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Oscillating sequences, MMA and MMLS flows and Sarnak's conjecture

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Abstract. We define an *oscillating sequence*, an important example of which is generated by the Möbius function in number theory. We also define a *minimally mean attractable (MMA) flow* and a *minimally mean-L-stable (MMLS) flow*. One of the main results is that any oscillating sequence is linearly disjoint from all MMA and MMLS flows. In particular, this confirms Sarnak's conjecture for all MMA and MMLS flows. We provide several examples of flows that are MMA and MMLS. These examples include flows defined by all p-adic polynomials of integral coefficients, all p-adic rational maps with good reduction, all automorphisms of the 2-torus with zero topological entropy, all diagonalizable affine maps of the 2-torus with zero topological entropy, all Feigenbaum maps, and all orientation-preserving circle homeomorphisms.

1. *Introduction*

Consider a pair $\mathcal{X} = (X, T)$, where X is a compact metric space with metric $d(\cdot, \cdot)$ and $T : X \to X$ is a continuous map. We call $\mathcal{X}$ a *flow* or a *dynamical system* because we will consider iterations $\{T^n\}_{n=0}^{\infty}$. A sequence of complex numbers $\xi(n)$, $n = 1, 2, \ldots$, is said to be *observable* in $\mathcal{X}$ if it can be realized as the image of the orbit of a point in X under some continuous function. Precisely, if there are a continuous function $f : X \to \mathbb{C}$ and a point $x \in X$ such that $f(T^n x) = \xi(n)$. Suppose that $\mathbf{c} = (c_n)$, $n = 1, 2, \ldots$, is a sequence of complex numbers. Following the idea of Sarnak [**46**], we have the following definition.

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Definition 1. (Disjointness) We say that the sequence $\mathbf{c} = (c_n)$ is *linearly disjoint* from the flow $\mathcal{X}$ if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n \xi(n) = 0 \tag{1}$$

for any observable $\xi(\cdot)$ in $\mathcal{X}$.

An interesting example is the sequence $\mathbf{u} = (\mu(n))$ generated by the Möbius function $\mu : \mathbb{N} \to \{-1, 0, 1\}$, where $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers. Recall that, by definition, $\mu(1) = 1$; $\mu(n) = (-1)^r$ if n is a product of r distinct primes; and $\mu(n) = 0$ if n is not square-free. We call $\mathbf{u}$ in this paper the Möbius sequence. Sarnak conjectured that the Möbius sequence is linearly disjoint from all flows $\mathcal{X}$ with zero topological entropy (see [**46**, **47**]). This is now called Sarnak's conjecture and it remains open in full generality, although it been studied for several special cases [**2**, **3**, **13**, **22**, **25**, **31**, **35**, **38**].

As pointed out in [**46**, **47**], Sarnak's conjecture has a connection with many important problems in number theory because of the extremely important role played by the Möbius function there. For example, an important property satisfied by the Möbius function is that

$$\sum_{n=1}^{N} \mu(n) = o(N); \tag{2}$$

this is equivalent to the prime number theorem. A stronger statement (conjecture) is that for $\epsilon > 0$,

$$\sum_{n=1}^{N} \mu(n) = O_\epsilon(N^{(1/2)+\epsilon}); \tag{3}$$

this is equivalent to the Riemann hypothesis. Another important property satisfied by the Möbius function is Davenport's theorem in [**10**]:

$$\sum_{n=1}^{N} \mu(n) e^{2\pi i \alpha n} = o(N) \tag{4}$$

for any real number $0 \le \alpha < 1$. One more important property satisfied by the Möbius function is Hua's theorem in [**26**]:

$$\sum_{n=1}^{N} \mu(n) e^{2\pi i \alpha n^k} = o(N) \tag{5}$$

for any real number $0 \le \alpha < 1$ and any integer $k \ge 2$. The reader who is interested in these properties may refer to [**36**] for more delicate estimates for all $k \ge 1$.

In this paper, we will not only consider the Möbius sequence but also a sequence $\mathbf{c} = (c_n)$ of complex numbers satisfying the oscillating condition:

$$\sum_{n=1}^{N} c_n e^{-2\pi i n t} = o_t(N) \quad \text{for all } t \in [0, 1). \tag{6}$$

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In order to present our ideas more clearly, we also assume that all oscillating sequences in this paper satisfy the growth condition:

$$\sum_{n=1}^{N} |c_n|^{\lambda} = O(N) \quad \text{for some } \lambda > 1. \tag{7}$$

We call such a sequence an *oscillating sequence*. There are many oscillating sequences generated by arithmetic functions. These include the Möbius function and the Liouville function defined as $l(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the number of prime factors of n, counted with multiplicity (see [**9**, **11**]).

In this paper, we define the concepts of *minimally mean attractable (MMA)* and *minimally mean-L-stable (MMLS)* flows and prove that any oscillating sequence is linearly disjoint from all MMA and MMLS flows.

One of the most interesting parts of this paper is our proofs of the MMA and MMLS properties for several important examples of flows. These examples include flows defined by all p-adic polynomials of integral coefficients, all p-adic rational maps with good reduction, automorphisms of the 2-torus with zero topological entropy, all diagonalizable affine maps of the 2-torus with zero topological entropy, all Feigenbaum maps, and all orientation-preserving circle homeomorphisms.

We organize our paper as follows. In §2, we define oscillating sequences (Definition 2). In §3, we define MMLS flows (Definition 5) and MMA flows (Definition 6). In the same section, we give a review of mean-L-stability and an equivalent notion, mean-equicontinuity. In §4, we prove that any oscillating sequence is linearly disjoint from all MMA and MMLS flows (Theorem 1). In particular, we confirm Sarnak's conjecture for all MMA and MMLS flows as a corollary (Corollary 1). Another corollary is that any oscillating sequence is linearly disjoint from all equicontinuous flows (Corollary 2). In §5, we discuss flows defined by p-adic polynomials and p-adic rational maps. All the polynomials with p-adic integral coefficients and all the p-adic rational maps with good reduction define equicontinuous flows on the ring of p-adic integers and on the projective line $\mathbb{P}^1(\mathbb{Q}_p)$, respectively. Thus, they are MMA and MMLS. Therefore, any oscillating sequence is linearly disjoint from all flows defined by all p-adic polynomials and all flows defined by p-adic rational maps with good reduction (Corollaries 4, 5, and 6). In §6, we explain how to use our method to study affine maps and automorphisms of the 2-torus with zero topological entropy. We first prove that in the diagonalizable case, any oscillating sequence is linearly disjoint from all flows defined by affine maps of the 2-torus with zero topological entropy (Corollary 7) since these flows are equicontinuous (Proposition 8). We then prove that all flows defined by automorphisms of the 2-torus with zero topological entropy are MMA and MMLS (Theorem 2) (they are not necessarily equicontinuous). Thus, any oscillating sequence is linearly disjoint from all flows defined by automorphisms of the 2-torus with zero topological entropy (Corollary 8). Furthermore, we give an example of an affine map of the 2-torus with zero topological entropy and an example of an oscillating sequence that is not linearly disjoint from the flow defined by this affine map example (Example 7). Notice that Liu and Sarnak proved that the Möbius sequence is linearly disjoint from the flow defined by this affine map example in [**35**]. We further note that in order to be linearly disjoint from all flows with zero topological

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entropy, an oscillating sequence should be oscillating not only in the first order but also in any higher order (Remark 8). The Möbius sequence has this property, as shown by Hua in [**26**] (see equation (5)). In §7, we prove that all flows defined by Feigenbaum maps are MMA and MMLS (Theorem 6) (they are not globally equicontinuous, but when restricted to their minimal Cantor set attractors, they are equicontinuous). Thus, any oscillating sequence is linearly disjoint from all flows defined by Feigenbaum maps (Corollary 10). In §8, we prove that all flows defined by Denjoy counter-examples are MMA and MMLS (Theorem 8). It is known that these flows are not equicontinuous on the whole circle. Furthermore, in the same theorem, we also show that they are not equicontinuous on their non-wandering sets, which are Cantor sets. Thus, any oscillating sequence is linearly disjoint from all flows defined by Denjoy counter-examples (Corollary 11). In the same section, we give a detailed explanation showing that any oscillating sequence is linearly disjoint from all flows defined by orientation-preserving circle homeomorphisms which are not Denjoy counter-examples. Thus, we finally prove that any oscillating sequence is linearly disjoint from all flows defined by orientation-preserving circle homeomorphisms. We would like to note that we just learned that Karagulyan [**31**] proved Sarnak's conjecture for flows defined by all orientation-preserving circle homeomorphisms. This result is now a corollary of Theorems 1 and 9 (Corollary 12).

