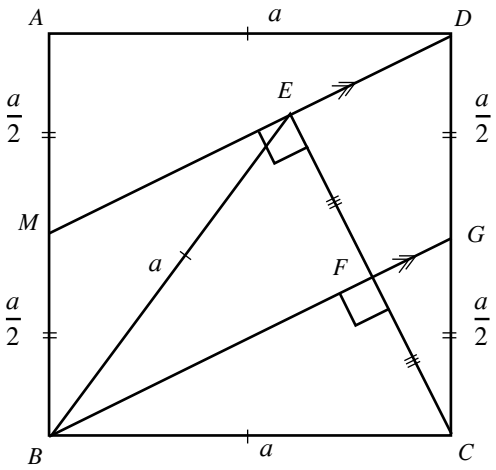


FIGURE 3

VICTOR OXMAN
Western Galilee College, Acre Israel
MOSHE STUPEL
Shaanan College, Haifa Israel

10.1017/mag.2021.126
© The Mathematical Association 2021

105.45 British flag theorem for isosceles trapezia

The British flag theorem [1, 2] is an interesting theorem on a rectangle with an arbitrary point. Lardner [1] calls this theorem 'the most useful and remarkable theorem which may be inferred' from the results in Book II of Euclid's *Elements*.

We can consider a rectangle as a particular case of an isosceles trapezium. In this Note, we shall give a generalisation of the British flag theorem for an isosceles trapezium with proof using Stewart's theorem [3]. We also give a further extension for this generalisation with proof using the dot product of two vectors (see [4]).

Here we denote by XY the Euclidean distance between two points X and Y; $\overrightarrow{XY}$ the Euclidean vector connecting an initial point X with a terminal point Y [5].

Theorem 1 (British flag theorem): Let $ABCD$ be a rectangle and P be any point in its plane. Then we have the equation

$$PA^2 + PC^2 = PB^2 + PD^2. \tag{1}$$

https://doi.org/10.1017/mag.2021.127 Published online by Cambridge University Press

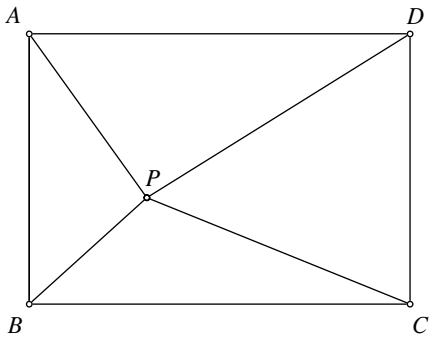


FIGURE 1

Theorem 2 (Stewart's theorem): Let ABC be a triangle and P a point lying on side BC. Then we have the equation

$$PB \times AC^2 + PC \times AB^2 = BC \times PA^2 + BC \times PB \times PC. \qquad (2)$$

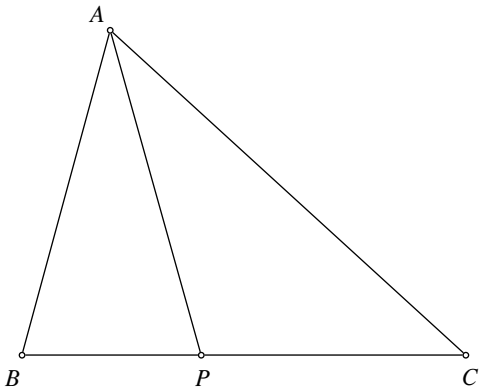


FIGURE 2

We now give a generalised theorem.

