Proc. Camb. Phil. Soc. (1974), **76**, 313
PCPS 76–30
Printed in Great Britain

Potential problems involving an annulus

By D. L. CLEMENTS
Department of Applied Mathematics, University of Western Ontario, Canada
and E. R. LOVE
Department of Mathematics, University of Melbourne, Australia

(*Received* 5 *July* 1972; *Revised* 30 *August* 1973)

1. *Introduction.* In electrostatic, elastostatic and hydrodynamic problems involving a plane annulus in space, two mixed boundary value potential problems are frequently encountered. For reasons given below, we consider here the axisymmetric cases of them; and we describe them using cylindrical polar coordinates (r, θ, z) so chosen that the annulus is $a < r < b$, $z = 0$.

'Neumann problem'. *Given* σ_0, V_1 *and* V_2, *functions of* r *only, to find an axially symmetric function* $V(r, z)$ *which is harmonic in the half space* $z > 0$ *and satisfies the boundary conditions*

$$\frac{\partial V}{\partial z} = -\sigma_0(r) \quad for \quad a < r < b \quad and \quad z = 0,$$

$$V = V_1(r) \quad for \quad r < a \quad and \quad z = 0, \qquad V = V_2(r) \quad for \quad r > b \quad and \quad z = 0,$$

$$V \to 0 \quad as \quad (r^2 + z^2)^{\frac{1}{2}} \to \infty.$$

Although this is a mixed boundary value problem, not a proper Neumann problem, it is convenient to follow the nomenclature used by Collins(3) for a special case of it. A similar remark applies also to the next problem.

'Dirichlet problem'. *Given* V_0, σ_1 *and* σ_2, *functions of* r *only, to find an axially symmetric function* $V(r, z)$ *which is harmonic in the half space* $z > 0$ *and satisfies the boundary conditions*

$$V = V_0(r) \quad for \quad a < r < b \quad and \quad z = 0,$$

$$\frac{\partial V}{\partial z} = -\sigma_1(r) \quad for \quad r < a \quad and \quad z = 0, \qquad \frac{\partial V}{\partial z} = -\sigma_2(r) \quad for \quad r > b \quad and \quad z = 0,$$

$$V \to 0 \quad as \quad (r^2 + z^2)^{\frac{1}{2}} \to \infty.$$

These problems have been discussed by several authors, including Gubenko and Mossakovskii(2), Collins(3), Noble(4), Cooke(5) and Williams(6). Some consider only zero boundary values outside the annulus; some consider more general non-axisymmetric problems. All conclude with simultaneous Fredholm equations of the second kind of varying degrees of complexity.

https://doi.org/10.1017/S0305004100048969 Published online by Cambridge University Press

In this paper, we extend to an annulus the method of Copson(1) for a disc. The Neumann problem is reduced to the uncoupled pair of equations (24) and (25), namely

$$f_+(x)+\frac{2}{\pi}\int_0^k \frac{1}{1-x^2t^2}f_+(t)\,dt = g_+(x) \quad \text{for} \quad 0<x<k,$$

$$f_-(x)-\frac{2}{\pi}\int_0^k \frac{1}{1-x^2t^2}f_-(t)\,dt = g_-(x) \quad \text{for} \quad 0<x<k,$$

where f_+ and f_- are unknown functions, g_+ and g_- are known functions, and $0<k<1$; k^2 is the radius ratio a/b of the annulus. The Dirichlet problem is reduced to a similar pair of equations, (49) and (50), differing from these only by having an extra factor $-xt$ in the kernels; however, the meanings of the known and unknown functions in them are naturally different.

These equations have some advantages over equations previously presented. They have very simple kernels which are symmetric, continuous and bounded. The interval of integration is finite, and the simpler theories of integral equations are applicable to them. In fact they can be solved by iteration, with uniform convergence of the iterates, for all values of $a/b = k^2$ up to 0·90 for the Neumann problem and up to 0·97 for the Dirichlet problem.

As well as leading to the new equations, our method for the Neumann problem seems to be more elementary than earlier treatments; further the known functions $g_\pm$ are more simply constructed from the physical data. In contrast, our method for the Dirichlet problem is probably no simpler than other methods. Its inclusion here is justified rather on the ground that it avoids some possibly divergent integrals which occur in the axisymmetric case of several earlier treatments; and on the ground that the equations obtained can be solved by iteration for the wide range of radius ratios mentioned above.

Our method might have been expressed in terms of fractional integrals; but the variety of operators needed seems to be too great for that mode of presentation to be illuminating here. It is also possible to use our method for the non-axisymmetric cases; but its advantages in simplicity and elegance tend to be lost in that context. Despite great differences, our treatment of the Neumann problem has points of contact with (3) and (5), and our treatment of the Dirichlet problem with (2), (4), (5) and (6). These connections are indicated in sections 5, 7 and 8.

2. *The 'Neumann problem'*. We aim to meet the requirements by a representation of V of the form

$$V(r,z) = \frac{1}{2\pi}\int_0^\infty \sigma(\rho)\,\rho\,d\rho \int_{-\pi}^{\pi} \frac{d\phi}{(z^2+r^2+\rho^2-2r\rho\cos\phi)^{\frac{1}{2}}}, \tag{1}$$

where σ is a partially unknown function, sufficiently small at infinity and sufficiently smooth elsewhere for $V(r,z)$ to be axially harmonic in $z>0$ and small at infinity, and for $-\sigma(r)$ to be its z-derivative on $z=0$.

We write $V(r)$ for $V(r,0)$, and suppose throughout that r and all other named

https://doi.org/10.1017/S0305004100048969 Published online by Cambridge University Press

independent variables take positive values only. The requirements of the Neumann problem are now that

$$\sigma(r) = \sigma_0(r) \quad \text{for} \quad a < r < b, \tag{2}$$

$$V(r) = V_1(r) \quad \text{for} \quad r < a, \qquad V(r) = V_2(r) \quad \text{for} \quad r > b. \tag{3}$$

Copson's Integral (A 1) (see Appendix) provides other expressions for the inner integral in (1) when $z = 0$; it gives

$$\tfrac{1}{2}\pi V(r) = \int_0^b \sigma(\rho)\rho\, d\rho \int_0^{\min(r,\rho)} \frac{ds}{(r^2-s^2)^{\frac{1}{2}}(\rho^2-s^2)^{\frac{1}{2}}} + \int_b^\infty \sigma(\rho)\rho\, d\rho \int_{\max(r,\rho)}^\infty \frac{ds}{(s^2-r^2)^{\frac{1}{2}}(s^2-\rho^2)^{\frac{1}{2}}}. \tag{4}$$

