RISK-BASED CAPITAL FOR VARIABLE ANNUITY UNDER STOCHASTIC INTEREST RATE*

BY

JINDONG WANG AND WEI XU

ABSTRACT

Interest rate is one of the main risks for the liability of the variable annuity (VA) due to its long maturity. However, most existing studies on the risk measures of the VA assume a constant interest rate. In this paper, we propose an efficient two-dimensional willow tree method to compute the liability distribution of the VA with the joint dynamics of the mutual fund and interest rate. The risk measures can then be computed by the backward induction on the tree structure. We also analyze the sensitivity and impact on the risk measures with regard to the market model parameters, contract attributes, and monetary policy changes. It illustrates that the liability of the VA is determined by the long-term interest rate whose increment leads to a decrease in the liability. The positive correlation between the interest rate and mutual fund generates a fat-tailed liability distribution. Moreover, the monetary policy change has a bigger impact on the long-term VAs than the short-term contracts.

KEYWORDS

Variable annuity, value at risk, conditional tail expectation, GMWB, GMDB, GMMB, two-factor willow tree method, stochastic interest rate, Hull–White model.

1. INTRODUCTION

Variable annuity (VA) is an insurance contract allowing the policyholder to share financial returns from equity investment, but it also provides some guaranteed minimal benefits to the policyholder for the protection from the

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downside risk. The most popular guarantees embedded in the VA contract are the guaranteed minimum withdrawal benefit (GMWB), guaranteed minimum death benefit (GMDB), and guaranteed minimum maturity benefit (GMMB), which will be introduced in details in Section 2. Since the invention of these guarantees in the late 1990s, VAs are traded popularly all over the world. The total VA assets in US amount to 2.158 trillion dollars at the end of 2018. (Data Source: Insurance Information Institute.)

The VA also generates a lot of attention in the academic community as well since its birth in 1990s. However, most research is carried out on the policyholder's side. That is, how to evaluate a VA contract, or how to determine the premium charge so that the fair value of a VA contract equals to the initial investment from the policyholder. The first GMDB model was proposed by Milevsky and Posner (2001). Later, this model was extended to roll-up and ratchet options in Ulm (2008) and Ulm (2010). Among these guarantees, the GMWB is the most complicated which presumably receives the most attention from researchers. Milevsky and Salisbury (2006) first proposed a model to evaluate the GMWB from the policyholder's point of view. Dai *et al.* (2008) consider the optimal withdrawal strategy in the GMWB pricing and derive a Hamilton–Jacobi–Bellman (HJB) equation resulting in a singular stochastic control problem. Huang and Forsyth (2012) present the rigorous convergence proof of the penalty approximation schemes for solving the stochastic control models. A recent work by Peng *et al.* (2012) studies the pricing of GMWBs under stochastic interest rates. Other revisions of the singular stochastic control models under the geometric Brownian motion can be found in Bauer *et al.* (2008), Huang *et al.* (2012), Yang and Dai (2013), Feng and Volmer (2014). Recent works show the extension to stochastic models, other than GBM, such as jump diffusion, CEV, and Lévy processes, in the design of numerical schemes and implementation of hedging strategies (Luo and Shevchenko, 2015; Bacinello *et al.*, 2016; Shevchenko and Luo, 2017; Costabile, 2017; Kang and Ziveyi, 2018; Dong *et al.*, 2019). However, the literatures on the study of the VA from the insurer's perspective are very limited. The study on the insurer's side focuses on the liability of the VA contracts, rather than its market fair value. Feng and Volmer (2014) build up a framework to evaluate the liability of the GMWB and determine the premium charge from the insurer's side. Kolkiewicz and Liu (2012), Kelani and Quittard-Pinon (2017), Mackay (2014) study the hedging strategy for the VA. Due to a large number of existing VA contracts, the liability is crucial for the insurer to manage the risk and calculate the regulatory capitals.

The calculation of risk measures for the liability is challenging because the whole distribution of the liability has to be determined, instead of the expectation in pricing. Thus, the nested simulations are the common approach in the insurance industry, even though it is notorious for its time demanding (Bauer *et al.*, 2008). Thus, some efficient numerical methods are proposed to evaluate the liability for each individual guaranteed benefit under the Black–Scholes framework. Feng and Huang (2012) derive an analytic expression

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of the risk measures for the insurer's gross liability and net liability for the GMMB or GMDB riders. Feng and Volmer (2014) also propose two spectral approaches, called the spectral expansion approach and Green's function approach, to accelerate the calculation of the risk measures of the GMDB or GMMB. Recently, Cui *et al.* (2019) propose a much faster method for the risk measures computation based on the Hermite series expansion for the GMMB or GMDB. As for the GMWB, Feng and Volmer (2014) and Feng and Vecer (2017) propose the partial differential equation (PDE) method to estimate the risk measures for the net liability of GMWB and compare the performance with Monte Carlo simulations without considering the mortality risk of the policyholder. Dong *et al.* (to appear) consider all these guarantees, GMWB, GMDB, and GMMB, embedded in one VA contract and studied the net liability of the VA. They also propose an efficient method to compute the value at risk (VaR) and conditional tail expectation (CTE) for the risk-based capital (RBC). Due to the long maturity of the VA, usually from 10 to 30 years, all these existing researches show that the interest rate has a big influence on the liability, but none of them takes the interest rate risk into consideration in their models. Only a constant interest rate is assumed. Thus, in this paper, we consider the interest rate as another risk factor in the VA besides the market risk and propose an efficient method to compute the two common risk measures, VaR and CTE, and the liability distribution for the insurance companies.

To our best knowledge, this paper is the first article studied on the RBC of VAs with the consideration of both the stochastic interest rate and the dynamics of the mutual fund. We propose a two-dimensional willow tree structure to describe the correlated fund investment account values and the interest rate so that the net liability distribution of the VA can be evaluated efficiently. We adopt the Hull–White model (Hull and White, 1990) to describe the stochastic short rate because it can easily match the market interest rate term structure, even though our method can also be extended to other short rate models, such as Vasicek (Vasicek, 1977), CIR (Cox *et al.*, 1985) and Black-Karasinski (Black and Karasinski, 1991) models. Then, the Gaussian copula is employed to approximate the conditional joint distribution of the interest rate and account value at time t given the interest rate and investment account value at $t - \Delta t$. Later, we compare our method with the benchmark, Monte Carlo method, on the efficiency and accuracy. The risk measures of VAs are thoroughly studied on two types of mutual funds, α-type and β-type, although our method is not limited to these two types. The α-type fund provides a fixed expected return no matter how the risk-free interest rate varies in the future. The β-type returns a fixed risk premium besides the risk-free interest rate. Then, we study the sensitivity of the VaR and CTE on parameters of interest rate and mutual fund and the VA contract attributes. We also take the US Federal Reserve benchmark rate hikes in the past two years as scenarios to study the impacts on the risk measures of VAs when the monetary policy changes. The yields term structure of US treasure bonds in three days after the benchmark rate hike announcement are used to depict the scenarios. Finally, we decompose

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the liability cumulative density function (CDF) curve for each benefit of the VAs to study the impact of mortality risk on the overall risk measures. Based on these numerical experiments, we have several interesting observations. First, the positive correlation between the interest rate and mutual fund increases the fatness of the tail of the liability, but the impact decreases for the long maturity VAs. Second, the increase of benefit charges cannot catch up the increase of the liability, but it has a side effect of losing customers. Third, the monetary policy change has a bigger impact on the long-term contracts than on the short-term ones. Lastly, the mortality risk plays a key role in the risk measures, but it performs differently on VAs owned by different-aged policyholders.

The rest of the paper is organised as follows. We first define the gross and net liability of the GMDB, GMMB, and GMWB in a VA contract in Section 2. The net liability of a VA is aggregated with all liabilities of these three guarantees. After that, in Section 3, we give a brief introduction of Hull–White model and the equity-linked mechanism of VA with GMWB. Then, a two-dimensional willow tree is formed to describe the correlated interest rate and investment account value. In Section 4, the computing scheme is proposed to evaluate the net liability of the VA under the two-dimensional willow tree framework. Numerical experiments are carried out in Section 5 to show the accuracy and efficiency of our proposed method. We also analyze the sensitivity of VaRs and CTEs in terms of the stochastic model parameters, contract attributes, and monetary policy changes. Finally, the paper is concluded with some remarks in Section 6.

2. Insurer's liability

Once the insurer writes a VA contract with guaranteed benefits, they have the liability of the future payments to the policyholder. In this section, we formulate the gross liability and net liability for each guaranteed benefit, such as the GMDB, GMMB, and GMWB, individually first. Then, we aggregate the net liability with all three benefits to evaluate the net liability for the VA contract from the insurer's perspective. In fact, we propose a unified algorithm to evaluate VaR and CTE for VAs with various guarantees, but the VA contract is not necessary to have all these guarantees. The proposed method can still be used for VAs with one or two guarantees introduced in this paper to make the contract profitable.

2.1. GMDB

The liability of GMDB is generally dependent on the survivorship of the policyholder. We adopt the actuarial notations in Feng and Huang (2012) for modeling the mortality risk.

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- τ_x, the future lifetime of a policyholder at age x.
- ${}_tp_x$, the probability that an x-year-old policyholder will be alive t years.
- ${}_{t_2}q_{x+t_1}$, the probability that an $(x+t_1)$-year-old policyholder will die in $(t_1, t_2]$.

The GMDB ensures that the designated beneficiary receives the greater of a guaranteed base and the investment account value $F(\tau_x)$ at the time of the policyholder's death. Since the investigation and administration work delays the benefit payment, just similar in Feng and Huang (2012), we assume that the benefit is paid at the coming policy anniversary after the policyholder's death. The guaranteed base for the GMDB is often in a roll-up type, that is, the guaranteed base increases in a rate δ annually up to the maturity. Usually, the roll-up rate δ is less than the risk-free interest rate r, that is, $0 \le \delta \le r$, which implies that insurer's own return on bonds investment. Therefore, the gross liability of the GMDB, the present value of the payment for a GMDB at τ_x, is in the form of

$$D(\tau_x)(e^{\delta\tau_x}G_d(\tau_x) - F(\tau_x))^+\mathbb{I}(\tau_x \le T),$$

where $G_d(\tau_x)$ is the benefit base for the GMDB at τ_x, $(e^{\delta\tau_x}G_d(\tau_x) - F(\tau_x))^+ = \max\{e^{\delta\tau_x}G_d(\tau_x) - F(\tau_x), 0\}$, $D(\tau_x)$ is the discount factor at τ_x to the inception time t_0, and $\mathbb{I}(\cdot)$ is the indicator function. The accumulated charge generated by the GMDB rider till τ_x is $\epsilon_d \int_0^{\tau_x\wedge T} D(s)F(s)ds$ where ϵ_d is the fee charged for the GMDB rider and $\tau_x \wedge T = \min\{\tau_x, T\}$. Hence, the net liability of the GMDB is

$$L_0^D = D(\tau_x)(e^{\delta\tau_x}G_d(\tau_x) - F(\tau_x))^+\mathbb{I}(\tau_x \le T) - \epsilon_d \int_0^{\tau_x\wedge T} D(s)F(s)ds.$$

2.2. GMMB

The GMMB rider guarantees that the policyholder receives the greater of a guaranteed base, $G_e(T)$, and the account value $F(T)$ if the policyholder is alive at the maturity T. The insurer has to offset the amount to the guaranteed base if the account value is not enough at maturity. Thus, the gross liability of the GMMB is

$$D(T)(G_e(T) - F(T))^+\mathbb{I}(\tau_x > T).$$

Then, the net liability can be determined as

$$L_0^M = D(T)(G_e(T) - F(T))^+\mathbb{I}(\tau_x > T) - \epsilon_e \int_0^{\tau_x\wedge T} D(s)F(s)ds,$$

where ϵ_e is the charge rate for the GMMB rider.

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2.3. GMWB

The GMWB ensures that the policyholder can withdraw certain amount from the account annually even if the account is depleted. There are some new versions of GMWBs, such as step-ups, bonuses prior to withdrawal, or optimal step-ups, to reflect the investment growth after withdrawals. Details of GMWBs and the market shares can be referred to Drinkwater *et al.* (2013, 2014, 2015). According to SOA and LIMRA's 2015 experience study (Drinkwater *et al.*, 2015), the majority of policyholders withdraw the maximum amount allowed in their contracts without penalty annually. Thus, in this paper, we assume that the policyholder only withdraws at the maximum amount, that is, the policyholder only withdraws w per unit time from the account until the guaranteed base G_w is exhausted. This assumption is also mainly taken in practice in practitioner's simulations (Feng and Vecer, 2017).

There is no liability to the insurer until the time when the account value hits zero, that is,

$$\tau_0 = \inf\{t : \ F(t) \leq 0\},$$

where the account value $F(t)$ is defined in a continuous withdrawal form as in (3.3), which represents the net account value after the premium and withdrawal. In other words, if the account is depleted prior to the maturity T, the withdrawal w must be paid by the insurer's own capital. Therefore, the gross liability of the GMWB is

$$\int_{\tau_0 \wedge T}^{T} D(s) w ds.$$

Since the insurer charges at the rate of, ϵ_w, only when the account value is positive, the net liability of the GMWB is

$$L_0^W = \int_{\tau_0 \wedge T}^{T} D(s) w ds - \epsilon_w \int_0^{\tau_0 \wedge T} D(s) F(s) ds,$$

where $\epsilon_w \int_0^{\tau_0 \wedge T} D(s) F(s) ds$ represents the present value of the GMWB charges until the account value reaches zero.