Theorem 3 (Generalised theorem): Let $ABCD$ be an isosceles trapezium with bases AD, BC, and P be any point. Let the ratio $\frac{AD}{BC} = k$. Then we have the equation

$$PA^2 + kPC^2 = PD^2 + kPB^2. \qquad (3)$$

https://doi.org/10.1017/mag.2021.127 Published online by Cambridge University Press

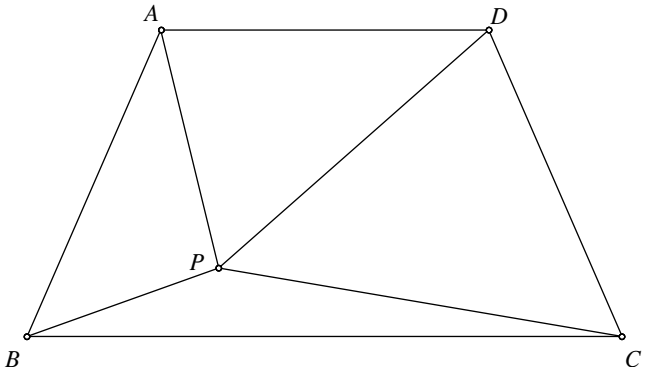


FIGURE 3

If $k = 1$ then $ABCD$ is a rectangle and we obtain Theorem 1.

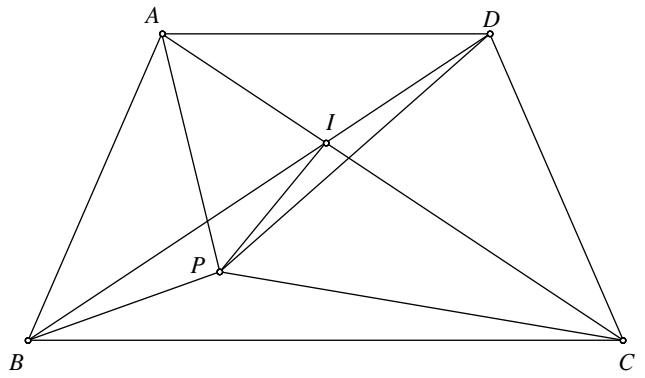


FIGURE 4

Proof of Theorem 3: Let I be the intersection of the two diagonals AC and BD. Applying Stewart's theorem for triangle PAC with I on side AC, we have

$$IA \times PC^2 + IC \times PA^2 = AC \times PI^2 + AC \times IA \times IC. \quad (4)$$

Applying Stewart's theorem for triangle PBD with I on side BD, we have

$$ID \times PB^2 + IB \times PD^2 = BD \times PI^2 + BD \times ID \times IB. \quad (5)$$

Since $ABCD$ is an isosceles trapezium, we have

$$AC = BD, \quad IA = ID \quad \text{and} \quad IB = IC. \quad (6)$$

From (4), (5), and (6), we obtain

$$IA \times PC^2 + IC \times PA^2 = ID \times PB^2 + IB \times PD^2. \quad (7)$$

https://doi.org/10.1017/mag.2021.127 Published online by Cambridge University Press

Divide both sides of (7) by IB ($IB = IC$) and note that

$$\frac{IA}{IC} = \frac{ID}{IB} = \frac{AD}{BC} = k,$$

to get

$$PA^2 + kPC^2 = PD^2 + kPB^2. \tag{8}$$

This completes the proof.

Corollary 1: Let $ABCD$ be an isosceles trapezium and P be any point not lying on the perpendicular bisector of bases AD and BC. If

$$PA^2 + PC^2 = PB^2 + PD^2 \tag{9}$$

then $ABCD$ is a rectangle.

Proof: Let $\frac{AD}{BC} = k$. From (9) and applying Theorem 3, we have

$$PC^2 - PB^2 = PD^2 - PA^2 = k\left(PC^2 - PB^2\right)$$

or

$$(1 - k)\left(PC^2 - PB^2\right) = 0. \tag{10}$$

Because $PC \neq PB$ (since P does not lie on the perpendicular bisector of bases AD and BC), from (10), we get $k = 1$. Thus, $ABCD$ is a rectangle. We finish proof.

Corollary 2: Let $ABCD$ be an isosceles trapezium and P any point not lying on the perpendicular bisector of bases AD and BC. We have the equation

$$\frac{PA^2 - PD^2}{PB^2 - PC^2} = \frac{AD}{BC}.$$

The proof is trivial from (3).

We can consider a point P as four vertices of a trapezium which coincide. We also give here further extension of Theorem 3.

