109.21 e = 2.71828 (to 6 significant figures): An elementary demonstration

The Taylor series expansion of the exponential function is given by

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots . \tag{1}$$

This holds for all the real values of x.

In particular $x = 1$ yields

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \ldots$$

$$> 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40320} + \frac{1}{362880}$$

$$\Rightarrow e - 2.5 > \frac{60480 + 15120 + 3024 + 504 + 72 + 9 + 1}{362880}$$

$$= \frac{79210}{362880} = \frac{7921}{36288} > 0.218281$$

$$\Rightarrow e > 2.718281.$$

Taking $x = -1$, in (1), we get

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots + \frac{(-1)^n}{n!} + \ldots .$$

As every term is numerically smaller than the preceding term, the right-hand side can be written as a series with only positive terms in the following manner.

$$e^{-1} = \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{4!} - \frac{1}{5!}\right) + \ldots + \left(\frac{1}{(2n)!} - \frac{1}{(2n+1)!}\right) + \ldots$$

$$> \left(\frac{1}{2} - \frac{1}{6}\right) + \left(\frac{1}{24} - \frac{1}{120}\right) + \left(\frac{1}{720} - \frac{1}{5040}\right) + \left(\frac{1}{40320} - \frac{1}{362880}\right)$$

$$= \frac{181440 - 60480 + 15120 - 3024 + 504 - 72 + 9 - 1}{362880}$$

$$= \frac{133496}{362880} = \frac{16687}{45360}$$

$$\Rightarrow e < \frac{45360}{16687} < 2.718284.\infty$$

Thus, we have shown that $2.718281 < e < 2.718284$

$$\Rightarrow e = 2.71828 \text{ (to 6 significant figures)}.$$

Note: All the calculations are purposely done manually to avoid calculators completely. The reason being – the use of calculators would make, all our

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efforts done above, ridiculous owing to the fact that simply pressing the 'exp' key and then 1, one can get value of e to larger number of significant figures.

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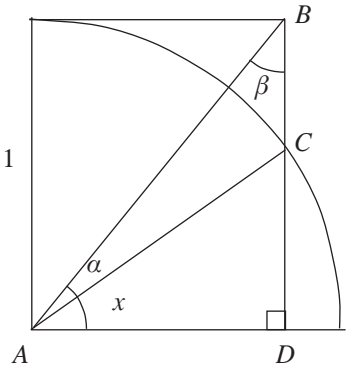
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109.22 A constant function

This Note was inspired by my attempt in a Calculus I course at the University of Zanjan to find a "proof without words simplification" of the composed quantity $\tan^{-1}(\cos x)$. After some effort with students, we generated the following diagram, including the quantity under question as the angle β.



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