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The defect of factor maps

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Abstract. The defect of a factor map out of a totally disconnected dynamical system is a number in $[0, \infty]$ which gives a quantitative measure of how well the dynamical behaviour of the factor agrees with that of the extension. The larger the defect, the more they differ. In this paper we obtain the basic general properties of the defect and calculate it in a series of specific cases.

0. *Introduction*

In an earlier work [**Th**] the author pursued the connection between the topological entropy of a dynamical system and the covering dimension of compact spaces which reveals itself when the definition of topological entropy is phrased in terms of the C^*-algebra of continuous functions. The result was invariants for dynamical systems which comprise of both the topological entropy of the given map and the dimension of the space on which it acts, and it seemed to deviate from the topological entropy only when the set of periodic points formed subsets of positive covering dimension. These invariants are typically somewhat more difficult to calculate than the topological entropy itself. However, at the end of that study, a numerical invariant for factor maps out of a dynamical system acting on a totally disconnected space was introduced. This invariant was called the *defect* of the factor map because it gives a measure of how well the dynamical behaviour of the totally disconnected dynamical system agrees with that of the factor, in particular as far as the periodic points are concerned. The intention of the present paper is to advertise this invariant for factor maps a little further. The hope is to convince the reader that the defect of a factor map is not terribly difficult to calculate and that it does give an interesting numerical indication of how well the dynamical systems are related.

The paper is organized as follows. In §1 we give the definition of the defect and derive a series of its general properties. Most of these properties are analogous to corresponding properties for the topological entropy. In §2 we calculate the defect of factor maps out of certain one-sided finite type subshifts, called ‘full’ and ‘left-resolving’, respectively. In both cases the defect is determined by the ambiguity of the factor map over the periodic points, but with the difference that the exponential of the defect is always a natural number in the left-resolving case, while in the full case it can be a quite general rational

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power of a natural number. By exploiting this we construct factor maps between totally disconnected spaces whose defect is an arbitrary number in $[0, \infty]$. Compared to the one-sided case, our calculations are more complete in the case of two-sided subshifts which we consider in §3. For a factor map mapping a two-sided irreducible finite type subshift onto a sofic subshift, the defect is ∞ when the map is not finite-to-one, and equals the logarithm of the maximal number of pre-images of a periodic point if it is finite-to-one. In §4 we calculate the defect of the factor maps coming from Markov partitions of expansive endomorphisms and homeomorphisms. The result is again that the defect equals the logarithm of the maximal number of pre-images of a periodic point. Thus, for example, the Markov partitions for a hyperbolic automorphism of $\mathbb{T}^2$ constructed by Adler and Weiss [**AW**] gives a factor map with defect $\log 3$, while the Markov partitions constructed by Bowen [**Bo1**] give factor maps with defect $2 \log 2$. Finally, in §5 we calculate the defect of the factor map $\pi : (Y, \psi) \to (X, \varphi)$ corresponding to a metrizable totally disconnected extension of a uniquely ergodic map φ. In this case the defect equals

$$\int \log \#\pi^{-1}(x)\, d\mu(x),$$

where μ is the unique φ-invariant Borel probability measure.

1. *Definitions and general results*

Let X be a compact Hausdorff space and $\varphi : X \to X$ a continuous map. Such a pair (X, φ) will be referred to as a *dynamical system*. When $\mathcal{F} = \{F_i : i \in I\}$ is a finite collection of subsets of X we set

$$q_k(x, \mathcal{F}) = \#\left\{(i_1, i_2, \ldots, i_k) \in I^k : x \in \bigcap_{j=1}^{k} \varphi^{-j+1}(F_{i_j})\right\}$$

for all $x \in X, k \in \mathbb{N}$, and

$$q_k(\varphi, \mathcal{F}) = \max_{x \in X} q_k(x, \mathcal{F})$$

for all $k \in \mathbb{N}$. Sometimes it is more useful to consider $q_k(\varphi, \mathcal{F})$ as the maximal number of elements in a subset $J \subset I^k$ such that

$$\bigcap_{(i_1, i_2, \ldots, i_k) \in J} F_{i_1} \cap \varphi^{-1}(F_{i_2}) \cap \cdots \cap \varphi^{-k+1}(F_{i_k})$$

is not empty. In any case, it is clear that $q_{k+n}(\varphi, \mathcal{F}) \leq q_k(\varphi, \mathcal{F}) q_n(\varphi, \mathcal{F})$ for all $k, n \in \mathbb{N}$. If $\bigcup_{i \in I} F_i$ contains a non-empty φ-invariant set (e.g. if $\mathcal{F}$ is a cover of X), we have that $q_k(\varphi, \mathcal{F}) \geq 1$ for all k and hence we can define $Q(\varphi, \mathcal{F})$ by

$$Q(\varphi, \mathcal{F}) = \lim_{n \to \infty} \frac{1}{n} \log\ q_n(\varphi, \mathcal{F}).$$

It is a useful fact that this limit equals $\inf_{n \in \mathbb{N}} (1/n) \log q_n(\varphi, \mathcal{F})$.

Consider now a compact totally disconnected space Y and a continuous map $\psi : Y \to Y$ such that (X, φ) is a factor of (Y, ψ), i.e. there is a continuous surjection $\pi : Y \to X$ such that $\varphi \circ \pi = \pi \circ \psi$. A partition of Y is a finite cover of Y consisting of mutually disjoint open subsets in Y. We define the *defect* of the factor map by

$$D(\pi) = \sup_{\mathcal{P}} Q(\varphi, \pi(\mathcal{P})),$$

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where we take the supremum over all partitions $\mathcal{P}$ of Y. It is an important observation that $Q(\varphi, \pi(\mathcal{P}_1)) \leq Q(\varphi, \pi(\mathcal{P}_2))$ when $\mathcal{P}_2$ is a refinement of $\mathcal{P}_1$, see [**Th**, Lemma 6.6].

LEMMA 1.1. *Let (X_1, φ_1), (X_2, φ_2), (X_3, φ_3) and (X_4, φ_4) be dynamical systems with X_1 and X_2 totally disconnected. Let $\pi_1 : (X_1, \varphi_1) \to (X_2, \varphi_2), \pi_2 : (X_2, \varphi_2) \to (X_3, \varphi_3), \pi_3 : (X_3, \varphi_3) \to (X_4, \varphi_4)$ be factor maps. Then*

$$D(\pi_3 \circ \pi_2 \circ \pi_1) \geq D(\pi_2)$$

and

$$D(\pi_3 \circ \pi_2 \circ \pi_1) = D(\pi_2)$$

when π_1 and π_3 are injective.

Proof. Let $\mathcal{P}$ be a partition of X_2 and consider $\mathcal{P}' = {\pi_1}^{-1}(\mathcal{P})$. For every $x \in X_3$ we have that

$$q_k(x, \pi_2(\mathcal{P})) = q_k(x, \pi_2 \circ \pi_1(\mathcal{P}')) \leq q_k(\pi_3(x), \pi_3 \circ \pi_2 \circ \pi_1(\mathcal{P}')) \leq q_k(\varphi_4, \pi_3 \circ \pi_2 \circ \pi_1(\mathcal{P}'))$$

for all $k \in \mathbb{N}$, showing that

$$Q(\varphi_2, \pi_2(\mathcal{P})) \leq Q(\varphi_4, \pi_3 \circ \pi_2 \circ \pi_1(\mathcal{P}')) \leq D(\pi_3 \circ \pi_2 \circ \pi_1).$$

It follows that $D(\pi_2) \leq D(\pi_3 \circ \pi_2 \circ \pi_1)$. The equality in the case when π_1 and π_3 are conjugacies is trivial. □

LEMMA 1.2. *Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map between dynamical systems with Y totally disconnected. If $\#\pi^{-1}(x) \leq d$ for all $x \in X$ it follows that*

$$D(\pi) \leq \log d.$$

Proof. For any partition $\mathcal{P} = \{P_1, P_2, \ldots, P_N\}$ of Y we have that $q_k(x, \pi(\mathcal{P})) \leq d^k$ for all $k \in \mathbb{N}$ and $x \in X$. □

PROPOSITION 1.3. *Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map between dynamical systems with Y totally disconnected. Let L_1 and L_2 be closed φ-invariant subsets such that $X = L_1 \cup L_2$. Then $\pi|_{\pi^{-1}(L_i)} : (\pi^{-1}(L_i), \psi) \to (L_i, \varphi)$ are factor maps and*

$$\max\{D(\pi|_{\pi^{-1}(L_1)}), D(\pi|_{\pi^{-1}(L_2)})\} = D(\pi).$$

Proof. Let $\mathcal{P} = \{P_i : i = 1, 2, \ldots, N\}$ be a partition of $\pi^{-1}(L_j)$. Since

$$\{Q \cap \pi^{-1}(L_j) : Q \text{ closed and open in } Y\}$$

is a base for the topology of $\pi^{-1}(L_j)$, we see that there is a partition $\mathcal{R} = \{R_i : i = 1, 2, \ldots, N\}$ in Y such that $P_i = R_i \cap \pi^{-1}(L_j)$ for all i. Since

$$Q(\varphi, \pi(\mathcal{P})) \leq Q(\varphi, \pi(\mathcal{R})) \leq D(\pi),$$

we conclude that $D(\pi|_{\pi^{-1}(L_j)}) \leq D(\pi)$, $j = 1, 2$. Conversely, if $\mathcal{R}$ is a partition of Y, it is clear that

$$q_n(\varphi, \mathcal{R}) \leq \max\{q_n(\varphi, \mathcal{R} \cap \pi^{-1}(L_1)), q_n(\varphi, \mathcal{R} \cap \pi^{-1}(L_2))\}$$

for all n, and the desired identity follows. □

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THEOREM 1.4. *Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map between dynamical systems with Y totally disconnected. Let Ω denote the set of non-wandering points for φ. Then*

$$D(\pi|_{\pi^{-1}(\Omega)}) = D(\pi).$$

Proof. By Proposition 1.3 it suffices to prove that $D(\pi|_{\pi^{-1}(\Omega)}) \geq D(\pi)$. Let $\mathcal{P} = \{P_i : i = 1, 2, \ldots, N\}$ be a partition of Y and let $\epsilon > 0$. We seek a partition $\mathcal{R}$ of Y which refines $\mathcal{P}$ and satisfies

$$Q(\varphi, \pi(\mathcal{R})) \leq Q(\varphi|_\Omega, \pi(\mathcal{P} \cap \pi^{-1}(\Omega))) + \epsilon.$$

Since $Q(\varphi, \pi(\mathcal{P})) \leq Q(\varphi, \pi(\mathcal{R}))$ because $\mathcal{R}$ refines $\mathcal{P}$, and since $\mathcal{P}$ and $\epsilon > 0$ are arbitrary, this will yield the desired conclusion. So it suffices to construct $\mathcal{R}$.