Changing the order of integration (as we shall do again and again),

$$\tfrac{1}{2}\pi V(r) = \int_0^{\min(r,b)} \frac{ds}{(r^2-s^2)^{\frac{1}{2}}} \int_s^b \frac{\sigma(\rho)\rho}{(\rho^2-s^2)^{\frac{1}{2}}} d\rho + \int_{\max(r,b)}^\infty \frac{ds}{(s^2-r^2)^{\frac{1}{2}}} \int_b^s \frac{\sigma(\rho)\rho}{(s^2-\rho^2)^{\frac{1}{2}}} d\rho. \tag{5}$$

For $r < a$, (5) can be written

$$\tfrac{1}{2}\pi V(r) = \int_0^r \frac{f_1(s)+h_1(s)}{(r^2-s^2)^{\frac{1}{2}}} ds + \int_b^\infty \frac{f_2(s)}{(s^2-r^2)^{\frac{1}{2}}} ds, \tag{6}$$

where

$$f_1(s) = \int_s^a \frac{\sigma(\rho)\rho}{(\rho^2-s^2)^{\frac{1}{2}}} d\rho \quad \text{and} \quad h_1(s) = \int_a^b \frac{\sigma_0(\rho)\rho}{(\rho^2-s^2)^{\frac{1}{2}}} d\rho \quad \text{for} \quad s < a, \tag{7}$$

$$f_2(s) = \int_b^s \frac{\sigma(\rho)\rho}{(s^2-\rho^2)^{\frac{1}{2}}} d\rho \quad \text{and} \quad h_2(s) = \int_a^b \frac{\sigma_0(\rho)\rho}{(s^2-\rho^2)^{\frac{1}{2}}} d\rho \quad \text{for} \quad s > b. \tag{8}$$

We have imposed (2) in (7) and (8); h_1 and h_2 are thus known functions. (It is convenient to define h_2 here, although it is not needed until (13) below.)

Now imposing part of (3), (6) can be written

$$\int_0^r \frac{f_1(s)+h_1(s)}{(r^2-s^2)^{\frac{1}{2}}} ds = \tfrac{1}{2}\pi V_1(r) - \int_b^\infty \frac{f_2(s)}{(s^2-r^2)^{\frac{1}{2}}} ds \quad \text{for} \quad r < a. \tag{9}$$

Solving this as an Abel's integral equation for f_1+h_1, and rearranging,

$$f_1(r) + \frac{2}{\pi}\frac{d}{dr}\int_0^r \frac{t\,dt}{(r^2-t^2)^{\frac{1}{2}}} \int_b^\infty \frac{f_2(s)}{(s^2-t^2)^{\frac{1}{2}}} ds = g_1(r) \quad \text{for} \quad r < a, \tag{10}$$

where, using (7), g_1 is the known function given by

$$g_1(r) = -\int_a^b \frac{\sigma_0(\rho)\rho}{(\rho^2-r^2)^{\frac{1}{2}}} d\rho + \frac{d}{dr}\int_0^r \frac{V_1(t)\,t}{(r^2-t^2)^{\frac{1}{2}}} dt \quad \text{for} \quad r < a. \tag{11}$$

Changing the order of integration in (10), using (A 3) and then differentiating, we obtain

$$f_1(r) + \frac{2}{\pi}\int_b^\infty \frac{s}{s^2-r^2} f_2(s)\, ds = g_1(r) \quad \text{for} \quad r < a. \tag{12}$$

https://doi.org/10.1017/S0305004100048969 Published online by Cambridge University Press

Equations (4) and (5) are equally true with b replaced by a. Supposing that $r > b$ this gives, parallel to (6),

$$\tfrac{1}{2}\pi V(r) = \int_0^a \frac{f_1(s)}{(r^2-s^2)^{\frac{1}{2}}}\,ds + \int_r^\infty \frac{f_2(s)+h_2(s)}{(s^2-r^2)^{\frac{1}{2}}}\,ds; \tag{13}$$

here we have used (7) and (8). Imposing the remaining requirement (3) of the Neumann problem, (13) becomes, like (9),

$$\int_r^\infty \frac{f_2(s)+h_2(s)}{(s^2-r^2)^{\frac{1}{2}}}\,ds = \tfrac{1}{2}\pi V_2(r) - \int_0^a \frac{f_1(s)}{(r^2-s^2)^{\frac{1}{2}}}\,ds \quad \text{for} \quad r > b. \tag{14}$$

Applying to (14) a standard formula for solving it as an infinite Abel's integral equation for $f_2 + h_2$, and rearranging,

$$f_2(r)+h_2(r) = -\frac{d}{dr}\int_r^\infty \frac{V_2(t)\,t}{(t^2-r^2)^{\frac{1}{2}}}\,dt + \frac{2}{\pi}\frac{d}{dr}\int_r^\infty \frac{t\,dt}{(t^2-r^2)^{\frac{1}{2}}}\int_0^a \frac{f_1(s)}{(t^2-s^2)^{\frac{1}{2}}}\,ds.$$

Formal simplification of this on the same pattern as led from (10) to (12) is pointless, however, because it leads to a divergent inner integral.

Instead we rewrite (14), adding to each side

$$\frac{2}{\pi}\int_r^\infty \frac{s\,ds}{(s^2-r^2)^{\frac{1}{2}}(s^2-a^2)}\int_0^a f_1(t)\,dt = \frac{1}{(r^2-a^2)^{\frac{1}{2}}}\int_0^a f_1(t)\,dt, \tag{15}$$

an equation which follows from (A 4). In place of (14) we thus have

$$\int_r^\infty \left(f_2(s)+h_2(s)+\frac{2}{\pi}\,\frac{s}{s^2-a^2}\int_0^a f_1(t)\,dt\right)\frac{ds}{(s^2-r^2)^{\frac{1}{2}}} = \tfrac{1}{2}\pi V_2(r) + \int_0^a \left(\frac{1}{(r^2-a^2)^{\frac{1}{2}}} - \frac{1}{(r^2-s^2)^{\frac{1}{2}}}\right) f_1(s)\,ds \tag{16}$$

