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Convex dynamics with constant input

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Abstract. In Adler *et al* [Convex dynamics and applications. *Ergod. Th. & Dynam. Sys.* **25** (2005), 321–352] certain piecewise linear maps were defined in terms of a convex polytope. When the convex polytope is a simplex, the resulting map has a dual nature. On one hand it is defined on $\mathbb{R}^N$ and acts as a piecewise translation. On the other it can be viewed as a translation on the N-torus. What relates its two roles? A natural answer would be that there exists an invariant fundamental set into which all orbits under piecewise translation eventually enter. We prove this for $N = 1$ and for acute and right triangles—i.e. non-obtuse triangles. We leave open the case of obtuse triangles and higher-dimensional simplices. Another question not treated is the connectivity of the invariant fundamental sets which arise.

1. *Introduction*

Let P be a polytope in $\mathbb{R}^N$; and for each γ in P let ϕ_γ denote the map from $\mathbb{R}^N$ to $\mathbb{R}^N$ defined by

$$\phi_\gamma(x) = x + (\gamma - v(x)),$$

where $v(x)$ is the closest vertex of P to x in Euclidean norm with any fixed tie-breaking rule, which is needed to select one of the vertices if there are two or more with the same shortest distance to the point x. Except for the case P an interval, there are an infinite number of possible tie-breaking rules due to the fact that there is a continuum of points without a unique closest vertex. For the interval case there are only two tie-breaking rules, one of which we select in discussing this case in §2. Otherwise the tie-breaking rule will remain unspecified.

ORBIT BOUNDING THEOREM. *For any compact set K there exists a bounded convex set Q containing K such that for all γ in P, $\phi_\gamma Q \subset Q$. Furthermore, Q can be constructed so that for any compact subset P_0 of the interior of P the set Q is globally absorbing for the family $\{\phi_\gamma \mid \gamma \in P_0\}$: i.e. given a compact subset X of $\mathbb{R}^N$ there exists an n_0 (depending on P_0) such that if $n > n_0$ then $\phi_{\gamma_1} \cdots \phi_{\gamma_n}(X) \subset Q$ for $\gamma_1, \ldots, \gamma_n \in P_0$.*

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This result was proved in [**1**] (see Theorems 1.1 and 6.1) and in §5 of the present work where a brief description of its connection to digital printing is provided. Also see [**1**] for more details about that application and the role played by convexity.

In the orbit bounding theorem the affine space $\mathbb{R}^N$ is identified with the vector space $\mathbb{R}^N$ by an arbitrary choice of an origin. We shall make this choice specific later in the paper.

In contrast with the generality of the above theorem, we shall be concerned with the special case where:

(1) P is a simplex of positive volume with vertices $v_0, v_1, \ldots, v_N$;

(2) P_0 consists of a single point γ in the interior of P (when γ lies on the boundary of P, the dynamics of ϕ is strikingly different, and we leave it to the reader to examine this case).

In fact, after a brief discussion of the setup for arbitrary dimension, we study in §2 the case $N = 1$ and are able to give complete analysis of it. The balance of the paper is devoted to proving a theorem on the existence of what we call 'an absorbing and invariant fundamental set' for a restricted case when $N = 2$, the restriction being that P is a non-obtuse triangle.

Once γ has been chosen, we drop the subscript and write $\phi = \phi_\gamma$. For this work the terms *absorbing set* and *invariant set* will mean the following.

Definition 1.1. A set X will be called *absorbing* if every orbit enters X and none leaves it; i.e. for each $x \in \mathbb{R}^N$ there exists n such that $\phi^n x \in X$, and $\phi X \subset X$.

Definition 1.2. A set X is called *invariant* if $\phi X = X$.

Remark. We shall coin a term to distinguish the property $\phi X \subset X$ from $\phi X = X$. If $\phi X \subset X$, we call the set X *forward-inclusive*. In the orbit bounding theorem the set Q is forward-inclusive for all ϕ_γ, $\gamma \in P$. Maps like ϕ, which are piecewise translations, enjoy some easily proven properties on invariant sets which they may not on merely forward-inclusive ones; for instance, they are one-to-one almost everywhere and Lebesgue measure preserving.

Let $\mathcal{L}$ be the lattice (discrete group) generated by the vectors $(v_1 - v_0)$, $(v_2 - v_0), \ldots, (v_N - v_0)$. If two points x, x' differ by an integral combination of the vectors $(v_i - v_0)$, $i = 1, \ldots, N$, then their images ϕx, $\phi x'$ also differ by an integral combination of these vectors, the two integral combinations being the same only if $v(x) = v(x')$. As a map on $\mathbb{R}^N$ the map ϕ is defined as piecewise translations, but on the N-torus $\mathbb{T}^N = \mathbb{R}^N/\mathcal{L}$ its natural projection acts simply as a translation by γ. What is the relation of these two roles? A possible answer immediately presents itself.

CONJECTURE. *There is a fundamental† set for the lattice which is absorbing and invariant.*

We shall prove the conjecture for P a line segment and for P a non-obtuse triangle. In [**1**, §1.3] we alluded to a forthcoming proof for simplices in any dimension—a premature expectation. We leave as open problems the cases of obtuse triangles and higher-

† We recall that two points which are lattice translates of one another are called *equivalent;* and by a *fundamental set* of a lattice we mean a set no two points of which are equivalent and whose lattice translates cover $\mathbb{R}^N$.

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dimensional simplices. Related interesting results where obtained in [**3**] in the case of ergodic $\Sigma\Delta$ quantizations.

The standard lift for translation on the torus $\mathbb{T}^N$ is a translation on $\mathbb{R}^N$. The conjecture, if true, provides a lift of the same translation consisting of piecewise translation acting injectively on a bounded set.

If the conjecture fails in the case of obtuse triangles it could still hold for simplices satisfying some kind of acuteness condition.

If a fundamental set F is partitioned into a family of subsets and each subset is translated by a different lattice vector, the resulting union of translates F' is again a fundamental set: for if the translates of F' did not cover $\mathbb{R}^N$, then neither would the translates of F; and if two points of F' were lattice translates of one another, then so would two points of F. We formulate this as follows.

LEMMA 1.1. (Basic lemma) *For any γ the image of a fundamental set under the map ϕ is again a fundamental set.*

The minimal set for a translation on $\mathbb{T}^N$ which absorbs all the trajectories is the torus itself. Therefore any absorbing set for any lift to $\mathbb{R}^N$ of a translation must contain a fundamental set for the lattice defining the torus. In particular, this holds for invariant absorbing sets.