Let $\{U_\alpha\}$ be a net of open subsets in X such that $\alpha < \beta \Rightarrow \overline{U_\beta} \subset U_\alpha$ and $\Omega = \bigcap_\alpha \overline{U_\alpha}$. Choose $n \in \mathbb{N}$ so large that $(1/n) \log q_n(\varphi|_\Omega, \pi(\mathcal{P}) \cap \Omega) \leq Q(\varphi|_\Omega, \pi(\mathcal{P}) \cap \Omega) + \epsilon$. For each $\iota = (i_1, i_2, \ldots, i_n) \in \{1, 2, \ldots, N\}^n$ set

$$W_\iota = (\pi(P_{i_1}) \cap \Omega) \cap \varphi^{-1}(\pi(P_{i_2}) \cap \Omega) \cap \cdots \cap \varphi^{-n+1}(\pi(P_{i_n}) \cap \Omega)$$

and

$$W_\iota^\alpha = (\pi(P_{i_1}) \cap \overline{U_\alpha}) \cap \varphi^{-1}(\pi(P_{i_2}) \cap \overline{U_\alpha}) \cap \cdots \cap \varphi^{-n+1}(\pi(P_{i_n}) \cap \overline{U_\alpha}).$$

Then

$$\bigcap_{\iota \in F} W_\iota = \bigcap_\alpha \bigcap_{\iota \in F} W_\iota^\alpha$$

for any finite subset $F \subset \{1, 2, \ldots, N\}^n$. By compactness there is, therefore, an α_0 such that

$$\bigcap_{\iota \in F} W_\iota = \emptyset \Rightarrow \bigcap_{\iota \in F} W_\iota^\alpha = \emptyset$$

for all finite subsets $F \subset \{1, 2, \ldots, N\}^n$ and all $\alpha > \alpha_0$. It follows that

$$\begin{aligned} Q(\varphi, \pi(\mathcal{P}) \cap U_\alpha) &\leq \frac{1}{n} \log q_n(\varphi, \pi(\mathcal{P}) \cap U_\alpha) = \frac{1}{n} \log q_n(\varphi, \pi(\mathcal{P}) \cap \Omega) \\ &\leq Q(\varphi|_\Omega, \pi(\mathcal{P}) \cap \Omega) + \epsilon, \end{aligned}$$

for all $\alpha > \alpha_0$. Let $\beta > \alpha > \alpha_0$. For each $i \in \{1, 2, \ldots, N\}$ we can find a closed and open subset R_i of Y such that

$$\pi^{-1}(\overline{U_\beta}) \cap P_i \subset R_i \subset \pi^{-1}(U_\alpha) \cap P_i. \tag{1.1}$$

Then $\pi(y) \notin \overline{U_\beta}$ for all $y \in Y \setminus \bigcup_{i=1}^N R_i$, so in particular $\pi(y)$ is wandering for φ for such y. By using the compactness of $Y \setminus \bigcup_{i=1}^N R_i$ we find a finite cover $\{V_j : j \in J\}$ of $Y \setminus \bigcup_{i=1}^N R_i$ such that each V_j is a closed and open subset of P_i for some i and such that $\varphi^k(\pi(V_j)) \cap \pi(V_j) = \emptyset$ for all $k \in \mathbb{N}$. By passing to a refinement if necessary, we can also assume that the V_j's are mutually disjoint. Then $\mathcal{R} = \{R_i : i = 1, 2, \ldots, N\} \cup \{V_j : j \in J\}$ is a partition of Y which refines $\mathcal{P}$. Set $\mathcal{R}' = \{R_i : i = 1, 2, \ldots, N\}$ and $M = \#J$. Because $\varphi^k(\pi(V_j)) \cap \pi(V_j) = \emptyset$ for all $k \in \mathbb{N}$ and all $j \in J$, we find from an elementary combinatorial consideration that

$$q_m(\varphi, \pi(\mathcal{R})) \leq \max\{X_j(m) : j = 0, 1, 2, \ldots, M\},$$

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where $X_0(m) = q_m(\varphi, \pi(\mathcal{R}'))$ and

$$X_j(m) = \max\left\{\prod_{i=1}^{j} q_{n_i}(\varphi, \pi(\mathcal{R}')) : n_1, n_2, \ldots, n_j \in \mathbb{N}, \sum_{i=1}^{j} n_i = m - j\right\},$$

$j = 1, 2, \ldots, M$. Let $\delta > 0$ be arbitrary and set $\rho = Q(\varphi, \pi(\mathcal{R}'))$. Choose $K \in \mathbb{N}$ so large that $(1/r)\log q_r(\varphi, \pi(\mathcal{R}')) \leq \rho + \delta,\ r \geq K$. When $n_1, n_2, \ldots, n_j$ are elements in $\mathbb{N}$ that sum to $m - j$ we have that

$$\prod_{i=1}^{j} q_{n_i}(\varphi, \pi(\mathcal{R}')) \leq \mathcal{K}^d e^{(\rho+\delta)(m-j)},$$

where d is the numbers of i's for which $n_i \leq K$ and $\mathcal{K} = \max\{q_l(\varphi, \pi(\mathcal{R}')) : l = 1, 2, \ldots, K\}$. Thus

$$q_m(\varphi, \pi(\mathcal{R})) \leq \mathcal{K}^M e^{(\rho+\delta)m}$$

for all sufficiently large m and hence $Q(\varphi, \pi(\mathcal{R})) \leq \rho + \delta$. Since $\delta > 0$ was arbitrary, we conclude that

$$Q(\varphi, \pi(\mathcal{R})) \leq Q(\varphi, \pi(\mathcal{R}')).$$

It follows from (1.1) that $Q(\varphi, \pi(\mathcal{R}')) \leq Q(\varphi, \pi(\mathcal{P}) \cap U_\alpha)$ and hence that $Q(\varphi, \pi(\mathcal{R})) \leq Q(\varphi|_\Omega, \pi(\mathcal{P}) \cap \Omega) + \epsilon$ since $\alpha > \alpha_0$. □

THEOREM 1.5. *Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map between dynamical systems with Y totally disconnected. Set $X_\infty = \bigcap_{k=1}^{\infty} \varphi^k(X)$. Then*

$$D(\pi|_{\pi^{-1}(X_\infty)}) = D(\pi).$$

Proof. Proposition 1.3 gives the inequality $D(\pi|_{\pi^{-1}(X_\infty)}) \leq D(\pi)$, so we concentrate on the reverse. Let $\mathcal{P} = \{P_i : i = 1, 2, \ldots, N\}$ be a partition of Y. Then $\mathcal{P}' = \mathcal{P} \cap \pi^{-1}(X_\infty)$ is a partition of $\pi^{-1}(X_\infty)$ and $Q(\varphi|_{X_\infty}, \pi(\mathcal{P}')) = Q(\varphi, \pi(\mathcal{P}) \cap X_\infty)$. Let $\epsilon > 0$ and choose $n \in \mathbb{N}$ such that

$$\frac{1}{n}\log q_n(\varphi, \pi(\mathcal{P}) \cap X_\infty) \leq Q(\varphi, \pi(\mathcal{P}) \cap X_\infty) + \epsilon.$$

Since $X_\infty = \bigcap_{k=1}^{\infty} \varphi^k(X)$, a compactness argument as used in the proof of Theorem 1.4 gives a $K \in \mathbb{N}$ such that

$$q_n(\varphi, \pi(\mathcal{P}) \cap X_\infty) = q_n(\varphi, \pi(\mathcal{P}) \cap \varphi^K(X)).$$

Thus $Q(\varphi, \pi(\mathcal{P}) \cap \varphi^K(X)) \leq Q(\varphi, \pi(\mathcal{P}) \cap X_\infty) + \epsilon$. Since

$$\begin{aligned} q_m(x, \pi(\mathcal{P})) &\leq q_K(\varphi, \pi(\mathcal{P}))q_{m-K}(\varphi^K(x), \pi(\mathcal{P})) \\ &\leq q_K(\varphi, \pi(\mathcal{P}))q_{m-K}(\varphi, \pi(\mathcal{P}) \cap \varphi^K(X)) \end{aligned}$$

for all $x \in X$ and $m > K$, we see that $Q(\varphi, \pi(\mathcal{P})) \leq Q(\varphi, \pi(\mathcal{P}) \cap \varphi^K(X))$. Since $\epsilon > 0$ was arbitrary, we find that

$$Q(\varphi, \pi(\mathcal{P})) \leq Q(\varphi, \pi(\mathcal{P}) \cap X_\infty) = Q(\varphi|_{X_\infty}, \pi(\mathcal{P}')) \leq D(\pi|_{\pi^{-1}(X_\infty)}).$$

The conclusion follows because $\mathcal{P}$ was an arbitrary partition of Y. □

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Remark 1.6. For each $n \in \mathbb{N}$ set $d_n(\varphi) = \sup_{x\in X} \prod_{j=1}^{n} \#\pi^{-1}(\varphi^j(x))$ and note that $D_\infty(\pi) = \lim_{n\to\infty}(1/n)\log d_n(\varphi)$ exists. This quantity is also a measure of π's deviation from being injective and it is easily seen that $D(\pi) \leq D_\infty(\pi)$ and that $D_\infty(\pi)$ is exactly the quantity we get if, in the definition of $D(\pi)$, we take the supremum over all partitions *before* taking the limit over n. In many cases $D(\pi) = D_\infty(\pi)$, but not in general, and the conclusions of Theorem 1.4 and 1.5 fail for D_∞. To see this consider $Y = \{0\}\cup\{1/n : n \in \mathbb{Z}\}$ endowed with the selfmap $\psi(1/n) = 1/(n+1), n \geq 0, \psi(1/n) = 1/(n-1), n \leq 0, \psi(0) = 0$, and $X = \{0\} \cap \{1/n : n \in \mathbb{N}\}$ endowed with the selfmap $\varphi(1/n) = 1/(n+1), \varphi(0) = (0)$. If we define a factor map $\pi : (Y, \psi) \to (X, \varphi)$ by $\pi(0) = 0$ and $\pi(1/n) = 1/|n|$, then $D(\pi) = 0$ by either Theorem 1.4 or Theorem 1.5. But $D_\infty(\pi) = \log 2$. However, it may be that we always have $D(\pi) = D_\infty(\pi|_{\pi^{-1}(\Omega)})$.

Let $\mathrm{Per}_n(\varphi) = \{x \in X : \varphi^n(x) = x\}$ and set $\mathrm{Per}(\varphi) = \bigcup_{n=1}^{\infty} \mathrm{Per}_n(\varphi)$. As a general lower bound for the defect we have the following.

PROPOSITION 1.7. *Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map between dynamical systems with Y totally disconnected. Then*

$$D(\pi) \geq \frac{1}{p}\sum_{j=0}^{p-1} \log(\#\pi^{-1}(\varphi^j(x)))$$

for all $x \in \mathrm{Per}_p(\varphi)$.

Proof. Let $x \in X$ be of period p and let $M_j \subset \pi^{-1}(\varphi^j(x))$ be a finite subset. Choose a partition $\mathcal{P}$ of Y which separates the points of $\bigcup_{j=0}^{p-1} M_j$. Then $D(\pi) \geq Q(\varphi, \pi(\mathcal{P}))$. Since $q_{mp}(x, \pi(\mathcal{P})) \geq \left(\prod_{j=0}^{p-1} \#M_j\right)^m$ for all $m \in \mathbb{N}$, the conclusion follows. □

COROLLARY 1.8. *Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map between dynamical systems with Y totally disconnected. Assume that $\#\pi^{-1}(x) = d$ for all $x \in X$ and that $\mathrm{Per}(\varphi) \neq \emptyset$. Then*

$$D(\pi) = \log d.$$

Proof. Combine Lemma 1.2 and Proposition 1.7. □

Next we consider the continuity properties of the defect under inverse limits. Consider a sequence $\pi_n : (Y_n, \psi_n) \to (X_n, \varphi_n), n \in \mathbb{N}$, of factor maps such that each Y_n is totally disconnected. Assume that we are given continuous maps $\lambda_n^X : X_{n+1} \to X_n$ and $\lambda_n^Y : Y_{n+1} \to Y_n$ such that $\varphi_n \circ \lambda_n^X = \lambda_n^X \circ \varphi_{n+1}$, $\psi_n \circ \lambda_n^Y = \lambda_n^Y \circ \psi_{n+1}$ for all n. We can then define the inverse limit dynamical systems $(X_\infty, \varphi_\infty)$ and (Y_∞, ψ_∞), where X_∞ and Y_∞ are the inverse limit spaces $X_\infty = \varprojlim(X_n, \lambda_n^X)$ and $Y_\infty = \varprojlim(Y_n, \lambda_n^Y)$, respectively. Note that Y_∞ is totally disconnected because the Y_n's are. We make the following assumption:

(A) for all $(x, y) \in X_{n+1} \times Y_n$ such that $\lambda_n^X(x) = \pi_n(y)$ there is a $z \in Y_{n+1}$ such that $\lambda_n^Y(z) = y$ and $\pi_{n+1}(z) = x$.