for $r > b$. Formally solving this Abel's integral equation,

$$f_2(r)+h_2(r)+\frac{2}{\pi}\frac{r}{r^2-a^2}\int_0^a f_1(t)\,dt = -\frac{d}{dr}\int_r^\infty \frac{V_2(t)\,t}{(t^2-r^2)^{\frac{1}{2}}}\,dt - \frac{2}{\pi}\frac{d}{dr}\int_r^\infty \frac{t\,dt}{(t^2-r^2)^{\frac{1}{2}}}\int_0^a \left(\frac{1}{(t^2-a^2)^{\frac{1}{2}}} - \frac{1}{(t^2-s^2)^{\frac{1}{2}}}\right) f_1(s)\,ds \tag{17}$$

$$= -\frac{d}{dr}\int_r^\infty \frac{V_2(t)\,t}{(t^2-r^2)^{\frac{1}{2}}}\,dt - \frac{2}{\pi}\int_0^a \left(\frac{r}{r^2-s^2} - \frac{r}{r^2-a^2}\right) f_1(s)\,ds. \tag{18}$$

The simplification from (17) to (18) uses (A 5) and does not involve a divergent integral.

Rewriting (12) and (18) we have, for the unknown functions f_1 and f_2, the simultaneous integral equations

$$f_1(r) + \frac{2}{\pi}\int_b^\infty \frac{s}{s^2-r^2}\,f_2(s)\,ds = g_1(r) \quad \text{for} \quad r < a, \tag{19}$$

$$f_2(r) + \frac{2}{\pi}\int_0^a \frac{r}{r^2-s^2}\,f_1(s)\,ds = g_2(r) \quad \text{for} \quad r > b; \tag{20}$$

https://doi.org/10.1017/S0305004100048969 Published online by Cambridge University Press

where g_1 and g_2 are known functions given, using (11), (18) and (8), by

$$g_1(r) = -\int_a^b \frac{\sigma_0(t)\,t}{(t^2-r^2)^{\frac{1}{2}}}\,dt + \frac{d}{dr}\int_0^r \frac{V_1(t)\,t}{(r^2-t^2)^{\frac{1}{2}}}\,dt \quad \text{for} \quad r < a, \tag{21}$$

$$g_2(r) = -\int_a^b \frac{\sigma_0(t)\,t}{(r^2-t^2)^{\frac{1}{2}}}\,dt - \frac{d}{dr}\int_r^\infty \frac{V_2(t)\,t}{(t^2-r^2)^{\frac{1}{2}}}\,dt \quad \text{for} \quad r > b. \tag{22}$$

Equations (19) to (22) constitute an intermediate formulation of the Neumann problem, comparable with the various sets of simultaneous integral equations obtained by other writers. Comments on these will be made in section 4.

3. *Further reduction of the 'Neumann problem'.* We now reduce equations (19) and (20) by writing

$$c = \sqrt{(ab)} \quad \text{and} \quad k = \sqrt{(a/b)}, \tag{23}$$

together with $r = cx$ and $s = c/t$ in (19), and $r = c/x$ and $s = ct$ in (20). The equations so obtained can be written

$$f_1(cx) + \frac{2}{\pi}\int_0^k \frac{1}{1-x^2t^2}\frac{f_2(c/t)}{t}\,dt = g_1(cx) \quad \text{for} \quad 0 < x < k,$$

$$\frac{f_2(c/x)}{x} + \frac{2}{\pi}\int_0^k \frac{1}{1-x^2t^2} f_1(ct)\,dt = \frac{g_2(c/x)}{x} \quad \text{for} \quad 0 < x < k;$$

then by adding and subtracting these we obtain the uncoupled equations

$$f_+(x) + \frac{2}{\pi}\int_0^k \frac{1}{1-x^2t^2} f_+(t)\,dt = g_+(x) \quad \text{for} \quad 0 < x < k, \tag{24}$$

$$f_-(x) - \frac{2}{\pi}\int_0^k \frac{1}{1-x^2t^2} f_-(t)\,dt = g_-(x) \quad \text{for} \quad 0 < x < k, \tag{25}$$

where $f_\pm$ are unknown functions, and $g_\pm$ known functions, given by

$$f_\pm(x) = f_1(cx) \pm \frac{f_2(c/x)}{x}, \qquad g_\pm(x) = g_1(cx) \pm \frac{g_2(c/x)}{x}. \tag{26}$$

If (24) and (25) have been solved, f_1 and f_2 are readily found from (26), and then σ by inverting the Abel transforms in (7) and (8). Then (1) gives the required potential function $V(r, z)$.

A sufficient condition for existence of unique continuous solutions of both (24) and (25), and for approximating them by iteration with uniform convergence, is that $\|K\| < 1$, where

$$\|K\| = \sup_{0<x<k} \int_0^k |K(x,t)|\,dt \qquad \text{and} \qquad K(x,t) = \frac{2}{\pi}\frac{1}{1-x^2t^2}. \tag{27}$$

It is easily shown that

$$\|K\| = \frac{1}{\pi k}\log\frac{1+k^2}{1-k^2}, \tag{28}$$

whence $\|K\| < 1$ for all a and b such that $a/b = k^2 \leqslant 0{\cdot}903$.

https://doi.org/10.1017/S0305004100048969 Published online by Cambridge University Press

4. *Equations* (19) *to* (22) *related to previous investigations.* If we eliminate f_2 between (19) and (20) we obtain for f_1 an integral equation of the same form as the one given by Collins ((3), last equation but one on p. 245). His derivation of it comes, however, from very different sources, and his unknown function g seems not to have any simple connexion with our f_1 and f_2.

The axisymmetric Neumann problem we have discussed is the case $n = 0$ and $\alpha = \frac{1}{2}$ of Cooke's paper (5), for which he gives two distinct reductions. Both reductions lead to equations not unlike (19) and (20), but a little less simple; for instance his equation (23) may be compared with our (19) and (21) by re-writing it, in the particular case, as

$$f_1(x) + \frac{2}{\pi}\int_b^\infty \frac{t}{t^2-x^2}\left(\frac{t^2-a^2}{a^2-x^2}\right)^{\frac{1}{2}} f_2(t)\,dt$$

$$= -\frac{2}{\pi}\int_a^b \frac{t}{t^2-x^2}\left(\frac{t^2-a^2}{a^2-x^2}\right)^{\frac{1}{2}} \sigma_0(t)\,dt - \frac{2}{\pi x}\frac{d}{dx}\int_x^a \frac{s\,ds}{(s^2-x^2)^{\frac{1}{2}}}\frac{d}{ds}\int_0^s \frac{V_1(t)\,t}{(s^2-t^2)^{\frac{1}{2}}}\,dt$$

for $x < a$. Similar comparisons may be made of our equations (20) and (22) with Cooke's (22) (with $(x^2-a^2)^{-\alpha}$ amended to $(x^2-b^2)^{-\alpha}$); and of our equations with his (30) and (31), which are the outcome of his second reduction. More will be said in section 7 about this second reduction.