2. *Warm-up exercise for P an interval*

THEOREM. *Let P be the interval $[v_0, v_1]$ and $\mathcal{L}$ the lattice in $\mathbb{R}^1$ generated by the vector $(v_1 - v_0)$. For $\gamma \in (v_0, v_1)$ define a map ϕ by*

$$\phi(x) = x + (\gamma - v(x)), \quad x \in \mathbb{R}^1,$$

where $v(x)$ is the closest endpoint of P to x in Euclidean norm with any fixed tie-breaking rule. Then there is a fundamental set F for the lattice which is absorbing and invariant.

In other words, the connection of ϕ as a rotation of the circle to ϕ as piecewise translations on the line is via the existence of an invariant absorbing fundamental set.

Proof. For the sake of ease of description we take P to be the unit interval, in which case $v_0 = 0$, $v_1 = 1$, and $\mathcal{L} = \mathbb{Z}$. Furthermore, in specifying $\phi = \phi_\gamma$ we take for the tie-breaking rule $v(1/2) = 0$ so that

$$\phi(x) = \begin{cases} x + \gamma & \text{if } x \le 1/2, \\ x + \gamma - 1 & \text{if } x > 1/2. \end{cases}$$

We leave to the reader the straightforward modifications to the proof to accommodate an arbitrary interval and the other tie-breaking rule. The choice of the tie-breaking rule is arbitrary and only affects the boundary of the resulting fundamental set.

We assume that γ is any point in $(0, 1)$ but have drawn Figure 2.1 with γ to the left of $1/2$.

The following assertions are easy to check. If $x \le \gamma + 1/2$, then $\phi x \le \gamma + 1/2$; and if $x > \gamma - 1/2$, then $\phi x > \gamma - 1/2$. So the half open interval $(\gamma - 1/2, \gamma + 1/2]$ is

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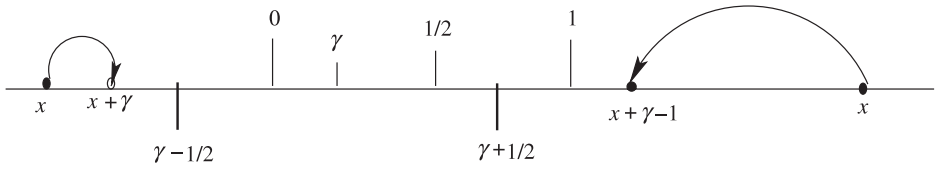


FIGURE 2.1. Interval.

invariant. We claim this interval is also absorbing: for if $x \leq \gamma - 1/2 < 1/2$, then there exists n such that $\phi^n(x) = x + n\gamma > \gamma - 1/2$; and if $x\ \gamma + 1/2 > 1/2$, then there exists n such that $\phi^n(x) = x - n(1-\gamma) \leq \gamma + 1/2$.

Since length of the half open interval $(\gamma - 1/2, \gamma + 1/2]$ is 1, it is a fundamental set for $\mathbb{Z}$. Thus $F = (\gamma - 1/2, \gamma + 1/2]$ is the absorbing invariant fundamental set promised by the theorem. □

In the next section we shall consider the question in two dimensions. However, there is a big difference which leads to difficulties. In one dimension there is only one shape for the invariant fundamental set—namely, an interval. In two dimensions there are an infinite number of different shapes and no one description fits them all.

3. *Main theorem for P a triangle*

MAIN THEOREM. *Let P be a non-obtuse triangle in $\mathbb{R}^2$ with vertices v_0, v_1, v_2 and $\mathcal{L}$ the lattice generated by the vectors $(v_1 - v_0)$, $(v_2 - v_0)$. For $\gamma \in P^\circ$ define a map ϕ by*

$$\phi(x) = x + (\gamma - v(x)), \quad x \in \mathbb{R}^2,$$

where $v(x)$ is the vertex of P closest to x in Euclidean norm with any fixed tie-breaking rule. Then there is a fundamental set F for the lattice which is absorbing and invariant.

As previously mentioned, we leave the tie-breaking rule unspecified; the proof will at once cover all the ways of choosing one.

3.1. *Facts for all triangles.*

First we shall establish some facts for all triangles, not just the non-obtuse ones.

Notation. We use the convention of viewing subscripts as elements of $\mathbb{Z}/3\mathbb{Z}$ with modulo 3 addition.

We take as *origin* o of the affine space $\mathbb{R}^2$ *the center of the circumscribed circle of* P and identify a point x of the affine space with the vector $x - o$. This applies in particular to the scalar products of the form $(v - w) \cdot x = (v - w) \cdot (x - o)$ or $v \cdot w = (v - o) \cdot (w - o)$.

A side of triangle P and its opposite vertex will share the same index; thus s_i is the segment (v_{i+1}, v_{i+2}). Denote by $s_i^\perp$ an outward normal to side s_i. Any one will do, say the unit normal. In the case where P is an acute triangle, the origin—i.e. the intersection of the perpendicular bisectors—lies in the interior of P, and we could take the vector $s_i^\perp$ to be $v_{i+1} + v_{i+2} \neq 0$; but in the case where P is a right triangle the midpoint of a side is the origin, and one of these vectors will vanish.

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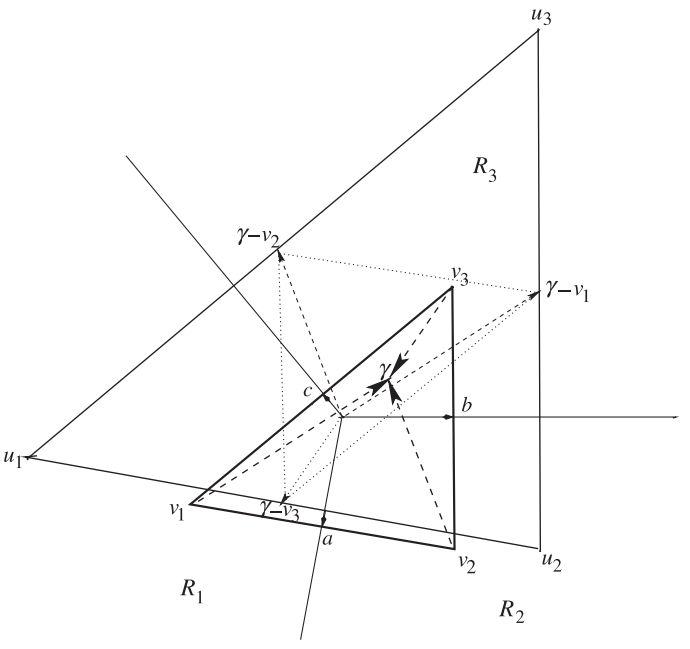


FIGURE 3.1. Choice of ϕ.