There is then a factor map $\pi_\infty : (Y_\infty, \psi_\infty) \to (X_\infty, \varphi_\infty)$ given by

$$\pi_\infty((y_i)) = (\pi_i(y_i)), \quad (y_i) \in Y_\infty.$$

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THEOREM 1.9. *$D(\pi_k) \leq D(\pi_{k+1})$ and*

$$\lim_{k\to\infty} D(\pi_k) = D(\pi_\infty).$$

Proof. Let $\mathcal{R} = \{R_j : j \in R\}$ be a partition of Y_k and set $\mathcal{R}' = \lambda_k^{Y^{-1}}(\mathcal{R})$. By using assumption (A) it is easily seen that $Q(\varphi_k, \pi_k(\mathcal{R})) = Q(\varphi_{k+1}, \pi_{k+1}(\mathcal{R}'))$, leading to the conclusion that $D(\pi_k) \leq D(\pi_{k+1})$. Similarly, we see that for any point (x_i) in X_∞, $(x_i) \in \pi_\infty(\rho_k^{-1}(R_j)) \Leftrightarrow x_k \in \pi_k(R_j)$, $j \in R$, where $\rho_k : Y_\infty \to Y_k$ is the projection. It follows that $q_n(\varphi_\infty, \pi_\infty(\rho_k^{-1}(\mathcal{R}))) = q_n(\varphi_k, \pi_k(\mathcal{R}))$ for all n and hence that

$$Q(\varphi_\infty, \pi_\infty(\rho_k^{-1}(\mathcal{R}))) = Q(\varphi_k, \pi_k(\mathcal{R})).$$

Thus $D(\pi_\infty) \geq \lim_{k\to\infty} D(\pi_k)$. Let $\mathcal{P} = \{P_j : j \in J\}$ be a partition in Y_∞. Then there is a $m \in \mathbb{N}$ and a partition $\mathcal{R} = \{R_j : j \in R\}$ of Y_k such that $\rho_m^{-1}(\mathcal{R})$ refines $\mathcal{P}$ (cf. e.g. [**G**, Lemma 3]). Hence

$$Q(\varphi_\infty, \pi_\infty(\mathcal{P})) \leq Q(\varphi_\infty, \pi_\infty(\rho_m^{-1}(\mathcal{R}))) = Q(\varphi_m, \pi_m(\mathcal{R})) \leq \lim_{k\to\infty} D(\pi_k). \qquad \square$$

Remark 1.10. We used (A) to guarantee that the map π_∞ was surjective. Sometimes this is the case for other reasons and then the inequality

$$D(\pi_\infty) \leq \sup_i D(\pi_i),$$

also holds when (A) is not satisfied.

THEOREM 1.11. *Let $\pi_n : (Y_n, \psi_n) \to (X_n, \varphi_n)$, $n \in \mathbb{N}$, be a sequence of factor maps with each Y_n totally disconnected. Then the defect of the infinite product factor map*

$$\pi_\infty : \left(\prod_{i=1}^\infty Y_i, \prod_{i=1}^\infty \psi_i\right) \to \left(\prod_{i=1}^\infty X_i, \prod_{i=1}^\infty \varphi_i\right)$$

is

$$D(\pi_\infty) = \sum_{i=1}^\infty D(\pi_i).$$

Proof. An obvious application of Theorem 1.9 gives that it suffices to prove the sum formula for two factor maps. Thus we let $\pi_1 : (Y_1, \psi_1) \to (X_1, \varphi_1)$ and $\pi_2 : (Y_2, \psi_2) \to (X_2, \varphi_2)$ be factor maps with Y_1 and Y_2 totally disconnected and consider

$$\pi_1 \times \pi_2 : (Y_1 \times Y_2, \psi_1 \times \psi_2) \to (X_1 \times X_2, \varphi_1 \times \varphi_2).$$

If $\mathcal{P}$ is a product partition of $Y_1 \times Y_2$, i.e. is of the form

$$\{P_i^1 \times P_j^2 : (i, j) \in I^1 \times I^2\},$$

where $\mathcal{P}^j = \{P_i^j : i \in I^j\}$ is a partition of Y_j, $j = 1, 2$, then it is easy to see that

$$Q(\varphi_1 \times \varphi_2, \pi_1 \times \pi_2(\mathcal{P})) = Q(\varphi_1, \mathcal{P}^1) + Q(\varphi_2, \mathcal{P}^2).$$

Since an arbitrary partition of $Y_1 \times Y_2$ admits a product partition as a refinement, we see that $D(\pi_1 \times \pi_2) = D(\pi_1) + D(\pi_2)$. $\square$

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The defect is a little less well-behaved with respect to fibered products, but we have the following.

THEOREM 1.12. *Let $\pi_1 : (Y_1, \psi_1) \to (X, \varphi)$ and $\pi_2 : (Y_2, \psi_2) \to (X, \varphi)$ be factor maps with Y_1 and Y_2 totally disconnected. Consider the factor map*

$$\pi : (\{(y_1, y_2) \in Y_1 \times Y_2 : \pi_1(y_1) = \pi_2(y_2)\}, \psi_1 \times \psi_2) \to (X, \varphi)$$

given by $\pi(y_1, y_2) = \pi_1(y_1)$. Then

$$\max\{D(\pi_1), D(\pi_2)\} \leq D(\pi) \leq D(\pi_1) + D(\pi_2).$$

Proof. Set $Z = \{(y_1, y_2) \in Y_1 \times Y_2 : \pi_1(y_1) = \pi_2(y_2)\}$. If $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ is a partition of Y_1, consider $\mathcal{P}' = \{P_1 \times Y_2 \cap Z, P_2 \times Y_2 \cap Z, \ldots, P_m \times Y_2 \cap Z\}$. Then $\pi(\mathcal{P}') = \pi_1(\mathcal{P})$, and hence $D(\pi) \geq Q(\varphi, \pi_1(\mathcal{P}))$, proving that $D(\pi) \geq D(\pi_1)$. By symmetry $\max\{D(\pi_1), D(\pi_2)\} \leq D(\pi)$. If instead $\mathcal{P}$ is a partition of Z then there are partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of Y_1 and Y_2, respectively, such that $(\mathcal{P}_1 \times \mathcal{P}_2) \cap Z$ is a refinement of $\mathcal{P}$. It is a straightforward consideration that $Q(\varphi, \pi(\mathcal{P})) \leq Q(\varphi, \pi_1(\mathcal{P}_1)) + Q(\varphi, \pi_2(\mathcal{P}_2)) \leq D(\pi_1) + D(\pi_2)$. □

Theorem 1.12 is the best general result about the defect of fibered products in the sense that there are (easily constructed) examples such that $\max\{D(\pi_1), D(\pi_2)\} = D(\pi) < D(\pi_1) + D(\pi_2)$ and others where $\max\{D(\pi_1), D(\pi_2)\} < D(\pi) = D(\pi_1) + D(\pi_2)$.

2. *On the defect of factor maps between one-sided subshifts*

Throughout this section we let $\pi : (\Sigma_1, \sigma) \to (\Sigma_2, \sigma)$ be a factor map between one-sided subshifts. We will assume that Σ_1 is of finite type, i.e. is given by a square matrix over $\{0, 1\}$ with no non-zero columns. By passing to a higher block representation of Σ_1 we may assume that π is a 1-block map. This will not change the defect by Lemma 1.1. Hence we assume that there is a map $I_1 \to I_2$, which we also denote by π, from the states I_1 of Σ_1 to the states I_2 of Σ_2 such that

$$\pi((x_i))_j = \pi(x_j), \quad j \in \mathbb{N}.$$

Let W_n^i denote the words of length n in $\Sigma_i, n \in \mathbb{N}, i = 1, 2$. π induces a map $\pi : W_n^1 \to W_n^2$ in the obvious way. For every $w \in W_n^1$, set $C_w = \{(i_j) \in \Sigma_1 : (i_0, i_1, \ldots, i_{n-1}) = w\}$ and let $\mathcal{C}_n = \{C_w : w \in W_n^1\}$ be the corresponding partition of $\Sigma_1, i = 1, 2$. Let $n, m \in \mathbb{N}$. For every $w = (x_0, x_1, \ldots, x_{n+m-1}) \in W_{n+m}^2$, set

$$A_{n,m}(w) = \prod_{i=0}^{n-1} \#\pi^{-1}(x_i, x_{i+1}, \ldots, x_{i+m})$$

and

$$A_{n,m}(\pi) = \max\{A_{n,m}(w) : w \in W_{n+m}^2\}.$$

Then $A_{n+k,m}(\pi) \leq A_{n,m}(\pi) A_{k,m}(\pi)$ and hence

$$\lim_{n \to \infty} \frac{1}{n} \log A_{n,m}(\pi)$$

exists for all m.

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Definition 2.1. We say that π is full when the following holds, When $i_0 \in I_1$ and $(\pi(i_0), j_1) \in W_2^2$, then there is a $i_1 \in I_1$ such that $\pi(i_0, i_1) = (j_0, j_1)$.

LEMMA 2.2. *Assume that π is full. Then*

$$Q(\sigma, \pi(\mathcal{C}_k)) = \lim_{n\to\infty} \frac{1}{n} \log A_{n,k-1}(\pi)$$

for each $k \in \mathbb{N}$.

Proof. Since π is full we have that

$$\#\{w \in W_k^1 : x \in \pi(C_w)\} = \#\pi^{-1}(x_0, x_1, \ldots, x_{k-1})$$

for all $x = (x_i) \in \Sigma_2$. Thus $q_n(x, \pi(\mathcal{C}_k)) = A_{n,k-1}(x_0, x_1, \ldots, x_{n+k-2})$. From this we see that

$$q_n(\sigma, \pi(\mathcal{C}_k)) = A_{n,k-1}(\pi), \quad n \in \mathbb{N},$$

and hence that $Q(\sigma, \pi(\mathcal{C}_k)) = \lim_{n\to\infty}(1/n) \log A_{n,k-1}(\pi)$. □

LEMMA 2.3. *Assume that π is full. Set $K = \#I_1$, i.e. K is the number of states in Σ_1. Then there is a $j \in \{1, 2, \ldots, K\}$ and a j-periodic element $\chi = (x_i) \in \Sigma_1$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \log A_{n,0}(\pi) = \frac{1}{j} \sum_{i=0}^{j-1} \log \#\pi^{-1}(\pi(x_i)) \le \frac{1}{j} \sum_{i=0}^{j-1} \log \#\pi^{-1}(\sigma^i(\pi(\chi))).$$

Proof. If (y_i) is any periodic point in Σ_1, say of period p, then

$$A_{np,0}(\pi(y_0, y_1, \ldots, y_{np-1})) = A_{p,0}(\pi(y_0, y_1, \ldots, y_{p-1}))^n$$

and hence

$$\lim_{n\to\infty} \frac{1}{n} \log A_{n,0}(\pi) = \lim_{n\to\infty} \frac{1}{np} \log A_{np,0}(\pi) \ge \frac{1}{p} \sum_{i=0}^{p-1} \log \#\pi^{-1}(\pi(y_i)).$$

For each $j \in \{1, \ldots, K\}$ let Γ_j denote the set of loops of length j, i.e. the set of elements $\gamma = (x_1, x_2, \ldots, x_{j+1}) \in W_{j+1}^1$ such that $x_1 = x_{j+1}$. Set

$$B(\gamma) = \prod_{i=1}^{j} \#\pi^{-1}(\pi(x_i)) \quad \text{and} \quad l(\gamma) = j$$

for $\gamma = (x_1, x_2, \ldots, x_{j+1}) \in \Gamma_j$. Set $L = \max\{A_{i,0}(w) : w \in W_i^2,\ i = 1, 2, \ldots, K\}$. Let $m > K$. For each $w \in W_m^1$ there are elements $\gamma_i \in \bigcup_{j=1}^{K} \Gamma_j$, $i = 1, 2, \ldots, M$, such that

$$A_{m,0}(\pi(w)) \le L \prod_{i=1}^{M} B(\gamma_i) \tag{2.1}$$

$$\sum_{i=1}^{M} l(\gamma_i) \le m. \tag{2.2}$$

We prove this by induction on m. Assume that it holds for all $w' \in W_n^1$, $K < n \le m-1$. To prove it for $w \in W_m^1$, we observe that a word in Σ_1 of length $> K$ must

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contain an element from Γ_j for some $j \in \{1, 2, \ldots, K\}$ as a sub-word. In particular, $w = w_1 \gamma w_2$ where $\gamma \in \cup_{j=1}^K \Gamma_j$ and w_1, w_2 are (possibly empty) words in Σ_1. Let $\gamma = (x_1, x_2, \ldots, x_{j+1})$ and set $w' = w_1 x_1 w_2 \in W^1_{m-j}$. Then

$$A_{m,0}(\pi(w)) = A_{m-j,0}(\pi(w'))B(\gamma).$$

If $m - j \leq K$ we have that $A_{m,0}(\pi(w)) \leq LB(\gamma)$. If instead $m - j > K$ we may apply the induction hypothesis to get $\gamma_i \in \cup_{j=1}^K \Gamma_j$, $i = 1, 2, \ldots, M'$, such that $A_{m-j,0}(\pi(w')) \leq L \prod_{i=1}^{M'} B(\gamma_i)$ and $\sum_{i=1}^{M'} l(\gamma_i) \leq m - j$. Since $\sum_{i=1}^{M'} l(\gamma_i) + l(\gamma) \leq m - j + j = m$, this completes the induction.