2.4. Net liability of VA

In the previous subsections, the net liability for each guarantee benefit is determined individually. Now, we aggregate them to estimate the net liability of the VA. There are two hitting times in the VA contract. One is the death time of the policyholder, τ_x. The other is the time when the account value hits zero, τ_0. When $\tau_x \leq \tau_0 \leq T$, that is, the policyholder dies before the account value

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hits zero, the VA contract is terminated at τ_x, so there is no obligation for the GMMB and GMWB. The net liability of the VA is, then,

$$L_0 = D(\tau_x)(e^{\delta\tau_x} G_d(\tau_x) - F(\tau_x))^+ - \epsilon_x \int_0^{\tau_x} D(s)F(s)ds,$$

where ϵ_x is the total charges for the GMDB, GMMB, and GMWB, that is, $\epsilon_x = \epsilon_d + \epsilon_e + \epsilon_w$. When $\tau_0 < \tau_x \leq T$, which means the account value hits zero before the policyholder dies prior to the maturity. Then, the net liability of the VA is

$$L_0 = D(\tau_x)e^{\delta\tau_x} G_d(\tau_x) + \int_{\tau_0}^{\tau_x} D(s)wds - \epsilon_x \int_0^{\tau_0} D(s)F(s)ds.$$

When the policyholder is alive at the maturity T, that is, $\tau_x > T$, the combined net liability can be written as

$$L_0 = D(T)(G_e(T) - F(T))^+ + \int_{\tau_0 \wedge T}^{T} D(s)wds - \epsilon_x \int_0^{\tau_0 \wedge T} D(s)F(s)ds.$$

Therefore, the net liability of a VA can be written into a general mathematical expression to cover all above three circumstances, that is,

$$\begin{aligned} L_0 = {} & D(\tau_x)(e^{\delta\tau_x} G_d(\tau_x) - F(\tau_x))^+ \mathbb{I}(\tau_x \leq T) + D(T)(G_e(T) - F(T))^+ \mathbb{I}(\tau_x > T) \\ & + \mathbb{I}(\tau_0 < \tau_x) \int_{\tau_0 \wedge T}^{\tau_x \wedge T} D(s)wds - \epsilon_x \int_0^{(\tau_0 \wedge \tau_x) \wedge T} D(s)F(s)ds. \end{aligned} \tag{2.1}$$

The benefit base for each guarantee, such as $G_d(t)$, $G_e(t)$, and $G_w(t)$, can be defined as a function of time t according to the guarantees embedded in the VA contract. For example, $G_w(t)$ usually equals to the initial investment F_0; $G_e(t)$ can be defined as a function, $G_e^0 - wt$, and $G_d(t)$ can be defined as $(F_0 - wt)e^{\delta t}$, where G_e^0 is the initial value of the GMMB, w is the continuous withdrawal ratio, and δ is the roll-up rate. These bases can be defined in other formats as long as they are deterministic functions of time t.

3. TWO-DIMENSIONAL WILLOW TREE FRAMEWORK

There are two key risk factors to determine the liability of the VA contract, interest rate, and value of the mutual fund. In Subsection 3.1, we give a brief introduction on the nonarbitrage short rate model adopted in this paper, Hull–White model, and the equity-linked mechanism of the investment account value. The liability of a VA can be evaluated on these two factors. Although only the Hull–White model is considered here, our method is also applicable to other popular short rate models, such as Vasicek (Vasicek, 1977), CIR (Cox *et al.*, 1985), and Black-Karasinski (Black and Karasinski, 1991) models. Then,

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TABLE 1

TERM STRUCTURE OF INTEREST RATE AND THE DISCOUNT FACTOR.

Maturity (year)	t_1	t_2	$\cdots$	t_N
Yield (%)	y_1	y_2	$\cdots$	y_N
Discount factor	$D(t_1)$	$D(t_2)$	$\cdots$	$D(t_N)$

we form a one-dimensional willow tree to depict the Hull–White model in Subsection 3.2. Finally, a two-dimensional willow tree is constructed based on the interest rate tree structure for the correlated dynamics of stochastic interest rate and equity-linked investment account value in Subsection 3.3 through the Gaussian copula. Compared to the existing tree methods, such as the binomial tree, the willow tree can be recombining for various stochastic processes, such as GBM, CIR process, and Lévy processes, with a constant number of tree nodes at each discrete time. In other words, the willow tree requires less tree nodes than the existing tree methods. Moreover, due to the multinodes at each time, the conditional expectation, which is the key for the risk measures estimation as shown in Section 4, can be estimated more accurately on the willow tree than on the other tree methods.

3.1. Dynamics of interest rate and investment account value

Given an interest rate structure as in Table 1, the Hull–White model Hull and White (1990) is an Ornstein–Uhlenbeck process

$$dr(t) = (\theta(t) - ar(t))dt + \sigma_1 dW(t), \tag{3.1}$$

where $\theta(t)$ is a function of t, and both a and σ_1 are constant. Also, the Hull–White model (3.1) can be rewritten as

$$r(t) = r^*(t) + \phi(t), \text{ and } dr^*(t) = -ar^*(t)dt + \sigma_1 dW^1,$$

where $\phi(t)$ is a piecewise constant function on each interval $[t_{i-1}, t_i]$ to match the term structure in Table 1. Then, the discount factor at time t under the Hull–White model, $D(t)$, can be defined as the corresponding zero coupon bond value at the inception time.

Given the risk free interest rate in (3.1), the mutual fund policyholder invested is driven by a diffusion process as

$$dS_t = \mu(r, t)S_t\, dt + \sigma_2 S_t\, dW^2, \qquad t > 0, \tag{3.2}$$

where $\mu(r, t)$ is the expected return of the mutual fund, which is a function of the risk-free interest rate $r \equiv r(t)$ and t, σ_2 is a constant volatility, and W^2 is the Brownian motion correlated with W^1, that is, $\mathbb{E}[dW^1 dW^2] = \rho dt$. The insurer charges a fixed percentage of the investment account value as a mortality and expense (M&E) fee to cover the premium of the guarantees and its

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administration cost. If there is no withdrawal from the account, the account value, $F(t)$, at time t evolves proportionally to the mutual fund. That is,

$$F(t) = F_0 \frac{S_t}{S_0} e^{-\epsilon t}, \qquad t \geq 0,$$

or in the stochastic differential equation form as

$$dF(t) = (\mu(r, t) - \epsilon)F(t)dt + \sigma_2 F(t)dW^2,$$

where F_0 is the initial investment on the mutual fund when the contract is drawn up. Assuming that the policyholder only withdraws at the maximum allowed amount, that is, constant at each discrete withdrawal dates, then the account value can evolve as

$$dF(t) = (\mu(r, t) - \epsilon)F(t)dt - wdt + \sigma_2 F(t)dW^2, \tag{3.3}$$

where w is the withdrawal rate per time unit. In other words, the account value $F(t)$ in (3.3) is the net account value after the premium and withdrawal. The stochastic model of $F(t)$ is commonly used in many studies of VAs, such as Feng and Vecer (2017), Dai *et al*. (2008).

In summary, two stochastic processes are employed to describe the interest rate $r(t)$ and the account value $F(t)$ in the physical probability measure, that is,

$$\begin{cases} r(t) = r^*(t) + \phi(t), \quad dr^*(t) = -ar^*(t)dt + \sigma_1 dW^1 \\ dF(t) = (\mu(r, t) - \epsilon)F(t)dt - wdt + \sigma_2 F(t)dW^2, \end{cases} \tag{3.4}$$

with the correlation $\mathbb{E}(dW^1 dW^2) = \rho dt$. The rest of this section is to illustrate how to build up a two-dimensional willow tree structure to depict the two correlated stochastic processes (3.4).

3.2. One-dimensional willow tree for interest rate

Before forming a two-dimensional willow tree for the dynamics in (3.4), we build up a one-dimensional willow tree for the Hull–White short rate model. Then, the two-dimensional tree can be formed on it afterward.

The willow tree method was first proposed by Curran (2001), then improved by Xu *et al*. (2013). A graphic illustration of the tree structure for the short rate is displayed in Figure 1. The nodes represent the possible interest rate $r(t)$ at each time period, whereas the edges represent the transition probability from the short rate at t_n to the one possible rate at t_{n+1}. The main advantages of the tree structure are efficiency and flexibility. First, the computational complexity of the willow tree is linear in the number of time steps, in contrast to the quadratic complexity of other lattice methods (Cox *et al*., 1979; Yang and Dai, 2013). Second, the willow tree can be formed for a wide range of one-factor stochastic models, besides the geometric Brownian motion, such as jump diffusion and Lévy processes (Yao and Xu, 2020). Therefore, the derivative

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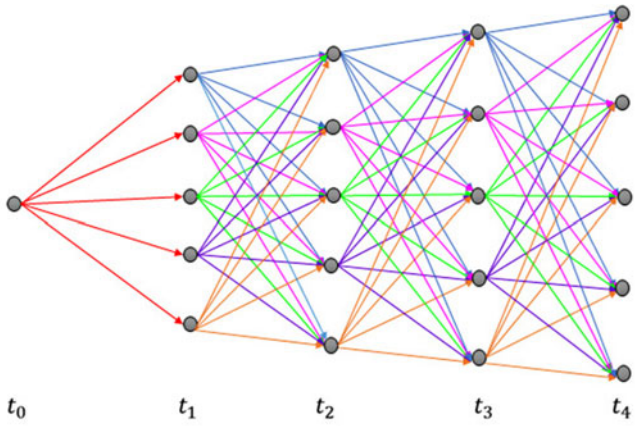


FIGURE 1: Graphical depiction of the willow tree lattice with five possible interest rates and four discrete time periods.

pricing on the willow tree is independent of models, that is, the pricing scheme developed for a specific stochastic model under a willow tree is also applicable to other stochastic processes as long as the corresponding willow tree can be formed. The pricing and tree construction are so well separated that the cost of implementation and maintenance can be minimized. Readers can refer to Xu *et al.* (2013), Xu and Yin (2014), Lu *et al.* (2017), Yao and Xu (2020), Ma *et al.* (2020) for details about willow tree construction, convergence analysis, and applications to option pricing under various stochastic models.

Given the Hull–White short rate model as in (3.4), it is easy to demonstrate that $r^*(t)$ follows a normal distribution with mean 0 and variance $\frac{\sigma^2}{2a}(1-e^{-2at})$, that is,

$$r^*(t) \sim \mathcal{N}\left(0, \frac{\sigma^2}{2a}\left(1-e^{-2at}\right)\right).$$

Divide the interval $[0,\ T]$ identically into N subintervals with $t_n = n \cdot \Delta t$, then the distribution of $r^*(t)$ at time t_n can be discretely approximated on the willow tree nodes in the form of

$$(r^*)_i^n = \sigma\sqrt{\frac{1-e^{-2at_n}}{2a}}z_i, \quad i = 1, 2, \cdots, m,$$

where z_i are the sampled discrete values of the standard normal distribution as in Xu *et al.* (2013). These m discrete approximates $(r^*)_i^n$ form the tree nodes at each discrete time t_n. Then, the transition probability of $r^*(t)$ between t_n and t_{n+1} on the willow tree can be determined by the conditional probability of $r^*(t_{n+1})$ given $(r^*)_i^n$, that is,

$$q_{ij}^n = \mathrm{P}(B_j^{n+1} \le r^*(t_{n+1}) \le B_{j+1}^{n+1} \mid (r^*)_i^n), \qquad j = 1, 2, \cdots, m,$$

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where $B_j^{n+1} = ((r^*)_{j-1}^{n+1} + (r^*)_j^{n+1})/2$ with $(r^*)_0^{n+1} = -\infty$ and $(r^*)_{m+1}^{n+1} = +\infty$. Based on the results in Wang and Xu (2018), the conditional probability density function of $\mathrm{P}(B_j^{n+1} \le r^*(t_{n+1}) \le B_{j+1}^{n+1} \,|(r^*)_i^n)$ is the solution of the Fokker–Planck equation

$$\frac{\partial p}{\partial t} = \frac{\partial (ar^* p)}{\partial r^*} + \frac{1}{2}\frac{\partial^2 (\sigma^2 p)}{\partial^2 r^*},$$

which can be approximated in the path-integral representation as

$$q(r^*(t_{n+1})|(r^*)_i^n) = \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} \exp\left\{-\frac{(r^*(t_{n+1}) - (r^*)_i^n + a(r^*)_i^n \Delta t)^2}{2\sigma^2 \Delta t}\right\}. \tag{3.5}$$

Thus, the corresponding probability can be calculated as

$$q_{ij}^n = \mathrm{P}(B_j^{n+1} \le r^*(t_{n+1}) \le B_{j+1}^{n+1} |(r^*)_i^n) = \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} \int_{B_j^{n+1}}^{B_{j+1}^{n+1}} e^{-\frac{(x-(r^*)_i^n + a(r^*)_i^n \Delta t)^2}{2\sigma^2 \Delta t}} dx. \tag{3.6}$$

In summary, the transition probabilities between t_n and t_{n+1} form an $m \times m$ matrix, whose entry, q_{ij}^n, represents the transition probability from $(r^*)_i^n$ to $(r^*)_j^{n+1}$, whereas the transition probabilities between t_0 and t_1 form an $m \times 1$ vector with each entry, q_i, representing the transition probability from r_0^* to $(r^*)_i^1$, that is, the transition probabilities from r_0^* to $(r^*)_i^1$ can be estimated as

$$q_i = \mathrm{P}(B_j^1 \le r^*(t_1) \le B_{j+1}^1 | r_0^*) = \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} \int_{B_j^1}^{B_{j+1}^1} e^{-\frac{(x-r_0^* + ar_0^* \Delta t)^2}{2\sigma^2 \Delta t}} dx,$$

with $r_0^* = 0$.