2. *Oscillating sequences*

In classical analysis (see [**29**]), a sequence of complex numbers $(b_n)_{n\in\mathbb{Z}}$ is called a generalized almost periodic sequence if for any $t \in [0, 1)$ the following limit exists:

$$s(t) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} b_n e^{-2\pi int}.$$

Almost periodic and generalized almost periodic sequences were studied by Bohr, Hartman, *et al*. The spectrum of a generalized almost periodic sequence is the set of $t \in [0, 1)$ such that $s(t) \neq 0$. In general, the spectrum is countable. We are interested in those generalized almost periodic sequences with empty spectrum and defined only on positive integers. Thus, we introduce the following definition.

Definition 2. (Oscillation) Let $\mathbf{c} = (c_n)$, $n = 1, 2, \ldots$, be a sequence of complex numbers. We say that $\mathbf{c}$ is an *oscillating sequence* if the Cesaro means

$$\sigma_N(t) = \frac{1}{N} \sum_{n=1}^{N} c_n e^{-2\pi int}$$

converge to zero as N tends to infinity for every $0 \leq t < 1$.

Remark 1. We assume that all oscillating sequences in this paper satisfy the growth condition (7) for the convenience of our proofs.

If we only assume that $\lim \sigma_N(t)$ exists, then we define

$$\mathcal{Z}(\mathbf{c}) = \left\{ t \in [0, 1) \;\middle|\; \lim_{N\to\infty} \sigma_N(t) = 0 \right\}.$$

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The statement that $\mathbf{c}$ is an oscillating sequence means that $\mathcal{Z}(\mathbf{c}) = [0, 1)$. In general, the complement of $\mathcal{Z}(\mathbf{c})$ is called the spectrum of $\mathbf{c}$.

Let us give several examples of oscillating sequences and some counter-examples.

Example 1. The sequence of complex numbers $\mathbf{c} = (e^{2\pi i n\alpha})$ for a fixed $0 \le \alpha < 1$ is not an oscillating sequence because $\mathcal{Z}(\mathbf{c}) = [0, 1)\backslash\{\alpha\}$. Note that the spectrum is the one-point set $\{\alpha\}$.

Example 2. The sequence of complex numbers $\mathbf{c} = (e^{2\pi i c n \log n})$, for any fixed constant $c > 0$, is an oscillating sequence because the Cesaro means

$$\sigma_N(t) = \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i c n \log n} e^{-2\pi i n t} = O\left(\frac{1}{\sqrt{N}}\right)$$

uniformly on $0 \le t < 1$ [**52**, Vol. 1, p. 197].

Example 3. The sequence of complex numbers $\mathbf{c} = (e^{2\pi i n^2\alpha})$ for any fixed irrational number α is an oscillating sequence because the sequence

$$\{n^2\alpha - nt \pmod 1\}_{n=0}^{\infty}$$

is uniformly distributed on the unit interval $[0, 1]$ for any fixed $0 \le t < 1$. So, the Cesaro means

$$\sigma_N(t) = \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i (n^2\alpha - nt)} = o_t(1).$$

PROPOSITION 1. *Consider the sequence of complex numbers* $\mathbf{c} = (e^{2\pi i n^2\alpha})$, *where* $0 \le \alpha = p/q < 1$ *is a rational number with* $(p, q) = 1$. *The spectrum of* $\mathbf{c}$ *is the set of rational numbers* r/s *such that* $s \mid q$ *and*

$$\sum_{k=0}^{q-1} e^{2\pi i (k^2(p/q) - k(r/s))} \ne 0.$$

Proof. We have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i n^2\alpha} e^{-2\pi i n t} = \sum_{k=0}^{q-1} e^{2\pi i (k^2(p/q) - kt)} \lim_{N\to\infty}\frac{1}{N}\sum_{m:mq+k\le N} e^{-2\pi i m q t}.$$

If t is irrational or $t = r/s$ is rational with $s \nmid q$, the last sum is bounded and the above limit is zero. Therefore, such a t is not in the spectrum. If $t = r/s$ is rational with $s \mid q$, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i n^2\alpha} e^{-2\pi i n (r/s)} = \frac{1}{q}\sum_{k=0}^{q-1} e^{2\pi i (k^2(p/q) - k(r/s))}. \qquad \square$$

It is easy to check that

$$\mathcal{Z}(\{e^{2\pi i n^2\cdot(1/2)}\}) = [0, 1)\backslash\{\tfrac{1}{2}\},$$
$$\mathcal{Z}(\{e^{2\pi i n^2\cdot(1/3)}\}) = \mathcal{Z}(\{e^{2\pi i n^2\cdot(2/3)}\}) = [0, 1)\backslash\{0, \tfrac{1}{3}, \tfrac{2}{3}\},$$
$$\mathcal{Z}(\{e^{2\pi i n^2\cdot(1/4)}\}) = \mathcal{Z}(\{e^{2\pi i n^2\cdot(3/4)}\}) = [0, 1)\backslash\{0, \tfrac{1}{2}\}.$$

The following example is due to Davenport [**10**] (see also [**24**]).

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Example 4. The Möbius sequence $\mathbf{u} = (\mu(n))$ is an oscillating sequence. Actually a stronger result holds for the Cesaro means

$$\sigma_N(t) = \frac{1}{N}\sum_{n=1}^{N} \mu(n)e^{-2\pi int} = O\left(\frac{1}{\log^h N}\right)$$

for any $h > 0$, where the estimate is uniform on $0 \le t < 1$.

Another example can be obtained from the following proposition. Let (ξ_n) be a sequence of independent and identically distributed real random variables such that

$$\mathbb{E}e^{\lambda\xi_1} \le e^{\lambda^2/2} \quad \text{for all } \lambda \in \mathbb{R}.$$

Such a sequence is called a subnormal sequence.

PROPOSITION 2. *Suppose that (ξ_n) is a subnormal sequence. Let (u_n) be a sequence of real numbers such that $u_n = O(n^\tau)$ for some $0 < \tau < 1/2$. Then almost surely the random sequence $\mathbf{c} = (u_n\xi_n)$ is an oscillating sequence.*

Proof. Actually, as a consequence of the Salem–Zygmund inequality on the uniform estimate of random trigonometric polynomials and as we refer to [**29**, p. 73], where the Rademacher random sequence was considered but the proof for subnormal sequences is the same, we have almost surely that the Cesaro means

$$\sigma_N(t) = \frac{1}{N}\sum_{n=1}^{N} u_n\xi_n e^{-2\pi int} = O\left(\frac{\sqrt{\log N}}{N^{(1/2)-\tau}}\right),$$

where the estimate is uniform on $0 \le t < 1$. This proves the proposition. □

From [**17**], we can get other oscillating sequences similar to those in Proposition 2. The next example is due to Daboussi and Delange [**9**, p. 254].