Set

$$X = \max\left\{\frac{1}{l(\gamma)} \log B(\gamma) : \gamma \in \cup_{j=1}^K \Gamma_j\right\}.$$

From (2.1) and (2.2) we conclude that

$$A_{m,0}(\pi) \leq Le^{mX}.$$

It follows that $\lim_{m\to\infty}(1/m) \log A_{m,0}(\pi) \leq X$. Let $\gamma = (x_1, x_2, \ldots, x_{j+1}) \in \Gamma_j$ be an element such that $(1/l(\gamma)) \log B(\gamma) = X$. Then $\chi = (x_1, x_2, \ldots, x_j)^\infty \in \Sigma_1$ is an $l(\gamma)$-periodic point in Σ_1 such that

$$\lim_{m\to\infty} \frac{1}{m} \log A_{m,0}(\pi) \leq \frac{1}{j} \sum_{i=0}^{j-1} \log \#\pi^{-1}(\pi(x_i)).$$

Finally, note that because π is full $\#\pi^{-1}(\pi(x_i)) \leq \#\pi^{-1}(\sigma^i(\pi(\chi)))$ for all $i = 0, 1, \ldots, j-1$. Thus

$$\frac{1}{j} \sum_{i=0}^{j-1} \log \#\pi^{-1}(\pi(x_i)) \leq \frac{1}{j} \sum_{i=0}^{j-1} \log \#\pi^{-1}(\sigma^i(\pi(\chi))). \qquad \square$$

THEOREM 2.4. *Assume that Σ_1 is a finite-type subshift and that π is a full 1-block map. Then*

$$\begin{aligned} D(\pi) &= \sup\left\{\frac{1}{p} \sum_{j=0}^{p-1} \log \#\pi^{-1}(\sigma^j(y)) : y \text{ a } p\text{-periodic point in } \Sigma_2,\ p \in \mathbb{N}\right\} \\ &= \sup\left\{\frac{1}{p} \sum_{j=0}^{p-1} \log \#\pi^{-1}(\sigma^j(\pi(x))) : x \text{ a } p\text{-periodic point in } \Sigma_1,\ p \in \mathbb{N}\right\}. \end{aligned}$$

Proof. By Proposition 1.7 we know that

$$\begin{aligned} &\sup\left\{\frac{1}{p} \sum_{j=0}^{p-1} \log \#\pi^{-1}(\sigma^j(\pi(x))) : x \text{ a } p\text{-periodic point in } \Sigma_1,\ p \in \mathbb{N}\right\} \\ &\leq \sup\left\{\frac{1}{p} \sum_{j=0}^{p-1} \log \#\pi^{-1}(\sigma^j(y)) : y \text{ a } p\text{-periodic point in } \Sigma_2,\ p \in \mathbb{N}\right\} \leq D(\pi), \end{aligned}$$

so it suffices to show that

$$D(\pi) \leq \sup\left\{\frac{1}{p} \sum_{j=0}^{p-1} \log \#\pi^{-1}(\sigma^j(\pi(x))) : x \text{ a } p\text{-periodic point in } \Sigma_1,\ p \in \mathbb{N}\right\}.$$

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To obtain this inequality it suffices to pick a $k \in \mathbb{N}$ and find a p-periodic point $\chi \in \Sigma_1$ such that

$$Q(\sigma, \pi(\mathcal{C}_k)) \leq \frac{1}{p} \sum_{j=0}^{p-1} \log \#\pi^{-1}(\sigma^j(\pi(\chi)))$$

or, by Lemma 2.2, such that

$$\lim_{n\to\infty} \frac{1}{n} \log A_{n,k-1}(\pi) \leq \frac{1}{p} \sum_{j=0}^{p-1} \log \#\pi^{-1}(\sigma^j(\pi(\chi))).$$

To this end let π' denote the 1-block map between the k-block representations of Σ_1 and Σ_2 defined by π. Then

$$A_{n,k-1}(\pi) = A_{n,0}(\pi')$$

and hence $\lim_{n\to\infty}(1/n) \log A_{n,k-1}(\pi) = \lim_{n\to\infty}(1/n) \log A_{n,0}(\pi')$. We therefore get the desired element $\chi \in \Sigma_1$ by applying Lemma 2.3 to π'. □

Example 2.5. Let $k, m \in \mathbb{N}$ and write $k = d_1 d_2 \cdots d_m$, where $d_1, d_2, \ldots, d_m \in \mathbb{N}$. A cyclic permutation on $\{1, 2, \ldots, m\}$ can be realized as the one-sided subshift defined by the graph $\mathcal{G}_1 : 1 \to 2 \to 3 \to \cdots \to m \to 1$. We define another one-sided subshift by specifying a graph $\mathcal{G}_2$ in the following way. For each $j = 1, 2, \ldots, m$ we take a set G_j with d_j elements and we choose functions $\lambda_j : G_j \to G_{j+1}$, $j = 1, 2, \ldots, m-1$, and $\lambda_m : G_m \to G_1$. The vertices of the new graph are $\bigcup_{i=1}^m G_i$ and there is an edge from each $x \in G_j$ to $\lambda_j(x)$, $j = 1, 2, \ldots, m$, and no other edges. The one-sided subshift determined by this graph $\mathcal{G}_2$ factors onto the cyclic subshift given by $\mathcal{G}_1$ in an obvious way (take all elements of G_j to j) and the factor map is clearly full. By Theorem 2.4 the defect is $(1/m) \sum_{i=1}^m \log d_i = (1/m) \log k$.

This very simple example allows us to realize a dense set in $]0, \infty]$ as possible defect values, and by using the continuity results, Theorem 1.9 or Theorem 1.11, from the first section we can now show that any $t \in]0, \infty]$ can occur as the defect of a factor map between zero-dimensional dynamical systems. Any $t \in]0, \infty]$ can be written as an infinite sum $t = \sum_{i=1}^\infty (k_i/n_i) \log 2$ where $n_i, k_i \in \mathbb{N}$ for all i. From the preceding examples we get factor maps $\pi_i : (X_i, \varphi_i) \to (Y_i, \psi_i)$ between one-sided subshifts such that $D(\pi_i) = (k_i/n_i) \log 2$. By Theorem 1.11 the product factor map

$$\pi_\infty : \left(\prod_{i=1}^\infty Y_i, \prod_{i=1}^\infty \psi_i \right) \to \left(\prod_{i=1}^\infty X_i, \prod_{i=1}^\infty \varphi_i \right)$$

will have defect $D(\pi_\infty) = t$.

We now turn to the case where π is a left-resolving factor map. Recall that a 1-block map $\pi : \Sigma_1 \to \Sigma_2$ is left-resolving when the following holds. If $(x, y), (z, y)$ are words of length two in Σ_1 such that $\pi((x, y)) = \pi((z, y))$, then $x = z$.

THEOREM 2.6. *Assume that Σ_1 is a subshift of finite type and that π is a left-resolving 1-block map. It follows that*

$$\begin{aligned} D(\pi) &= \max_{y \in \Sigma_2} \log \#\pi^{-1}(y) \\ &= \max\{\log \#\pi^{-1}(y) : y \in \mathrm{Per}(\Sigma_2)\}. \end{aligned}$$

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Proof. Because π is left-resolving we have that π is bounded-to-one and $\pi^{-1}(\mathrm{Per}(\Sigma_2)) = \mathrm{Per}(\Sigma_1)$. By Lemma 1.2 and Proposition 1.7 we therefore have that

$$\max\{\log \#\pi^{-1}(\pi(x)) : x \in \mathrm{Per}(\Sigma_1)\} \leq D(\pi) \leq \max_{y\in\Sigma_2} \log \#\pi^{-1}(y).$$

Let $\mathcal{C}_0$ denote the cylinder cover of Σ_1 consisting of the sets

$$C_i = \{(i_j) \in \Sigma_1 : i_0 = i\}, \quad i \in I_1.$$

By using the fact that π is left-resolving one sees that $\max_{y\in\Sigma_2} \pi^{-1}(y) = N$, where $N-1$ is the order of the cover $\pi(\mathcal{C}_0)$. It therefore suffices to show that when $x = (x_j) \in \Sigma_2$ and $i_1, i_2, \ldots, i_N$, are distinct elements of I_1 such that

$$x \in \bigcap_{j=1}^{N} \pi(\mathcal{C}_{i_j}),$$

then there are N distinct periodic points v_j, $j = 1, 2, \ldots, N$, in Σ_1 such that $\pi(v_j) = \pi(v_1)$ for all j. This is done as follows. Let t be a state which occurs infinitely often in x. Take an element $z^1 \in \mathcal{C}_{i_1}$ such that $\pi(z^1) = x$. Every time t occurs in x, say as $t = x_s$, the state $t' = z^1_s$ satisfies

$$\pi(t') = t.$$

We say that t' sits over t in z^1. Since there are only a finite number of states in Σ_1 and there are infinitely many occurrences of t in x, we see that there are infinitely many occurrences of t in x such that it is the same state in z^1 which sits over each of them. We restrict our attention to these occurrences of t. Take an element $z^2 \in \mathcal{C}_{i_2}$ such that $\pi(z^2) = x$. Among the occurrences of t we have restricted our attention to there are infinitely many for which it is the same state in z^2 which sits over t. Restrict attention to these and continue through N steps. This gives us infinitely many occurrences of t in x, say as $t = x_{r_i}, r_1 < r_2 < r_3 < \cdots$, and for each $j \in \{1, 2, \ldots, N\}$ an element $z^j \in \mathcal{C}_{i_j}$ such that $\pi(z^j) = x$ and $z^j_{r_i} = z^j_{r_1}$ for all i. It will be enough to consider $i = 1, 2$. Note that for each j,

$$(z^j_{r_1}, z^j_{r_1+1}, \ldots, z^j_{r_2})$$

defines a loop in $W^1_{r_2-r_1+1}$ such that

$$\pi((z^j_{r_1}, z^j_{r_1+1}, \ldots, z^j_{r_2})) = (t, x_{r_1+1}, \ldots, x_{r_2-1}, t).$$

Since the i_j's are mutually distinct and π is left resolving, $(z^j_{r_1}, \ldots, z^j_{r_2})$ and $(z^k_{r_1}, \ldots, z^k_{r_2})$ cannot agree in any coordinate when $k \neq j$. Therefore

$$v_j = (z^j_{r_1}, z^j_{r_1+1}, \ldots, z^j_{r_2-1})^\infty \in \Sigma_1, \quad j = 1, 2, \ldots, N,$$

are N distinct $(r_2 - r_1)$-periodic points such that $\pi(v_j) = \pi(v_1)$ for all j. □

3. *On the defect of factor maps between two-sided subshifts*

Compared to the one-sided case, the defect is often easier to calculate for factor maps out of two-sided finite-type subshifts thanks to the following results.