Equations presented in (2), (4), (6) may be thought to show some resemblance to (19) and (20); but they refer not to the Neumann but to the Dirichlet problem. They will be discussed in sections 7 and 8.

The method used in section 3 is mentioned by Cooke ((5), section 7) as being due to Noble ((4), p. 361).

5. *The 'Dirichlet problem'.* This includes the famous problem of the electrified annular disc, which is the case $\sigma_1 = 0 = \sigma_2$ if we interpret V as potential and σ_0 as surface density of charge on the disc, with suitable units. As before we represent $V(r, z)$ by (1) and write $V(r)$ for $V(r, 0)$. To fulfil the conditions of the Dirichlet problem, we hope to make

$$V(r) = V_0(r) \quad \text{for} \quad a < r < b, \tag{29}$$

$$\sigma(r) = \sigma_1(r) \quad \text{for} \quad r < a, \qquad \sigma(r) = \sigma_2(r) \quad \text{for} \quad r > b. \tag{30}$$

We assume that V_0 is differentiable, with derivative absolutely continuous, in $[a, b]$. It is then expressible in the form

$$V_0(r) = V_3(r) + V_4(r) \quad \text{for} \quad a \leqslant r \leqslant b, \tag{31}$$

where V_3 and V_4 are also differentiable with derivatives V_3' and V_4' absolutely continuous in $[a, b]$, and further

$$V_3(a) = 0 = V_3'(a), \qquad V_4(b) = 0 = V_4'(b). \tag{32}$$

We extend these functions outside $[a, b]$ by defining

$$V_3(r) = 0 \quad \text{for} \quad r < a, \qquad V_4(r) = 0 \quad \text{for} \quad r > b. \tag{33}$$

https://doi.org/10.1017/S0305004100048969 Published online by Cambridge University Press

These properties ensure the existence of subsequent expressions involving V_3 and V_4. That such a dissection of V_0 exists, is demonstrated by

$$V_3(r) = \tfrac{1}{2}V_0(r) - \{\tfrac{1}{2}V_0(a) + A\}\left(\frac{b-r}{b-a}\right)^2 + A\left(\frac{b-r}{b-a}\right)^3 + \{\tfrac{1}{2}V_0(b) + B\}\left(\frac{r-a}{b-a}\right)^2 - B\left(\frac{r-a}{b-a}\right)^3 \quad \text{for} \quad a \leqslant r \leqslant b,$$

where $A = V_0(a) + \tfrac{1}{2}(b-a)\,V_0'(a), \qquad B = V_0(b) - \tfrac{1}{2}(b-a)\,V_0'(b),$

and of course $V_4(r) = V_0(r) - V_3(r) \quad \text{for} \quad a \leqslant r \leqslant b.$

We continue to associate the subscripts 0, 1 and 2 with functions defined on (a, b), $(0, a)$ and (b, ∞) respectively; and now we also associate the subscripts 3 and 4 with functions defined on $(0, b)$ and (a, ∞).

We express the unknown function σ_0, the restriction of σ to $[a, b]$, as a sum

$$\sigma_0(r) = \sigma_3(r) + \sigma_4(r) \quad \text{for} \quad a \leqslant r \leqslant b, \tag{34}$$

where σ_3 on $(0, b)$ and σ_4 on (a, ∞) are unknown functions which are to satisfy (37) and (38) below. Copson's Integral (A 1) with (1), (30) and (34) gives

$$\begin{aligned}\tfrac{1}{2}\pi V(r) = {} & \int_0^a \{\sigma_1(\rho) - \sigma_3(\rho)\}\rho\,d\rho \int_{\max(r,\rho)}^{\infty} \frac{ds}{(s^2-r^2)^{\frac{1}{2}}(s^2-\rho^2)^{\frac{1}{2}}} \\ & + \int_0^b \sigma_3(\rho)\rho\,d\rho \int_0^{\min(r,\rho)} \frac{ds}{(r^2-s^2)^{\frac{1}{2}}(\rho^2-s^2)^{\frac{1}{2}}} \\ & + \int_b^{\infty} \{\sigma_2(\rho) - \sigma_4(\rho)\}\rho\,d\rho \int_0^{\min(r,\rho)} \frac{ds}{(r^2-s^2)^{\frac{1}{2}}(\rho^2-s^2)^{\frac{1}{2}}} \\ & + \int_a^{\infty} \sigma_4(\rho)\rho\,d\rho \int_{\max(r,\rho)}^{\infty} \frac{ds}{(s^2-r^2)^{\frac{1}{2}}(s^2-\rho^2)^{\frac{1}{2}}}\end{aligned} \tag{35}$$

$$\begin{aligned} = {} & \int_r^{\infty} \frac{ds}{(s^2-r^2)^{\frac{1}{2}}} \int_0^{\min(a,s)} \frac{\sigma_1(\rho) - \sigma_3(\rho)}{(s^2-\rho^2)^{\frac{1}{2}}}\rho\,d\rho + \int_0^{\min(r,b)} \frac{ds}{(r^2-s^2)^{\frac{1}{2}}} \int_s^b \frac{\sigma_3(\rho)}{(\rho^2-s^2)^{\frac{1}{2}}}\rho\,d\rho \\ & + \int_0^r \frac{ds}{(r^2-s^2)^{\frac{1}{2}}} \int_{\max(b,s)}^{\infty} \frac{\sigma_2(\rho) - \sigma_4(\rho)}{(\rho^2-s^2)^{\frac{1}{2}}}\rho\,d\rho + \int_{\max(r,a)}^{\infty} \frac{ds}{(s^2-r^2)^{\frac{1}{2}}} \int_a^s \frac{\sigma_4(\rho)}{(s^2-\rho^2)^{\frac{1}{2}}}\rho\,d\rho. \end{aligned} \tag{36}$$