Next, we need to determine the piecewise constant function $\phi(t)$ on the willow tree of $r^*(t)$ by the forward induction to match the interest rate term structure as shown in Table 1. Starting from the time period $(0, t_1]$, the price of the Arrow–Debreu security at t_1 on $(r^*)_i^1$, whose value is 1 at $(r^*)_i^1$ and 0 otherwise (Arrow, 1964), is defined as $A_i^1 = q_i e^{-((r^*)_i^1 + \phi_1)\Delta t}$, where q_i is the transition probability from r_0^* to $(r^*)_i^1$. Given the term structure, the value of ϕ_1 is the solution of the following equation:

$$\sum_{i=1}^m A_i^1 = \sum_{i=1}^m q_i e^{-((r^*)_i^1 + \phi_1)\Delta t} = e^{-y_1 t_1}. \tag{3.7}$$

Solving the above equation, we can uniquely determine ϕ_1 as

$$\phi_1 = \frac{\ln\left(\sum_{i=1}^m q_i e^{-(r^*)_i^1 \Delta t}\right) + y_1 t_1}{\Delta t}.$$

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The m tree nodes for the Hull–White model at t_1 are $r_i^1 = (r^*)_i^1 + \phi_1$, for $i = 1, 2, \cdots, m$. Then, the price of the Arrow–Debreu security at time t_2 can be defined as

$$A_i^2 = \sum_{j=1}^{m} q_{ji}^1 A_j^1 e^{-((r^*)_i^2 + \phi_2)\Delta t}$$

where q_{ji}^1 is the probability from $(r^*)_j^1$ at time t_1 to $(r^*)_i^2$ at time t_2. The value of ϕ_2 is the unique solution of the equation similar as the one in (3.7) based on the term structure. Thus, keeping rolling forward on the willow tree, we can obtain the piecewise constant function value for ϕ_n, for $n = 2, 3, \cdots, N$, to match the given term structure. The nodes for the Hull–White short rate model can be updated as $r_i^n = (r^*)_i^n + \phi_n$. Nonetheless, the transition probability between r_j^n and r_i^{n+1} is still the same as the one between $(r^*)_j^n$ and $(r^*)_i^{n+1}$. Based on the willow tree structure of the interest rate, the discount factor at t to the inception, $D(t)$, in Table 1 can be evaluated as the corresponding zero coupon bond value, which can be calculated by the backward induction on the tree. The details of the other short rate models under the willow tree can be referred to Wang and Xu (2018).

3.3. Two-dimensional willow tree

Based on the one-factor willow tree for the interest rate, $r(t)$, we are ready to form a two-dimensional tree to describe the dynamics of $(F(t), r(t))$ as in (3.4). Given the interest rate $r(t)$, the first four moments of the account value at t, $F(t)$, can be evaluated as in Theorem 3.1.

Theorem 3.1. *Assuming the account value $F(t)$ and interest rate $r(t)$ as the SDE in (3.4), the first four moments of $F(t)$ can be evaluated as follows, given the interest rate $r(t)$,*

$$\mathbb{E}[F(t)|r(t)] = \frac{w}{\mu(r,t) - \epsilon} + \left(F_0 - \frac{w}{\mu(r,t) - \epsilon}\right) e^{(\mu(r,t) - \epsilon)t},$$

$$\begin{aligned}\mathbb{E}[F^2(t)|r(t)] = {} & \frac{w}{\mu(r,t) - \epsilon}\left(\frac{\sigma^2}{2(\epsilon - \mu(r,t)} - \frac{w}{\mu(r,t) - \epsilon}\right) \\ & + \left(F_0 - \frac{w}{\mu(r,t) - \epsilon}\right)\left(\frac{\sigma^2}{\epsilon - \mu(r,t)} + 2\frac{w}{\mu(r,t) - \epsilon}\right) e^{(\mu(r,t) - \epsilon)t} \\ & + \left[\left(F_0 - \frac{w}{\mu(r,t) - \epsilon}\right)^2 \right. \\ & \left. + \frac{\sigma^2}{\epsilon - \mu(r,t)}\left(\frac{w}{2(\mu(r,t) - \epsilon)} - F_0\right)\right] e^{2(\mu(r,t) - \epsilon)t},\end{aligned}$$

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$$\mathbb{E}[F^3(t)|r(t)] = \frac{3w^2\sigma^2 - 2w^3 - w\sigma^4}{2(\epsilon - \mu(r,t))^3}$$
$$+ \left(F_0 - \frac{w}{\mu(r,t) - \epsilon}\right) \frac{-9w\sigma^2 + 6w^2 + 3\sigma^4}{2(\mu(r,t) - \epsilon)^2} e^{(\mu(r,t)-\epsilon)t}$$
$$+ \left[\left(F_0 - \frac{w}{\mu(r,t) - \epsilon}\right)^2 \frac{-3w + 3\sigma^2}{\epsilon - \mu(r,t)}\right.$$
$$\left. + \left(\frac{w}{2(\mu(r,t) - \epsilon)} - F_0\right) \frac{-3w\sigma^2 + 3\sigma^4}{(\mu(r,t) - \epsilon)^2}\right] e^{2(\mu(r,t)-\epsilon)t}$$
$$+ \left[F_0^3 + \frac{3w^2\sigma^2 - 4w^3 + w\sigma^4}{2(\epsilon - \mu(r,t))^3} - \left(F_0 - \frac{w}{\mu(r,t) - \epsilon}\right)^2 \frac{-3w + 3\sigma^2}{\epsilon - \mu(r,t)}\right.$$
$$\left. - F_0 \frac{-3w\sigma^2 + 6w^2 - 3\sigma^4}{2(\mu(r,t) - \epsilon)^2}\right] e^{3(\mu(r,t)-\epsilon)t},$$

$$\mathbb{E}[F^4(t)|r(t)] = \frac{-12w^3\sigma^2 + 4w^4 + 11w^2\sigma^4 - 3w\sigma^6}{4(\mu(r,t) - \epsilon)^4}$$
$$+ \left(F_0 - \frac{w}{\mu(r,t) - \epsilon}\right) \frac{12w^2\sigma^2 - 4w^3 - 11w\sigma^4 + 3\sigma^6}{(\epsilon - \mu(r,t))^3} e^{(\mu(r,t)-\epsilon)t}$$
$$+ \left[\left(F_0 - \frac{w}{\mu(r,t) - \epsilon}\right)^2 \frac{6w^2 - 15w\sigma^2 + 9\sigma^4}{(\mu(r,t) - \epsilon)^2}\right.$$
$$\left. + \left(\frac{w}{2(\mu(r,t) - \epsilon)} - F_0\right) \frac{6w^2\sigma^2 - 15w\sigma^4 + 9\sigma^6}{(\mu(r,t) - \epsilon)^3}\right] e^{2(\mu(r,t)-\epsilon)t}$$
$$+ \left[F_0^3 \frac{-4w + 6\sigma^2}{\epsilon - \mu(r,t)} + \frac{-18w^3\sigma^2 + 8w^4 + 7w^2\sigma^4 + 3w\sigma^6}{(\mu(r,t) - \epsilon)^4}\right.$$
$$- \left(F_0 - \frac{w}{\mu(r,t) - \epsilon}\right)^2 \frac{12w^2 - 30w\sigma^2 + 18\sigma^4}{(\mu(r,t) - \epsilon)^2}$$
$$\left. - F_0 \frac{24w^2\sigma^2 - 12w^3 - 3w\sigma^4 - 9\sigma^6}{(\epsilon - \mu(r,t))^3}\right] e^{3(\mu(r,t)-\epsilon)t}$$
$$+ \left[F_0^4 - \frac{48w^2\sigma^2(\epsilon - \mu(r,t)) - 20w^3(\epsilon - \mu(r,t)) + 25w^2\sigma^4 + 3w\sigma^6}{4(\mu(r,t) - \epsilon)^4}\right.$$

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$$+F_0\frac{18w^2\sigma^2-8w^3-7w\sigma^4-3\sigma^6}{(\epsilon-\mu(r,t))^3}$$

$$+\left(F_0-\frac{w}{\mu(r,t)-\epsilon}\right)^2\frac{6w^2-15w\sigma^2+9\sigma^4}{(\mu(r,t)-\epsilon)^2}$$

$$-F_0^3\frac{-4w+6\sigma^2}{\epsilon-\mu(r,t)}\Bigg]e^{4(\mu(r,t)-\epsilon)t}.$$

Proof. The derivation of the first four moments of $F(t)$ is same as the one presented in Wang and Xu (2018). □

Once the first four moments of $F(t)$ are evaluated by Theorem 3.1, the Johnson curve (Johnson, 1949) can be employed to map a standard normal distribution random variable into the random variable with the first four moments of $F(t)$.

At a discrete time t_n, given one possible interest rate $r_j^n, j=1,2,\cdots,m_r$ on the willow tree, m_F possible account values, $F_l^n, l=(j-1)m_F+1,\cdots,jm_F$, can be generated via the Johnson curve (Johnson, 1949) matching the first four moments of $F(t_n)$ given r_j^n, that is,

$$F_l^n=a_3+a_4g^{-1}\left(\frac{z_i-a_1}{a_2}\right),\quad l=(j-1)m_F+1,,\cdots,jm_F,\quad ,i=l-(j-1)m_F \tag{3.8}$$

where parameters a_1, a_2, a_3, a_4, and function $g^{-1}(\cdot)$ can be determined by the first four moments of $F(t_n)$ through an efficient algorithm (Hill and Holder, 1976), and z_i is a discrete value sampled from the standard normal distribution (Xu *et al.*, 2013).

Therefore, there are $M=m_F*m_r$ nodes on the two-dimensional willow tree at t_n. Each node on the tree is a pair (F_l^n, r_j^n) representing the account value and interest rate for $l=1,2,\cdots,M, j=\lceil l/m_F\rceil$, and $n=1,2,\cdots,N$ with the initial node (F_0,r_0) at t_0. Figure 2 shows a two-factor willow tree with nine possible nodes at each discrete time with $m_r=3$, and $m_F=3$. The next step to build up the tree is to determine the edge between two tree nodes, that is, the transition probability from (F_l^n, r_j^n) to (F_h^{n+1}, r_c^{n+1}).

Just as the one for the one-dimensional willow tree, the transition probability between (F_l^n, r_j^n) and (F_h^{n+1}, r_c^{n+1}) is the conditional joint probability of $F(t_{n+1})$ and $r(t_{n+1})$ given (F_l^n, r_j^n), that is,

$$p_{lh}^n=P(A_k^{n+1}\le F(t_{n+1})\le A_{k+1}^{n+1}, B_c^{n+1}\le r(t_{n+1})\le B_{c+1}^{n+1}|F_l^n, r_j^n),\quad l,h=1,2,\cdots,M \tag{3.9}$$

where $A_k^{n+1}=(F_{(c-1)m_F+k}^{n+1}+F_{(c-1)m_F+k-1}^{n+1})/2$, $A_{k+1}^{n+1}=(F_{(c-1)m_F+k}^{n+1}+F_{(c-1)m_F+k+1}^{n+1})/2$, $B_c^{n+1}=(r_{c-1}^{n+1}+r_c^{n+1})/2$, and $B_{c+1}^{n+1}=(r_c^{n+1}+r_{c+1}^{n+1})/2$, where $k=h-(c-1)m_F$. It

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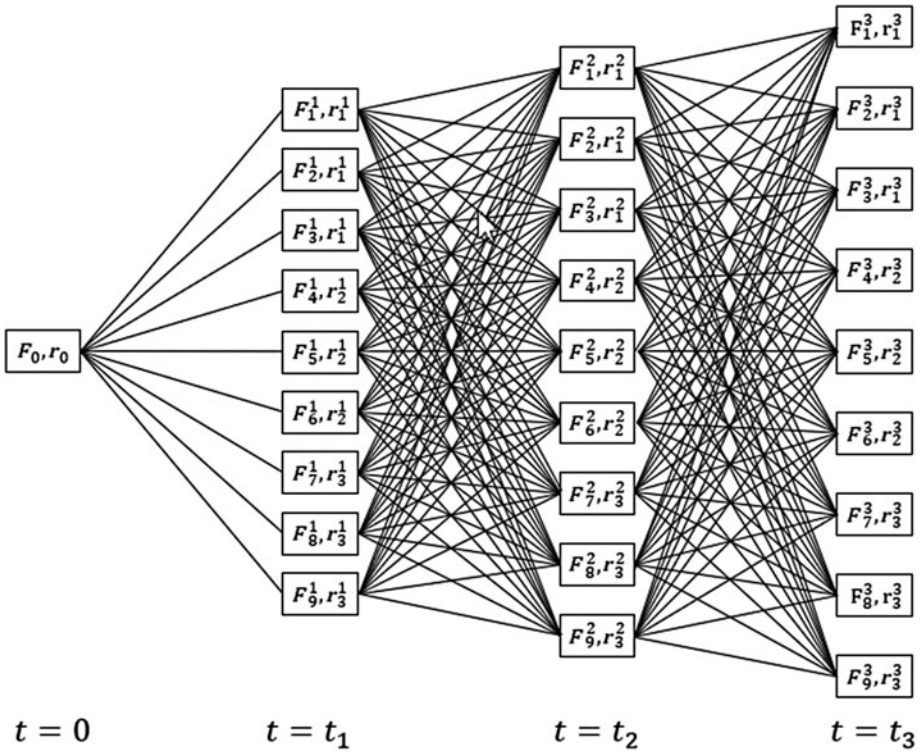


FIGURE 2: A two-dimensional willow tree with nine pairs of (F_l^n, r_j^n).

is easy to show that the marginal distributions of $F(t_{n+1})$ and $r(t_{n+1})$ given (F_l^n, r_j^n) can be approximated by a normal distribution, that is, $F(t_{n+1}) \approx F_l^n + (\mu(r_j^n, t_n) - \epsilon)F_l^n \Delta t - w\Delta t + \sigma_2 F_l^n \sqrt{\Delta t}\phi$ where $\phi \sim N(0, 1)$, so the Gaussian copula, which is widely used in the quantitative finance to model and minimize tail risk and portfolio-optimization applications (Low *et al.*, 2013), can be used to approximate the conditional joint distribution of $(F(t_{n+1}), r(t_{n+1}))$. The Gaussian copula is a distribution over the unit cube $[0, 1] \times [0, 1]$ constructed from a multivariate normal distribution over $\mathbb{R}^2$ by using the probability integral transform.

Theorem 3.2. *Given a* 2×2 *correlation matrix R for random variables X and Y, the Gaussian copula with parameter matrix, R, can be written as*

$$C_R^{Gauss}(x, y) = \Phi_R(\Phi^{-1}(x), \Phi^{-1}(y)),$$

where Φ^{-1} *is the inverse cumulative distribution function of a standard normal and* Φ_R *is the joint cumulative distribution function of a multivariate normal distribution with mean vector zero and covariance matrix equal to the correlation matrix R. The density function of joint random variables (X, Y) can be written as*

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$$c_R^{\text{Gauss}}(x,y)=\frac{1}{\sqrt{\det R}}\exp\left(-\frac{1}{2}\begin{pmatrix}\Phi^{-1}(x)\\ \Phi^{-1}(y)\end{pmatrix}^T\cdot\left(R^{-1}-I\right)\cdot\begin{pmatrix}\Phi^{-1}(x)\\ \Phi^{-1}(y)\end{pmatrix}\right) \quad (3.10)$$

where I is the identity matrix.