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THEOREM 3.1. *Consider two-sided subshifts,* (Σ_1, σ) *and* (Σ_2, σ) *with* Σ_1 *of finite type, and a factor map* $\pi : (\Sigma_1, \sigma) \to (\Sigma_2, \sigma)$. *Assume that* $\pi : (\Sigma_1, \sigma) \to (\Sigma_2, \sigma)$ *is boundedly finite-to-one. Then*

$$\begin{aligned} D(\pi) &= \max\{\log \#\pi^{-1}(y) : y \in \mathrm{Per}(\Sigma_2)\} \\ &= \max\{\log \#\pi^{-1}(\pi(x)) : x \in \mathrm{Per}(\Sigma_1)\}. \end{aligned}$$

Proof. We may assume that π is a 1-block map. By Proposition 1.7 we have that

$$\max\{\log \#\pi^{-1}(\pi(x)) : x \in \mathrm{Per}(\Sigma_1)\} \le \max\{\log \#\pi^{-1}(y) : y \in \mathrm{Per}(\Sigma_2)\} \le D(\pi),$$

so it suffices to prove that $\max\{\log \#\pi^{-1}(\pi(x)) : x \in \mathrm{Per}(\Sigma_1)\} \ge D(\pi)$. For each $K \in \mathbb{N}$, set

$$C_\iota = \{(i_j) \in \Sigma_1 : (i_{-k}, i_{-k+1}, \ldots, i_k) = \iota\}$$

for all $\iota \in I_1^{2K+1}$, where I_1 is the set of states of Σ_1. Then $\mathcal{C}_K = \{C_\iota : \iota \in I_1^{2K+1}\}$ is a partition of Σ_1 and $D(\pi) = \lim_{K\to\infty} Q(\sigma, \pi(\mathcal{C}_K))$. It therefore suffices to fix $K \in \mathbb{N}$ and $\epsilon > 0$ and find $m \ge e^{Q(\sigma,\pi(\mathcal{C}_K))-\epsilon}$ distinct periodic points $a_1, a_2, \ldots, a_m \in \Sigma_1$ such that $\pi(a_i) = \pi(a_1),\ i = 1, 2, \ldots, m$. By passing to the $(2K+1)$-block representation of Σ_1 it suffices, for this purpose, to consider the case $K = 0$. Let $A \in \mathbb{N}$ such that $\#\pi^{-1}(y) \le A$ for all $y \in \Sigma_2$. There is a natural number $n > (2/\epsilon)A!2^{\#I_1}2^A \log \#I_1$ such that

$$\frac{1}{n}\log q_n(\sigma, \pi(\mathcal{C}_0)) > Q(\sigma, \pi(\mathcal{C}_0)) - \frac{\epsilon}{2}.$$

Then there is a point $y \in \Sigma_2$ such that

$$\frac{1}{n}\log q_n(y, \pi(\mathcal{C}_0)) > Q(\sigma, \pi(\mathcal{C}_0)) - \frac{\epsilon}{2}.$$

For each $j = 0, 1, 2, \ldots, n-1$, set $F_j = \{i \in I_1 : \sigma^j(y) \in \pi(C_i)\}$ and $\#F_j = M_j$. Then

$$q_n(y, \pi(\mathcal{C}_0)) = \prod_{j=0}^{n-1} M_j$$

and

$$\frac{1}{n}\sum_{j=0}^{n-1} \log M_j > Q(\sigma, \pi(\mathcal{C}_0)) - \frac{\epsilon}{2}.$$

Note that $\log M_j \le \log \#I_1$ so that

$$\#\{j \in \{0, 1, 2, \ldots, n-1\} : \log M_j > Q(\sigma, \pi(\mathcal{C}_0)) - \epsilon\} \ge \frac{\epsilon}{2}\frac{n}{\log \#I_1} \ge A!2^{\#I_1}2^A.$$

Set $\mathcal{M}_0 = \{j \in \{0, 1, 2, \ldots, n-1\} : \log M_j > Q(\sigma, \pi(\mathcal{C}_0)) - \epsilon\}$. There is a subset $F \subset I_1$ such that $F = F_j$ for more than $A!2^A$ of the j's in $\mathcal{M}_0$. Set $\mathcal{M}_1 = \{j \in \mathcal{M}_0 : F_j = F\}$. By re-indexing we may assume that $F = \{1, 2, \ldots, m\}$. Note that $m = \#F \ge e^{Q(\sigma,\pi(\mathcal{C}_0))}e^{-\epsilon}$. For each $j \in \mathcal{M}_1$, pick $a_1^j, a_2^j, \ldots, a_m^j \in \Sigma_1$ such that $a_k^j \in C_k,\ \pi(a_k^j) = \sigma^j(y),\ k = 1, 2, \ldots, m$. Then

$$G_j = \{\sigma^{-j}(a_i^j) : i = 1, 2, \ldots, m\}$$

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is a subset of $\pi^{-1}(y)$ for all $j \in \mathcal{M}_1$. Since $\#\pi^{-1}(y) \leq A$, there are more than $A!$ of the j's for which the G_j's are the same. Let $G = \{a_1^0, a_2^0, \ldots, a_m^0\}$ be this common subset of $\pi^{-1}(y)$ and let $\mathcal{M}_2 = \{j \in \mathcal{M}_1 : G_j = G\}$. For each $j \in \mathcal{M}_2$ there is a permutation ρ_j of $\{1, 2, \ldots, m\}$ such that

$$\sigma^{-j}(a_i^j) = a^0_{\rho_j(i)}, \quad i = 1, 2, \ldots, m,\ j \in \mathcal{M}_2.$$

Since there are more than $A! \geq m!$ elements in $\mathcal{M}_2$, there must be two elements $j_1 < j_2$ in $\mathcal{M}_2$ such that $\rho_{j_1} = \rho_{j_2}$. Set $b_i = \sigma^{-j_1}(a_i^{j_1})$, $i = 1, 2, \ldots, m$. Then the j_1th coordinate of b_i is the same as the j_2th coordinate, so

$$b_i|_{[j_1, j_2]}, \quad i = 1, 2, \ldots, m,$$

are m distinct periodic words in Σ_1 such that $\pi(b_i|_{[j_1,j_2]}) = y|_{[j_1,j_2]}$, $i = 1, 2, \ldots, m$. Set

$$a_i = (b_i|_{[j_1, j_2-1]})^\infty, \quad i = 1, 2, \ldots, m. \qquad \square$$

COROLLARY 3.2. *Consider two-sided subshifts, (Σ_1, σ) and (Σ_2, σ) with Σ_1 of finite type, and a factor map $\pi : (\Sigma_1, \sigma) \to (\Sigma_2, \sigma)$. Assume that Σ_1 is irreducible. Then $D(\pi) = \infty$ when π is not finite-to-one and*

$$\begin{aligned} D(\pi) &= \max\{\log \#\pi^{-1}(y) : y \in \mathrm{Per}(\Sigma_2)\} \\ &= \max\{\log \#\pi^{-1}(\pi(x)) : x \in \mathrm{Per}(\Sigma_1)\} \end{aligned}$$

when it is.

Proof. If π is not finite-to-one, there is at least one 'point diamond' for π, cf. [**LM**, Theorem 1.16, Chap. 8]. This shows that there is a periodic point of Σ_2 with infinitely many pre-images and hence $D(\pi) = \infty$ by Proposition 1.7. If π is finite-to-one it is boundedly finite-to-one by [**LM**, Corollary 1.18, Chap. 8]. Apply Theorem 5.1. $\square$

4. *The defect of some factor maps arising from Markov and other partitions*

We begin with a brief sketch of how to get totally disconnected extensions (or symbolic dynamics) of higher-dimensional dynamical systems by a method which covers many cases. Let (X, φ) be a dynamical system with X a metric space. We will assume that φ is open at most points. To be precise, we will say that φ is *open at* x when there is an open neighbourhood V of x such that $\varphi|_V$ is open, i.e. such that $W \subset V$ open $\Rightarrow \varphi(W)$ is open. The assumption we will make is that:

(A) $\{x \in X : \varphi$ is open at $x\}$ is a dense subset of X.

Let $\mathcal{V} = \{V_i : i = 1, 2, \ldots, N\}$ be subsets of X such that

(B) $\bigcup_{i=1}^N V_i$ is dense and open in X.

It follows then that $\varphi^{-j}(\bigcup_{i=1}^N V_i)$ is open and dense in X for all $j \in \mathbb{N}$. Assume, furthermore, that

(C) $\lim_{k\to\infty} \mathrm{diam}(\bigcap_{j=0}^k \varphi^{-j}(V_{i_j})) = 0$ for all $(i_j) \in \{1, 2, \ldots, N\}^{\mathbb{N}}$,

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when φ is not invertible and that

(C′) $\lim_{k\to\infty} \operatorname{diam}(\bigcap_{j=-k}^{k} \varphi^{-j}(V_{i_j})) = 0$ for all $(i_j) \in \{1, 2, \ldots, N\}^{\mathbb{Z}}$,

when φ is invertible.

Under these assumptions there is an obvious continuous map

$$\pi : \left\{ (i_j) \in \{1, 2, \ldots, N\}^{\mathbb{N}} : \bigcap_{k=1}^{\infty} \overline{\bigcap_{j=0}^{k} \varphi^{-j}(V_{i_j})} \neq \emptyset \right\} \to X$$

in the non-invertible case and

$$\pi : \left\{ (i_j) \in \{1, 2, \ldots, N\}^{\mathbb{Z}} : \bigcap_{k=1}^{\infty} \overline{\bigcap_{j=-k}^{k} \varphi^{-j}(V_{i_j})} \neq \emptyset \right\} \to X$$

in the invertible case. Let $\Sigma_{\mathcal{V}}$ be the closure in $\{1, 2, \ldots, N\}^{\mathbb{N}}$ of

$$\left\{ (i_j) \in \{1, 2, \ldots, N\}^{\mathbb{N}} : \bigcap_{j=0}^{\infty} \varphi^{-j}(V_{i_j}) \neq \emptyset \right\}$$

in the non-invertible case, and the closure in $\{1, 2, \ldots, N\}^{\mathbb{Z}}$ of

$$\left\{ (i_j) \in \{1, 2, \ldots, N\}^{\mathbb{Z}} : \bigcap_{j=-\infty}^{\infty} \varphi^{-j}(V_{i_j}) \neq \emptyset \right\}$$

in the invertible case. In both cases $\Sigma_{\mathcal{V}}$ are subshifts (one-and two-sided, respectively) and the factor map

$$\pi : (\Sigma_{\mathcal{V}}, \sigma) \to (X, \varphi)$$

is defined as the restrictions of π to $\Sigma_{\mathcal{V}}$. Define continuous functions $g_k : \Sigma_{\mathcal{V}} \to \mathbb{R}^+$ by

$$g_k((i_j)) = \operatorname{diam}\left(\overline{\bigcap_{j=0}^{k} \varphi^{-j}(V_{i_j})} \right)$$

in the non-invertible case. The sequence $\{g_k\}$ converges monotonically to zero, so by Dini's theorem we have that

$$\sup_{(i_j)\in\Sigma_{\mathcal{V}}} \operatorname{diam}\left(\overline{\bigcap_{j=0}^{k} \varphi^{-j}(V_{i_j})} \right) \xrightarrow{k\to\infty} 0. \tag{4.1}$$

Similarly, we have that

$$\sup_{(i_j)\in\Sigma_{\mathcal{V}}} \operatorname{diam}\left(\overline{\bigcap_{j=-k}^{k} \varphi^{-j}(V_{i_j})} \right) \xrightarrow{k\to\infty} 0 \tag{4.2}$$

in the invertible case.

The assumptions (A), (B) and (C) ((C′) in the invertible case) will be in force throughout this section. We consider first the least typical case where $\mathcal{V}$ consists of closed sets (and hence covers X).