PROPOSITION 3. *Let $\mathbf{c} = (c_n)$ be a multiplicative arithmetical function such that $|c_n| \le 1$ for all n. Then $\mathbf{c}$ is an oscillating sequence if and only if for any Dirichlet character χ and any real number u, we have*

$$\sum_p \frac{1}{p}(1 - Re(\chi(p)c_p p^{-iu})) = \infty,$$

where the sum is taken over prime numbers p.

This classical result of Daboussi and Delange was generalized to asymptotic orthogonality of multiplicative functions to irrational nilsequences by Frantzikinakis and Host [**21**].

An oscillating sequence shares the following property, which says that all its arithmetic subsequences are oscillating.

PROPOSITION 4. *Suppose that $\mathbf{c} = (c_n)$ is an oscillating sequence. Then, for any $q \ge 2$ and any $r = 1, 2, \ldots, q$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{1\le n\le N \\ n\equiv r \pmod q}} c_n e^{-2\pi itn} = 0 \quad \textit{for all } t \in [0, 1). \tag{8}$$

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Proof. The idea comes from [**9**]. The proof is based on the fact that

$$\frac{1}{q}\sum_{k=1}^{q} e^{2\pi ika/q} = 1 \text{ or } 0$$

according to $a \equiv 0$ or $\not\equiv 0 \pmod q$. It follows that

$$\begin{aligned}\frac{1}{N}\sum_{\substack{1\le n\le N\\ n\equiv r \pmod q}} c_n e^{-2\pi itn} &= \frac{1}{N}\sum_{1\le n\le N} c_n e^{-2\pi itn}\left(\frac{1}{q}\sum_{k=1}^{q} e^{2\pi ik(n-r)/q}\right)\\ &= \frac{1}{q}\sum_{k=1}^{q} e^{-(2\pi ikr/q)}\left(\frac{1}{N}\sum_{1\le n\le N} c_n e^{-2\pi i(t-(k/q))n}\right) \to 0\end{aligned}$$

as $N \to \infty$, since $\mathbf{c}$ is an oscillating sequence. □

We would like to note that the proof of Proposition 4 actually shows that, in general, (8) holds for any t as long as $(t - k/q) \pmod 1 \in \mathcal{Z}(\mathbf{c})$ for all $k = 1, 2, \ldots, q$.

In [**30**], Kahane and Saias studied completely multiplicative functions with zero sum. These functions share another kind of oscillating property.

3. *Minimal mean attractability and minimal mean-L-stability*

In this section we define a class of flows so that any oscillating sequence is linearly disjoint from them. We start with equicontinuous flows.

Definition 3. (Equicontinuity) A flow $\mathcal{X} = (X, T)$ is said to be *equicontinuous* if the family $\{T^n\}_{n=0}^{\infty}$ is equicontinuous. That means that for every $\epsilon > 0$, there is a $\delta > 0$ such that whenever $x, y \in X$ with $d(x, y) < \delta$, we have $d(T^n x, T^n y) < \epsilon$ for $n = 0, 1, 2, \ldots$.

It is well known that a flow $\mathcal{X} = (X, T)$ with T being surjective is equicontinuous if and only if there exists a compatible metric ρ on X such that T acts on X as an isometry, i.e. $\rho(Tx, Ty) = \rho(x, y)$ for all $x, y \in X$. Thus, if T is surjective and equicontinuous, then T must be a homeomorphism. Moreover, it is also known that for an equicontinuous flow $\mathcal{X}$, if T is a homeomorphism, then T acts as a minimal system on the closure $\overline{\mathrm{Orb}(x)}$ of the forward orbit $\mathrm{Orb}(x) = \{T^n x\}_{n=0}^{\infty}$ of any $x \in X$ (see [**5**, **45**]). Thus, for an equicontinuous flow $\mathcal{X}$, when T is a transitive homeomorphism, it is always conjugate to a minimal rotation on a compact abelian metric group. In this case, let m be the unique Haar probability measure on X. Then the measurable dynamical system (X, T, m) has discrete spectrum. An equicontinuous flow $\mathcal{X}$ such that T is a homeomorphism can be decomposed into minimal subsystems. Sarnak's conjecture holds for such a flow. Actually, we will prove that Sarnak's conjecture holds for a much larger class of flows, which we will call minimally mean attractable and minimally mean-L-stable flows. We give a more detailed description now.

We first weaken the equicontinuity condition following Fomin in [**20**]. Remember that $\mathbb{N} = \{1, 2, \ldots, \}$ is the set of natural numbers. Let E be a subset of $\mathbb{N}$. The upper density of E is, by definition,

$$\overline{D}(E) = \limsup_{n\to\infty} \frac{\#(E \cap [1, n])}{n}.$$

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Definition 4. (MLS) A flow $\mathcal{X} = (X, T)$ is said to be *mean-L-stable* (MLS for short) (here L recalls the sense of Lyapunov) if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d(x, y) < \delta$ implies that $d(T^n x, T^n y) < \epsilon$ for all $n = 0, 1, 2, \ldots$, except for a subset of natural numbers with the upper density less than ϵ.

Fomin proved that a minimal and MLS flow is uniquely ergodic in [**20**]. Oxtoby proved that a transitive and MLS flow is uniquely ergodic in [**43**]. Li, Tu and Ye further proved that every ergodic invariant measure on an MLS flow has discrete spectrum in [**34**].

Any flow $\mathcal{X}$ admits a minimal subflow; for example, the restriction to the ω-limit set of any point.

Definition 5. (MMLS) We say that a flow $\mathcal{X} = (X, T)$ is *minimally MLS* (MMLS for short) if for every minimal subset $K \subseteq X$, the subflow $\mathcal{K} = (K, T)$ is MLS.

Following [**34**], we say that the flow $\mathcal{X}$ is *mean-equicontinuous* at a point $x \in X$ if for every $\epsilon > 0$, there is $\delta > 0$ such that for every $y \in X$ with $d(y, x) < \delta$, we have

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} d(T^k x, T^k y) < \epsilon. \tag{9}$$

We say that the flow $\mathcal{X}$ is mean-equicontinuous if it is mean-equicontinuous at every point $x \in X$. That is, by the compactness of X, the above inequality (9) holds for all $x, y \in X$ such that $d(y, x) < \delta$. The following proposition is proved by Li, Tu and Ye in [**34**].

PROPOSITION 5. *A flow $\mathcal{X}$ is mean-equicontinuous if and only if it is MLS.*

Let $C(X)$ be the space of all continuous functions $f : X \to \mathbb{C}$ with maximum norm

$$\|f\|_\infty = \max_{x\in X} |f(x)|.$$

The following result plays an important role in our proof of Theorem 1. We prove it under the growth condition (7) in order to avoid more complicated arguments.

PROPOSITION 6. (Mean-equicontinuity) *Suppose that the flow $\mathcal{X}$ is MLS. Suppose that $\mathbf{c} = (c_n)$ is a sequence satisfying the growth condition (7). Then, for any continuous function $f \in C(X)$,*

$$S_N f(x) = \frac{1}{N} \sum_{n=1}^{N} c_n f(T^n x), \quad N = 1, 2, \ldots,$$

is an equicontinuous sequence in $C(X)$.