Thus, given (F_l^n, r_j^n), the two conditional random variables $F(t_{n+1})$ and $r(t_{n+1})$ are correlated with the matrix $\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}$. Then, applying the Gaussian copula theory in Theorem 3.2, the conditional probability of $(F(t_{n+1}), r(t_{n+1}))$ in (3.9) can be evaluated as, given (F_l^n, r_c^n),

$$\begin{aligned}p_{lh}^n &= P(A_k^{n+1}\le F(t_{n+1})\le A_{k+1}^{n+1}, B_c^{n+1}\le r(t_{n+1})\le B_{c+1}^{n+1}|F_l^n, r_j^n)\\ &=\frac{1}{2\pi\,\Delta t\sqrt{(1-\rho^2)\sigma_1^2(F_l^n)^2\sigma_2^2}}\int_{A_k^{n+1}}^{A_{k+1}^{n+1}}\int_{B_c^{n+1}}^{B_{c+1}^{n+1}}\\ &\exp\left[-\frac{1}{2\left(1-\rho^2\right)}\left(\frac{\left(x-F_l^n-(\mu(r_j^n,t_n)-\epsilon)F_l^n\Delta t+w\Delta t\right)^2}{\sigma_1^2(F_l^n)^2\Delta t}\right.\right.\\ &-\frac{2\rho\left(x-F_l^n-(\mu(r_j^n,t_n)-\epsilon)F_l^n\Delta t+w\Delta t\right)\left(y-(r^*)_j^n+a(r^*)_j^n\Delta t\right)}{\sqrt{\sigma_1^2(F_l^n)^2\Delta t\sigma_2^2\Delta t}}\\ &\left.\left.+\frac{\left(y-(r^*)_j^n+a(r^*)_j^n\Delta t\right)^2}{\sigma_2^2\Delta t}\right)\right]dxdy, \end{aligned} \quad (3.11)$$

which is the transition probability from (F_l^n, r_j^n) to (F_h^{n+1}, r_c^{n+1}). Likewise, the transition probability from (F_0, r_0) to (F_l^1, r_j^1) can be determined as

$$\begin{aligned}p_l^n &= P(A_k^{n+1}\le F(t_1)\le A_{k+1}^{n+1}, B_c^{n+1}\le r(t_1)\le B_{c+1}^{n+1}|F_0, r_0)\\ &=\frac{1}{2\pi\,\Delta t\sqrt{(1-\rho^2)\sigma_1^2(F_0)^2\sigma_2^2}}\int_{A_k^{n+1}}^{A_{k+1}^{n+1}}\int_{B_c^{n+1}}^{B_{c+1}^{n+1}}\\ &\exp\left[-\frac{1}{2\left(1-\rho^2\right)}\left(\frac{(x-F_0-(\mu(r_0,t_0)-\epsilon)F_0\Delta t+w\Delta t)^2}{\sigma_1^2(F_0)^2\Delta t}\right.\right.\\ &-\frac{2\rho\,(x-F_0-(\mu(r_0,t_0)-\epsilon)F_0\Delta t+w\Delta t)\left(y-r_0^*+ar_0^*\Delta t\right)}{\sqrt{\sigma_1^2(F_0)^2\Delta t\sigma_2^2\Delta t}}\\ &\left.\left.+\frac{\left(y-r_0^*+ar_0^*\Delta t\right)^2}{\sigma_2^2\Delta t}\right)\right]dxdy \end{aligned} \quad (3.12)$$

where $c=\lceil l/m_F\rceil$, $k=l-(c-1)m_F$ at time t_1.

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Now, the two-dimensional willow tree for $(F(t), r(t))$ is formed. If the account value $F(t)$ hits zeros, no further charges will be taken from the account. In other words, the account value cannot be negative. Thus, we have to adjust value of $F(t)$ on the tree as

$$F_l^n = \max\{F_l^n,\ 0\},$$

to guarantee the nonnegativity of the account. In the next section, we will introduce how to estimate the liability distribution of the VA under the two-dimensional willow tree framework. Then, the common risk measures VaR and CTE can be determined by the liability distribution of the VA.

4. VaR AND CTE FOR THE NET LIABILITY

The VaR and CTE are quantile risk measures, which are often used by firms and regulators in the financial industry to gauge the amount of capital needed to cover possible losses. The National Association of Insurance Commissioners (NAIC) requires variable annuity writers to use 70% CTE to determine the reserves and 90% CTE to determine the RBC in the US market (Feng and Vecer, 2017). The VaR for the liability is defined by $\alpha(0 \le \alpha \le 1)$ as

$$\mathrm{VaR}_\alpha = \inf\{y :\ \mathrm{P}(L_0 \le y) \ge \alpha\}.$$

Since the liability L_0 is a continuous random variable determined by $F(t)$ and $r(t)$, thus we can compute the quantile risk measure VaR_α by solving the following equation

$$\mathrm{P}(L_0 > \mathrm{VaR}_\alpha) = 1 - \alpha.$$

Different from VaR, CTE quantifies the expected value of the loss given that an event outside a specified probability level has occurred. Thus, it can be defined for $\alpha(0 \le \alpha \le 1)$ as

$$\mathrm{CTE}_\alpha = \mathbb{E}[L_0 | L_0 > \mathrm{VaR}_\alpha],$$

where $\mathbb{E}[X|Y]$ is the conditional expectation of X given Y. In this section, we propose an efficient and accurate two-dimensional willow tree method to compute VaR_α and CTE_α for the net liability of the VA, L_0, at time 0.

We first define a function $v(t, x, y)$ for a fixed number K as

$$\begin{aligned} v(t, x, y) &\equiv \mathrm{P}(L_0 \le K \mid F(t) = x, A(t) = y) \\ &= \mathbb{E}[\mathbb{I}(L_0 \le K) \mid F(t) = x, A(t) = y], \end{aligned}$$

where $A(t)$ is the present value of the accumulated charge at time t, that is, $A(t) \equiv \epsilon_x \int_0^t D(s)F(s)dt$. That is, given the account value x and accumulated charge y at time t, $v(t, x, y)$ is the probability of the net liability of the VA no

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larger than K. The probability of the net liability not exceeding K at the inception is $v(0, F_0, 0)$, that is, $v(0, F_0, 0) = \mathrm{P}(L_0 \le K)$. Therefore, the VaR of the net liability at the confidence level, α, can be defined as the smallest K such that

$$v(0, F_0, 0) = \mathrm{P}(L_0 \le K) = \alpha. \tag{4.1}$$

In other words, evaluating VaR_α is mathematically equivalent to solving (4.1) for K. Once the VaR_α is known, the corresponding CTE_α can be evaluated as

$$\mathrm{CTE}_\alpha = \frac{1}{1-\alpha}\mathbb{E}[L_0 \cdot \mathbb{I}(L_0 > \mathrm{VaR}_\alpha)].$$

Another function $u(t, x, y)$ is then introduced at time t as

$$u(t, x, y) \equiv \mathbb{E}[L_0 \cdot \mathbb{I}(L_0 > \mathrm{VaR}_\alpha) \mid F(t) = x,\ A(t) = y], \tag{4.2}$$

in order that $\mathrm{CTE}_\alpha = u(0, F_0, 0)/(1-\alpha)$. Therefore, the evaluation of the CTE_α is transformed to a function evaluation of $u(0, F_0, 0)$ given VaR_α. Before solving (4.1) for VaR_α given a specific confidence level α, we first propose an algorithm to evaluate $v(t, x, y)$ on the two-dimensional willow tree structure given K.

Assuming that the account value $F(t)$ and interest rate $r(t)$ follow the joint dynamics of (3.4), the present value of accumulated charge is

$$A(t) = \epsilon_x \int_0^t D(s)F(s)ds,$$

can be approximated in a discrete form as

$$A^n = \epsilon_x \sum_{i=1}^{n} D(t_i)F^i, \qquad n = 1, 2, \cdots, N,$$

where ϵ_x is the premium charge for the guarantees in the VA contract, $F^n \equiv F(t_n)$ and $A^n \equiv A(t_n)$ for simplification. Thus, A^{n+1} can be updated recursively as

$$A^{n+1} = A^n + \epsilon_x D(t_{n+1})F^{n+1}.$$

Based on the willow structure as shown in Subsection 3.3, there are M possible account values, F_l^n, at each discrete time t_n, $l = 1, \cdots, M$, among which F_M^n is the largest account value, whereas F_1^n is the smallest assuming the tree nodes are organized in an increasing order. Thus, the range for the present value of accumulated charges from t_1 to t_n can be determined as $[0, A_{max}^n]$ where $A_{max}^n = \epsilon_x \sum_{i=1}^n D(t_i)F_M^i$. Therefore, at time t_n, Q possible present values of accumulated charges at t_n are considered, that is,

$$A_k^n = k\frac{A_{max}^n}{Q}, \qquad \text{for} \qquad k = 1, 2, \cdots, Q.$$

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For simplicity, we denote v_{lk}^n as the function value of $v(t_n, F_l^n, A_k^n)$ at time t_n given the account value F_l^n and present value of accumulated capital A_k^n in the rest of this paper. Next, we will show how to compute $v(0, F_0, 0)$ given a fixed K through the backward induction on the willow tree of $(F(t), r(t))$.

We start the induction at the maturity $t_N \equiv T$. Given the present value of accumulated charges at t_N, A_k^N, $k = 1, 2, \cdots, Q$, and the account value F_l^N, the function value v_{lk}^N can be computed as, for $l = 1, 2, \cdots, M$,

$$\begin{aligned} v_{lk}^N &= \mathbb{E}[\mathbb{I}(L_0 \le K) \mid F^N = F_l^N, A^N = A_k^N] \\ &= {}_{t_N}q_{x+t_{N-1}} \mathbb{E}[\mathbb{I}(L_0 \le K) \mid F^N = F_l^N, A^N = A_k^N, d^N = 1] \\ &\quad + (1 - {}_{t_N}q_{x+t_{N-1}}) \mathbb{E}[\mathbb{I}(L_0 \le K) \mid F^N = F_l^N, A^N = A_k^N, d^N = 0], \end{aligned} \tag{4.3}$$

where $d^N = 1$ represents the policyholder dies in $(t_{N-1}, t_N]$, that is, $\tau_x \in (t_{N-1}, t_N]$ with the probability ${}_{t_N}q_{x+t_{N-1}}$, whereas $d^N = 0$ represents the policyholder survives at t_N with the probability $(1 - {}_{t_N}q_{x+t_{N-1}})$.

Given $F^N = F_l^N$, $A^N = A_k^N$, and $d^N = 1$, the net liability of the VA contract at t_N can be evaluated as

$$L_0 = D(T)(e^{\delta t_N} G_d(\tau_x) - F_l^N)^+ - A_k^N.$$

Once the policyholder dies, VA is terminated at t_N, and the benefits of the GMMB and GMWB are not necessary to be paid to the policyholder. Thus, the conditional expectation in the first term of (4.3) can be estimated as

$$\mathbb{E}[\mathbb{I}(L_0 \le K) \mid F^N = F_l^N, A^N = A_k^N, d^N = 1] = \begin{cases} 1 & \text{if } L_0 \le K \\ 0 & \text{otherwise} \end{cases}. \tag{4.4}$$

If the policyholder is alive at t_N, that is, $d^N = 0$, then the liability of the VA is determined by the GMMB, that is,

$$L_0 = D(t_N)(G_e(T) - F_l^N)^+ - A_k^N.$$

Similarly, the corresponding conditional expectation can be evaluated as

$$\mathbb{E}[\mathbb{I}(L_0 \le K) \mid F^N = F_l^N, A^N = A_k^N, d^N = 0] = \begin{cases} 1 & \text{if } L_0 \le K \\ 0 & \text{otherwise} \end{cases}. \tag{4.5}$$

Inserting both conditional expectations (4.4) and (4.5) into v_{lk}^N in (4.3), we can obtain the function value $v(t, x, y)$ at t_N on the tree node F_l^N given the accumulated charge A_k^N.