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PROPOSITION 4.1. *Assume that $\mathcal{V}$ consists of closed subsets. Then the defect of the factor map $\pi : (\Sigma_{\mathcal{V}}, \sigma) \to (X, \varphi)$ in the preceding construction is $D(\pi) = \infty$ when $Q(\varphi, \mathcal{V}) > 0$ and $D(\pi) = 0$ when $Q(\varphi, \mathcal{V}) = 0$.*

Proof. We give the proof in the invertible case, the non-invertible case being quite similar. Since $\mathcal{V}$ consists of closed subsets,

$$\Sigma_{\mathcal{V}} = \left\{ (i_k) \in \{1, 2, \ldots, N\}^{\mathbb{Z}} : \bigcap_{j=-\infty}^{\infty} \varphi^{-j}(V_{i_k}) \neq \emptyset \right\}.$$

Let $\mathcal{C}_k$ be the partition of $\Sigma_{\mathcal{V}}$ given by the cylinders obtained by fixing the middle $2k+1$ coordinates. By definition

$$\begin{aligned} \pi(\mathcal{C}_k) &= \vee_{j=-k}^{k} \varphi^{-j}(\mathcal{V}) \\ &= \{\varphi^k(V_{i_{-k}}) \cap \varphi^{k-1}(V_{i_{-k+1}}) \cap \cdots \cap \varphi^{-k}(V_{i_k}) : (i_{-k}, \ldots, i_k) \in \{1, 2, \ldots, N\}^{2k+1}\}. \end{aligned}$$

For each $x \in X$ we let $M(x) = \#\{i : x \in V_i\}$. An elementary combinatorial consideration shows that

$$q_{n+1}(x, \vee_{j=-k}^{k} \varphi^{-j}(\mathcal{V})) = \prod_{d=-k}^{k+n} M(\varphi^d(x))^{F(d)},$$

where $F(d) \in \mathbb{N}$ is the number of ways to write $d = a + b$ with $a \in \{-k, -k+1, \ldots, k\}$ and $b \in \{0, 1, \ldots, n\}$. Since $F(d) \leq 2k+1$ for all d and $F(d) = 2k+1$ when $n + k \leq d \leq n - k$, we see that

$$K \prod_{d=-k}^{k+n} M(\varphi^d(x))^{2k+1} \leq \prod_{d=-k}^{k+n} M(\varphi^d(x))^{F(d)} \leq \prod_{d=-k}^{k+n} M(\varphi^d(x))^{2k+1}$$

for some n-independent constant K. But $\prod_{d=-k}^{k+n} M(\varphi^d(x)) = q_{2k+n+1}(\varphi^{-k}(x), \mathcal{V})$, so we see that

$$Q(\varphi, \pi(\mathcal{C}_k)) = (2k+1) Q(\varphi, \mathcal{V}).$$

This proves the proposition because $D(\pi) = \lim_{k\to\infty} Q(\varphi, \pi(\mathcal{C}_k))$. □

So we see that for this type of extension there is a dichotomy, and which of the two possibilities occur depends on whether or not $Q(\varphi, \mathcal{V})$ is zero. This number is of course zero when $\bigcup_{i \neq j} V_i \cap V_j = \emptyset$. Otherwise, set

$$m(n) = \sup_{x \in X} \#\left\{ k \in \{0, 1, 2, \ldots, n-1\} : \varphi^k(x) \in \bigcup_{i \neq j} V_i \cap V_j \right\}.$$

It is easily seen that

$$\log 2 \limsup_{n\to\infty} \frac{m(n)}{n} \leq Q(\varphi, \mathcal{V}) \leq \log N \liminf_{n\to\infty} \frac{m(n)}{n},$$

so we see that

$$D(\pi) = 0 \Leftrightarrow \lim_{n\to\infty} \frac{m(n)}{n} = 0,$$

which shows that the dichotomy of Proposition 4.1 is controlled by recurrence properties of the overlaps of the closed cover.

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Example 4.2. Let $\varphi : \mathbb{T} \to \mathbb{T}$ be an irrational rotation and $\{V_i : i = 1, 2, \ldots, N\}$ a closed cover of $\mathbb{T}$ such that (C′) holds. Note that condition (C′) holds for quite a variety of closed covers. As is well known, $(1/n)\#\{i \in \{0, 1, 2, \ldots, n-1\} : \varphi^i(x) \in F\}$ converges, uniformly in x, to the Lebesgue measure $m(F)$ for every closed subset $F \subset \mathbb{T}$ whose boundary has measure zero. So if the boundaries of the V_i's are reasonable, Proposition 4.1 shows that the defect of the corresponding extension is ∞ if $m(\bigcup_{i \neq j} V_i \cap V_j) > 0$ and 0 if $m(\bigcup_{i \neq j} V_i \cap V_j) = 0$. See also Theorem 5.2 below.

Most applications of symbolic dynamics to dynamical systems use a totally disconnected extension which does not come from a closed cover. In fact, the V_i's are mostly open subsets, as for example in the case of a finite-type subshift extension arising from a Markov partition where the role of the V_i's are played by the interior of the Markov partitions. We make the following additional assumption:

(D) each V_i is open.

For each $k \in \mathbb{N}$, $(i_1, i_2, \ldots, i_k) \in \{1, 2, \ldots, N\}^k$, set

$$F_{i_1, i_2, \ldots, i_k} = \overline{V_{i_1} \cap \varphi^{-1}(V_{i_2}) \cap \cdots \cap \varphi^{-k+1}(V_{i_k})}$$

and let

$$\mathcal{F}_k = \{F_{i_1, i_2, \ldots, i_k} : (i_1, i_2, \ldots, i_k) \in \{1, 2, \ldots, N\}^k\}.$$

In the invertible case we set

$$F_{i_{-k}, i_{-k+1}, \ldots, i_k} = \overline{\bigcap_{j=-k}^{k} \varphi^{-j}(V_{i_j})}$$

for all $(i_{-k}, i_{-k+1}, \ldots, i_k) \in \{1, 2, \ldots, N\}^{2k+1}$ and

$$\mathcal{F}_{(-k,k)} = \{F_{i_{-k}, i_{-k+1}, \ldots, i_k} : (i_{-k}, i_{-k+1}, \ldots, i_k) \in \{1, 2, \ldots, N\}^{2k+1}\}.$$

LEMMA 4.3. *$\pi(\mathcal{C}_k) = \mathcal{F}_k$ and $\pi(\mathcal{C}_{(-k,k)}) = \mathcal{F}_{(-k,k)}$, in the invertible case, where $\mathcal{C}_k$ denotes the cylinder cover of $\Sigma_{\mathcal{V}}$ obtained by fixing the first k coordinates in the non-invertible case and $\mathcal{C}_{(-k,k)}$ denotes the cylinder cover obtained by fixing the middle $2k+1$ coordinates in the invertible case. In particular, it follows that*

$$D(\pi) = \lim_{k \to \infty} Q(\varphi, \mathcal{F}_k)$$

in the non-invertible case and

$$D(\pi) = \lim_{k \to \infty} Q(\varphi, \mathcal{F}_{(-k,k)})$$

in the invertible case.

Proof. We only give the proof in the non-invertible case, but note that the invertible case is quite similar, and in fact slightly simpler. Let $(r_1, r_2, \ldots, r_k) \in \{1, 2, \ldots, N\}^k$ and set

$$C = \{(i_j) \in \Sigma_{\mathcal{V}} : i_j = r_j, \ \ j = 1, 2, \ldots, k\}.$$

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We must show that $\pi(C) = F_{r_1,r_2,\dots,r_k}$. If $x \in F_{r_1,r_2,\dots,r_k}$ we can choose a sequence $x_n \in V_{r_1} \cap \varphi^{-1}(V_{r_2}) \cap \cdots \cap \varphi^{-k+1}(V_{r_k})$ such that $\lim_{n\to\infty} x_n = x$. For each n,

$$V_{r_1} \cap \varphi^{-1}(V_{r_2}) \cap \cdots \cap \varphi^{-k+1}(V_{r_k}) \cap \left\{ y \in X : d(x_n, y) < \frac{1}{n} \right\}$$

is open and contains an element x'_n from $\bigcap_{j=k}^{\infty} \varphi^{-j}(\bigcup_{i=1}^{N} V_i)$ because this set is dense by Baire's theorem. Thus $\pi^{-1}(x'_n) \cap C \neq \emptyset$. Take $\chi_n \in \pi^{-1}(x'_n) \cap C$ and let (χ_{n_i}) be a convergent subsequence of (χ_n). Then $\chi = \lim_{i\to\infty} \chi_{n_i} \in C$ and $\pi(\chi) = \lim_{i\to\infty} x'_{n_i} = x$. Thus $F_{r_1,r_2,\dots,r_k} \subset \pi(C)$. Conversely, let $\chi \in C$. Then there is a sequence $(\chi_n) = ((i^n_j))$ in C such that $\chi_n \to \chi$ and

$$\bigcap_{j=0}^{\infty} \varphi^{-j}(V_{i^n_j}) \neq \emptyset$$

for all n, so we see that $\pi(\chi) = \lim_{n\to\infty} \pi(\chi_n) \in F_{r_1,r_2,\dots,r_k}$. □

Example 4.4. Let $\varphi : [0,1] \to [0,1]$ be continuous and let $0 < x_1 < x_2 < \cdots < x_{N-1} < 1$ be points in $[0,1]$ such that $V_1 = [0, x_1[$, $V_N =]x_{N-1}, 1]$ and $V_i =]V_{i-1}, V_i[$, $i = 2, 3, \dots, N-1$. In this case the defect of the extension $\pi : (\Sigma_{\mathcal{V}}, \sigma) \to ([0,1], \varphi)$ is $D(\pi) = q \log 2$, where $q \in \mathbb{Q}$ is determined as follows. Let S_0 denote the set of periodic points in $\{x_1, x_2, \dots, x_{N-1}\}$ and for each $s \in S_0$ let p_s denote the minimal period of s and $n_s = \#\{m \in \{0, 1, 2, \dots, p_s - 1\} : \varphi^m(s) \in S_0\}$. Then

$$q = \max\left\{ \frac{n_s}{p_s} : s \in S_0 \right\}.$$

For a proof of this we refer to Proposition 8.2 of [**Th**] where the argument was given in the case of a continuous self-map of the circle, but carries over without change. □

Next we want to add the Markov condition for the non-invertible case:

(E) for every $i, j \in \{1, 2, \dots, N\}$, $i \neq j \Rightarrow V_i \cap V_j = \emptyset$ and
$\varphi(V_i) \cap V_j \neq \emptyset \Rightarrow \varphi(\overline{V_i}) \supset V_j$.

As is well known and easily verified, in the presence of assumption (E) we have that $\Sigma_{\mathcal{V}}$ is the one-sided finite-type subshift of $\{1, 2, \dots, N\}^{\mathbb{N}}$ determined by the transition matrix

$$A_{ij} = \begin{cases} 1, & \text{when } \varphi(V_i) \cap V_j \neq \emptyset, \\ 0, & \text{when } \varphi(V_i) \cap V_j = \emptyset. \end{cases}$$

LEMMA 4.5. *Assume that (A), (B), (C), (D) and (E) holds. Assume in addition that φ is locally injective and injective on $\overline{V_i}$ for all i. Then π is left resolving in the following sense.*

If $(r_i), (s_i) \in \Sigma_{\mathcal{V}}$, $\pi((r_i)) = \pi((s_i))$ and $r_k = s_k$ for some $k \in \mathbb{N}$, then $r_i = s_i$, $i = 0, 1, 2, \dots, k$.

Proof. To get a contradiction assume that $r_{k-1} \neq s_{k-1}$. Set $a = r_k = s_k$, $x = \pi((r_i))$ and $v = \varphi^{k-1}(x)$. By construction of $\Sigma_{\mathcal{V}}$ there are sequences $(x_n) \subset V_{r_{k-1}} \cap \varphi^{-1}(V_a)$, $(y_n) \subset V_{s_{k-1}} \cap \varphi^{-1}(V_a)$ such that $\lim_n x_n = \lim_n y_n = v$. Let W be an open neighbourhood of

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v such that φ is injective on W. We may assume that $(y_n) \subset W$. Since $\varphi(\overline{V_{r_{k-1}}}) \supset V_a$, there are elements $z_n \in \overline{V_{r_{k-1}}}$ such that $\varphi(z_n) = \varphi(y_n)$. Since φ is injective on W and $\overline{V_{r_{k-1}}} \cap V_{s_{k-1}} = \emptyset$ we have that $z_n \notin W$. By compactness there is a convergent subsequence (z_{n_i}), say $z = \lim_i z_{n_i} \in \overline{V_{r_{k-1}}}$. Then $\varphi(z) = \lim_i \varphi(y_{n_i}) = \varphi(v)$. As $z \neq v$ because $z \notin W$, this contradicts the assumption that φ is injective on $\overline{V_{r_{k-1}}}$. Hence $r_{k-1} = s_{k-1}$. Repeat this argument k times. □