Equations that express perpendicularity of bisectors and equidistance of vertices from our chosen origin are

$$(v_i - v_j) \cdot (v_i + v_j) = 0$$
$$v_i \cdot v_i = v_j \cdot v_j.$$

See Figure 3.1.

The following are two equivalent ways of specifying the triangle P: first, as the intersection of the three open half spaces; second, as the intersection of cones,

$$p \in P \iff \text{for all } i,\ (p - v_{i+1}) \cdot s_i^{\perp} \leq 0, \tag{3.1}$$

$$p \in P \iff \text{for all } i,\ \text{there exists } a_i, b_i \geq 0 \text{ such that} \tag{3.2}$$
$$p - v_i = a_i(v_{i+1} - v_i) + b_i(v_{i+2} - v_i).$$

We shall be using (3.1) with $p = \gamma \in P^{\circ}$, in which case the inequality will be strict. In (3.2) 'for all i' could be replaced by 'for any pair of distinct indices i_1, i_2 among $\{0, 1, 2\}$'.

A vertex v_i of triangle P also satisfies the half space specification

$$(v_i - v_{i+1}) \cdot s_i^{\perp} = (v_i - v_{i+2}) \cdot s_i^{\perp} < 0.$$

Furthermore, $\gamma \in P^{\circ}$ satisfies

$$(\gamma - v_i) \cdot s_i^{\perp} > 0. \tag{3.3}$$

We define the quantity θ_i by

$$\theta_i = (\gamma - v_{i+1}) \cdot s_i^{\perp} \tag{3.4}$$

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and note by (3.1) that θ_i is negative and also satisfies

$$\theta_i = (\gamma - v_{i+2}) \cdot s_i^{\perp} < 0. \tag{3.5}$$

The *Voronoï cell* V_i° of the vertex v_i is the set of all x such that the distance from x to v_i is smaller than that of x to any other vertex, i.e.

$$V_i^{\circ} = \bigcap_{j \neq i} \{x \mid \|x - v_i\| < \|x - v_j\|\}.$$

By adding points on the boundaries of Voronoï cells according to the tie-breaking rule we obtain a partition of the space, an element of which we call a *Voronoï domain* inspired by its role as a domain of definition.

Definition 3.1. The *Voronoï domain* V_i of v_i is defined as

$$V_i = \{x \mid v(x) = v_i\},$$

where $v(x)$ is the closest vertex of P to x in Euclidean norm with a fixed tie-breaking rule.

A *simple property* of the mapping ϕ, and one we shall use repeatedly, is the following:

$$\text{if } p \in V_i, \ p' \in V_{i+1}, \text{ and } p' - p = v_{i+1} - v_i, \text{ then } \phi p' - \phi p = 0. \tag{3.6}$$

We can specify a Voronoï cell two ways. Firstly, as the intersection of a pair of open half spaces

$$V_i^{\circ} = \{x \mid (v_i - v_{i+1}) \cdot x > 0 \text{ and } (v_i - v_{i+2}) \cdot x > 0\}, \tag{3.7}$$

where the set of x with $0 = (v_i - v_{i+1}) \cdot x \equiv (v_i - v_{i+1}) \cdot (x - o)$ describes the median between the vertices v_i and v_{i+1} as o is equidistant to both vertices. Secondly, as a cone

$$x \in V_i^{\circ} \iff x = o + \alpha s_{i+1}^{\perp} + \beta s_{i+2}^{\perp} \quad \text{for some } \alpha, \beta > 0, \tag{3.8}$$

as the vectors $s^{\perp}$ are parallel to the medians. Similarly, for the closure of a Voronoï cell or domain replace the sharp inequalities $>$ by $\geq$ in the above equations. The boundaries of the Voronoï domains are

$$\partial V_i = \{o + a s_j^{\perp} \mid a \geq 0, \ j = i + 1, i + 2\}.$$

LEMMA 3.1. *Let $S(R)$ be a disk of radius R centered at the origin—i.e. the intersection of the perpendicular bisectors of the sides of P. If R is large enough, then $\phi S(R) \subset S(R)$.*

Proof. The lines tangent to the circle $\partial S(R)$ at $s_{i+1}^{\perp} \cap \partial S(R)$ and $s_{i+2}^{\perp} \cap \partial S(R)$ are parallel to sides s_{i+1} and s_{i+2}. The vector $\gamma - v_i$ based at the point of intersection of s_{i+1} and s_{i+2} lies strictly between these lines. If the vector is translated so that its base lies on $V_i \cap \partial S(R)$, then the endpoint of the vector lies on a chord parallel to $\gamma - v_i$ or possibly an extension of it. But if R_0 is large enough, then the length of these chords will be greater than $\|\gamma - v_i\|$ for $i = 0, 1, 2$, and the endpoint will lie in $S(R)$. Therefore if $x \in V_i \cap \partial S(R_0)$, then $\phi(x) = x + \gamma - v_i \in S(R_0)$. Consequently for $R > R_0 + \operatorname{diam}(P)$, if $x \in S(R)$, then $\phi(x) \in S(R)$. □

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We call attention to the fact that in the orbit bounding theorem the set Q is not a disk. That large disks are forward-inclusive is a consequence of the fact that the point γ is a compact set chosen in the interior of P.

Stronger results, which we shall not use, could be established. For example, large disks are also absorbing and undergo uniform contraction by the action of ϕ (see [**1**, §6]).

The angle at v_i is acute if and only if the angle between $s_j^\perp$ and $s_k^\perp$ is obtuse, in which case $s_j^\perp \cdot s_k^\perp < 0$; and the angle at v_i is a right angle if and only if the angle between $s_j^\perp$ and $s_k^\perp$ is also, in which case $s_j^\perp \cdot s_k^\perp = 0$.

In every triangle at least two of its angles are not obtuse. The hypothesis of the following theorem makes use of this fact.