Proof. Suppose that $\|f\|_\infty \neq 0$, otherwise it is trivial. Take a $\lambda > 1$ satisfying (7). Let $\lambda' > 1$ be the number such that $(1/\lambda) + (1/\lambda') = 1$. There is a positive constant C (we can take C to be the supremum of $((1/N)\sum_{n=1}^{N} |c_n|^\lambda)^{1/\lambda}$) such that for any $x, y \in X$, we have

$$|S_N f(x) - S_N f(y)| \le C\left(\frac{1}{N} \sum_{n=0}^{N-1} |f(T^n x) - f(T^n y)|^{\lambda'}\right)^{1/\lambda'}. \tag{10}$$

This is a direct consequence of the Hölder inequality and (7).

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By the uniform continuity of f, for any $\epsilon > 0$ there exists $\eta > 0$ such that for any $x, y \in X$ with $d(x, y) < \eta$,

$$|f(x) - f(y)| < \frac{\epsilon}{2C}. \tag{11}$$

We can take $\eta \leq (\epsilon/4C\|f\|_\infty)$. Since $\mathcal{X}$ is MLS, there is a $\delta' > 0$ such that $x, y \in X$ with $d(x, y) < \delta'$ implies that $d(T^n x, T^n y) < \eta$ for all positive integers n except for a set, denoted by E, of upper density less than η. There exists an integer N^* such that

$$\frac{\#(E \cap [1, N])}{N} < \eta \quad \text{for all } N \geq N^*.$$

Thus, by (10), if $N \geq N^*$ we have

$$|S_N f(x) - S_N f(y)| < C\left(2\|f\|_\infty \eta + \frac{\epsilon}{2C}\right) = \epsilon. \tag{12}$$

In the above inequality, we first split the sum in (10) into two sums according to $d(T^n x, T^n y) \geq \eta$ or $< \eta$.

The family $\{S_1 f, \ldots, S_{N^*} f\}$ of a finite number of continuous functions on the compact metric space X is equicontinuous. Thus, there exists δ'' such that

$$d(x, y) < \delta'' \Rightarrow \max_{1 \leq N \leq N^*} |S_N f(x) - S_N f(y)| < \epsilon. \tag{13}$$

Therefore, we prove the proposition by combining (12) and (13). □

In [**1**, p. 575], Auslander decomposed an MLS flow into a star-closed decomposition. In this decomposition each component contains precisely one minimal set and all invariant measures concentrate on some minimal sets. This implies that from the measure-theoretical point of view, the union of these minimal sets, which support invariant measures, is an attractor: all points outside the union of these minimal sets are attracted into minimal sets eventually. We will develop this idea into a new concept called the minimal mean attractability.

Definition 6. (MMA) Suppose that $\mathcal{X} = (X, T)$ is a flow. Suppose that K is a closed subset of X and $T : K \to K$ is minimal. We say that $x \in X$ is *mean attracted* to K if for any $\epsilon > 0$, there is a point $z = z_{\epsilon,x} \in K$ (depending on x and ϵ) such that

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} d(T^n x, T^n z) < \epsilon. \tag{14}$$

The *basin of attraction* of K, denoted as $\mathrm{Basin}(K)$, is defined to be the set of all points x which are mean attracted to K. We call $\mathcal{X}$ *minimally mean attractable* (MMA for short) if

$$X = \bigcup_K \mathrm{Basin}(K), \tag{15}$$

where K varies among all minimal subsets of X.

Remark 2. It is trivial that $K \subset \mathrm{Basin}(K)$. The union in the above definition could be uncountable.

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Recall that a point $x \in X$ is attracted to K if

$$\lim_{n\to\infty} d(T^n x, K) = 0.$$

In general, this cannot imply that x is mean attracted to K. However, in the case that K is a periodic cycle, then K consists of a finite number of points. In this case, x is attracted to K if and only if it is mean attracted to K. The following proposition sharpens this observation.

PROPOSITION 7. *Let $x \in X$. Suppose that the ω-limit set $\omega(x) = \{x_0, x_1, \ldots, x_{k-1}\}$ is a periodic cycle of period $k \geq 1$. Then, for any $\epsilon > 0$ sufficiently small, there exist an integer $\tau \geq 0$ and $z \in \omega(x)$ such that*

$$d(T^{\tau+m}x, T^{\tau+m}z) < \epsilon \quad \textit{for all } m \geq 0.$$

Proof. Let $K = \omega(x)$. Take any $\epsilon > 0$ sufficiently small so that the balls $B(x_j, \epsilon)$, $j = 0, 1, \ldots, k-1$, are disjoint. For any j and any $\tau \geq 0$, consider the subset of the natural numbers

$$R_\tau(x_j) = \{n \geq \tau : T^n x \in B(x_j, \epsilon)\}.$$

For any fixed $0 \leq j < k$,

$$\lim_{R_\tau(x_j)\ni n\to\infty} T^n x = x_j.$$

By the continuity of T, we have

$$\lim_{R_\tau(x_j)+1\ni n\to\infty} T^n x = x_{j+1 \,(\mathrm{mod}\, k)}.$$

This implies that we can find a $\tau_j \geq 1$ such that

$$R_\tau(x_j) + 1 \subset R_\tau(x_{j+1}) \quad \text{for all } \tau \geq \tau_j. \tag{16}$$

Let $\tau_{\max} = \max_{0\leq j<k} \tau_j$. Take $\tau \geq \tau_{\max} \in R_{\tau_{\max}}(x_0)$. Then

$$d(T^\tau x, x_0) < \epsilon$$

and, from (16),

$$d(T^{m+\tau}x, T^m x_0) = d(T^{m+\tau}x, x_{m \,(\mathrm{mod}\, k)}) < \epsilon \quad \text{for all } m \geq 0.$$

We take $z \in K$ such that $T^\tau z = x_0$; then we have that

$$d(T^{\tau+m}x, T^{\tau+m}z) = d(T^{\tau+m}x, x_{m \,(\mathrm{mod}\, k)}) < \epsilon \quad \text{for all } m \geq 0. \qquad \square$$

Example 5. The flow

$$([-1, 1], T(x) = -x^2)$$

is MMA and MMLS. This flow is actually equicontinuous on any closed interval $[a, b] \subset (-1, 1)$ but not equicontinuous on $[-1, 1]$.

Proof. The points -1 and 0 are the only two fixed points. Since $T'(-1) = 2$ and $T'(0) = 0$, -1 is the repelling fixed point and 0 is the attracting fixed point. We have that

$$\mathrm{Basin}(\{-1\}) = \{-1, 1\} \quad \text{and} \quad \mathrm{Basin}(\{0\}) = (-1, 1).$$

Thus, we have

$$[-1, 1] = \mathrm{Basin}(\{-1\}) \cup \mathrm{Basin}(\{0\}).$$

This implies that $\mathcal{X}$ is MMA and MMLS. $\square$

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The next example is different (see Remark 3). Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the standard 2-torus. Consider the automorphism

$$T_t(x, y) = (x + ty \pmod 1, y) : \mathbb{T}^2 \to \mathbb{T}^2$$

for a fixed integer t.

Example 6. The flow

$$\mathcal{X} = (\mathbb{T}^2, T_t)$$

is MMA and MMLS. Actually, we will prove that all automorphisms of $\mathbb{T}^2$ with zero topological entropy are MMA and MMLS (see Theorem 2).