LEMMA 4.6. *Assume that (A), (B), (C), (D) and (E) hold. Assume in addition that φ is locally injective and injective on $\overline{V_i}$ for all i. If $i_1, i_2, \ldots, i_m$ are distinct elements of $\{1, 2, \ldots, N\}$ and $x \in \bigcap_{j=1}^m \overline{V_{i_j}}$, then there are distinct elements $i'_1, i'_2, \ldots, i'_m \in \{1, 2, \ldots, N\}$ such that*

$$\varphi(x) \in \bigcap_{j=1}^m \overline{V_{i'_j}}.$$

Proof. To get a contradiction assume that $\varphi(x)$ meets at most $m-1$ of the $\overline{V_i}$'s. Choose an open neighbourhood W of x such that φ is injective on W and so small that $\varphi(W)$ has non-trivial intersection with at most $m-1$ of the $\overline{V_i}$'s. Choose sequences $\{z_n^j\} \subset W \cap V_{i_j}$, $j = 1, 2, \ldots, m$, such that $\varphi(z_n^j) \in \bigcup_{i=1}^N V_{i_j}$ for all j, n and such that $\lim_n z_n^j = x$ for all j. Since $\varphi(W)$ meets at most $m-1$ of the F_i's, we can find two elements of $\{1, 2, \ldots, m\}$, say 1, 2, and an i_0 such that

$$\varphi(z_n^1),\ \varphi(z_n^2) \in V_{i_0}$$

for infinitely many n's. Let $n_1 < n_2 < \cdots$ be an increasing sequence in $\mathbb{N}$ such that $\varphi(z_{n_i}^1), \varphi(z_{n_i}^2) \in V_{i_0}$ for all $i \in \mathbb{N}$. Since $\varphi(\overline{V_{i_1}}) \supset V_{i_0}$, there are elements $z_i \in \overline{V_{i_1}}$ such that $\varphi(z_i) = \varphi(z_{n_i}^2)$. Since $\overline{V_{i_1}} \cap V_{i_2} = \emptyset$ and φ is injective on W we see that $z_i \notin W$. Passing to a subsequence we can assume that $\{z_i\}$ converges, say to z. Then $z \in \overline{V_{i_1}} \setminus W$ and $\varphi(z) = \lim_i \varphi(z_{n_i}^2) = \varphi(x)$, contradicting the assumption that φ is injective on $\overline{V_{i_1}}$. □

THEOREM 4.7. *Assume that φ is an expansive endomorphism and that (A), (B), (C), (D) and (E) hold. Then*

$$D(\pi) = \max_{x \in X} \log \#\pi^{-1}(x) = \max_{y \in \mathrm{Per}(\Sigma_\nu)} \log \#\pi^{-1}(\pi(y)).$$

Alternatively,

$$D(\pi) = \log M,$$

where $M-1$ is the order of the cover $\mathcal{F}_k$ for any $k \in \mathbb{N}$ so large that φ is injective on each member of $\mathcal{F}_k$.

Proof. We prove first that $D(\pi) = \max_{x \in X} \log \#\pi^{-1}(x)$. Let $\delta > 0$ be so small that φ is injective on each ball of radius $\leq \delta$. Choose $k \in \mathbb{N}$ so large that $\mathrm{diam}(G) < \delta$ for all $G \in \mathcal{F}_k$. This is possible by (4.1). Consider the cover

$$V_{i_1} \cap \varphi^{-1}(V_{i_2}) \cap \cdots \cap \varphi^{-k+1}(V_{i_k}), \quad (i_1, i_2, \ldots, i_k) \in \{1, 2, \ldots, N\}^k.$$

This cover satisfies (B), (C), (D), (E) and φ is injective on the closure of each member. The finite-type subshift extension obtained from this cover is the k-block representation

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of $\Sigma_{\mathcal{V}}$ so it suffices to prove that

$$D(\pi) = \sup_{x \in X} \log \#\pi^{-1}(x) = \log M,$$

where $M-1$ is the order of the cover $\{\overline{V_i} : i = 1, 2, \ldots, N\}$ under the additional assumption that φ is injective on each $\overline{V_i}$. We show first that $\#\pi^{-1}(y) \leq M$ for all $y \in X$. Let $x^1, x^2, \ldots, x^{M+1} \in \pi^{-1}(y)$ and let $\epsilon > 0$. It suffices to show that $d(x^i, x^j) < \epsilon$ for some $i \neq j$. Alternatively, it suffices to fix a $k \in \mathbb{N}$ and show that $x_l^i = x_l^j, l = 0, 1, 2, \ldots, k$, for some $i \neq j$. But this follows immediately from Lemma 4.5 and the fact that $\varphi^k(y)$ lies in no more than M of the $\overline{V_i}$'s. Thus $D(\pi) \leq \log M$ by Lemma 1.2. On the other hand, $D(\pi) \geq Q(\varphi, \pi(\mathcal{C}_0))$ and this quantity equals $\log M$ by Lemma 4.6. To prove that $\max_{x \in X} \#\pi^{-1}(x) = M$ is attained at the image of a periodic point, we may assume that $2\,\mathrm{diam}(V_i)$ is less than an expansive constant for φ for all i. Consider $x \in \bigcap_{k=1}^{M} \overline{V_{i_k}}$, where the i_k's are mutually distinct. Then there are M elements $y^1, y^2, \ldots, y^M \in \pi^{-1}(x)$ such that $y_0^k = i_k,\ k = 1, 2, \ldots, M$. By the argument used in the proof of Theorem 2.6, there are natural numbers $a < b$ such that $y_a^k = y_b^k,\ k = 1, 2, \ldots, M$. Since $\Sigma_{\mathcal{V}}$ is a subshift of finite type, $z^k = (y_a^k, y_{a+1}^k, \ldots, y_{b-1}^k)^\infty$, with $z_0^k = y_a^k, k = 1, 2, \ldots, M$, define periodic points in $\Sigma_{\mathcal{V}}$. Since $y_a^k \neq y_a^l$, when $k \neq l$, by Lemma 4.5, the z_k's are mutually distinct. Note that $\varphi^j(\pi(z_k)), \varphi^{a+j}(x) \in \overline{V_{y_j^k}},\ j = 0, 1, \ldots, b-a-1$, for all k, so that $d(\varphi^j(\pi(z_k)), \varphi^j(\pi(z_l))) \leq 2 \max_i \mathrm{diam}(V_i)$ for all $k, l,\ j = 0, 1, \ldots, b-a-1$. Since $b-a$ is a common period for $\pi(z_k),\ k = 1, 2, \ldots, M$, we see that $\sup_{n \in \mathbb{N}} d(\varphi^n(\pi(z_k)), \varphi^n(\pi(z_l)))$ is less than an expansive constant for φ, proving that $\pi(z_k) = \pi(z_l),\ k \neq l$. □

Example 4.8. It follows from Theorem 4.7 that a connected Markov partition for the map $z \to z^2$ on the circle $\mathbb{T}$ gives rise to a factor map which invariably has defect $\log 2$. However, when the Markov partition is not connected the defect value can be something else. This is, for example, the case in an example constructed by R. F. Williams; see Example 3.5 of [**AKS**]. The defect is $2 \log 2$ for this example. The order of the cover in this example is 3, so that it also illustrates the necessity of increasing k in Theorem 4.7 to make φ injective on each member of $\mathcal{F}_k$. In Williams' example this happens when $k = 2$.

Next we turn to Markov partitions of expansive homeomorphisms.

THEOREM 4.9. *Assume that φ is an expansive homeomorphism and that $\mathcal{V}$ consists of the interiors of the elements of a Markov partition for φ. Then*

$$D(\pi) = \max\{\log \#\pi^{-1}(y) : y \in \mathrm{Per}(\varphi)\} = \max\{\log \#\pi^{-1}(\pi(x)) : x \in \mathrm{Per}(\Sigma_{\mathcal{V}})\}.$$

Proof. The proof is essentially the same as the proof of Theorem 3.1. By passing to a block representation of $\Sigma_{\mathcal{V}}$ we may assume that $2 \max_i \mathrm{diam}(V_i)$ is less than an expansive constant for φ. By Proposition 1.7 we have that

$$\max\{\log \#\pi^{-1}(\pi(x)) : x \in \mathrm{Per}(\Sigma_{\mathcal{V}})\} \leq \max\{\log \#\pi^{-1}(y) : y \in \mathrm{Per}(\varphi)\} \leq D(\pi),$$

so it suffices to prove that $\max\{\log \#\pi^{-1}(\pi(x)) : x \in \mathrm{Per}(\Sigma_{\mathcal{V}})\} \geq D(\pi)$. For each $K \in \mathbb{N}$, set

$$C_\iota = \{(i_j) \in \Sigma_{\mathcal{V}} : (i_{-k}, i_{-k+1}, \ldots, i_k) = \iota\}$$

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for all $\iota \in \{1, 2, \ldots, N\}^{2K+1}$. Then $\mathcal{C}_K = \{C_\iota : \iota \in \{1, 2, \ldots, N\}^{2K+1}\}$ is a partition of $\Sigma_\mathcal{V}$ and $D(\pi) = \lim_{K\to\infty} Q(\varphi, \pi(\mathcal{C}_K))$. It therefore suffices to fix $K \in \mathbb{N}$ and $\epsilon > 0$ and find $m \geq e^{Q(\varphi,\pi(\mathcal{C}_K))-\epsilon}$ distinct periodic points $a_1, a_2, \ldots, a_m \in \Sigma_\mathcal{V}$ such that $\pi(a_i) = \pi(a_1),\ i = 1, 2, \ldots, m$. By passing to the $(2K+1)$-block representation of $\Sigma_\mathcal{V}$ it suffices, for this purpose, to consider the case $K = 0$. It is well known that π is boundedly finite-to-one, so we can proceed as in the proof of Theorem 3.1, with only the obvious changes, to obtain $m \geq e^{Q(\varphi,\pi(\mathcal{C}_K))-\epsilon}$ elements, $b^i,\ i = 1, 2, \ldots, m$, in $\Sigma_\mathcal{V}$ and integers $j_1 < j_2$ such that $\pi(b^i) = \pi(b^1),\ i = 1, 2, \ldots, m$, and $b^i_{j_1} = b^i_{j_2},\ b^i_{j_1} \neq b^k_{j_1},\ i, k = 1, 2, \ldots, m, i \neq k$. Let a_i be the $(j_2 - j_1)$-periodic element of $\Sigma_\mathcal{V}$ whose $(j_1 + l)$th coordinate is $b^i_{j_1+l},\ l = 0, 1, \ldots, j_2 - j_1 - 1$. Then the a_i's are distinct periodic elements of $\Sigma_\mathcal{V}$. Set $y = \pi(b^i),\ i = 1, 2, \ldots, m$. Then $\varphi^j(y), \varphi^j(a_i) \in \overline{V_{b^i_j}},\ j = j_1, j_1+1, \ldots, j_2$, so $d(\varphi^j(y), \varphi^j(a_i)) \leq \max_k \operatorname{diam}(V_k),\ i = 1, 2, \ldots, m,\ j = j_1, j_1 + 1, \ldots, j_2$. Since each $\pi(a_i)$ is $(j_2 - j_1)$-periodic, it follows that $\sup_{j\in\mathbb{Z}} d(\varphi^j(\pi(a_i)), \varphi^j(\pi(a_k)))$ is less than an expansive constant for φ, so that $\pi(a_i) = \pi(a_1),\ i = 1, 2, \ldots, m$. □

Example 4.10. Let φ be a hyperbolic automorphism of $\mathbb{T}^2$ and $\mathcal{P} = \{P_1, P_2, \ldots, P_N\}$ a Markov partition for φ as described, for example, in [**AKS**, Construction 3.3]. (This construction is apparently due to R. Bowen, [**Bo1**].) If $\pi : (\Sigma_A, \sigma) \to (\mathbb{T}^2, \varphi)$ is the finite-type subshift extension obtained from this Markov partition, then $\#\pi^{-1}(x) = 1$ for all φ-periodic points except $(1, 1)$, and $\#\pi^{-1}((1, 1)) = 4$. Hence

$$D(\pi) = 2 \log 2$$

by Theorem 4.9. In this case $D(\pi) = \max_{x\in\mathbb{T}^2} \log \#\pi^{-1}(x)$. If instead $\mathcal{P}_1$ is the Markov partition constructed by Adler and Weiss in [**AW**] and π_1 the corresponding factor map, then $\max_{x\in\operatorname{Per}(\varphi)} \#\pi_1^{-1}(x) = \#\pi_1^{-1}((1, 1)) = 3$, so that $D(\pi_1) = \log 3$.