Then, we go backward on the tree to the time t_{N-1}, given the accumulated charges A_k^{N-1} and the account value F_l^{N-1}, the value of v_{lk}^{N-1}, for $l = 1, 2, \cdots, M$, is in the form of

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$$\begin{aligned} v_{lk}^{N-1} &= \mathbb{E}[\mathbb{I}(L_0 \le K) \mid F^{N-1} = F_l^{N-1}, A^{N-1} = A_k^{N-1}] \\ &= {}_{t_{N-1}}q_{x+t_{N-2}}\mathbb{E}[\mathbb{I}(L_0 \le K) \mid F^{N-1} = F_l^{N-1}, A^{N-1} = A_k^{N-1}, d^{N-1} = 1] \\ &\quad + (1 - {}_{t_{N-1}}q_{x+t_{N-2}})\mathbb{E}[\mathbb{I}(L_0 \le K) \mid F^{N-1} = F_l^{N-1}, A^{N-1} = A_k^{N-1}, d^{N-1} = 0]. \end{aligned} \tag{4.6}$$

Similar to the computation scheme at t_N, we first consider the scenario that the policyholder dies in $(t_{N-2}, t_{N-1}]$, that is, $d^{N-1} = 1$. Once it occurs, the contract terminates at t_{N-1}. The net liability L_0 is then

$$L_0 = D(t_{N-1})(e^{\delta t_{N-1}} G_d(\tau_x) - F_l^{N-1})^+ - A_k^{N-1}.$$

The conditional expectation can be evaluated as

$$\mathbb{E}[\mathbb{I}(L_0 \le K) \mid F^{N-1} = F_l^{N-1}, A^{N-1} = A_k^{N-1}, d^{N-1} = 1] = \begin{cases} 1 & \text{if } L_0 \le K \\ 0 & \text{otherwise} \end{cases}.$$

If the policyholder is alive at t_{N-1} with a positive account value, that is, $F_l^{N-1} > 0$, the VA contract continues. When the account value evolves from F_l^{N-1} at t_{N-1} to F_h^N at t_N, then the present value of accumulated charges, via the forward-shooting grid technique (Barraquand and Pudet, 1996), is updated as

$$\bar{A}_{hk}^N = A_k^{N-1} + D(t_N) \cdot \epsilon_x \cdot F_h^N.$$

Since $\bar{A}_{hk}^N$ must fall into the range $[0, A_{max}^N]$, there exists an index k^* such that $A_{k^*}^N \le \bar{A}_{hk}^N \le A_{k^*+1}^N$. The corresponding function value $\bar{v}_{hk}^N \equiv v(t_N, F_h^N, \bar{A}_{hk}^N)$, which implies that

$$\bar{v}_{hk}^N = \mathbb{E}[\mathbb{I}(L_0 \le K) \mid F^N = F_h^N, F^{N-1} = F_l^{N-1}, A^{N-1} = A_k^{N-1}, d^{N-1} = 0],$$

can be estimated by a linear interpolation of $v_{jk^*}^N$ and $v_{jk^*+1}^N$, that is,

$$\bar{v}_{hk}^N = \alpha_{hk}^N v_{hk^*+1}^N + (1 - \alpha_{hk}^N) v_{hk^*}^N, \qquad \text{where} \qquad \alpha_{hk}^N = \frac{\bar{A}_{hk}^N - A_{k^*}^N}{A_{k^*+1}^N - A_{k^*}^N}.$$

On the other hand, due to the tower rule, we have

$$\begin{aligned} &\mathbb{E}[\mathbb{I}(L_0 \le K) \mid F^{N-1} = F_l^{N-1}, A^{N-1} = A_k^{N-1}, d^{N-1} = 0] \\ &= \mathbb{E}[\mathbb{E}[\mathbb{I}(L_0 \le K) \mid F^N, F^{N-1} = F_l^{N-1}, A^{N-1} = A_k^{N-1}, d^{N-1} = 0] \\ &\qquad \mid F^{N-1} = F_l^{N-1}, A^{N-1} = A_k^{N-1}, d^{N-1} = 0] \\ &= \textstyle\sum_{h=1}^M p_{lh}^{N-1} \bar{v}_{hk}^N, \end{aligned}$$

where p_{lh}^{N-1} is the transition probability from (F_l^{N-1}, r_j^{N-1}) to (F_h^N, r_c^N) on the two-dimensional willow tree.

Otherwise, if the policyholder is alive at t_{N-1}, but the account value hits zero, that is, $\tau_x > \tau_0 = t_{N-1}$, then there is no future charge by the insurer. That

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is, the account value is kept at 0 while the present value of accumulated charges is fixed at A_k^{N-1} till the maturity. Thus. we just need to consider the net liability of VA at each possible VA termination time in future, that is, the policyholder dies prior to the maturity or survives at the maturity. The value of $\tilde{v}_{lk}^{N-1}$ can be evaluated as

$$\begin{aligned}\tilde{v}_{lk}^{N-1} &= \mathbb{E}[\mathbb{I}(L_0 \le K) \,|F^{N-1}=0, A^{N-1}=A_k^{N-1}, d^{N-1}=0] \\ &= {}_{t_N}q_{x+t_{N-1}}\mathbb{E}[\mathbb{I}(L_0 \le K) \,|F^{N}=0, F^{N-1}=0, A^{N-1}=A_k^{N-1}, d^{N-1}=0, d^N=1] \\ &\quad + (1-{}_{t_N}q_{x+t_{N-1}})\mathbb{E}[\mathbb{I}(L_0 \le K) \,|F^{N}=0, F^{N-1}=0, A^{N-1}=A_k^{N-1}, \\ &\quad\quad d^{N-1}=0, d^N=0]).\end{aligned}$$

In general, the value of $\tilde{v}_{lk}^{n}$ at time t_n, when the account value hits zero and policyholder is alive at t_n, can be evaluated as

$$\begin{aligned}\tilde{v}_{lk}^{n} &= \mathbb{E}[\mathbb{I}(L_0 \le K) \,|F^{n}=0, A^{n}=A_k^{n}, d^n=0] \\ &= \textstyle\sum_{u=n+1}^{N}\mathrm{P}(d^u=1|d^n=0)\mathbb{E}[\mathbb{I}(L_0 \le K) \,|F^{u}=0, F^{n}=0, A^{n}=A_k^{n}, d^n=0, d^u{=}1] \\ &\quad + \mathrm{P}(d^N=0|d^n=0)\mathbb{E}[\mathbb{I}(L_0 \le K) \,|F^{N}=0, F^{n}=0, A^{n}=A_k^{n}, d^n=0, d^N=0],\end{aligned}$$

where $\mathrm{P}(d^u=1|d^n=0)$ is the probability of policyholder dying in $(t_{u-1}, t_u]$ given he/she is alive at t_n, and $\mathrm{P}(d^N=0|d^n=0)$ represents the conditional probability of the policyholder surviving at the maturity t_N. Based on the definition of the conditional probability, these two conditional probabilities can be evaluated as

$$\mathrm{P}(d^u=1|d^n=0)=\frac{{}_{t_{u-1}}p_x - {}_{t_u}p_x}{{}_{t_n}p_x} \qquad \text{and} \qquad \mathrm{P}(d^N=0|d^n=0)=\frac{{}_{t_N}p_x}{{}_{t_n}p_x}.$$

When $d^u=1$, that is, the policyholder dies in $(t_{u-1}, t_u]$, the VA contract is terminated at t_u. Thus, the net liability L_0 can be computed by (2.1) with $\tau_x = t_u$ and $\tau_0 = t_n$ for $u = n+1, \cdots, N$ as

$$L_0 = D(\tau_x)e^{\delta\tau_x}G_d(\tau_x) + w\sum_{i=n+1}^{u}D(t_i) - A_k^n.$$

Then, the conditional expectation is calculated as

$$\mathbb{E}[\mathbb{I}(L_0 \le K) \,|F^{u}=0, F^{n}=0, A^{n}=A_k^{n}, d^n=0, d^u=1] = \begin{cases} 1 & \text{if } L_0 \le K \\ 0 & \text{otherwise} \end{cases}.$$

When $d^N=0$, it means that the policyholder is alive survives at the maturity T, that is, $\tau_x > T$ and $\tau_0 = t_n$. The net liability L_0 can still be evaluated by (2.1) as

$$L_0 = D(T)G_e(T) + w\sum_{i=n+1}^{N}D(t_i) - A_k^n.$$

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The conditional expectation is calculated as

$$\mathbb{E}[\mathbb{I}(L_0 \le K) \,|F^N = 0, F^n = 0, A^n = A_k^n, d^n = 0, d^N = 0] = \begin{cases} 1 & \text{if } L_0 \le K \\ 0 & \text{otherwise} \end{cases}.$$

We keep rolling the willow tree backward till t_0 so that the function value $v(0, F_0, 0)$ is obtained for a fixed K. At time t_0, the function $v(0, F_0, 0)$ can be evaluated as

$$\begin{aligned} v(0, F_0, 0) &= \mathbb{E}[\mathbb{I}(L_0 \le K) \mid F^0 = F_0, A^0 = 0, d^0 = 0] \\ &= \textstyle\sum_{h=1}^{M} p_h \bar{v}_{hk}^1. \end{aligned}$$

where p_h is the transition probability from (F_0, r_0) to (F_h^1, r_c^1) since the policyholder must be alive at the initial time, that is, $d^0 = 0$, and the initial investment must be positive, that is, $F^0 = F_0 > 0$. In order to determine the VaR given a confidence level α, the bisection method or other nonlinear equation solver can be employed to solve the Equation (4.1) for VaR_α

$$v(0, G, 0) = P(L_0 \le \text{VaR}_\alpha) = \alpha.$$

Since the dynamics for $F(t)$ and $r(t)$ are unchanged, we just need to roll backward on the same willow tree to find VaR_α for a specific α.

After we obtain the VaR_α, the CTE_α can be determined by the function value of $u(t, x, y)$ defined in (4.2) at $t = 0, x = F_0$, and $y = 0$. The computing process is similar to the one we introduced for $v(t, x, y)$, but replaced $\mathbb{E}[\mathbb{I}(L_0 \le K)]$ with $\mathbb{E}[L_0 \cdot \mathbb{I}(L_0 > \text{VaR}_\alpha)]$ for $u(t, x, y)$, that is,

$$\mathbb{E}[L_0 \cdot \mathbb{I}(L_0 > \text{VaR}_\alpha) \,|F^n, A^n] = \begin{cases} L_0 & \text{if } L_0 > \text{VaR}_\alpha \\ 0 & \text{otherwise} \end{cases}.$$

Note that given VaR_α, only one backward induction is needed on the willow tree to compute CTE_α. Thus, the computational cost for the CTE is much lower than the one for the VaR.

5. NUMERICAL EXPERIMENTS

In this section, we apply the two-dimensional willow tree method to calculate the risk measures of VAs compared with the Monte Carlo stimulations as the benchmark to show the accuracy and efficiency of our proposed method. The joint dynamics of account value and interest rate are defined as

$$\begin{cases} r(t) = r^*(t) + \phi(t), \quad dr^*(t) = -ar^*(t)dt + \sigma_1 dW^1 \\ dF(t) = (\mu(r, t) - \epsilon)F(t)dt - wdt + \sigma_2 F(t)dW^2, \end{cases}$$

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TABLE 2

MORTALITY RATES OF MEN AT THE AGE OF 45, 55, AND 65 YEARS IN 20 YEARS.

Age	45		55		65	
t	${}_{t+1}q_{45+t}$	${}_{t}p_{45}$	${}_{t+1}q_{55+t}$	${}_{t}p_{55}$	${}_{t+1}q_{65+t}$	${}_{t}p_{65}$
0	0.00363	1.00000	0.00691	1.00000	0.01753	1.00000
1	0.00392	0.99637	0.00757	0.99309	0.01932	0.98247
2	0.00418	0.99246	0.00828	0.98557	0.02122	0.96349
3	0.00438	0.98832	0.00906	0.97741	0.02323	0.94304
4	0.00457	0.98399	0.00991	0.96856	0.02538	0.92114
5	0.00478	0.97949	0.01086	0.95896	0.02785	0.89776
6	0.00504	0.97481	0.01192	0.94854	0.03059	0.87276
7	0.00538	0.96990	0.01311	0.93724	0.03343	0.84606
8	0.00580	0.96468	0.01444	0.92495	0.03633	0.81777
9	0.00632	0.95908	0.01590	0.91159	0.03942	0.78806
10	0.00691	0.95302	0.01753	0.89710	0.04299	0.75700
11	0.00757	0.94644	0.01932	0.88137	0.04715	0.72446
12	0.00828	0.93927	0.02122	0.86435	0.05184	0.69030
13	0.00906	0.93149	0.02323	0.84600	0.05711	0.65451
14	0.00991	0.92305	0.02538	0.82635	0.06305	0.61713
15	0.01086	0.91391	0.02785	0.80538	0.06978	0.57822
16	0.01192	0.90398	0.03059	0.78295	0.07738	0.53787
17	0.01311	0.89321	0.03343	0.75900	0.08596	0.49625
18	0.01444	0.88150	0.03633	0.73362	0.09557	0.45360
19	0.01590	0.86877	0.03942	0.70697	0.10625	0.41025
20	0.01753	0.85495	0.04299	0.67910	0.11800	0.36666

with the correlation ρ, that is, $\mathbb{E}(dW^1 dW^2) = \rho dt$. We adopt the life table published by the US social security administration (Bell and Miller, 2005)[1] at calendar year 2010 to model the mortality risk of the policyholder in Table 2. The term structure of interest rate in Hainaut and MacGilchrist (2010) is adopted to model the interest risk as shown in Table 3. The definition of L_0 in (2.1) is in a continuous form to make the notations simple and clear, but we only consider the VAs with discrete anniversary in this paper. Thus, the discretization represents the discrete anniversaries not the continuous time.

Two types of mutual funds, α-type and β-type, are considered in the numerical experiments, although our method is applicable to mutual funds with other forms of expected returns. The α-type is the fund whose expected return is fixed, that is, $\mu(r, t) = \alpha$, whereas the β-type is the one with a floating expected return, which has a fixed spread with the risk-free interest rate, that is, $\mu(r, t) = r(t) + \beta$. In order to make a fair comparison, we assume that these two funds have the same expected return at the inception. The parameters for interest rate and mutual fund and the VA contract attributes, such as benefit bases and M&E fee, etc., are all specified in Table 4.

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TABLE 3

INTEREST RATE TERM STRUCTURE GIVEN IN HAINAUT AND MACGILCHRIST (2010).

Maturity (year)	Rate (%)	Maturity (year)	Rate (%)
1	3.81	6	4.93
2	4.12	7	5.04
3	4.38	8	5.15
4	4.60	9	5.22
5	4.78	10	5.28

TABLE 4

PARAMETERS OF INTEREST RATE AND MUTUAL FUND, AND CONTRACT ATTRIBUTES OF A 10-YEAR VA EMBEDDED WITH GMDB, GMWB AND GMMB HELD BY A 65-YEAR MALE. M&E CHARGE RATE ϵ INCLUDES THE TOTAL BENEFITS CHARGE ϵ_x AND THE ADMINISTRATION COST.