Proof. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the circle. Every circle $\mathbb{T}_y = \mathbb{T} \times \{y\}$ (y being fixed) is T_t-invariant. The restriction of T_t on each $\mathbb{T}_y$ is a one-dimensional rotation with the rotation number $\rho_y = ty \pmod 1$. If ρ_y is irrational, then $T_t : \mathbb{T}_y \to \mathbb{T}_y$ is minimal. If ρ_y is rational, then every point $(x, y) \in \mathbb{T}_y$ is a periodic point and the restriction of T_t on each of these periodic orbits $\mathrm{Orb}(x, y)$ is minimal. In this case, $\mathbb{T}_y = \bigcup_{(x,y)\in\mathbb{T}_y} \mathrm{Orb}((x, y))$. Therefore, we have the minimal decomposition

$$\mathbb{T}^2 = \Bigg(\bigcup_{ty:\text{irrational}} \mathbb{T}_y \Bigg) \cup \Bigg(\bigcup_{ty:\text{rational}} \bigcup_{(x,y)\in\mathbb{T}_y} \mathrm{Orb}((x, y)) \Bigg).$$

So, this flow is MMA.

Since T_t is isometric on each $\mathbb{T}_y$, it is equicontinuous on each of its minimal subsets. So, it is MMLS.

As we shall see in §6, any automorphism T on $\mathbb{T}^2$ with zero topological entropy is (linearly) topologically conjugate to a T_t for some integer t. So, the flow $\mathcal{X} = (\mathbb{T}^2, T)$ is also MMA and MMLS. □

Remark 3. The flow $(\mathbb{T}^2, T_t)$ is not MLS (equivalently, not mean-equicontinuous) because the rotation number of $T_t|_{\mathbb{T}_y}$ varies with y.

4. *Linear disjointness*

Recall that $\mathcal{X} = (X, T)$ is a flow if X is a compact metric space with metric d and $T : X \to X$ is a continuous map. A sequence $(\xi(n))$ is said to be *observable* in $\mathcal{X}$ if there exist a continuous function $f \in C(X)$ and a point $x \in X$ such that $\xi(n) = f(T^n x)$ for all $n \geq 1$. Recall (Definition 1) that a given sequence of complex numbers $\mathbf{c} = (c_n)$ is *linearly disjoint* from $\mathcal{X}$ if for any observable sequence $(\xi(n))$ in $\mathcal{X}$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n \xi(n) = 0. \tag{17}$$

The first main result in this paper is the following.

THEOREM 1. (MMA/MMLS and disjointness) *Any oscillating sequence $\mathbf{c} = (c_n)$ is linearly disjoint from all MMA and MMLS flows $\mathcal{X} = (X, T)$. More precisely, for any continuous function $f \in C(X)$, we have that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n f(T^n x) = 0$$

for every $x \in X$ and the limit is uniform on each minimal subset.

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Proof. By hypothesis, X has the decomposition in equation (15). So, we only need to prove that for any minimal subset $K \subset X$, we have

$$\text{for all } x \in \text{Basin}(K), \quad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n f(T^n x) = 0. \tag{18}$$

For this purpose, we define

$$S_N f(x) = \frac{1}{N} \sum_{n=1}^{N} c_n f(T^n x), \quad N = 1, 2, \ldots. \tag{19}$$

We first prove that the sequence $\mathcal{S} = \{S_N f(x)\}_{N=1}^{\infty}$ converges to 0 uniformly on K. The sequence $\mathcal{S}$ is uniformly bounded in $C(K)$ by $C\|f\|_\infty$ for some constant $C > 0$ and is equicontinuous by Proposition 6. So, it is precompact in $C(K)$ by the Arzela–Ascoli theorem. Therefore, every subsequence of $\mathcal{S}$ has a convergent subsequence. What we have to prove is the claim that the limit of any subsequence which converges uniformly on K must be zero on K.

Let us prove this claim by contradiction. Suppose that some subsequence $S_{N_j} f$ converges uniformly to a continuous function $g \in C(K)$ which is not identically zero. Assume that $|g(x_0)| > 0$ for some $x_0 \in K$; we can actually assume that

$$|g(x)| > 0 \quad \text{for all } x \in B(x_0, K, r) = \{x \in K \mid d(x, x_0) < r\} \tag{20}$$

for some $r > 0$ by the continuity of g. On the other hand, we consider the probability measures on K:

$$\nu_N = \frac{1}{N} \sum_{n=1}^{N} \delta_{T^n x_0},$$

where δ_y denotes the Dirac measure concentrated at y. Let ν be a weak limit of a subsequence of the sequence (ν_N). Then it is T-invariant. Since K is minimal, by the Gottschalk theorem (see [**23**, Lemma 4]), x_0 is almost periodic in the sense that for every $\epsilon > 0$, the set of n such that $d(x_0, T^n x_0) < \epsilon$ is relatively dense. This implies that $\nu(B(x_0, \epsilon)) > 0$ for any $\epsilon > 0$ (the support of ν is actually the whole minimal set K).

Now consider the measure-preserving dynamical system

$$(K, \mathcal{B}(K), T, \nu),$$

where $\mathcal{B}(K)$ is the Borel σ-field of K. Let σ_f be the spectral measure of f with respect to the T-invariant probability measure ν on K. That is, it is the finite Borel measure defined on the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ characterized by its Fourier coefficients,

$$\widehat{\sigma}_f(n) = \langle f, f \circ T^n \rangle \quad \text{if } n \geq 0 \quad \text{and} \quad \widehat{\sigma}_f(n) = \langle f \circ T^{-n}, f \rangle \quad \text{if } n < 0.$$

Then, by the spectral lemma (see [**32**, pp. 94–95]), we have

$$\|S_N f\|^2_{L^2(\nu)} = \int_{\mathbb{T}} \left| \frac{1}{N} \sum_{n=1}^{N} c_n e^{2\pi i n t} \right|^2 d\sigma_f(t).$$

Since $\mathbf{c} = (c_n)$ is an oscillating sequence, by the Lebesgue dominated convergence theorem, the right-hand side of the above equality converges to 0. This implies that $S_N f$

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converges to zero in $L^2(\nu)$-norm. In particular, $S_{N_j} f$ converges to zero in $L^2(\nu)$-norm. Consequently, there is a subsequence N_j' of N_j such that $S_{N_j'} f$ converges to zero ν-almost everywhere, which contradicts (20) and the fact that $\nu(B(x_0, K, r)) > 0$. The contradiction implies that the sequence $\mathcal{S}$ converges uniformly on K to 0.

We have just proved the limit (18) for every $x \in K$. Now we are going to prove (18) for a general point $x \in \mathrm{Basin}(K)$. Let $\eta > 0$ be an arbitrarily small number. By the uniform continuity of f, there is a $\epsilon > 0$ such that $d(u, v) < \epsilon$ implies that $|f(u) - f(v)| < \eta$. We assume that $2\|f\|_\infty \epsilon \le \eta$. For $x \in \mathrm{Basin}(K)$, by the definition of $\mathrm{Basin}(K)$, there exists $z = z_{\epsilon,x} \in K$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d(T^n x, T^n z) < \epsilon^2,$$

which implies that $\overline{D}(E) \le \epsilon$, where $E = \{n \ge 0 \mid d(T^n x, T^n z) \ge \epsilon\}$ because

$$\epsilon \#(E \cap [1, N]) \le \sum_{n=1}^{N} d(T^n x, T^n z).$$

Write

$$S_N f(x) = S_N f(x) - S_N f(z) + S_N f(z).$$

As $z \in K$, we have proved that $S_N f(z)$ tends to zero. On the other hand, following the exact same proof of Proposition 6, we show that

$$|S_N f(x) - S_N f(z)| \le C(2\|f\|_\infty \epsilon + \eta) \le 2C\eta.$$

Since $\eta > 0$ is arbitrary, we have thus proved (18) for every $x \in \mathrm{Basin}(K)$. This completes the proof. □

A special case of Theorem 1 is Sarnak's conjecture for all MMA and MMLS flows.