Example 4.11. Again let φ be a hyperbolic automorphism of the 2-torus $\mathbb{T}^2$ and fix $t \in]0, \infty]$. Here we want to indicate how to obtain a totally disconnected extension of $(\mathbb{T}^2, \varphi)$ such that the corresponding factor map has defect t. The point of departure for this is the Markov partition $\mathcal{P} = \{P_1, P_2, \ldots, P_N\}$ for φ which was constructed by Bowen in [**Bo1**]. Let $q \in \mathbb{T}^2$ be a φ-periodic point with minimal period Q and let $k \in \mathbb{N}$. Let $\mathcal{P}(k, q)$ be the family of open sets in $\mathbb{T}^2$ which contains each $q \operatorname{int}(P_i)$ k times, $i = 1, 2, \ldots, N$. Since we may assume that the diameter of each P_i is less than an expansive constant for φ, with respect to a translation invariant metric on $\mathbb{T}^2$, we have that $\mathcal{P}(k, q)$ meets condition (C′) above. By direct calculations, using the special properties of the boundary of the Markov partition, one can show that the corresponding factor map $\pi(k, q) : (\Sigma_{k,q}, \sigma) \to (\mathbb{T}^2, \varphi)$ has defect

$$D(\pi(k, q)) = \frac{2k}{Q} \log 2,$$

where Q is the minimal period of q. Choose a sequence q_n of φ-periodic points, together with natural numbers k_n such that

$$\frac{k_n}{Q_n} \leq \frac{k_{n+1}}{Q_{n+1}} \quad \text{and} \quad \lim_{n\to\infty} \frac{2k_n}{Q_n} \log\ 2 = t.$$

This can be done such that the orbits of the q_n's are mutually disjoint. Then the fibered product of $\pi_{k_i,q_i} : (\Sigma_{k_i,q_i}, \sigma) \to (\mathbb{T}^2, \varphi),\ i = 1, 2, \ldots, m$, gives a factor map with defect

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$(2k_m/Q_m)\log 2$. It then follows from Theorem 1.9 that the infinite fibered product of the whole sequence defines an extension of $(\mathbb{T}^2, \varphi)$ whose factor map has defect t. The defect value zero can be obtained from a subshift extension (not of finite type) by Theorem 7.1 of [**Th**].

5. *The defect of factor maps onto uniquely ergodic dynamical systems*

The purpose of this section is to calculate the defect of a factor map $\pi : (Y, \psi) \to (X, \varphi)$ in the case where Y is metrizable and φ is uniquely ergodic. For this purpose we introduce for each finite Borel cover $\mathcal{V} = \{V_i : i \in I\}$ of X the Borel function

$$A_{\mathcal{V}}(x) = \#\{i \in I : x \in V_i\}, \quad x \in X.$$

Since Y is metrizable we can apply the Lebesgue covering lemma (cf., for example, [**W**, p. 18]) to find a sequence $\{\mathcal{P}_n\}$ of partitions of Y such that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ and such that every partition of Y is refined by $\mathcal{P}_N$ for some sufficiently large N. Then

$$\#\pi^{-1}(x) = \lim_{n\to\infty} A_{\pi(\mathcal{P}_n)}(x), \quad x \in X,$$

and the convergence is monotone. In particular, $x \to \#\pi^{-1}(x)$ is a Borel function.

PROPOSITION 5.1. *Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map with Y totally disconnected and metrizable. Then*

$$D(\pi) \geq \sup_{\mu} \int \log \#\pi^{-1}(x)\, d\mu(x),$$

where we take the supremum over all φ-invariant Borel probability measures μ on X.

Proof. Since

$$D(\pi) = \lim_{n\to\infty} Q(\varphi, \pi(\mathcal{P}_n))$$

and

$$\int \log \#\pi^{-1}(x)\, d\mu(x) = \lim_{n\to\infty} \int \log A_{\pi(\mathcal{P}_n)}\, d\mu,$$

it suffices to fix n and show that $Q(\varphi, \pi(\mathcal{P}_n)) \geq \int \log A_{\pi(\mathcal{P}_n)}\, d\mu$ for every φ-invariant Borel probability measure μ. To this end it suffices to show that

$$Q(\varphi, \mathcal{F}) \geq \int \log A_{\mathcal{F}}\, d\mu$$

whenever $\mathcal{F} = \{F_i : i \in I\}$ is a finite closed cover of X and μ is a φ-invariant Borel probability measure. Let $\epsilon > 0$. By a compactness argument used in the proof of Theorem 1.4, there is an open cover $\mathcal{U} = \{U_i : i \in I\}$ such that $F_i \subset U_i$ for all $i \in I$ and $Q(\varphi, \mathcal{U}) < Q(\varphi, \mathcal{F}) + \epsilon$. Hence $(1/n)\log q_n(\varphi, \mathcal{U}) < Q(\varphi, \mathcal{F}) + \epsilon$ for all sufficiently large n. For each $k \in \mathbb{N}$, set $L_k = \bigcup_J \bigcap_{j\in J} U_j$ where we take the union over all subsets $J \subset I$ of I of cardinality $\leq k$. Take continuous functions $g_k : X \to [0, 1]$ with support in L_k such that $\sum_{k=1}^{\#I} g_k(x) > 0$, $x \in X$. Now consider an ergodic φ-invariant Borel probability measure ν. Since

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\left(\sum_{k=1}^{\#I} g_k(\varphi^i(x))\right) = \int \log\left(\sum_{k=1}^{\#I} g_k\right) d\nu \quad \text{for } \nu\text{-almost all } x,$$

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there is an element $z \in X$ such that

$$\frac{1}{n}\sum_{i=0}^{n-1}\log\left(\sum_{k=1}^{\#I} g_k(\varphi^i(z))\right) \geq \int \log\left(\sum_{k=1}^{\#I} g_k\right) d\nu - \epsilon$$

for all sufficiently large n. Hence

$$\begin{aligned}\frac{1}{n}\log q_n(\varphi,\mathcal{U}) &\geq \frac{1}{n}\log q_n(z,\mathcal{U}) = \frac{1}{n}\log\left(\prod_{i=0}^{n-1}\#\{i \in I : \varphi^i(z) \in U_i\}\right)\\ &\geq \frac{1}{n}\sum_{i=0}^{n-1}\log\left(\sum_{k=1}^{\#I} g_k(\varphi^i(z))\right) \geq \int \log\left(\sum_{k=1}^{\#I} g_k\right) d\nu - \epsilon\end{aligned}$$

for all sufficiently large n. That is, we have that

$$Q(\varphi,\mathcal{F}) + \epsilon \geq \int \log\left(\sum_{k=1}^{\#I} g_k\right) d\nu - \epsilon.$$

Since this conclusion holds for all φ-ergodic ν and since an arbitrary φ-invariant Borel probability measure μ can be approximated by convex combinations of φ-ergodic Borel probability measures (in the weak*-topology of $C(X)^*$), it follows that

$$Q(\varphi,\mathcal{F}) + 2\epsilon \geq \int \log\left(\sum_{k=1}^{\#I} g_k\right) d\mu.$$

Now let $\{g_k^n\}$ be an increasing sequence of continuous functions such that $g_k^n \geq g_k$ and $g_k^n \uparrow 1_{L_k}$ for all k. Since $\sum_{k=1}^{\#I} 1_{L_k} = A_{\mathcal{U}}$, we conclude that

$$Q(\varphi,\mathcal{F}) + 2\epsilon \geq \lim_{n\to\infty}\int \log\left(\sum_{k=1}^{\#I} g_k^n\right) d\mu = \int \log A_{\mathcal{U}}\, d\mu.$$

Since $A_{\mathcal{U}} \geq A_{\mathcal{F}}$ and $\epsilon > 0$ was arbitrary, the proof is complete. □

It is clearly an interesting question as to whether we always have equality in Proposition 5.1. All calculations and results in this paper are consistent with a positive answer, but I have not been able to prove it, except in the case where φ is uniquely ergodic. If it turns out to be true in general it will present a striking resemblance with the variational principle for the topological entropy.

THEOREM 5.2. *Let $\pi : (Y,\psi) \to (X,\varphi)$ be a factor map where Y is totally disconnected and metrizable. Assume that (X,φ) is uniquely ergodic, and let μ be the unique φ-invariant Borel probability measure on X. Then*

$$D(\pi) = \int \log \#\pi^{-1}(x)\, d\mu(x).$$

Proof. By Proposition 5.1 it suffices to show that $D(\pi) \leq \int \log \#\pi^{-1}(x)\, d\mu(x)$. By the same reasoning as in the proof of Proposition 5.1 it suffices to show that

$$Q(\varphi,\mathcal{F}) \leq \int \log A_{\mathcal{F}}\, d\mu$$

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whenever $\mathcal{F} = \{F_i : i \in I\}$ is a finite closed cover of X. It is well known that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\varphi^i(x)) = \int f \, d\mu$$

uniformly in $x \in X$ for all $f \in C(X)$ because φ is uniquely ergodic. It is this property which allows us to painlessly perform the approximations of the previous proof from above instead of from below. Let $\epsilon > 0$. For each $k \in \mathbb{N}$, set $H_k = \bigcup_J \bigcap_{j\in J} F_j$ where we take the union over all subsets $J \subset I$ of I of cardinality $\leq k$, and let $f_k : X \to [0,1]$ be a continuous function such that $f_k(x) = 1,\ x \in H_k$. Then there is a $N \in \mathbb{N}$ so large so that

$$\frac{1}{n} \sum_{i=0}^{n-1} \log\left(\sum_{k=1}^{\#I} f_k(\varphi^i(x)) \right) \leq \int \log\left(\sum_{k=1}^{\#I} f_k \right) d\mu + \epsilon$$

for all $x \in X,\ n \geq N$. Since $q_m(x, \mathcal{F}) \leq \prod_{i=0}^{m-1} \sum_{k=1}^{\#I} f_k(\varphi^i(x))$ for all m, x, we see that

$$q_m(\varphi, \mathcal{F}) \leq \exp\left(m \int \log\left(\sum_{k=1}^{\#I} f_k \right) d\mu + \epsilon m \right)$$

for all $m \geq N$. Thus $Q(\varphi, \mathcal{F}) \leq \int \log(\sum_{k=1}^{\#I} f_k)\, d\mu + \epsilon$, or $Q(\varphi, \mathcal{F}) \leq \int \log(\sum_{k=1}^{\#I} f_k)\, d\mu$ since $\epsilon > 0$ was arbitrary. Because each H_k is closed we can find a decreasing sequence $\{f_k^n\}$ of continuous functions $f_k^n : X \to [0,1]$ with limit 1_{H_k}. Then $\lim_{n\to\infty} \sum_{k=1}^{\#I} f_k^n = A_{\mathcal{F}}$ and we see that

$$Q(\varphi, \mathcal{F}) \leq \lim_{n\to\infty} \int \log\left(\sum_{k=1}^{\#I} f_k^n \right) d\mu = \int \log\ A_{\mathcal{F}}\, d\mu. \qquad \square$$

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REFERENCES

[AW] R. Adler and B. Weiss. Similarity of automorphisms of the torus. *Mem. Amer. Math. Soc.* **219** (1970).

[AKS] J. Ashley, B. Kitchens and M. Stafford. Boundaries of Markov partitions. *Trans. Amer. Math. Soc.* **333** (1992), 177–201.

[Bo1] R. Bowen. *On Axiom A diffeomorphisms (CBMS Regional Conf. Ser. in Math. no. 35)*. Amer. Math. Soc., Providence, R.I., 1978.

[Bo2] R. Bowen. Markov partitions are not smooth. *Proc. Amer. Math. Soc.* **71** (1978), 130–132.

[BKM] M. Boyle, B. Kitchens and B. Marcus. A note on minimal covers for sofic systems. *Proc. Amer. Math. Soc.* **95** (1985), 44–51.

[G] L. W. Goodwyn. The product theorem for topological entropy. *Trans. Amer. Math. Soc.* **158** (1971). 445–452.

[LM] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, Cambridge, 1995.

[Th] K. Thomsen. Covering dimension and topological entropy in dynamical systems. *Preprint*, Aarhus, 1994.

[W] P. Walters. *An Introduction to Ergodic Theory*. Springer, New York, Heidelberg, Berlin, 1982.

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