Theorem 4 (Further extension): Let $ABCD$ and $A'B'C'D'$ be two directly similar isosceles trapeziums (see [6]) with the bases AD, BC, $A'D'$ and $B'C'$. Let the ratios $\frac{AD}{BC} = \frac{A'D'}{B'C'} = k$. Then we have the equation

$$AA'^2 + kCC'^2 = DD'^2 + kBB'^2 \tag{11}$$

https://doi.org/10.1017/mag.2021.127 Published online by Cambridge University Press

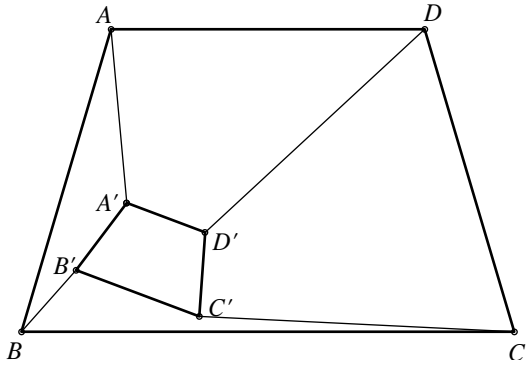


FIGURE 5

Proof: Let I be the intersection of two diagonals AC and BD; I' be the intersection of two diagonals $A'C'$ and $B'D'$. Let the vector $\vec{u} = \overrightarrow{I'I}$. Using the translation along vector $\vec{u}$: $A'B'C'D'$ transforms to $A''B''C''D''$ and I' transforms to I. Now we use Thales theorem and the assumption

$$\frac{AD}{BC} = \frac{A'D'}{B'C'} = k,$$

to get

$$\frac{IA}{IC} = k = \frac{IA''}{IC''}.$$

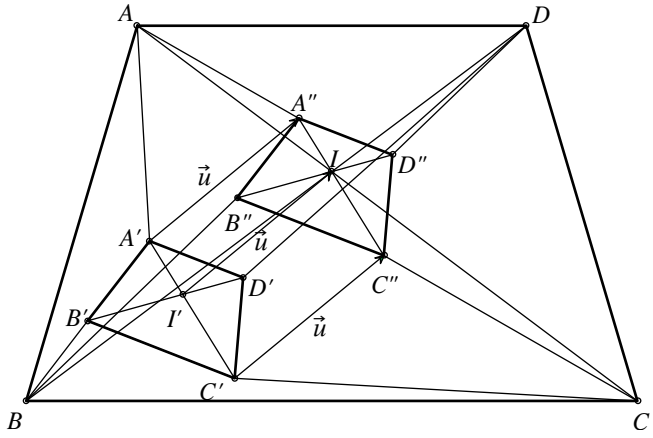


FIGURE 6

This implies that

$$\overrightarrow{AA''} = -k\,\overrightarrow{CC''}, \tag{12}$$

https://doi.org/10.1017/mag.2021.127 Published online by Cambridge University Press

and by the same reason, we have

$$\overrightarrow{DD''} = -k\overrightarrow{BB''}. \tag{13}$$

Using the scalar product of vectors, we have

$$\begin{aligned} AA'^2 + k\,CC'^2 &= \left(\overrightarrow{AA''} + \overrightarrow{A''A'}\right)^2 + k\left(\overrightarrow{CC''} + \overrightarrow{C''C'}\right)^2 \\ &= \left(\overrightarrow{AA''} - \vec{u}\right)^2 + k\left(\overrightarrow{CC''} - \vec{u}\right)^2 \\ &= AA''^2 + |\vec{u}|^2 - 2\vec{u}\cdot\overrightarrow{AA''} + k\left(CC''^2 + |\vec{u}|^2 - 2\vec{u}\cdot\overrightarrow{CC''}\right) \\ &= AA''^2 + k\,CC''^2 + (1+k)|\vec{u}|^2 - 2\vec{u}\cdot\left(\overrightarrow{AA''} + k\overrightarrow{CC''}\right) \\ &= AA''^2 + k\,CC''^2 + (1+k)|\vec{u}|^2, \qquad \text{(using (12))}. \end{aligned} \tag{14}$$