COROLLARY 1. (MMA/MMLS and Möbius) *The Möbius sequence* $\mathbf{u} = (\mu(n))$ *is linearly disjoint from all MMA and MMLS flows.*

Proof. The sequence $\mathbf{u}$ is an oscillating sequence due to Davenport's theorem (see Example 4). □

Another consequence of Theorem 1 is the following result.

COROLLARY 2. (Equicontinuity and disjointness) *Any oscillating sequence is linearly disjoint from all equicontinuous flows.*

Proof. Let $\mathcal{X} = (X, T)$ be an equicontinuous flow. The sequence of compact sets $\{T^n X\}_{n=1}^{\infty}$ decreases to a non-empty compact set, which we denote by X_∞. The subflow $T : X_\infty \to X_\infty$ is surjective and equicontinuous and thus a homeomorphism. The set X_∞ is decomposed into minimal subsets.

We claim that every point $x \in X$ is mean attracted to some minimal subset. In fact, the sequence $T^n x$ admits a subsequence $T^{n_j} x$ converging to a limit point x_∞, which must

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belong to X_∞. Recall the definition of equicontinuity for T: for any $\epsilon > 0$, there exists $\eta > 0$ such that for any $u, v \in X$ with $d(u, v) < \eta$,

$$d(T^n u, T^n v) < \frac{\epsilon}{2} \quad \text{for all } n = 1, 2, \ldots.$$

Take j_0 sufficiently large such that $d(T^{n_{j_0}} x, x_\infty) < \eta$. As $T : X_\infty \to X_\infty$ is a homeomorphism, there exists $z \in X_\infty$ such that $x_\infty = T^{n_{j_0}} z$. So, we have that

$$d(T^{n_{j_0}} x, T^{n_{j_0}} z) < \eta.$$

By the equicontinuity, we have

$$d(T^{n_{j_0}+n} x, T^{n_{j_0}+n} z) < \frac{\epsilon}{2} \quad \text{for all } n = 1, 2, \ldots.$$

It follows that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d(T^n x, T^n z) \le \frac{\epsilon}{2} < \epsilon.$$

Thus, we have proved that x is mean attracted to a minimal subset

$$K = \overline{\mathrm{Orb}(z)} = \overline{\mathrm{Orb}(x_\infty)},$$

where $\mathrm{Orb}(\cdot)$ means the forward orbit. So, we have proved that this flow is MMA. But it is equicontinuous on each minimal subset, so it is also MMLS. □

Remark 4. As one can see in the proof of Theorem 1, we do not really need that the spectrum of the sequence $\mathbf{c} = (c_n)$ is empty. What we really need is that the spectrum of $\mathbf{c}$ considered as a subset in the circle S^1 is disjoint from the support of σ_f in the circle S^1. In other words, the oscillating condition on $\mathbf{c}$ in Theorem 1 can be relaxed to the condition that

$$\{e^{2\pi i t} \mid t \in \mathcal{Z}(\mathbf{c})\} \subseteq \mathrm{supp}(\sigma_f).$$

As a direct consequence of the proof of Theorem 1 and Remark 4, we have the following result.

COROLLARY 3. (Non-empty spectrum case) *Let $R_\alpha(z) = e^{2\pi i\alpha} z$ be the rigid rotation on the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ in the complex plane, where $\alpha \in (0, 1)$ is irrational. Any sequence $\mathbf{c} = (c_n)$ satisfying the growth condition (7) such that $\alpha\mathbb{Z} \subset \mathcal{Z}(\mathbf{c})$ is linearly disjoint from the flow $\mathcal{X} = (S^1, R_\alpha)$.*

Proof. We repeat the proof of Theorem 1. It suffices to notice that the rotation R_α admits the Lebesgue measure as the unique invariant measure and the support of σ_f is contained in $\alpha\mathbb{Z}$. Actually, it is easy to check that

$$\sigma_f = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2 \delta_{e^{2\pi i\alpha n}},$$

where $\{\widehat{f}(n)\}_{n\in\mathbb{Z}}$ are the Fourier coefficients of f. □

Remark 5. The idea of using the spectral lemma in the proof of Theorem 1 comes from [**16**], where the ergodic Hilbert transforms associated with irrational rotations are studied.

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We will apply Theorem 1 to several examples. In §5, we apply the theorem to all flows defined by p-adic polynomials and by p-adic rational maps with good reduction. In §6, we apply the theorem to all flows defined by automorphisms and some affine maps of the 2-torus with zero topological entropy. In §7, we apply the theorem to all flows defined by Feigenbaum maps. In §8, we apply the theorem to all flows defined by orientation-preserving circle homeomorphisms.

5. *p-adic polynomials and p-adic rational maps*

We give here some examples of equicontinuous flows on the fields of p-adic numbers. These flows share a very nice minimal decomposition [**14**, **15**]. We can compare this decomposition with Auslander's star-closed decomposition [**1**].

Let $p \geq 2$ be a prime number and let $\mathbb{Z}_p$ be the ring of p-adic integers. Let $\mathbb{Z}_p[x]$ be the ring of all polynomials with coefficients in $\mathbb{Z}_p$. Then every polynomial $P \in \mathbb{Z}_p[x]$ defines a flow $\mathcal{X} = (\mathbb{Z}_p, P)$. It is trivial that P is a 1-Lipschitz function, i.e.

$$|P(x) - P(y)|_p \leq |x - y|_p \quad \text{for all } x, y \in \mathbb{Z}_p,$$

where $|\cdot|_p$ is the p-adic norm on $\mathbb{Z}_p$. So, the flow $\mathcal{X} = (\mathbb{Z}_p, P)$ is equicontinuous. A particular example is the adding machine $\mathrm{add}(x) = x + 1 \in \mathbb{Z}_p[x]$, which is actually an equicontinuous homeomorphism. Corollary 2 gives us the following result.

COROLLARY 4. (p-adic polynomial flows) *Any oscillating sequence* $\mathbf{c} = (c_n)$ *is linearly disjoint from any flow* $\mathcal{X} = (\mathbb{Z}_p, P)$ *defined by a* p*-adic polynomial* $P \in \mathbb{Z}_p[x]$.

It was proved in [**15**] that for a flow $\mathcal{X} = (\mathbb{Z}_p, P)$ in Corollary 4, we have the minimal decomposition

$$\mathbb{Z}_p = \mathcal{P} \sqcup \mathcal{M} \sqcup \mathcal{B},$$

where $\mathcal{P}$ is the finite set consisting of all periodic points of P, $\mathcal{M} = \bigsqcup_i \mathcal{M}_i$ is the union of all (at most countably many) clopen invariant sets such that each $\mathcal{M}_i$ is a finite union of balls and each subflow $P : \mathcal{M}_i \to \mathcal{M}_i$ is minimal, and points in $\mathcal{B}$ lie in the attracting basin of a periodic orbit or of a minimal subflow. Using a proof similar to that in [**15**, Theorem 1], points in $\mathcal{B}$ are mean attracted to a periodic cycle or a minimal set M_i.