Market parameters		Contract attributes	
Risk premium β	0.05	Initial investment F_0	1
Fixed expected return α	0.0881	M&E charge rate ϵ	0.02
Initial interest rate r_0	0.0381	Roll-up rate of GMDB δ	0.01
a in Hull–White model	0.07	Benefit base $G_w = G_e = G_d$	1
Volatility of $r(t)$ σ_1	0.0157	Benefit charge $\epsilon_w = \epsilon_e = \epsilon_d$	0.0035
Volatility of mutual fund σ_2	0.30	Total benefits charge ϵ_x	0.0105
		Withdrawal rate w	0.1
		Maturity T (years)	10
		Correlation ρ	0.4

We first consider a 10-year VA contract embedding with the GMDB, GMWB, and GMMB owned by a 65-year-old policyholder. Our proposed two-dimensional willow tree (WT) method is compared with the Monte Carlo (MC) method with one hundred thousand(100,000) paths on the calculation of the risk measures of VaR and CTE with 90% and 95% confidence levels. In order to further reduce the variance of the MC method, we run MC method ten times and record the average value as the benchmark result. Table 5 records the computed VaRs and CTEs for the VA contract through the WT and MC methods. It illustrates that the WT method can return the risk measures as accurate as the benchmark, but only requires less than one-tenth computational time of the MC method. Tables 6 and 7 record the computed probability of $L < 0.85$, that is, $\mathrm{P}(L < 0.85)$, with various values of m_F, m_r, and Q. It shows that computed probability is not sensitive to the values of m_r and Q, but increasing the value of m_F improves the accuracy of the result. Thus, in order to balance the computational time and the accuracy, we choose $m_F = 50$, $m_r = 10$, and $Q = 20$ for the rest experiments.

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TABLE 5

COMPUTED VARS AND CTES BY THE WT AND MC METHODS AND THE REQUIRED COMPUTATIONAL TIME (IN SECONDS) FOR 10-YEAR VA CONTRACTS HELD BY A 65-YEAR MALE WITH INVESTMENT ON α AND β TYPES MUTUAL FUNDS WITH $m_F = 50$, $m_r = 10$ AND $Q = 20$ IN THE WT METHOD.

	β-type		α-type	
$\mu(r,t)$	WT	MC (*std*)	WT	MC (*std*)
$\text{VaR}_{90\%}$	83.141	83.252 (0.023)	83.047	83.094 (0.024)
$\text{CTE}_{90\%}$	94.329	94.093 (0.087)	91.229	92.995 (0.129)
$\text{VaR}_{95\%}$	91.641	91.202 (0.015)	91.000	91.024 (0.025)
$\text{CTE}_{95\%}$	99.441	100.362 (0.109)	98.518	99.052 (0.178)
Times(s)	5.955	71.123	6.012	72.036

Then, we study the sensitivity of the risk measures of VAs with regard to the market parameters, the volatility of mutual fund, σ_1, the expected return α and β, and the correlation, ρ. Table 8 records the VaRs and CTEs of the 10-year VA contract in terms of σ_1 and the expected return of the funds. Generally, the risk measures of the VA invested in the α-type fund are lower than the ones for the β-type. The VaRs for these two types are close, but the corresponding CTEs are different, which implies that the liability of VAs invested in the β-type has a fatter tail than the one invested in the α-type. Compared to the expected return, the risk measure is more sensitive to the volatility of the mutual fund, σ_2. An increment of σ_2 from 0.25 to 0.35 leads to a 13% rise in the risk measures. Thus, hedging the risk of the mutual fund would be an effective way to reduce the liability of the VAs. Table 9 records the risk measures of the VA with respect to the correlation, ρ, between mutual fund and interest rate. Since the expected return of the α-type fund is independent of the interest rate $r(t)$, the risk measures are not sensitive to the correlation. In contrast, because of the high dependency in the expected return of the β-type, when the correlation is positive, the interest rate and the mutual fund move in the same direction, which implicitly increases the volatility of the investment account value. Thus, it leads to a large VaR and CTE under the positive correlation circumstance. It is vice versa for the negative correlation. Therefore, for the β-type fund, the positive correlation results in a fat-tailed liability distribution whereas the negative one generates a thin tail.

Next, we explore the sensitivity of the risk measures on the contract attributes. Table 10 lists the computed VaRs and CTEs in terms of rolling-up rate, δ, total benefit charge, ϵ_x, benefit base, G, and ages of the policyholder on VAs invested in the β-type funds. A large benefit base implies a large liability so

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TABLE 6

PROBABILITY OF $L \leq 0.85$ (IN PERCENTAGE %), THAT IS, $P(L < 0.85)$, FOR A 10-YEAR VARIABLE ANNUITY CONTRACT HELD BY A 65-YEAR-OLD WHO INVESTS IN AN α-TYPE FUND.

$m_r = 6$					
m_F \ Q	5	10	15	20	25
10	92.120	92.139	92.113	92.139	92.070
20	92.558	92.570	92.574	92.607	92.553
30	94.200	94.189	94.239	94.241	94.220
40	93.644	93.562	93.607	93.601	93.585
50	93.200	93.146	93.189	93.196	93.155
70	92.631	92.657	92.612	92.639	92.567
$m_r = 10$					
m_F \ Q	5	10	15	20	25
10	92.120	92.139	92.113	92.139	92.070
20	92.558	92.570	92.574	92.607	92.553
30	94.200	94.189	94.239	94.241	94.220
40	93.644	93.562	93.607	93.601	93.585
50	93.200	93.146	93.189	93.196	93.155
70	92.630	92.657	92.612	92.639	92.567
$m_r = 16$					
m_F \ Q	5	10	15	20	25
10	92.120	92.139	92.113	92.139	92.070
20	92.558	92.570	92.574	92.607	92.553
30	94.200	94.189	94.239	94.241	94.221
40	93.644	93.562	93.607	93.601	93.585
50	93.200	93.146	93.189	93.196	93.155
70	92.632	92.657	92.612	92.639	92.567
Benchmark(MC, std)			(92.81, 0.11)		

that the insurer has to reserve a large amount of capital. Another key contract attribute is the roll-up rate δ in the GMDB. In this experiment, we consider three cases for δ, less than the interest rate r_0, close to r_0, and larger than r_0. The first panel in Table 10 shows that VaRs and CTEs increase dramatically as δ hikes. It implies that when the roll-up rate δ is large, the mortality risk becomes a recognizable risk source of the VA. In addition, increasing the benefit charge does increase the revenue of the insurer, but the risk measures subside insignificantly. For example, when the benefit charge for each guarantee increases from 25 b.p. to 55 b.p., more than doubled, the 90% CTE only decreases about 1%, but doubling the premium has the side effect of losing current and potential

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TABLE 7

PROBABILITY OF $L \leq 0.85$ (IN PERCENTAGE %), THAT IS, $P(L < 0.85)$, FOR A 10-YEAR VARIABLE ANNUITY CONTRACT HELD BY A 65-YEAR-OLD WHO INVESTS IN A β-TYPE FUND.

$m_r = 6$					
m_F \ Q	5	10	15	20	25
10	91.195	91.203	91.252	91.264	91.281
20	93.079	93.105	93.054	93.088	93.075
30	91.856	91.873	91.833	91.854	91.846
40	91.984	92.004	91.968	91.998	91.997
50	91.502	91.524	91.488	91.516	91.543
70	91.791	91.818	91.780	91.810	91.832
$m_r = 10$					
m_F \ Q	5	10	15	20	25
10	91.303	91.311	91.363	91.361	91.391
20	92.497	92.512	92.469	92.493	92.482
30	92.364	92.383	92.343	92.369	92.363
40	91.895	91.917	91.881	91.911	91.907
50	91.603	91.626	91.598	91.627	91.656
70	91.563	91.586	91.561	91.578	91.604
$m_r = 16$					
m_F \ Q	5	10	15	20	25
10	91.298	91.283	91.323	91.337	91.355
20	92.767	92.785	92.743	92.773	92.762
30	92.277	92.296	92.255	92.284	92.272
40	91.807	91.827	91.791	91.816	91.846
50	91.745	91.768	91.737	91.764	91.791
70	91.958	91.575	91.628	91.569	91.595
Benchmark(MC, std)			(91.72, 0.0598)		

customers. Therefore, increasing benefit charge may not be a good choice for the insurer to manage the liability. The fourth panel lists the risk measures of VAs held by different-aged policyholders, which implies that the mortality risk has the same impact on the overall risk measures among different-aged policyholders. Later, we will decompose the overall risk measures into the one for each individual benefit in Figures 5 and 6. Then, the performance of the mortality risk is reexamined on the individual level. Table 11 records the corresponding sensitivities for the VA invested in the α-type fund. Same observations can be made in Table 11 as well.

So far, we study the sensitivity of the risk measures on the market parameters and contract attributes given a fixed interest rate term structure. Now,

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TABLE 8

COMPUTED VARS AND CTES (IN %) OF A 10-YEAR VA CONTRACT WITH VARIOUS MUTUAL FUND VOLATILITY σ_2 AND EXPECTED RETURNS α AND β.

β-type						
	0.25		0.3		0.35	
σ_2	WT	MC (*std*)	WT	MC (*std*)	WT	MC (*std*)
$VaR_{90\%}$	75.875	75.709 (0.047)	83.141	83.252 (0.023)	90.234	90.93 (0.028)
$CTE_{90\%}$	85.419	85.822 (0.109)	94.329	94.093 (0.087)	101.845	101.311 (0.173)
$VaR_{95\%}$	83.750	83.316 (0.027)	91.641	91.202 (0.015)	98.672	99.273 (0.032)
$CTE_{95\%}$	90.951	91.796 (0.092)	99.441	100.362 (0.109)	107.745	107.713 (0.217)
	0.03		0.05		0.07	
β	WT	MC (*std*)	WT	MC (*std*)	WT	MC (*std*)
$VaR_{90\%}$	86.500	86.354 (0.569)	83.141	83.252 (0.023)	82.266	82.268 (0.053)
$CTE_{90\%}$	96.922	96.551 (0.129)	94.329	94.093 (0.087)	91.834	91.669 (0.138)
$VaR_{95\%}$	92.841	92.176 (0.626)	91.641	91.202 (0.015)	90.391	90.710 (0.037)
$CTE_{95\%}$	101.568	102.587 (0.195)	99.441	100.362 (0.109)	98.095	98.269 (0.167)
α-type						
	0.25		0.3		0.35	
σ_2	WT	MC (*std*)	WT	MC (*std*)	WT	MC (*std*)
$VaR_{90\%}$	75.391	75.549 (0.017)	83.047	83.094 (0.024)	90.578	90.781 (0.037)
$CTE_{90\%}$	84.123	84.742 (0.099)	92.229	92.995 (0.129)	99.663	100.461 (0.099)
$VaR_{95\%}$	82.813	83.126 (0.027)	91.000	91.024 (0.025)	98.900	99.123 (0.027)
$CTE_{95\%}$	89.425	90.469 (0.128)	98.518	99.052 (0.178)	106.453	106.785 (0.08)

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TABLE 8

CONTINUED

	0.0681		0.0881		0.1081	
α	WT	MC (*std*)	WT	MC (*std*)	WT	MC (*std*)
$VaR_{90\%}$	84.375	84.149 (0.478)	83.047	83.094 (0.024)	81.250	81.800 (0.139)
$CTE_{90\%}$	94.804	95.714 (0.134)	92.229	92.995 (0.129)	89.910	90.757 (0.073)
$VaR_{95\%}$	91.250	91.44 (0.033)	91.000	91.024 (0.025)	89.750	90.383 (0.041)
$CTE_{95\%}$	100.645	101.437 (0.159)	98.518	99.052 (0.178)	96.620	97.259 (0.07)

we will study how the monetary policy impacts on the liability, especially on the RBC of VAs. In the past two years, the Federal Reserves of US raised the benchmark rate 25 base points almost every quarter. We take each Federal Reserve benchmark rate hike announcement as a monetary policy change scenario to analyze the impact. Table 12 lists the yields of US treasure bonds three days after the announcement. Then, we consider the risk measures of two VAs with maturity in 10 and 20 years, whose withdrawal rates, w, are 0.1 and 0.05, respectively, held by different-aged policyholders. Just as in our previous experiments, the policyholder can choose to invest in an α-type mutual fund with the expected return 7% annually or a β-type fund with the spread 5.99%, whose expected return is also 7%, on March 20, 2017. Figure 3 plots VaRs and CTEs of the 10-year contracts in terms of the monetary policy changes. The left column (subfigures a, c, and e) is for the policyholders who choose to invest in an α-type fund, while the right column (subfigures b, d, and f) is invested in the β-type. It illustrates that 90% VaRs and 95% VaRs in these subfigures are very close no matter which fund is invested or how old the policyholder is, but the CTEs are slightly different. The CTEs for the one invested in a β-type fund are a little bit larger than the ones for the α-type. In other words, the liability distribution for a VA invested in the β-type has a slightly fatter tail than the one invested in the α-type. The reason for this fatted tail is the positive correlation between the interest rate and mutual fund. Just as we observe in Table 9, the positive correlation increases the fatness of the tail for the β-type fund. Figure 4 plots the risk measures for the 20-year VA contracts held by the same group of policyholders. Compared to the 10-year contract, the risk measures, VaRs and CTEs, are decreased. The 20-year contract only allows the policyholder to withdraw 5% of the benefit base annually without penalty, whereas the withdrawal rate is 10% for the 10-year VA contract. A small withdrawal rate reduces the possibility of the account depletion, which also reduces the liability of the insurer. Both VaRs and CTEs for the VA invested in the

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TABLE 9

COMPUTED VARS AND CTES (IN %) OF A 10-YEAR VA CONTRACT WITH A 65-YEAR-OLD MALE POLICYHOLDER IN TERMS OF THE CORRELATION, ρ, BETWEEN THE INTEREST RATE AND MUTUAL FUND.