Analogously, we have

$$DD'^2 + k\,BB'^2 = DD''^2 + k\,BB''^2 + (1+k)|\vec{u}|^2. \tag{15}$$

Since $ABCD$ and $A'B'C'D'$ are two directly similar isosceles trapeziums, $ABCD$ and $A''B''C''D''$ are also two directly similar isosceles trapeziums. Therefore the two triangles IAD and $IA''D''$ are directly similar, hence there is a spiral similarity (see [7]) centre I transforming $D \to A$ and $D'' \to A''$. But $IA = ID$ and $IA'' = ID''$, thus this spiral similarity must be a rotation. This implies that

$$AA'' = DD'', \tag{16}$$

and similarly,

$$BB'' = CC''. \tag{17}$$

From (14), (15), (16), and (17), we obtain

$$AA'^2 + k\,CC'^2 = DD'^2 + k\,BB'^2. \tag{18}$$

This completes the proof.

Acknowledgements

The author would like to thank the anonymous reviewer for carefully reading of the manuscript with many suggestions. The author would like to thank the Editor Gerry Leversha for his invaluable help during the preparation of the manuscript.

References

1. D. Lardner, *The first six books of the Elements of Euclid*, H. G. Bohn 1848, p. 87.
2. J. W. Young and F. M. Morgan, *Elementary mathematical analysis*, Macmillan 1917, p. 304.
3. H. S. M. Coxeter and S. L. Greitzer, *Geometry revisited*, Math. Assoc. Amer. 1967, pp. 6-7.

https://doi.org/10.1017/mag.2021.127 Published online by Cambridge University Press

4. S. Lipschutz and M. Lipson, *Linear algebra* (4th ed.), Schaum Outlines, McGraw Hill 2009.
5. J. H. Heinbockel, *Introduction to tensor calculus and continuum mechanics*, Trafford Publishing 2001.
6. E. W. Weisstein, Directly Similar, MathWorld – A Wolfram Web Resource. https://mathworld.wolfram.com/DirectlySimilar.html
7. E. W. Weisstein, Spiral Similarity, MathWorld – A Wolfram Web Resource. https://mathworld.wolfram.com/SpiralSimilarity.html

QUANG HUNG TRAN
High School for Gifted Students,
Hanoi University of Science,
Vietnam National University at Hanoi, 182 Luong The Vinh,
Thanh Xuan, Hanoi, Vietnam
e-mail: *hungtq@hus.edu.vn*

10.1017/mag.2021.127
© The Mathematical Association 2021

105.46 Proximity of the incentre to the excentres and inequality for the circumcevians of the incentre

The incentre and the excentres of a triangle

In a triangle ABC with sides a, b, c, semiperimeter s, circumradius R and inradius r, let I_A, I_B, I_C denote the excentres of ABC. These are the centres of the excircles, the circles lying outside the triangle, touching one of its sides and the extensions of the other two. See Figure 1. For the sum of the distances of the incentre from the excentres, we give the following bound in terms of R and r.

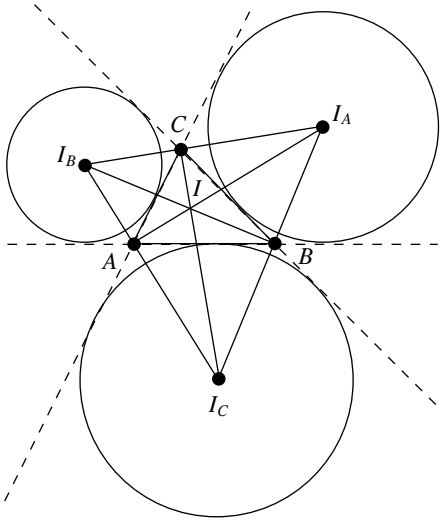


FIGURE 1: Excentral triangle

https://doi.org/10.1017/mag.2021.127 Published online by Cambridge University Press