More generally, we can consider flows defined by p-adic rational maps in the p-adic field. Let $\mathbb{Q}_p$ be the field of p-adic numbers. Any point in the projective line $\mathbb{P}^1(\mathbb{Q}_p)$ can be given in homogeneous coordinates by a pair $[x_1 : x_2]$ of points in $\mathbb{Q}_p$ which are not both zero. Two such pairs $[x_1 : x_2]$ and $[\lambda x_1 : \lambda x_2]$ with non-zero factor $\lambda \in \mathbb{Q}_p^*$ are identified. The field $\mathbb{Q}_p$ can be identified with the subset of $\mathbb{P}^1(\mathbb{Q}_p)$ given by

$$\{[x : 1] \in \mathbb{P}^1(\mathbb{Q}_p) \mid x \in \mathbb{Q}_p\}.$$

This subset covers all points in $\mathbb{P}^1(\mathbb{Q}_p)$ except for one: the point of infinity, which may be given as $\infty = [1 : 0]$. The spherical metric defined on $\mathbb{P}^1(\mathbb{Q}_p)$ is analogous to the standard spherical metric on the Riemann sphere as follows. If $u = [x_1; y_1]$ and $v = [x_2; y_2]$ are two points in $\mathbb{P}^1(\mathbb{Q}_p)$, we define

$$\rho(u, v) = \frac{|x_1 y_2 - x_2 y_1|_p}{\max\{|x_1|_p, |y_1|_p\} \max\{|x_2|_p, |y_2|_p\}}$$

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or, viewing $\mathbb{P}^1(\mathbb{Q}_p)$ as $\mathbb{Q}_p \cup \{\infty\}$, we define

$$\rho(z_1, z_2) = \frac{|z_1 - z_2|_p}{\max\{|z_1|_p, 1\} \max\{|z_2|_p, 1\}} \quad \text{if } z_1, z_2 \in \mathbb{Q}_p$$

and

$$\rho(z, \infty) = \begin{cases} 1 & \text{if } |z|_p \le 1, \\ 1/|z|_p & \text{if } |z|_p > 1. \end{cases}$$

We remark that the restriction of the spherical metric $\rho(\cdot\,, \cdot)$ on the ring $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x_p| \le 1\}$ of p-adic integers is the same as the metric induced by the absolute value $|\cdot|_p$.

A rational map $R \in \mathbb{Q}_p(x)$ induces a rational map defined on $\mathbb{P}^1(\mathbb{Q}_p)$, which we still denote as R. Rational maps R are always Lipschitz continuous with respect to the spherical metric $\rho(\cdot\,, \cdot)$ [**48**, Theorem 2.14]. Rational maps R with good reduction are 1-Lipschitz continuous [**48**, p. 59] in the sense that

$$\rho(R(u), R(v)) \le \rho(u, v) \quad \text{for all } u, v \in \mathbb{P}^1(\mathbb{Q}_p).$$

COROLLARY 5. (p-adic rational flows) *Any oscillating sequence* $\mathbf{c} = (c_n)$ *is linearly disjoint from any flow* $\mathcal{X} = (\mathbb{P}^1(\mathbb{Q}_p), R)$ *defined by a* p*-adic rational map* $R \in \mathbb{Q}_p(x)$ *with good reduction.*

It was proved in [**14**] that for a rational map R with good reduction of order at least 2, we have a minimal decomposition,

$$\mathbb{P}^1(\mathbb{Q}_p) = \mathcal{P} \sqcup \mathcal{M} \sqcup \mathcal{B},$$

similar to that of p-adic polynomials, of which we have given a detailed description.

One special case of Corollaries 4 and 5 is the following result.

COROLLARY 6. (Sarnak's conjecture for p-adic flows) *The Möbius sequence* $\mathbf{u} = (\mu(n))$ *is linearly disjoint from flows* $(\mathbb{Z}_p, P)$ *defined by all* p*-adic polynomials and flows* $(\mathbb{P}^1(\mathbb{Q}_p), R)$ *defined by all* p*-adic rational maps with good reduction.*

6. *Automorphisms and affine maps on the* 2*-torus*

Let

$$\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$$

be the 2-torus. Let $\mathrm{SL}(\mathbb{Z}, 2)$ be the space of all matrices A of integral entries such that $\det A = \pm 1$. Then all $A\mathbf{x}$ for $A \in \mathrm{SL}(\mathbb{Z}, 2)$ represent all automorphisms of $\mathbb{T}^2$. Affine maps on $\mathbb{T}^2$ are of the form

$$T_{A,b}(\mathbf{x}) = A\mathbf{x} + \mathbf{b},$$

where $A\mathbf{x}$ is an automorphism of $\mathbb{T}^2$ and $\mathbf{b}$ is an element in $\mathbb{T}^2$. In particular, $T_{A,0}(\mathbf{x}) = A\mathbf{x}$ means an automorphism of $\mathbb{T}^2$.

Given an affine map $T_{A,b}$ on $\mathbb{T}^2$, consider the flow $\mathcal{X}_{A,b} = (\mathbb{T}^2, T_{A,b})$. The entropy of $\mathcal{X}_{A,b}$ is equal to the logarithm of the sum of the modulus of all eigenvalues of A of modulus strictly larger than 1. Thus, its topological entropy is zero if and only if all eigenvalues of A are of modulus 1.

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Let $\|\cdot\|_{\mathbb{R}^2}$ be the Euclidean norm on $\mathbb{R}^2$. Recall that the induced metric on $\mathbb{T}^2$ is defined by

$$\|\mathbf{x}-\mathbf{y}\|_{\mathbb{T}^2}=\inf_{\mathbf{n}\in\mathbb{Z}^2}\|\mathbf{x}-\mathbf{y}-\mathbf{n}\|_{\mathbb{R}^2}.$$

Also recall that the Hermitian norm $\|\cdot\|_{\mathbb{C}^2}$ on $\mathbb{C}^2$ is defined by

$$\|\mathbf{z}\|_{\mathbb{C}^2}=z_1\bar{z}_1+z_2\bar{z}_2,\quad \mathbf{z}=(z_1,z_2)\in\mathbb{C}^2.$$

The space $\mathbb{R}^2$ is considered as a subspace of $\mathbb{C}^2$. It is clear that for $\mathbf{x}\in\mathbb{R}^2$, we have

$$\|\mathbf{x}\|_{\mathbb{T}^2}\le\|\mathbf{x}\|_{\mathbb{R}^2}=\|\mathbf{x}\|_{\mathbb{C}^2}.$$

The following proposition actually holds in higher dimensions, but we state it in the dimension-two case only.

PROPOSITION 8. *Let $A\in\mathrm{SL}(\mathbb{Z},2)$. Suppose that A is diagonalizable in the field of complex numbers and all its eigenvalues have modulus* 1. *Then there exists a constant C such that for all $\mathbf{n}\in\mathbb{Z}$ and all $\mathbf{x}\in\mathbb{T}^2$, we have*

$$\|A^n\mathbf{x}\|_{\mathbb{T}^2}\le C\|\mathbf{x}\|_{\mathbb{T}^2}.$$

Thus, the flow $\mathcal{X}_{A,b}$ for every $\mathbf{b}\in\mathbb{T}^2$ is equicontinuous.