	β-type				
ρ	Method	$VaR_{90\%}$	$CTE_{90\%}$	$VaR_{95\%}$	$CTE_{95\%}$
	WT	81.422	90.383	89.375	95.814
−0.8	MC	81.812	90.139	89.961	96.641
	(std)	(0.428)	(0.115)	(0.179)	(0.096)
	WT	81.922	90.909	90.391	96.931
−0.4	MC	82.207	91.021	90.53	97.459
	(std)	(0.075)	(0.139)	(0.056)	(0.173)
	WT	82.75	91.884	90.991	98.694
0	MC	82.871	92.457	90.928	98.766
	(std)	(0.026)	(0.1)	(0.016)	(0.105)
	WT	83.141	94.329	91.641	99.441
0.4	MC	83.252	94.093	91.202	100.362
	(std)	(0.023)	(0.087)	(0.015)	(0.109)
	WT	83.581	95.563	92.006	101.754
0.8	MC	83.57	95.554	91.434	101.892
	(std)	(0.05)	(0.189)	(0.042)	(0.243)
	α-type				
ρ	Method	$VaR_{90\%}$	$CTE_{90\%}$	$VaR_{95\%}$	$CTE_{95\%}$
	WT	83.047	92.229	91.000	98.518
−0.8	MC	83.111	93.102	91.04	99.158
	(std)	(0.038)	(0.142)	(0.027)	(0.164)
	WT	83.047	92.229	91.000	98.518
−0.4	MC	83.077	92.963	91.013	99.04
	(std)	(0.033)	(0.112)	(0.023)	(0.098)
	WT	83.047	92.229	91.000	98.518
0	MC	83.097	93.124	91.041	99.25
	(std)	(0.025)	(0.085)	(0.025)	(0.081)
	WT	83.047	92.229	91.000	98.518
0.4	MC	83.094	92.995	91.024	99.052
	(std)	(0.024)	(0.129)	(0.025)	(0.178)
	WT	83.047	92.229	91.000	98.518
0.8	MC	83.113	93.100	91.042	99.15
	(std)	(0.021)	(0.112)	(0.019)	(0.134)

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TABLE 10

COMPUTED 90% AND 95% VARS AND CTES (IN %) OF A 10-YEAR VA CONTRACT WITH VARIOUS CONTRACT ATTRIBUTES, SUCH AS THE ROLL-UP RATE δ, TOTAL BENEFIT CHARGE ϵ_x AND BENEFIT BASE G FOR THE β-TYPE FUND. WE SUPPOSE THAT THE BENEFIT BASE FOR ALL THREE BENEFITS ARE EQUAL, THAT IS, $G_e = G_d = G_w = G$. THE LAST PANEL IS THE RISK MEASURES FOR A 10-YEAR VA OWNED BY DIFFERENT AGES OF POLICYHOLDERS.

	0.01		0.03		0.05	
δ	WT	MC (*std*)	WT	MC (*std*)	WT	MC (*std*)
$\text{VaR}_{90\%}$	83.141	83.252 (0.023)	85.625	85.567 (0.36)	90.391	90.877 (0.043)
$\text{CTE}_{90\%}$	94.329	94.093 (0.087)	97.47	97.408 (0.125)	102.773	103.159 (0.235)
$\text{VaR}_{95\%}$	91.641	91.202 (0.015)	94.641	95.215 (0.43)	98.75	99.438 (0.039)
$\text{CTE}_{95\%}$	99.441	100.362 (0.109)	103.439	104.274 (0.144)	110.448	111.571 (0.288)
	0.0075		0.0105		0.0165	
ϵ_x	WT	MC (*std*)	WT	MC (*std*)	WT	MC (*std*)
$\text{VaR}_{90\%}$	83.297	83.668 (0.016)	83.141	83.252 (0.023)	82.828	82.336 (0.056)
$\text{CTE}_{90\%}$	94.786	94.317 (0.096)	94.329	94.093 (0.087)	93.937	93.298 (0.133)
$\text{VaR}_{95\%}$	91.797	91.478 (0.015)	91.641	91.202 (0.015)	91.028	90.578 (0.032)
$\text{CTE}_{95\%}$	99.718	100.536 (0.132)	99.441	100.362 (0.109)	99.231	99.732 (0.137)
	75%		100%		120%	
$G\backslash F_0$	WT	MC (*std*)	WT	MC (*std*)	WT	MC (*std*)
$\text{VaR}_{90\%}$	68.750	68.368 (0.025)	83.141	83.252 (0.023)	95.016	95.189 (0.039)
$\text{CTE}_{90\%}$	78.058	78.77 (0.138)	94.329	94.093 (0.087)	106.41	106.429 (0.134)
$\text{VaR}_{95\%}$	76.25	76.382 (0.021)	91.641	91.202 (0.015)	103.516	103.082 (0.026)
$\text{CTE}_{95\%}$	85.603	85.07 (0.188)	99.441	100.362 (0.109)	111.739	112.674 (0.11)

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TABLE 10

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	45		55		65	
age	WT	MC (*std*)	WT	MC (*std*)	WT	MC (*std*)
$VaR_{90\%}$	83.141	83.34 (0.036)	83.141	83.345 (0.026)	83.141	83.252 (0.023)
$CTE_{90\%}$	94.53	94.495 (0.2)	94.47	94.501 (0.151)	94.329	94.093 (0.087)
$VaR_{95\%}$	91.641	91.23 (0.026)	91.641	91.234 (0.022)	91.641	91.202 (0.015)
$CTE_{95\%}$	99.678	100.596 (0.212)	99.754	100.642 (0.175)	99.441	100.362 (0.109)

β-type fund are slightly larger than the ones for the α-type fund invested VAs. Again, it implies a fat tail for the liability of the VA invested in the β-type fund. However, compared to the 10-year VA contracts, the fatness of the tail for the β-type fund VA is reduced. It means that the impact on the positive correlation is weakened as the maturity rises. Moreover, when the monetary policy changes, the fluctuation of the risk measures for the 20-year VA is more dramatic than the one for 10-year VAs, which implies that the monetary policy change has a bigger impact on the long-term VAs than on the short-term VAs.

In the final experiment, we decompose the overall risk measures into the one for each guaranteed benefit in the VA so that we can observe the performance of the mortality risk among different- aged policyholders. We take the interest rate term structure on December 20, 2018, as an example for the decomposition. Figure 5 plots the liability cumulative distribution function (CDF) curves of the 10-year VAs contracts invested in the α- and β-type fund by the WT and MC methods. First of all, no matter which fund is invested, the CDF curves have the similar shape. Thus, we just take the CDF for VAs invested in the α-type as an example for analysis. The same observation can also be made on the β-type. When there is only GMDB embedded in the VA, the 90% VaRs for all these contracts are almost zeros. However, when the GMWB is added to the VA, the 90% VaR for VAs held by 45- and 55-year-old male increases to 0.35 while it becomes 0.55 for the VA held by the 65 year old. In other words, the mortality risk intensifies the liability of the GMWB for the old. Finally, when all GMDB, GMWB, and GMMB are embedded, the overall 90% VaR are close for all these three different-aged policyholders, but the contribution for each benefit is quire different. For the 45- or 55-year-old policyholder, the GMMB contributions about 0.7 to the 90% VaR out of 1.05 in overall, while it contributes 0.5 to the overall 90% VaR for a 65-year-old policyholder. It implies that for a short-term VA, the liability is mainly determined by the GMMB. Figure 6 plots the CDF curves for the 20-year VA contracts. Different from the 10-year VA contract, the liability of the GMDB contributes significantly

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TABLE 11

COMPUTED 90% AND 95% VARS AND CTES (IN %) OF A 10-YEAR VA CONTRACT WITH VARIOUS CONTRACT ATTRIBUTES, SUCH AS THE ROLL-UP RATE δ, TOTAL BENEFIT CHARGE ϵ_x AND BENEFIT BASE G FOR THE α-TYPE FUND. WE SUPPOSE THAT THE BENEFIT BASE FOR ALL THREE BENEFITS ARE EQUAL, THAT IS, $G_e = G_d = G_w = G$. THE LAST PANEL IS THE RISK MEASURES FOR A 10-YEAR VA OWNED BY DIFFERENT AGES OF POLICYHOLDERS.

	0.01		0.03		0.05	
δ	WT	MC	WT	MC	WT	MC
$VaR_{90\%}$	83.047	83.094	84.141	84.369	90.000	90.832
		(0.024)		(0.318)		(0.04)
$CTE_{90\%}$	92.229	92.995	96.033	96.62	101.903	102.751
		(0.129)		(0.158)		(0.183)
$VaR_{95\%}$	91.000	91.024	91.641	92.358	99.375	99.347
		(0.025)		(0.366)		(0.041)
$CTE_{95\%}$	98.518	99.052	103.125	103.264	110.569	111.061
		(0.178)		(0.218)		(0.245)
	0.0075		0.0105		0.0165	
ϵ_x	WT	MC (*std*)	WT	MC (*std*)	WT	MC (*std*)
$VaR_{90\%}$	83.516	83.58	83.047	83.094	81.953	82.107
		(0.011)		(0.024)		(0.035)
$CTE_{90\%}$	92.596	93.441	92.229	92.995	91.889	92.226
		(0.062)		(0.129)		(0.103)
$VaR_{95\%}$	90.625	91.377	91.000	91.024	89.750	90.300
		(0.009)		(0.025)		(0.036)
$CTE_{95\%}$	99.126	99.476	98.518	99.052	97.303	98.473
		(0.099)		(0.178)		(0.116)
	75%		100%		120%	
$G\backslash F_0$	WT	MC (*std*)	WT	MC (*std*)	WT	MC (*std*)
$VaR_{90\%}$	68.203	68.222	83.047	83.094	94.766	95.053
		(0.02)		(0.024)		(0.013)
$CTE_{90\%}$	77.35	77.782	92.229	92.995	105.191	105.466
		(0.087)		(0.129)		(0.07)
$VaR_{95\%}$	76.000	76.204	91.000	91.024	102.344	102.921
		(0.02)		(0.025)		(0.015)
$CTE_{95\%}$	83.62	83.967	98.518	99.052	110.838	111.464
		(0.112)		(0.178)		(0.088)

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TABLE 11

CONTINUED

	45		55		65	
$age(T=10)$	WT	MC (*std*)	WT	MC (*std*)	WT	MC (*std*)
$VaR_{90\%}$	83.047	83.23 (0.027)	83.047	83.21 (0.027)	83.047	83.094 (0.024)
$CTE_{90\%}$	92.034	93.553 (0.131)	92.943	93.474 (0.139)	92.229	92.995 (0.129)
$VaR_{95\%}$	90.703	91.11 (0.014)	90.703	91.096 (0.019)	91.000	91.024 (0.025)
$CTE_{95\%}$	98.709	99.454 (0.142)	98.67	99.439 (0.162)	98.518	99.052 (0.178)

TABLE 12

YIELDS (IN %) OF U.S. TREASURY BONDS FOR VARIOUS MATURITIES (IN YEARS). (DATA SOURCE: https://www.federalreserve.gov/releases/h15/).

	1/12	4/12	6/12	1	2	3	5	7	10	20
20-Mar-17	0.70	0.76	0.89	1.01	1.30	1.57	2.00	2.28	2.47	2.83
19-Jun-17	0.85	1.02	1.13	1.22	1.36	1.52	1.80	2.02	2.19	2.53
18-Dec-17	1.26	1.38	1.51	1.70	1.84	1.94	2.17	2.30	2.39	2.57
22-Mar-18	1.67	1.72	1.95	2.05	2.29	2.43	2.63	2.76	2.83	2.94
14-Jun-18	1.81	1.94	2.07	2.35	2.59	2.69	2.81	2.90	2.94	2.99
27-Sep-18	2.10	2.18	2.37	2.58	2.83	2.89	2.96	3.02	3.06	3.13
20-Dec-18	2.42	2.39	2.55	2.64	2.67	2.65	2.65	2.72	2.79	2.92

in the overall 90% VaRs for the 55- and 65-year-old policyholders. The 90% overall VaR for the 65 year old is about 0.85, which can be decomposed into 0.5 for the GMDB, 0.3 for the GMWB, and 0.05 for the GMMB. In other words, the mortality risk plays a key role in the liability of the long-term VA contract. In contrast, although the overall 90% VaR is also around 0.85 for the 45 year old, the corresponding decomposition is quite different, 0 for the GMDB, 0.38 for the GMWB, and 0.47 for the GMMB. The liability of GMMB still dominates the overall risk measures. The risk measure decomposition for the 55-year-old policyholder is in the between of the decomposition for the 45- and 65-year-old policyholders. The contributions of these three benefits are better balanced than the other two. Therefore, although the overall 90% VaRs for the 20-year VAs are almost same for different-aged policyholders, the contribution for each benefit is quite different. In summary, the mortality risk plays a key role in the overall VaRs for VAs. For the old policyholder, the mortality risk intensifies the liability of the GMWB. It also impacts the liability of the

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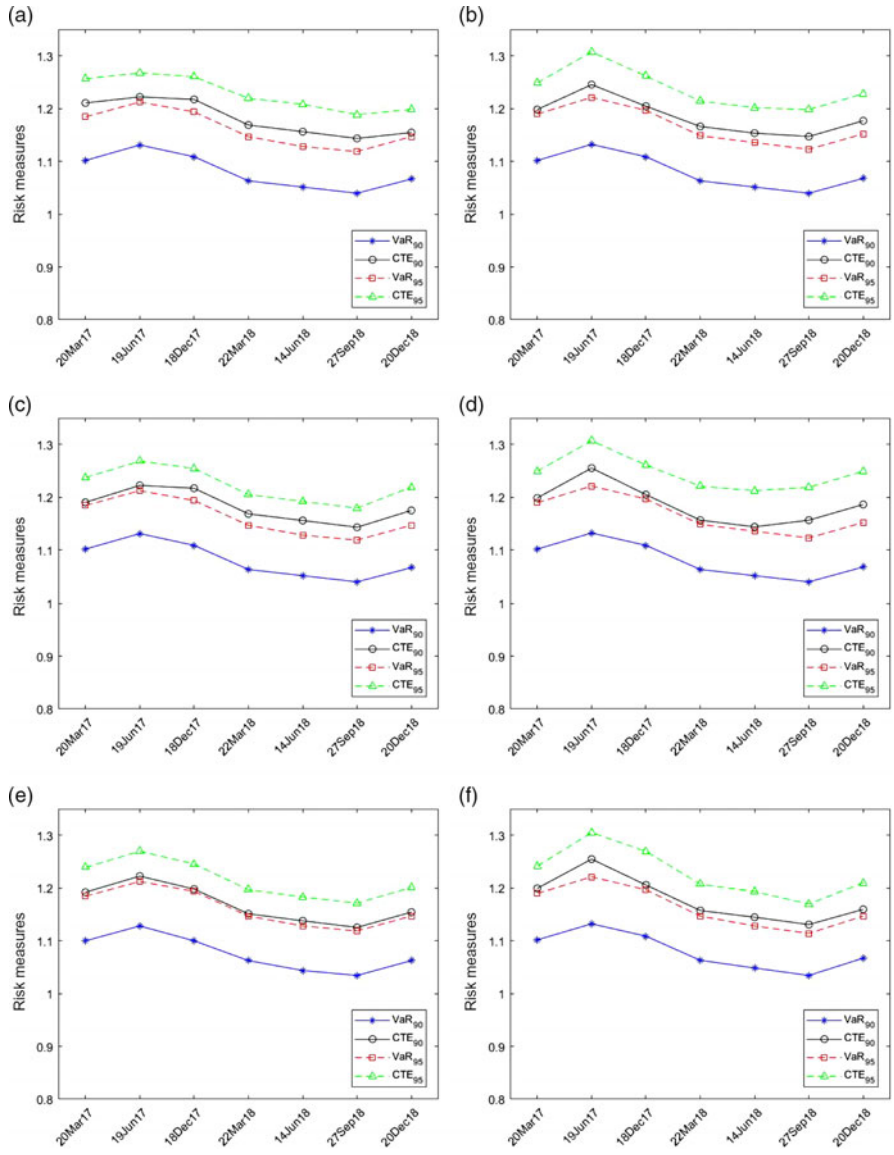


FIGURE 3: Computed VaRs and CTEs of 10-year contracts under different interest rate scenarios in Table 12. (a) 45 year old invested in α-type. (b) 45 year old invested in β-type. (c) 55 year old invested in α-type. (d) 55 year old invested in β-type. (e) 65 year old invested in α-type. (f) 65 year old invested in β-type

GMDB or GMMB depending on the length of VA's maturity. In contrast, the mortality risk of a young policyholder determines the liability of the GMMB in both long- and short-term contracts. The GMMB dominates the overall VaR of the VA held by the young. Therefore, the insurer should perform different strategies to hedge the mortality risk for different-aged policyholders.