We hope to satisfy the remaining boundary condition (29) on $a < r < b$ by equating the sum of the second and third terms in (36) to $\frac{1}{2}\pi V_3(r)$ in $r < b$, and the sum of the first and fourth to $\frac{1}{2}\pi V_4(r)$ in $r > a$; this will accord with (31). So we require that

$$\int_0^r \frac{ds}{(r^2-s^2)^{\frac{1}{2}}} \left(\int_b^{\infty} \frac{\sigma_2(\rho) - \sigma_4(\rho)}{(\rho^2-s^2)^{\frac{1}{2}}}\rho\,d\rho + \int_s^b \frac{\sigma_3(\rho)}{(\rho^2-s^2)^{\frac{1}{2}}}\rho\,d\rho \right) = \tfrac{1}{2}\pi V_3(r) \quad \text{for} \quad r < b, \tag{37}$$

$$\int_r^{\infty} \frac{ds}{(s^2-r^2)^{\frac{1}{2}}} \left(\int_0^a \frac{\sigma_1(\rho) - \sigma_3(\rho)}{(s^2-\rho^2)^{\frac{1}{2}}}\rho\,d\rho + \int_a^s \frac{\sigma_4(\rho)}{(s^2-\rho^2)^{\frac{1}{2}}}\rho\,d\rho \right) = \tfrac{1}{2}\pi V_4(r) \quad \text{for} \quad r > a. \tag{38}$$

https://doi.org/10.1017/S0305004100048969 Published online by Cambridge University Press

Solving these Abel's integral equations,

$$\int_b^\infty \frac{\sigma_2(\rho)-\sigma_4(\rho)}{(\rho^2-r^2)^{\frac{1}{2}}}\rho\,d\rho+\int_r^b \frac{\sigma_3(\rho)}{(\rho^2-r^2)^{\frac{1}{2}}}\rho\,d\rho=\chi_3(r)=\frac{d}{dr}\int_0^r \frac{V_3(t)\,t}{(r^2-t^2)^{\frac{1}{2}}}dt \quad \text{for} \quad r<b, \tag{39}$$

$$\int_0^a \frac{\sigma_1(\rho)-\sigma_3(\rho)}{(r^2-\rho^2)^{\frac{1}{2}}}\rho\,d\rho+\int_a^r \frac{\sigma_4(\rho)}{(r^2-\rho^2)^{\frac{1}{2}}}\rho\,d\rho=\chi_4(r)=-\frac{d}{dr}\int_r^\infty \frac{V_4(t)\,t}{(t^2-r^2)^{\frac{1}{2}}}dt \quad \text{for} \quad r>a, \tag{40}$$

χ_3 and χ_4 being known functions defined by the equalities on the right. The equalities on the left can in turn be treated as Abel's integral equations. Thus, solving (39),

$$\begin{aligned}\tfrac{1}{2}\pi r\sigma_3(r)&=-\frac{d}{dr}\int_r^b \frac{\chi_3(t)\,t}{(t^2-r^2)^{\frac{1}{2}}}dt+\frac{d}{dr}\int_r^b \frac{t\,dt}{(t^2-r^2)^{\frac{1}{2}}}\int_b^\infty \frac{\sigma_2(\rho)-\sigma_4(\rho)}{(\rho^2-t^2)^{\frac{1}{2}}}\rho\,d\rho\\&=-\frac{d}{dr}\int_r^b \frac{\chi_3(s)\,s}{(s^2-r^2)^{\frac{1}{2}}}ds-\int_b^\infty \{\sigma_2(\rho)-\sigma_4(\rho)\}\frac{\rho r}{\rho^2-r^2}\frac{(\rho^2-b^2)^{\frac{1}{2}}}{(b^2-r^2)^{\frac{1}{2}}}d\rho\end{aligned}$$

for $r<b$, using (A 2); and similarly for (40), using (A 2) with a slight difference. These equations can be written

$$\sigma_3(r)-\frac{2}{\pi}\int_b^\infty \frac{t}{t^2-r^2}\left(\frac{t^2-b^2}{b^2-r^2}\right)^{\frac{1}{2}}\sigma_4(t)\,dt=\omega_3(r) \quad \text{for} \quad r<b, \tag{41}$$

$$\sigma_4(r)-\frac{2}{\pi}\int_0^a \frac{t}{r^2-t^2}\left(\frac{a^2-t^2}{r^2-a^2}\right)^{\frac{1}{2}}\sigma_3(t)\,dt=\omega_4(r) \quad \text{for} \quad r>a, \tag{42}$$

where ω_3 and ω_4 are the known functions

$$\omega_3(r)=-\frac{2}{\pi}\int_b^\infty \frac{t}{t^2-r^2}\left(\frac{t^2-b^2}{b^2-r^2}\right)^{\frac{1}{2}}\sigma_2(t)\,dt-\frac{2}{\pi r}\frac{d}{dr}\int_r^b \frac{s\,ds}{(s^2-r^2)^{\frac{1}{2}}}\frac{d}{ds}\int_0^s \frac{V_3(t)\,t}{(s^2-t^2)^{\frac{1}{2}}}dt, \tag{43}$$

$$\omega_4(r)=-\frac{2}{\pi}\int_0^a \frac{t}{r^2-t^2}\left(\frac{a^2-t^2}{r^2-a^2}\right)^{\frac{1}{2}}\sigma_1(t)\,dt-\frac{2}{\pi r}\frac{d}{dr}\int_a^r \frac{s\,ds}{(r^2-s^2)^{\frac{1}{2}}}\frac{d}{ds}\int_s^\infty \frac{V_4(t)\,t}{(t^2-s^2)^{\frac{1}{2}}}dt, \tag{44}$$

and we have substituted for χ_3 and χ_4 from (39) and (40).

There is one slightly unsatisfactory element in equations (41) to (44) as a formulation of the Dirichlet problem. This is that there is a certain arbitrariness in V_3 and V_4, since they have only to satisfy the requirements associated with (31) to (33). This arbitrariness is no doubt reflected in σ_3 and σ_4, although their sum (34) should be unique. A similar arbitrariness is however present in most other treatments, since they use the 'Laurent series dissection' (see (58) and (60) below).

6. *Further reduction of the 'Dirichlet problem'.* The first stage in solving (41) and (42) for σ_3 and σ_4 is to restrict (41) to $r<a$ and (42) to $r>b$, and to consider only the resulting 'restricted equations (41) and (42)'. The second stage is to substitute their solutions into the integrals in (41) and (42), in order to obtain σ_3 and σ_4 in their whole domains.