LEMMA 3.2. *If* $s_i^\perp \cdot s_{i+1}^\perp \le 0$ *and* $s_i^\perp \cdot s_{i+2}^\perp \le 0$, *then every orbit enters the half space* $H_i \equiv \{x \mid (x - (\gamma - v_i)) \cdot s_i^\perp \le 0\}$.

The half space H_i is bounded by the line parallel to the edge (v_{i+1}, v_{i+2}) opposite to v_i and translated from the origin o by the vector $\gamma - v_i$ in such a way that it contains the original triangle (see Figure 3.2).

Proof. Suppose a point $\phi x \notin H_i$, i.e.

$$(x - (\gamma - v_i)) \cdot s_i^\perp > 0.$$

It then follows from (3.3) that

$$x \cdot s_i^\perp > 0.$$

Writing x as

$$x = a s_{i+1}^\perp + b s_{i+2}^\perp,$$

we have

$$0 < x \cdot s_i^\perp = a s_i^\perp \cdot s_{i+1}^\perp + b s_i^\perp \cdot s_{i+2}^\perp.$$

From the hypothesis it follows that either a or b or both are negative which implies $x \notin V_i$ according to (3.8).

Now suppose the orbit of x under ϕ never enters the half-space H_i, i.e.

$$(\phi^n x - (\gamma - v_i)) \cdot s_i^\perp > 0 \quad \text{for } n \ge 0.$$

Then from what has just been shown $\phi^n x \in V_{i+1} \cup V_{i+2}$. Thus

$$\phi^n x = x + n_1(\gamma - v_{i+1}) + n_2(\gamma - v_{i+2}),$$

where $n = n_1 + n_2$. Using (3.4) and (3.5), we have

$$(\phi^n x - x) \cdot s_i^\perp = n_1 \theta_{i+1} + n_2 \theta_{i+2} \to -\infty \quad \text{as } n \to \infty.$$

This contradicts the invariance of $S(R)$ established by Lemma 3.1 for large R. □

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LEMMA 3.3. *If* $s_i^\perp \cdot s_{i+1}^\perp \le 0$ *and* $s_i^\perp \cdot s_{i+2}^\perp \le 0$, *then no orbit leaves the half-space* $H_i \equiv \{x \mid (x - (\gamma - v_i)).s_i^\perp \le 0\}$.

Proof. Suppose $x \in H_i$. First, take $x \in V_i$ in which case by (3.8),

$$x = as_{i+1}^\perp + bs_{i+2}^\perp$$

where $a, b \ge 0$. Then

$$[\phi(x) - (\gamma - v_i)] \cdot s_i^\perp = (as_{i+1}^\perp + bs_{i+2}^\perp) \cdot s_i^\perp \le 0.$$

Second, take $x \in V_{i+1}$. Then

$$\phi(x) \cdot s_i^\perp = [(x + (\gamma - v_{i+1})] \cdot s_i^\perp.$$

Since $x \cdot s_i^\perp \le (\gamma - v_i) \cdot s_i^\perp$, using (3.4) we get

$$[\phi(x) - (\gamma - v_i)] \cdot s_i^\perp = (x + v_i - v_{i+1} \cdot s_i^\perp \le (\gamma - v_{i+1}) \cdot s_i^\perp = \theta_i < 0.$$

Thus $\phi(x)$ lies in H_i.

The proof for the remaining case $x \in V_{i+2}$ uses (3.5), otherwise it is entirely similar. □

3.2. *Facts for non-obtuse triangles.* From now on we assume that P is non-obtuse (see Figure 3.1) and thus enjoys additional properties such as

$$s_i^\perp \cdot s_{i+1}^\perp \le 0 \quad \text{for all } i,$$

and the origin o, which is the common extreme point of the closed Voronoï cells, belongs to P.

Consider the three extreme points (i.e. the $o + \gamma - v_i = \gamma - v_i$) of the closed images of the Voronoï cells. Figure 3.2 depicts a triangle similar to $\triangle v_0 v_1 v_2$, each side of which contains a different extreme point in its interior. This triangle† turns out to be $\triangle u_0 u_1 u_2$ whose vertices satisfy

$$u_i = \gamma + v_i - v_{i+1} - v_{i+2}. \tag{3.9}$$

When triangle $\triangle u_0 u_1 u_2$ is non-obtuse, each side determines a half space $H_i \equiv \{x \mid (x - u_i) \cdot s_{i+2}^\perp) \le 0\}$ which contains the image of the Voronoï domain V_{i+2}. Indeed, we shall see that, when it is non-obtuse, it is an absorbing set.

We have $\triangle u_0 u_1 u_2 = \triangle_0 \cup \triangle_{u_0} \cup \triangle_{u_1} \cup \triangle_{u_2}$ where

$$\triangle_0 \equiv \triangle(\gamma - v_0)(\gamma - v_1)(\gamma - v_2),$$
$$\triangle_{u_i} \equiv \triangle u_i(\gamma - v_{i+1})(\gamma - v_{i+2}).$$

Examining its vertices, one sees that $\triangle_0$ is a reflected translate of P and that because γ lies in the interior of P the origin o lies in the interior of $\triangle_0$. It follows that $\triangle_{u_0}$, $\triangle_{u_1}$, and $\triangle_{u_2}$ are congruent translates of P satisfying

$$\triangle_{u_i} = P + \gamma - v_{i+1} - v_{i+2}$$

† When the triangle symbol $\triangle$ appears subscripted with vertices—e.g. $\triangle u_0 u_1 u_2$—the set in question will be closed.