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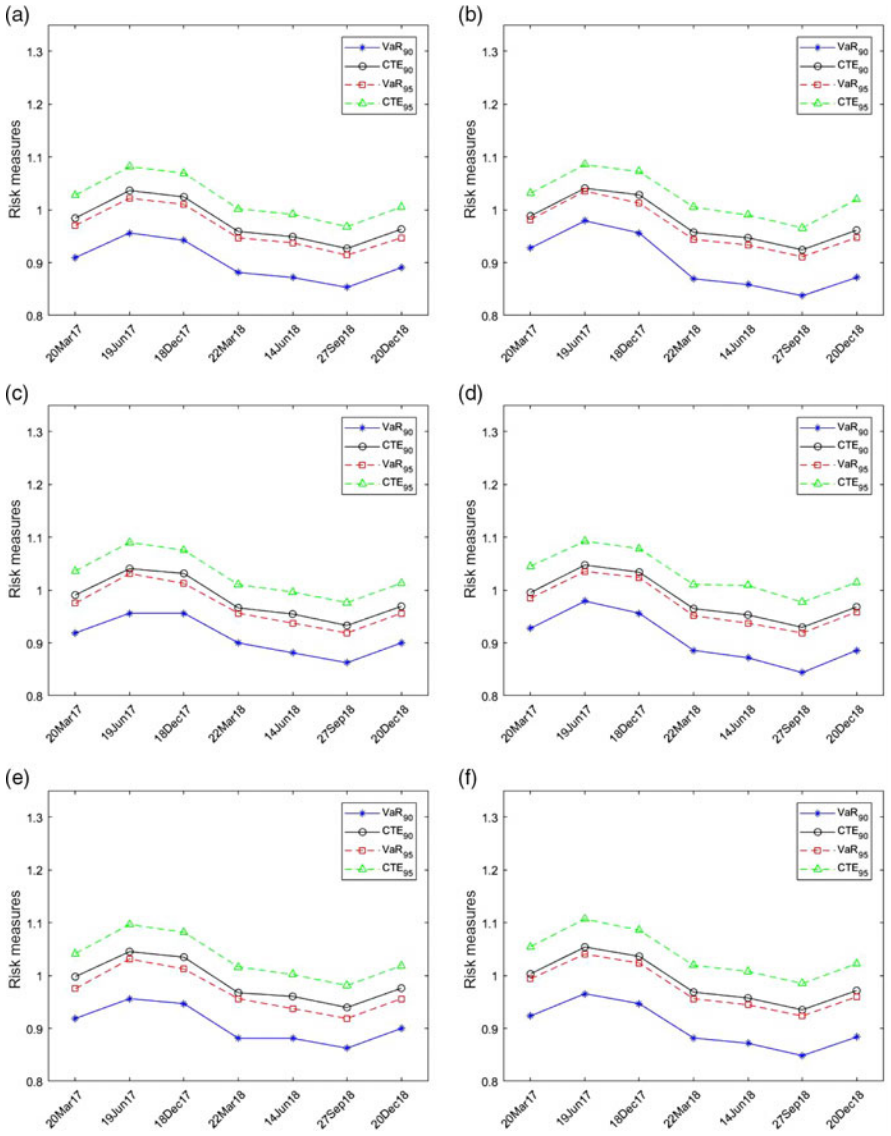


FIGURE 4: Computed VaRs and CTEs of 20-year contracts under different interest rate scenarios in Table 12. (a) 45 year old invested in α-type. (b) 45 year old invested in β-type. (c) 55 year old invested in α-type. (d) 55 year old invested in β-type. (e) 65 year old invested in α-type. (f) 65 year old invested in β-type.

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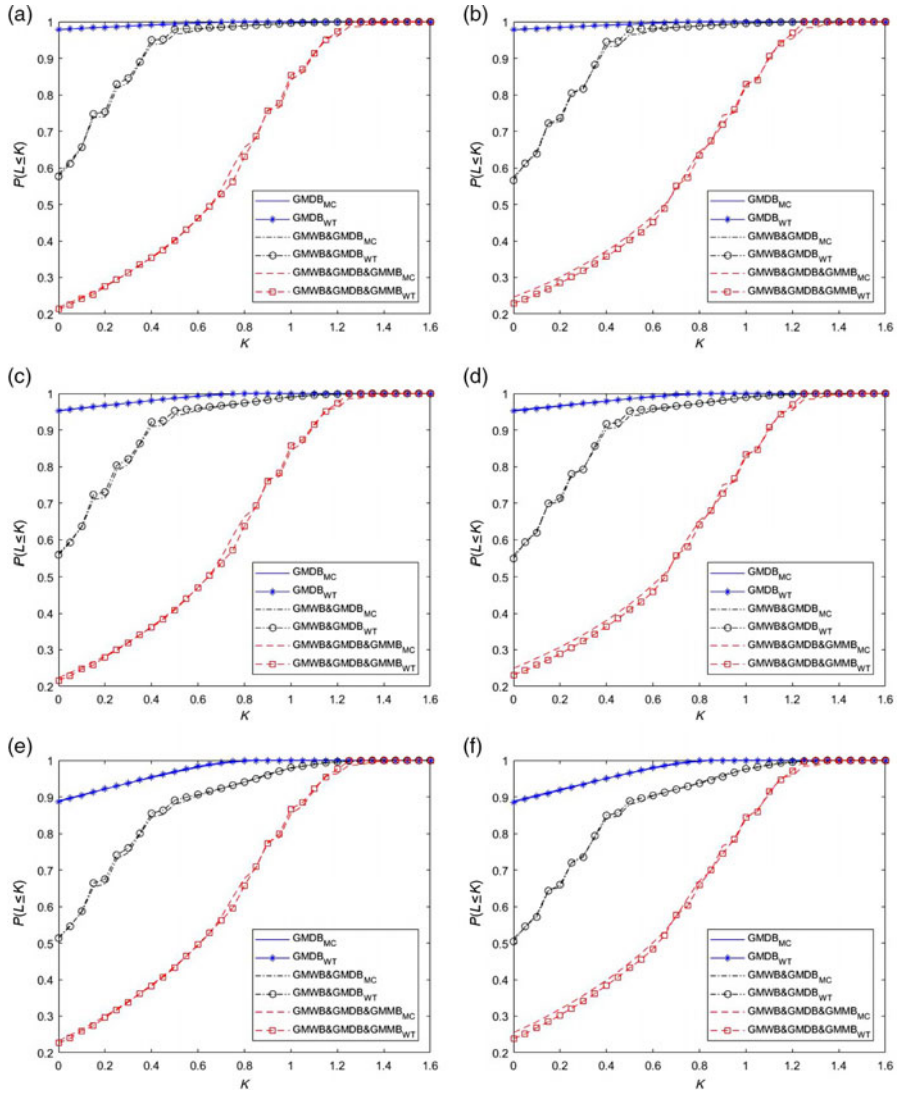


FIGURE 5: The cumulative distribution function (CDF) curves for 10-year VAs embedded with the GMDB, GMWB and GMMB with different aged policyholders invested in the α and β types fund. (a) 45 year old invested in α-type. (b) 45 year old invested in β-type. (c) 55 year old invested in α-type. (d) 55 year old invested in β-type. (e) 65 year old invested in α-type. (f) 65 year old invested in β-type.

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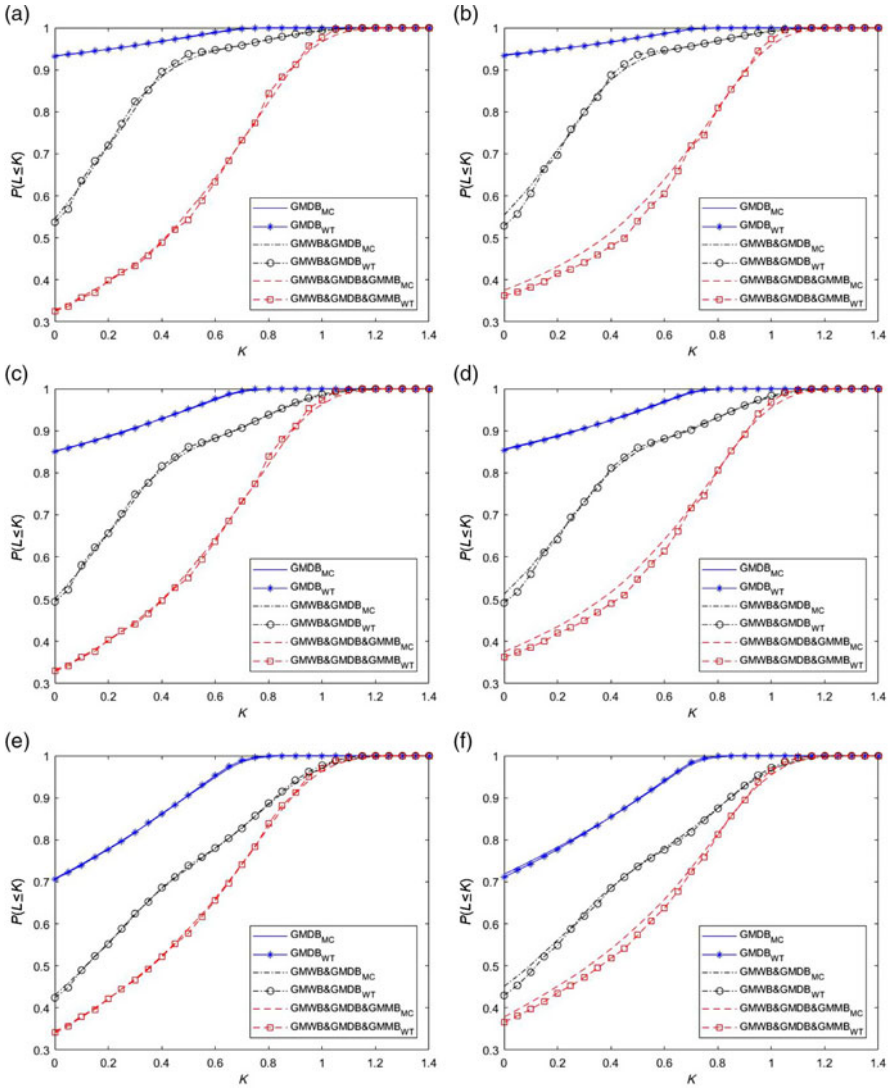


FIGURE 6: The cumulative distribution function (CDF) curves for 20-year VAs embedded with the GMDB, GMWB and GMMB with different aged policyholders invested in the α and β types fund. (a) 45 year old in α-type. (b) 45 year old in β-type. (c) 55 year old in α-type. (d) 55 year old in β-type. (e) 65 year old in α-type. (f) 65 year old in β-type.

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6. Conclusion

Variable annuities are popularly traded all over the world, but there are very limited studies on the risk measures of the liabilities from the insurer's perspective. In this paper, we propose an efficient two-dimensional willow tree method to describe the joint dynamics of the mutual fund and interest rate. Based on the tree structure, the liability distribution (including VaR and CTE) of the VA can be computed by the backward induction. We first analyze the sensitivity of risk measures (VaR and CTE) to the stochastic model parameters and contract attributes. It shows that the volatility of the mutual fund plays a key role in the risk measure. One of the effective ways to reduce the liability is to hedge the market risk of the mutual fund. Also, the positive correlation between the mutual fund and interest rate leads to a fat-tailed liability distribution, but the tail fatness is reduced for a long-term contract. Then, we take the US benchmark rate hikes in the past two years as monetary policy change scenarios to analyze the impact of the monetary policy change on the risk measures of the VAs. It turns out that the monetary policy has a bigger impact on the long-term VA than on the short-term contract. Finally, we decompose the overall risk measures into the one for each individual benefit. We find that the mortality risk contributes differently for different- aged policyholders even though the overall risk measures are all close. For the old policyholder, the mortality risk impacts the VA liability via the GMDB while it dominates the liability through the GMMB for the young. Therefore, the mortality risk is an important risk factor for the liability, so the insurer should play different hedging strategies among different-aged policyholders.

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Notes

1. http://s3.amazonaws.com/zanran_storage/www.socialsecurity.gov/ContentPages/10640324.pdf.

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JINDONG WANG
School of Mathematical Sciences
Peking University
Beijing, China
E-Mail: jdwang@pku.edu.cn

WEI XU (Corresponding author)
Department of Mathematics
Ryerson University
Toronto, ON, Canada
E-Mail: wei.xu@ryerson.ca

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