For the first stage, the restriction of r to $(0,a)$ and (b,∞) permits us to use the self-inverse monotonic substitutions

$$r=b\left(\frac{a^2-\rho^2}{b^2-\rho^2}\right)^{\frac{1}{2}}, \qquad t=b\left(\frac{a^2-\tau^2}{b^2-\tau^2}\right)^{\frac{1}{2}}, \tag{45}$$

https://doi.org/10.1017/S0305004100048969 Published online by Cambridge University Press

which map $(0,a)$ onto $(0,a)$ and (b,∞) onto (b,∞). Defining, in conjunction with (45), unknown functions f_1 and f_2, and known functions g_1 and g_2, by

$$\left.\begin{aligned} f_1(\rho) &= (b^2-\rho^2)^{-\frac{3}{2}}\sigma_3(r), \qquad g_1(\rho) = (b^2-\rho^2)^{-\frac{3}{2}}\omega_3(r), \qquad \text{for} \quad \rho < a, \\ f_2(\rho) &= (\rho^2-b^2)^{-\frac{3}{2}}\sigma_4(r), \qquad g_2(\rho) = (\rho^2-b^2)^{-\frac{3}{2}}\omega_4(r), \qquad \text{for} \quad \rho > b, \end{aligned}\right\} \tag{46}$$

the restricted equations (41) and (42) simplify (quite remarkably) to

$$f_1(\rho) - \frac{2}{\pi}\int_b^\infty \frac{\tau}{\tau^2-\rho^2} f_2(\tau)\,d\tau = g_1(\rho) \quad \text{for} \quad \rho < a, \tag{47}$$

$$f_2(\rho) - \frac{2}{\pi}\int_0^a \frac{\tau}{\rho^2-\tau^2}\frac{\tau}{\rho} f_1(\tau)\,d\tau = g_2(\rho) \quad \text{for} \quad \rho > b. \tag{48}$$

To uncouple equations (47) and (48) we write, as before

$$c = \sqrt{(ab)} \quad \text{and} \quad k = \sqrt{(a/b)} \tag{23}$$

together with $\rho = cx$ and $\tau = c/t$ in (47), and $\rho = c/x$ and $\tau = ct$ in (48). In similar fashion to our derivation of (24) and (25) we now obtain

$$f_+(x) - \frac{2}{\pi}\int_0^k \frac{xt}{1-x^2t^2} f_+(t)\,dt = g_+(x) \quad \text{for} \quad 0 < x < k, \tag{49}$$

$$f_-(x) + \frac{2}{\pi}\int_0^k \frac{xt}{1-x^2t^2} f_-(t)\,dt = g_-(x) \quad \text{for} \quad 0 < x < k, \tag{50}$$

where $f_\pm$ are unknown functions, and $g_\pm$ known functions, given by

$$f_\pm(x) = xf_1(cx) \pm \frac{f_2(c/x)}{x^2}, \qquad g_\pm(x) = xg_1(cx) \pm \frac{g_2(c/x)}{x^2}. \tag{51}$$

If (49) and (50) have been solved, f_1 and f_2 are readily found from (51).

We now come to the second stage mentioned above, in which we use these results to provide solutions for the unrestricted equations (41) and (42). Changing the variables in the integrals in these equations by

$$t = b\frac{(\tau^2-a^2)^{\frac{1}{2}}}{(\tau^2-b^2)^{\frac{1}{2}}} \quad \text{for} \quad \tau > b, \qquad t = b\frac{(a^2-\tau^2)^{\frac{1}{2}}}{(b^2-\tau^2)^{\frac{1}{2}}} \quad \text{for} \quad \tau < a \tag{52}$$

respectively, as in (45), and using (46), equations (41) and (42) become

$$\sigma_3(r) = \omega_3(r) + \frac{2}{\pi}\frac{b^2(b^2-a^2)^{\frac{3}{2}}}{(b^2-r^2)^{\frac{1}{2}}}\int_b^\infty \frac{b\tau}{(b^2-r^2)\tau^2 + (r^2-a^2)b^2} f_2(\tau)\,d\tau \quad \text{for} \quad r < b, \tag{53}$$

$$\sigma_4(r) = \omega_4(r) + \frac{2}{\pi}\frac{b^2(b^2-a^2)^{\frac{3}{2}}}{(r^2-a^2)^{\frac{1}{2}}}\int_0^a \frac{\tau^2}{(b^2-r^2)\tau^2 + (r^2-a^2)b^2} f_1(\tau)\,d\tau \quad \text{for} \quad r > a. \tag{54}$$

These equations both hold for $a < r < b$. Adding them we obtain, by (34), that $\sigma_0(r)$ in $a < r < b$ is equal to the sum of the right sides of (53) and (54). Here ω_3 and ω_4 are given by (43) and (44) in terms of the known functions σ_1 and σ_2, V_3 and V_4; and f_1 and f_2 are determined, as already indicated, by (49), (50) and (51).

Thus knowing σ_0, the desired potential $V(r,z)$ is given by (1).

https://doi.org/10.1017/S0305004100048969 Published online by Cambridge University Press

As at (27), a sufficient condition for existence of unique continuous solutions of both (49) and (50), and for approximating them by iteration with uniform convergence, is that $\|K\| < 1$, where

$$\|K\| = \sup_{0<x<k} \int_0^k |K(x,t)|\, dt \qquad \text{and} \qquad K(x,t) = \frac{2}{\pi} \frac{xt}{1-x^2t^2}. \tag{55}$$

It is easily shown that

$$\|K\| = \frac{1}{\pi k} \log \frac{1}{1-k^4}, \tag{56}$$

whence $\|K\| < 1$ for all a and b such that $a/b = k^2 \leqslant 0{\cdot}977$.

7. *Equations* (41) *to* (44) *related to previous investigations.* The axisymmetric Dirichlet problem we have just discussed is the case $\nu = 0$ of Noble's treatment of triple integral equations ((4), section 5). The left sides of our equations (41) and (42) may be transformed, apart from a known factor, into the corresponding terms of Noble's equations ((4), (51) and (50) with $\nu = 0$) by the substitutions

$$\sigma_3(r) = (b^2-r^2)^{-\frac{1}{2}}(a^2-r^2)^{-\frac{1}{2}} f_1(r), \qquad \sigma_4(r) = (r^2-b^2)^{-\frac{1}{2}}(r^2-a^2)^{-\frac{1}{2}} f_2(r), \tag{57}$$

provided that we restrict (41) to $r < a$ and (42) to $r > b$ (as in section 6 above) and observe that Noble uses b for the inner radius of the annulus and a for the outer. The remaining terms of Noble's equations are different from our (43) and (44), partly because of our dissection (31) and partly because Noble supposes that the known functions σ_1 and σ_2 both vanish.