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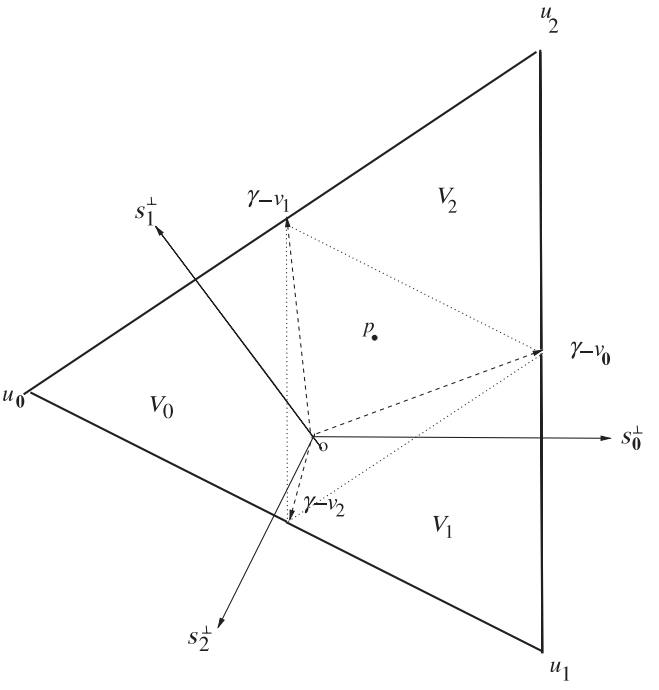


FIGURE 3.2. Invariant triangle.

and are related to each other by

$$\triangle_{u_i} = \triangle_{u_j} + (v_i - v_j). \tag{3.10}$$

Consequently, v_{i+2} is the midpoint of the side of $\triangle u_0 u_1 u_2$ having endpoints u_i, u_{i+1}, i.e.

$$(u_i + u_{i+1})/2 = \gamma - v_{i+2}; \tag{3.11}$$

and

$$u_i - u_{i+1} = 2(v_i - v_{i+1}). \tag{3.12}$$

Equation (3.9) now follows (3.11) and (3.12).

Remark. We have derived equation (3.9) from equations (3.11) and (3.12). Alternatively we could have derived (3.11) and (3.12) from (3.9) and obtained (3.9) from the algebraic relations implied by the definition of triangle $\triangle u_0 u_1 u_2$.

While it is true that the orbit of every point uniformly visits each V_i in the sense that there exists N depending only on $\|x\|$ such that $\phi^n x \in V_i$ for some $n < N$, we do not need the full strength of this assertion; we only need it for points in the triangle.

LEMMA 3.4. *Let $x \in \triangle u_0 u_1 u_2$. There exists N such that $\phi^n x \in V_i$ for some $n < N$.*

Proof. As in (3.3) and by Lemma 3.8 we have for $x \in \triangle u_0 u_1 u_2$,

$$(\phi^n x - u_i) \cdot s_i^\perp > 0. \tag{3.13}$$

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Take N_i large enough to satisfy

$$x \cdot s_i^{\perp} + N_i \cdot \theta_i < u_i \cdot s_i^{\perp}.$$

Suppose $\phi^n x \in (V_{i+1} \cup V_{i+2}) \cap \triangle u_0 u_1 u_2$ for $0 \le n \le N_i$. In other words,

$$\phi^{N_i} x = x + n_{i+1}(\gamma - v_{i+1}) + n_{i+2}(\gamma - v_{i+2}),$$

where $N_i = n_{i+1} + n_{i+2}$. From (3.4) and (3.13) we have

$$u_i \cdot s_i^{\perp} < \phi^{N_i} x \cdot s_i^{\perp} = x \cdot s_i^{\perp} + n_{i+1}\theta_i + n_{i+2}\theta_i < x \cdot s_i^{\perp} + N_i \theta_i,$$

a contradiction. Therefore $\phi^n x \in V_i$ for some $n \le N_i$. Now we can take $N = \max\{N_i \mid i = 0, 1, 2\}$. □

LEMMA 3.5. $u_i \in V_i^{\circ}$.

Proof. To prove that u_i lies in the Voronoï cell V_i° we shall show that u_i is closer to v_i than to v_j by showing the quantity $\delta = \|u_i - v_j\|^2 - \|u_i - v_j\|^2$ is strictly positive. Since $\gamma \in P^{\circ}$, we have

$$\gamma = \gamma_i v_i + \gamma_j v_j + \gamma_k v_k,$$

where $\gamma_i + \gamma_j + \gamma_j = 1,\ 0 < \gamma_i, \gamma_j, \gamma_j < 1,\ \{i, j, k\} = \{0, 1, 2\}$. We have

$$\begin{aligned}
\delta &= 2u_i \cdot (v_i - v_j) \\
&= 2(\gamma + v_i - v_j - v_k) \cdot (v_i - v_j) \\
&= 2((1 - \gamma_j)(v_i - v_j) + (1 - \gamma_k)(v_i - v_k)) \cdot (v_i - v_j) \\
&= 2(1 - \gamma_j)\|v_i - v_j\|^2 + (1 - \gamma_k)(v_i - v_k) \cdot (v_i - v_j).
\end{aligned}$$

Since P is non-obtuse, $(v_i - v_k) \cdot (v_i - v_j) \ge 0$, and so $\delta > 0$. □

LEMMA 3.6. *Exactly two midpoints of the sides of $\triangle u_0 u_1 u_2$ lie in the closure of one of the Voronoï domains.*

Proof. Let p denote the intersection of perpendicular bisectors of the sides of $\triangle u_0 u_1 u_2$. Suppose $p \in V_i$ (see Figure 3.2 where γ and p are in the same Voronoï domain, which they need not be)†. Then $\gamma - v_{i+1}$ and $\gamma - v_{i+2}$ also lie in $\overline{V_i}$; because by (3.7),

$$p \cdot (v_i - v_{i+1}) \ge 0 \quad \text{and} \quad p \cdot (v_i - v_{i+2}) \ge 0, \tag{3.14}$$

and by orthogonality,

$$(\gamma - v_{i+2} - p) \cdot (v_i - v_{i+1}) = (\gamma - v_{i+1} - p) \cdot (v_i - v_{i+2}) = 0. \tag{3.15}$$

From (3.15) and (3.14) it follows that $\gamma - v_{i+1}$ and $\gamma - v_{i+2}$ satisfy the closed version of criteria (3.7) for membership in $\overline{V_i}$. □

According to the previous lemma we may assume that $\gamma - v_0, \gamma - v_1 \in \overline{V_2}$ and $\gamma - v_2 \in \overline{V_1}$, so that for $i = 0, 1, 2, \gamma - v_i \notin V_0^{\circ}$.

† We recall that the trick in the plane geometry proof of concurrence of altitudes of a triangle, say $\triangle(\gamma - v_0)(\gamma - v_1)(\gamma - v_2)$, is to draw triangle $\triangle u_0 u_1 u_2$. See [**4**, pp. 46–48].

