}}\quad(0<\psi<\alpha),$$

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whilst that of (4) is

$$\sigma(\theta') = -\frac{1}{\pi}\tan^{n-1}\tfrac{1}{2}\theta'\sec^2\tfrac{1}{2}\theta'\frac{d}{d\theta'}\int_{\theta'}^{\alpha}\frac{\sin\psi\, S(\psi)\,d\psi}{(\cos\theta'-\cos\psi)^{\frac{1}{2}}} \quad (0<\theta'<\alpha).$$

Further, the value of $\sigma(\theta')$ at $\theta' = 0$ can also be shown to be given by this expression. This proves the results (1) and (2).

As an example we suppose the cap is raised to unit potential with no external field; then $n = 0$ and $f(\theta) = 1$. From (1) and (2) we find

$$S(\psi) = \frac{1}{2\pi a\sqrt{2}}\cos\tfrac{1}{2}\psi$$

and

$$4\pi a\sigma(\theta) = 1+\frac{2}{\pi}\left[\left(\frac{1+\cos\alpha}{\cos\theta-\cos\alpha}\right)^{\frac{1}{2}}-\tan^{-1}\left(\frac{1+\cos\alpha}{\cos\theta-\cos\alpha}\right)^{\frac{1}{2}}\right]$$

in agreement with Ferrers' result (3).

## REFERENCES

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