Cooke (5) considers more general problems than Noble, but uses the same method. Except for the terms involving V_3 and V_4, our equations (41) and (42) agree with the final equations ((5), (32) and (33)) of Cooke's second reduction in the case $n = 0$ and $\alpha = -\frac{1}{2}$. In contrast with Cooke and Noble, our equations are derived directly from (1) without the intervention of Bessel functions and triple integral equations.

In his second reduction ((5), sections 2 and 5) Cooke, like several other workers, makes the assumption that (in our notation)

$$V_0(r) = \sum_{n=0}^{\infty} a_n r^n + \sum_{n=1}^{\infty} a_{-n} r^{-n} \quad \text{for} \quad a < r < b. \tag{58}$$

This requires that $V_0(r)$ be infinitely differentiable (and more) in $a < r < b$, since it is to agree with a function regular in an annulus $a < |r| < b$ in the complex r-plane. This 'Laurent series dissection' is thus heavily restrictive; it excludes for instance such simple functions as $V_0(r) = \sqrt{r}$. In contrast our assumption, in (31) to (33), requires rather less than that $V_0(r)$ have continuous second derivative in $a \leqslant r \leqslant b$.

The Laurent series dissection of g in (5), sections 2 and 5 permits the possibility that $g_2(x) = 1/x$ for $x > a$. This makes Cooke's definition of G_1 at the top of page 311, following his (33), meaningless when $n = 0$ and $\alpha = -\frac{1}{2}$ because the integral in it is divergent. Further, for $n = 0$ and $0 > \alpha \geqslant -\frac{1}{2}$ that definition

$$G_1(s) = -\frac{d}{ds} \int_s^{\infty} \frac{x g_2(x)}{(x^2-s^2)^{-\alpha}} dx, \quad \text{where} \quad g_2(x) = \sum_{n=1}^{\infty} a_{-n} x^{-n}, \tag{59}$$

is indeed meaningless whenever the coefficient a_{-1} is non-zero and the other coefficients are small enough for term-by-term integration to be correct.

https://doi.org/10.1017/S0305004100048969 Published online by Cambridge University Press

The same objection may be made to the definition of g_2, which corresponds to Cooke's G_1, in (2), p. 479, equation (3·3); and to the definition of g_1 in (3), p. 238, an equation between (2·11) and (2·12). It may also be made to the definition of g_2 in (6), p. 319, equation (9), in the axisymmetric case $n = 0$.

On the other hand, Cooke's examples ((5), sections 8·1 and 8·2) use his theory only with $n = 1$, a non-axisymmetric case, and they are unaffected by the above observation. Also Cooke's first reduction ((5), section 4) and Noble's treatment ((4), section 5) do not use the Laurent series dissection and so do not involve the divergent integral.

The difficulty can be overcome by transferring the term $a_{-1}x^{-1}$ from g_2 to g_1, in Cooke's notation ((5), section 2); this removes most of the danger of divergence from the definition of G_1 on his p. 311, without introducing it into that of F_2 (on the same page). However, the restrictiveness of the Laurent series dissection still remains as a limitation on that approach. And with it is associated an unfortunate arbitrariness, in that one might have chosen to assume, in place of (58), that

$$V_0(r) = r^{-\lambda}\left(\sum_{n=0}^{\infty} a_n r^n + \sum_{n=1}^{\infty} a_{-n} r^{-n}\right) \quad \text{for} \quad a < r < b, \tag{60}$$

with λ some constant satisfying $0 < \lambda \leqslant 1$. Each such value of λ determines a different family of infinitely differentiable functions V_0, these families are all disjoint, and any of them could have been used in the various treatments mentioned without introducing the divergent integral.

8. *Equations* (47) *and* (48) *related to previous investigations.* The left sides of our equations (47) and (48) are formally identical with the corresponding terms in the equations ((2), p. 481, (4·5)) advanced by Gubenko and Mossakovskii for the Dirichlet problem in the case in which σ_1 and σ_2 both vanish. This is seen by rewriting those authors' symbols a, b, x, t, $u'_{1y}(x, 0)$ and $u'_{2x}(x, 0)$ as b, a, ρ, τ, $\rho f_1(\rho)$ and $\rho f_2(\rho)$ respectively. A similar remark applies to the equations of Williams ((6), p. 320, (18) and (23)), in the case which is common to his investigation and ours, that is, the case where his n and our σ_1 and σ_2 all vanish.

On the other hand, the right sides of our equations differ from the corresponding terms in (2) and (6). The differences do arise partly from using (31) instead of the Laurent series dissection, but also partly from the different way in which our unknown functions f_1 and f_2 are constructed from the entities in the original problem.

More explicitly, Williams's unknown functions h_1 and h_2 are defined in (6), (10) and (11), and also (1) to (4). For $n = 0$ they are, replacing his a, b, $f_1(t)$ and $f_2(t)$ by b, a, $\sigma_5(t)\,t$ and $\sigma_6(t)\,t$,

$$h_2(\rho) = \int_0^{\rho} \frac{\sigma_5(t)\,t}{(\rho^2 - t^2)^{\frac{1}{2}}}\,dt \quad \text{for} \quad \rho < a, \qquad h_1(\rho) = \int_{\rho}^{\infty} \frac{\sigma_6(t)\,t}{(t^2 - \rho^2)^{\frac{1}{2}}}\,dt \quad \text{for} \quad \rho > b, \tag{61}$$

where σ_5 and σ_6 are certain 'continuations' outside (a, b), like our σ_3 and σ_4 in (34), of functions forming a partition of the unknown σ_0 in (a, b). If we take as unknown functions

$$w_1(\rho) = h_2(\rho)/\rho, \qquad w_2(\rho) = h_1(\rho)/\rho, \tag{62}$$

https://doi.org/10.1017/S0305004100048969 Published online by Cambridge University Press

the final equations ((6), (18) and (23)) with $n = 0$ become, using ((6), (8) and (9)),