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LEMMA 3.7.

$$\triangle_{u_2}\backslash\{\gamma - v_1, \gamma - v_0\} \subset V_2, \tag{3.16}$$

$$\triangle_{u_1}\backslash\{\gamma - v_2\} \subset V_1 \cup V_2. \tag{3.17}$$

Proof. We have $u_2 \in V_2$ by Lemma 3.5, and the other two vertices of $\triangle_{u_2}$ are in $\overline{V_2}$ by assumption. Thus from the convexity of $\overline{V_2}$ and of triangles we have $\triangle u_2 \subset \overline{V_2}$. But the only possible points of $\triangle u_2$ which do not lie in the Voronoï domain V_2 are $\gamma - v_0$ and $\gamma - v_1$, and (3.16) follows.

The Voronoï domain V_0 is contained in the half space $H = \{x \mid x \cdot (v_0 - v_1) \geq 0\}$. The only point of $\triangle u_1$ which could possibly lie in H is $\gamma - v_2$, so (3.17) follows. □

LEMMA 3.8. *The set*

$$\triangle = (\triangle u_0 u_1 u_2 \backslash \{u_0, u_1, u_2, \gamma - v_0, \gamma - v_1, \gamma - v_2\}) \cup \{\phi(o)\}$$

is absorbing.

Proof. For any $p \in V_i \cap \triangle u_0 u_1 u_2$, the distance of the image ϕp from a side at vertex u_i is greater than the distance of p from that side—the increase being the positive distance of γ from the parallel side of P at vertex v_i. It follows that the corners u_0, u_1, u_2 and the elements of $\{\gamma - v_0, \gamma - v_1, \gamma - v_2\}\backslash\{\phi(o)\}$ do not have pre-images in $\triangle u_0 u_1 u_2$. The lemma then follows from Lemmas 3.2 and 3.3. □

The set $\triangle$ has the property that two equivalent points in it are *one-leg equivalent* and their locations are constrained—more precisely, we have the following lemma.

LEMMA 3.9. *Let $p, q \in \triangle$ be two distinct equivalent points. Then for some $i \neq j$ we have $p \in \triangle_{u_i}$ and $q \in \triangle_{u_j}$, which in turn implies that $p - q = v_i - v_j$. Conversely, if $p - q = v_i - v_j$, then $p \in \triangle_{u_i}$ and $q \in \triangle_{u_j}$.*

Proof. For each i the set $\triangle \cap \triangle_{u_i} \cup \triangle_0^\circ$ is a subset of a fundamental set. In particular,

$$\text{a point of } \triangle_0^\circ \text{ has no other equivalent point in } \triangle. \tag{3.18}$$

Also no two distinct equivalent points lie in $\triangle \cap \triangle_{u_i}$. Hence, if $p, q \in \triangle$, then $p, q \notin \triangle_0^\circ$ and do not lie the same $\triangle_{u_i}$.

The statement about the difference follows from (3.10).

To prove the converse, suppose $p \in \triangle_{u_k}$ and $q \in \triangle_{u_m}$. By what we have just proved $p - q = v_k - v_m$. It follows that $v_i - v_j = v_k - v_m$, which is only possible if $i = k$ and $j = m$. □

3.3. *Proof of the main theorem for non-obtuse triangles.* We are now in a position to be more specific about the absorbing invariant fundamental set F, the existence of which is the content of the main theorem, which we reformulate as:

> *there exists an integer $n > 0$ such that $F = \phi^n \triangle$ is an absorbing invariant fundamental set.*

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As a corollary to Lemma 3.8 we get that $\phi^n \triangle$ is absorbing for every non-negative integer n.

The set $\triangle$ contains a fundamental set—namely, the parallelogram spanned by u_0, $\gamma - v_2$, $\gamma - v_0$, $\gamma - v_1$ modulo removing appropriate boundary points. Thus by Lemma 1.1 we have that $\phi^n \triangle$ contains a fundamental set for any integer n.

All that remains to prove is that it does not contain anything more—namely,

there exists an n such that $\phi^n \triangle$ does not contain two equivalent points.

Once that is achieved in Lemma 3.11 below we get as corollaries that $\phi^n \triangle$ not only contains a fundamental set but also is itself one and that $\phi^n \triangle$ is not merely contained in its image $\phi(\phi^n \triangle)$ but is equal to it by virtue of the fact that both are fundamental sets.

We remark that in the case when the projection of ϕ on the torus is ergodic all its trajectories have to pass with uniform frequency through the projection of $\triangle_0^\circ$. By (3.18), ϕ does the same with respect to $\triangle_0^\circ$ itself. That means that after a uniform number of iterates there are no equivalent points to any trajectory starting in $\triangle$. This argument does not apply in general, however, and we provide a proof not referring to ergodicity.

Let N_0 be a minimal non-negative integer such that if $x \in \triangle$ then $\phi^n(x) \in V_0$ for some $n < N_0$. The existence of such N_0 independent of x follows from Lemma 3.4.

LEMMA 3.10. *Let $p, q \in \triangle$ and q be equivalent to p. If $p \in V_0$, then $\phi^{N_0}(p) = \phi^{N_0}(q)$.*

Proof. By Lemma 3.7 we have $p \in \triangle_{u_0}$. By Lemma 3.9, if $q \neq p$ then $q \in \triangle_{u_i}$ with $i \neq 0$ and $p - q = v_0 - v_i$. If $q \in \triangle_{u_i} \cap V_i$ then

$$\phi(p) - \phi(q) = (p + \gamma - v_0) - (q + \gamma - v_i) = (p - q) - (v_0 - v_i) = 0.$$

The remaining possibility is that $q \in \triangle_{u_1} \cap V_2$. Then $p - q = v_0 - v_1$ and $\phi(p) - \phi(q) = (p - q) - (v_0 - v_2) = v_2 - v_1$. Therefore, again by Lemma 3.9 and the fact that $p, q \neq o \in \triangle_0$, we have $\phi(p) \in \triangle_{u_2} \backslash \{(\gamma - v_1), (\gamma - v_2)\} \subset V_2$ and $\phi(q) \in \triangle_{u_1} \backslash \{(\gamma - v_2)\} \subset V_1 \cup V_2$, the set inclusions implied by (3.16) and (3.17) of Lemma 3.7.

Let $0 < M \leq N_0$ be minimal with $\phi^M(q) \notin V_2$. We claim that for all $1 \leq m \leq M$,

$$\begin{aligned} \phi^m(p) &\in \triangle_{u_2} \backslash \{(\gamma - v_1), (\gamma - v_0)\} \subset V_2, \\ \phi^m(q) &\in \triangle_{u_1} \backslash \{(\gamma - v_2)\} \subset V_1 \cup V_2, \\ \phi^m(q) - \phi^m(q) &= v_2 - v_1. \end{aligned}$$

The case $m = 1$ was demonstrated and starts the induction for any $m < M$. By minimality of M, if $m < M$, then $\phi^m(q) \in V_2$. So

$$\phi^{m+1}(p) - \phi^{m+1}(q) = \phi^m(p) + \gamma - v_2 - (\phi^m(q) + \gamma - v_2) = (v_2 - v_1).$$

As before Lemmas 3.9, and 3.7, and the fact that $\phi^m(p),\ \phi^m(q) \neq o$ gives us $\phi^{m+1}(p) \in \triangle_{u_2} \backslash \{(\gamma - v_1), (\gamma - v_0)\} \subset V_2$ and $\phi^{m+1}(q) \in \triangle_{u_1} \backslash \{(\gamma - v_2)\} \subset V_1 \cup V_2$. Finally, we have $\phi^M(q) \in V_1$ and $\phi^M(p) \in V_2$, which shows $M < N_0$ and $\phi^{M+1}(p) - \phi^{M+1}(q) = (\phi^M(p) - \phi^M(q)) - (v_2 - v_1) = 0$. □

LEMMA 3.11. *If $p, q \in \triangle$ are equivalent then $\phi^{2N_0}(p) = \phi^{2N_0}(q)$.*

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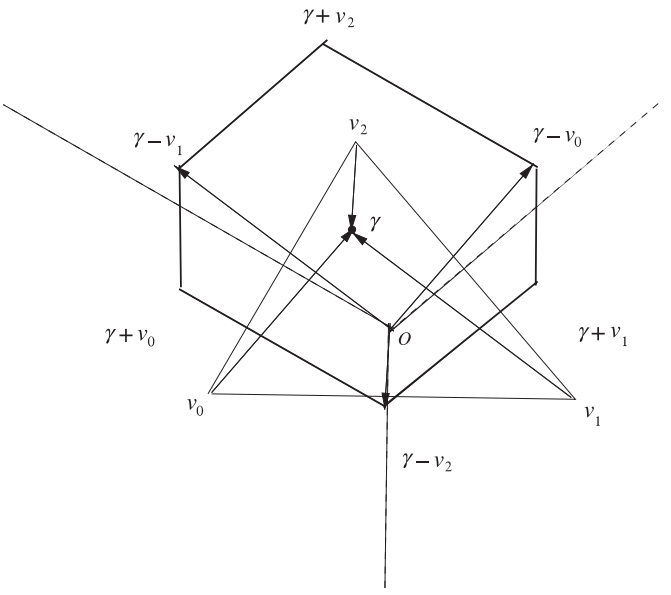


FIGURE 4.1. A simple absorbing fundamental set.

Proof. Using up some $n \leq N_0$ steps we can have $\phi^n(p) \in V_0$. Then we apply Lemma 3.10 to equivalent points $\phi^n(p)$ and $\phi^n(q)$. □

That concludes the proof of the main theorem.

4. *Questions and examples*

Figure 4.1 depicts one of the simpler types of absorbing fundamental sets which arise. Here $\triangle v_0 v_1 v_2$ is acute, and γ is chosen on the altitude from its closest vertex v_2. The absorbing fundamental set is a hexagon centered at γ with the appropriate boundary points. The Voronoï cells for the lattice generated by vectors $v_1 - v_0$, $v_2 - v_0$ which tessellate the plane are translates of the interior of this absorbing fundamental set. We leave this to the reader to verify.

In Figure 4.2 we select γ off the altitude and closer to vertex v_2. This is a sample of the more complicated absorbing fundamental sets we see as the distance from γ to v_2 decreases. Even so, such pictures seem to indicate that for non-obtuse triangles, the absorbing regions are connected, which we leave as one of the unsolved problems.

However, the primary unsolved one is a proof of the main theorem for obtuse triangles, if it is indeed true; or a counterexample, if not. The dynamics for obtuse triangles is more complicated than that for non-obtuse ones. One immediately loses the invariance of $\triangle$. Furthermore, for non-obtuse triangles, if two points are one-leg-of-P equivalent then their images under ϕ either are one-leg-of-P equivalent or coincide. There are three types of one-leg-of-P equivalences—namely those given by Lemma 3.9. In the obtuse case a fourth type of equivalence occurs as a image of one-leg-of-P equivalent points—namely,

$$p' = p \pm (v_{i+1} + v_{i+2} - 2v_i),$$

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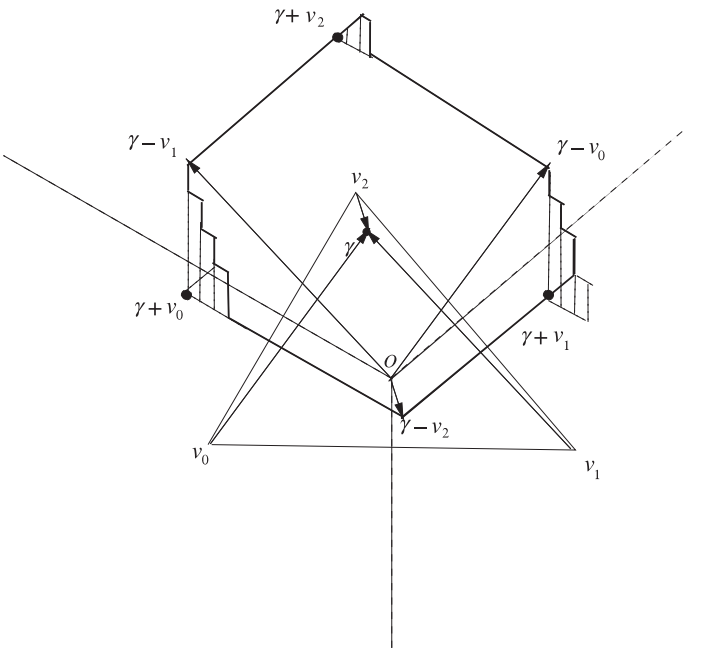


FIGURE 4.2. More complicated absorbing fundamental set.

where the angle at v_i is the obtuse one. This is the other diagonal of the standard parallelogram fundamental set generated by $(v_{i+1} - v_i)$ and $(v_{i+2} - v_i)$. In addition, while it is easy to construct an absorbing invariant figure small enough so that the only equivalences which need to be considered are of the four types, there does not seem to be one with a nice geometrical description.