$$w_1(\rho) - \frac{2}{\pi}\int_b^\infty \frac{\tau}{\tau^2-\rho^2} w_2(\tau)\,d\tau = -\frac{1}{2\pi^2}\frac{d}{d\rho}\int_0^b \log\left|\frac{\rho+s}{\rho-s}\right| \frac{ds}{s}\frac{d}{ds}\int_0^s \frac{t\phi_1(t)}{(s^2-t^2)^{\frac{1}{2}}}dt \quad \text{for} \quad \rho < a, \tag{63}$$

$$w_2(\rho) - \frac{2}{\pi}\int_0^a \frac{\tau}{\rho^2-\tau^2}\frac{\tau}{\rho} w_1(\tau)\,d\tau = -\frac{1}{2\pi^2\rho}\frac{d}{d\rho}\int_a^\infty \log\left|\frac{\rho+s}{\rho-s}\right| ds\frac{d}{ds}\int_s^\infty \frac{t\phi_2(t)}{(t^2-s^2)^{\frac{1}{2}}}dt \quad \text{for} \quad \rho > b, \tag{64}$$

where ϕ_1 and ϕ_2 are the Laurent series dissection of ϕ; ϕ being the function corresponding to our $2\pi V_0$.

However, it is difficult to be content with these equations. For instance, we may suppose that

$$\phi_1(t) = \frac{1}{b}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{t^2}{b^2}\right)^n = (b^2-t^2)^{-\frac{1}{2}} \quad \text{for} \quad 0 \leqslant t < b,$$

in accordance with the Laurent series assumption. We then find that the right side of (63) reduces, using (A 3) to evaluate the inner integral, to

$$-\frac{1}{2\pi^2}\frac{d}{d\rho}\int_0^b \frac{b}{b^2-s^2}\frac{1}{s}\log\left|\frac{\rho+s}{\rho-s}\right| ds$$

in which the integral is divergent.

APPENDIX: EVALUATIONS OF INTEGRALS

A 1. *Copson's Integral* ((1), Lemma 1). If $a \neq b$, $a > 0$ and $b > 0$,

$$\int_{-\pi}^{\pi} \frac{d\phi}{(a^2+b^2-2ab\cos\phi)^{\frac{1}{2}}} = 4\int_0^{\min(a,b)} \frac{dt}{(a^2-t^2)^{\frac{1}{2}}(b^2-t^2)^{\frac{1}{2}}} = 4\int_{\max(a,b)}^\infty \frac{ds}{(s^2-a^2)^{\frac{1}{2}}(s^2-b^2)^{\frac{1}{2}}};$$

the third integral follows from the second by the substitution $t = ab/s$.

A 2. If $0 \leqslant a < c \leqslant b$,

$$I = \int_a^c \frac{t\,dt}{(b^2-t^2)^{\frac{1}{2}}(t^2-a^2)^{\frac{1}{2}}} = \operatorname{arsin}\frac{(c^2-a^2)^{\frac{1}{2}}}{(b^2-a^2)^{\frac{1}{2}}},$$

$$\frac{\partial I}{\partial a} = -\frac{a}{b^2-a^2}\frac{(b^2-c^2)^{\frac{1}{2}}}{(c^2-a^2)^{\frac{1}{2}}}, \qquad \frac{\partial I}{\partial b} = -\frac{b}{b^2-a^2}\frac{(c^2-a^2)^{\frac{1}{2}}}{(b^2-c^2)^{\frac{1}{2}}}.$$

A 3. If $0 \leqslant c < a < b$,

$$I = \int_c^a \frac{t\,dt}{(b^2-t^2)^{\frac{1}{2}}(a^2-t^2)^{\frac{1}{2}}} = \operatorname{arsinh}\frac{(a^2-c^2)^{\frac{1}{2}}}{(b^2-a^2)^{\frac{1}{2}}},$$

$$\frac{\partial I}{\partial a} = \frac{a}{b^2-a^2}\frac{(b^2-c^2)^{\frac{1}{2}}}{(a^2-c^2)^{\frac{1}{2}}}, \qquad \frac{\partial I}{\partial b} = -\frac{b}{b^2-a^2}\frac{(a^2-c^2)^{\frac{1}{2}}}{(b^2-c^2)^{\frac{1}{2}}}.$$

A 4. If $0 \leqslant a < b < c$,

$$\int_b^c \frac{t\,dt}{(t^2-b^2)^{\frac{1}{2}}(t^2-a^2)} = \frac{1}{(b^2-a^2)^{\frac{1}{2}}}\operatorname{artan}\frac{(c^2-b^2)^{\frac{1}{2}}}{(b^2-a^2)^{\frac{1}{2}}}.$$

https://doi.org/10.1017/S0305004100048969 Published online by Cambridge University Press

A 5. If $0 \leqslant a < b < c$,

$$\int_c^\infty \frac{t}{(t^2-c^2)^{\frac{1}{2}}}\left(\frac{1}{(t^2-b^2)^{\frac{1}{2}}}-\frac{1}{(t^2-a^2)^{\frac{1}{2}}}\right)dt = \tfrac{1}{2}\log\frac{c^2-a^2}{c^2-b^2}.$$

REFERENCES

(1) Copson, E. T. On the problem of the electrified disc. *Proc. Edinburgh Math. Soc.* (II) **8** (1947), 14–19. MR 9 no. 318.

(2) Gubenko, V. S. and Mossakovskii, V. I. Pressure of an axially symmetric circular die on an elastic half space. *Prikl. Mat. Meh.* **24** (1960), 334–340 (Russian); translated in *J. Appl. Math. Mech.* **24** (1960), 477–486. MR 22 no. 8804.

(3) Collins, W. D. On the solution of some axisymmetric boundary value problems by means of integral equations. VIII. Potential problems for a circular annulus. *Proc. Edinburgh Math. Soc.* (II) **13** (1963), 235–246. MR 28 no. 439.

(4) Noble, B. The solution of Bessel function dual integral equations by a multiplying-factor method. *Proc. Cambridge Philos. Soc.* **59** (1963), 351–362. MR 26 no. 2842.

(5) Cooke, J. C. Some further triple integral equation solutions. *Proc. Edinburgh Math. Soc.* (II) **13** (1963), 303–316. MR 28 no. 4313.

(6) Williams, W. E. Integral equation formulation of some three-part boundary value problems. *Proc. Edinburgh Math. Soc.* (II) **13** (1963), 317–323. MR 28 no. 4314.

https://doi.org/10.1017/S0305004100048969 Published online by Cambridge University Press