There is apparently an even more dramatic complication for non-obtuse triangles. Computer simulations seem to indicate that the absorbing fundamental set need not be connected (see Figures 4.3 and 4.4).

Finally, if the invariant set is a fundamental set for the lattice generated by any triangle, then also each part of it (the intersection of the invariant set with the Voronoï domain) is a fundamental set, although with a different lattice. Furthermore, the same holds for any combination of two parts. In fact the following generalization holds for any simplex with bounded, invariant fundamental sets.

Let $\triangle$ be simplex in $\mathbb{R}^d$ with vertices $v_0, \ldots, v_d$ of full dimension (that is, $v_k - v_0$, $k = 1, \ldots, d$ are linearly independent). Given a partition of $\mathbb{R}^d$ into pairwise disjoint sets $V_0, \ldots, V_d$ and a $\gamma \in \mathbb{R}^d$, let ϕ_γ be defined piecewise by $\phi_\gamma(x) = x + \gamma - v_k$ for $x \in V_k$. If a set Q invariant under ϕ_γ is a bounded fundamental set for the lattice $L = \{\sum_k n_k(v_k - v_0), n_k \in \mathbb{Z}\}$, then for each choice of indices $I \subset \{1, \ldots, d\}$ the set $Q \cap \bigcup_{k \in I \cup \{0\}} V_k$ is a fundamental set for the lattice $L_I = \{\sum_{i \in I} n_i(v_i - v_0) + \sum_{j \notin I} n_j(v_j - \gamma), n_i, n_j \in \mathbb{Z}\}$.

The details will appear elsewhere [**2**].

Other problems for future consideration are the following: the question of existence of an absorbing invariant fundamental set for parallelograms; the question of higher-

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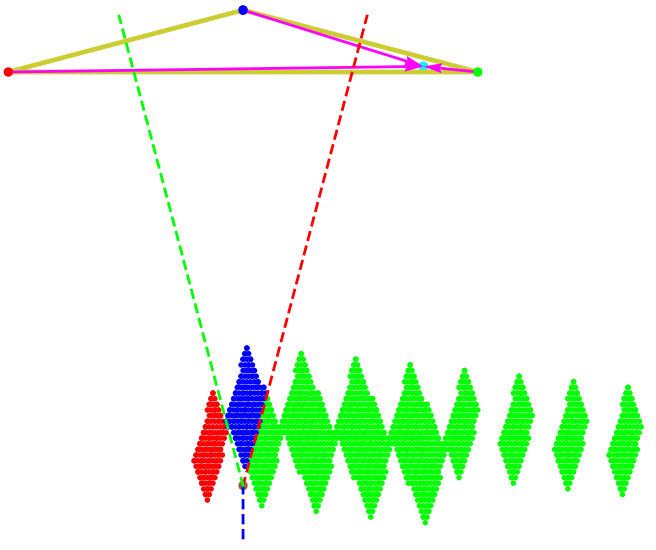


FIGURE 4.3. Obtuse triangle $(-1, 0)$, $(1, 0)$, $(0, 0.246)$ with constant input at $(0.768, 0.024)$.

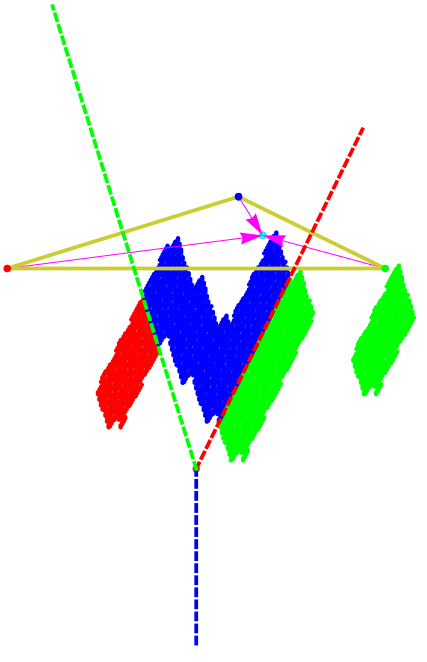


FIGURE 4.4. Obtuse triangle $(-1, 0)$, $(1, 0)$, $(0.224, 0.380)$ with constant input at $(0.352, 0.175)$.

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dimensional versions of the conjecture; and the dynamics of ϕ when P is a convex polytope whose vertices generate a discrete lattice. More precise questions concern the description of the symbolic dynamics that is generated in all those cases. On the opposite side of the spectrum, one finds questions such as characterizing the minimal invariant sets in the context of [**1**] and the minimal set of invariant vectors in the context of [**5**] and references therein.

5. *Addenda*

The orbit bounding theorem was established in a work by some of the present authors with others on a class of algorithms for digital printing (named error diffusion: see [**1**] and references therein). In this application, P is taken to be a polytope in a three-dimensional space, the coordinates of which represent the visual colors. The printable colors are at the extreme points of the polytope. The interior points provide an adequate range of colors to imitate a full color image. In grey scale pictures P is the unit interval, and the endpoints are black and white. In color printing P is most often the unit cube with vertices representing printable colors—black, white, cyan, magenta, yellow, red, green, and blue.

The manner in which the orbit bounding theorem is stated may conceal at first inspection that it deals with a time-dependent dynamical system since γ may well change at each iteration of ϕ_γ. A corollary is the fact that every orbit $\{x_n = \phi_{\gamma_n \ldots \gamma_1}(x_0) \mid n = 1, 2, \ldots\}$ is bounded, the proving of which was the purpose of the orbit bounding theorem in the first place. Time is identified with an ordering of pixels. At each step in time γ is an input chosen by some approximation scheme to imitate a color image, and the selection of the color to be printed is an optimization problem. The orbit bounding theorem is associated with a particular optimization procedure and leads via the bounding of orbits to the convergence of the average color printed to the average color to be imitated.

A more general optimization method involving a situation in which the polytope can also vary in a finite family was considered in [**5**] which also mentions some the work done after [**1**], including work in progress. Here there is no forward-inclusive set as in the orbit bounding theorem, but there exists a forward-inclusive set of error vectors that are the differences $x - v(x)$.

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