Proof. Let $A=PDP^{-1}$, where D is the diagonal matrix with the eigenvalues λ_1, λ_2 of A on the diagonal and P is an invertible complex matrix. Then

$$\|A^n\mathbf{x}\|_{\mathbb{R}^2}=\|PD^nP^{-1}\mathbf{x}\|_{\mathbb{R}^2}=\|PD^nP^{-1}\mathbf{x}\|_{\mathbb{C}^2}\le\|P\|_{\mathbb{C}^2}\|D^n\mathbf{z}\|_{\mathbb{C}^2},$$

where $\mathbf{z}=P^{-1}\mathbf{x}=(z_1,z_2)$. However,

$$\|D^n\mathbf{z}\|_{\mathbb{C}^2}^2=\sum_{j=1}^{2}|\lambda_j^n z_j|^2=\sum_{j=1}^{2}|z_j|^2=\|P^{-1}\mathbf{x}\|_{\mathbb{C}^2}^2.$$

Thus,

$$\|A^n\mathbf{x}\|_{\mathbb{R}^2}\le\|P\|_{\mathbb{C}^2}\|P^{-1}\|_{\mathbb{C}^2}\|\mathbf{x}\|_{\mathbb{R}^2}.$$

Let $\mathbf{n}\in\mathbb{Z}^2$ be such that $\|\mathbf{x}\|_{\mathbb{T}^2}=\|\mathbf{x}-\mathbf{n}\|_{\mathbb{R}^2}$. Replacing $\mathbf{x}$ by $\mathbf{x}-\mathbf{n}$ in the above inequality, we get

$$\|A^n(\mathbf{x}-\mathbf{n})\|_{\mathbb{R}^2}\le\|P\|_{\mathbb{C}^2}\|P^{-1}\|_{\mathbb{C}^2}\|\mathbf{x}\|_{\mathbb{T}^2}.$$

Since $A^n\mathbf{n}\in\mathbb{Z}^2$, we get that

$$\|A^n\mathbf{x}\|_{\mathbb{T}^2}\le C\|\mathbf{x}\|_{\mathbb{T}^2}$$

for $C=\|P\|_{\mathbb{C}^2}\|P^{-1}\|_{\mathbb{C}^2}$. This completes the proof. □

Combining Proposition 8 and Corollary 2, we have the following result.

COROLLARY 7. (Diagonalization disjointness) *Suppose that $A\in\mathrm{SL}(\mathbb{Z},2)$ is diagonalizable in the field of complex numbers and suppose that all eigenvalues of A have modulus* 1. *Then any oscillating sequence $\mathbf{c}=(c_n)$ is linearly disjoint from the flow $\mathcal{X}_{A,b}$ for every $\mathbf{b}\in\mathbb{T}^2$.*

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Now suppose that A is not diagonalizable. In this case, A has a double eigenvalue 1 or -1 and $\det A = 1$. A 2×2 square matrix is called a *modular matrix* if all entries are integers and its determinant is 1.

PROPOSITION 9. *Let M be a modular matrix with 1 or -1 as its double eigenvalue. There exists a modular matrix P such that*

$$P^{-1} M P = \pm T_t,$$

where T_t is a modular matrix of the form

$$T_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad t \in \mathbb{Z}.$$

Proof. If the double eigenvalue of M is -1, then $-M$ has double eigenvalue 1. So, we only need to prove the proposition in the case that the double eigenvalue of M is 1. In this case the trace of M is 2. So, we can write

$$M = \begin{pmatrix} a & b \\ c & 2-a \end{pmatrix}, \quad M - \mathrm{Id} = \begin{pmatrix} a-1 & b \\ c & 1-a \end{pmatrix}.$$

The fact that $\det(M - \mathrm{Id}) = 0$ means that

$$-bc = (a-1)^2. \tag{21}$$

If $b = 0$, we must have $a = 1$ by (21), so

$$M = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = P \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} P^{-1} \quad \text{with } P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then we prove the proposition in this case. If $c = 0$, we must have $a = 1$ and there is nothing to do.

Assume that $b \neq 0$ and $c \neq 0$. By (21), we have $a \neq 1$ and $c = -(a-1)^2/b$. We are looking for

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{Z}^2,$$

which is an eigenvector associated with 1, i.e. $(M - \mathrm{Id})\mathbf{x} = 0$. This equation is equivalent to

$$(a-1)x_1 + bx_2 = 0 \quad \text{and} \quad cx_1 + (1-a)x_2 = 0.$$

Let d be the gcd of $a-1$ and b. The above two equations become one equation

$$\frac{a-1}{d} x_1 + \frac{b}{d} x_2 = 0. \tag{22}$$

It has an integral solution,

$$\mathbf{x}_0 = \begin{pmatrix} b/d \\ -(a-1)/d \end{pmatrix}.$$

We then look for

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{Z}^2$$

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such that $(M - I)\mathbf{y} = t\mathbf{x}_0$ with t an integer to be determined. The first line of $(M - I)\mathbf{y} = t\mathbf{x}_0$ is

$$(a - 1)y_1 + by_2 = t\frac{b}{d}. \tag{23}$$

We claim that $b|d^2$. In fact, $b|(a - 1)^2$. Let us factorize $a - 1$ into primes:

$$a - 1 = \pm p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}.$$

Then we can write

$$b = \pm p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r}, \quad 0 \le \beta_j \le 2\alpha_j, \quad 1 \le j \le r.$$

So,

$$d = p_1^{\alpha_1 \wedge \beta} p_2^{\alpha_2 \wedge \beta_2} \cdots p_r^{\alpha_r \wedge \beta_r},$$

where $\alpha \wedge \beta = \min\{\alpha, \beta\}$. Then $b|d^2$ is equivalent to saying that $\beta_j \le 2(\alpha_j \wedge \beta_j)$ for all $1 \le j \le r$, which is true because $0 \le \beta_j \le 2\alpha_j$. This proves the claim.

Take $t = b^{-1}d^2 \in \mathbb{Z}$. Then equation (23) becomes

$$\frac{a - 1}{d} y_1 + \frac{b}{d} y_2 = 1. \tag{24}$$

By the Bézout theorem, equation (24) admits a solution $\mathbf{y}_0 \in \mathbb{Z}^2$ because $((a - 1)/d, b/d) = 1$.

Let $P = (\mathbf{x}_0, \mathbf{y}_0)$ be the 2×2 matrix with columns $\mathbf{x}_0$ and $\mathbf{y}_0$. Let $\mathbf{u} = (x_1, y_1)$ and $\mathbf{v} = (x_2, y_2)$ be the two rows of P. Equations (22) and (24) can be rewritten as

$$\mathbf{e}_2 = \frac{a - 1}{d}\mathbf{u} + \frac{b}{d}\mathbf{v} = (0, 1). \tag{25}$$

Then

$$\det P = \det(\mathbf{u}, \mathbf{v}) = \frac{d}{b}\det(\mathbf{u}, d^{-1}b\mathbf{v}) = \frac{d}{b}\det(\mathbf{u}, \mathbf{e}_2) = \frac{d}{b} \cdot \frac{b}{d} = 1.$$

This implies that P (as well as P^{-1}) is a positive modular matrix and

$$P^{-1}MP = T_t$$

for $t = b^{-1}d^2 \in \mathbb{Z}$. We have proved the proposition. □

Following Proposition 9, Example 6, and Theorem 1, we have the following result.

THEOREM 2. (MMA and MMLS for automorphism) *Suppose that* $A \in \mathrm{SL}(\mathbb{Z}, 2)$ *and suppose that all its eigenvalues have modulus* 1. *Then the flow* $\mathcal{X}_{A,0}$ *is MMA and MMLS.*

A consequence of Theorems 1 and 2 is the following result.

COROLLARY 8. (Automorphism disjointness) *Any oscillating sequence* $\mathbf{c} = (c_n)$ *is linearly disjoint from flows* $\mathcal{X}_{A,0}$ *for all* $A \in \mathrm{SL}(\mathbb{Z}, 2)$ *whose eigenvalues have modulus* 1.

The disjointness of the Möbius sequence $\mathbf{u} = (\mu(n))$ from flows in Corollary 8 was proved by Liu and Sarnak in [**35**].

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Remark 6. Let us make a remark about the number $t \in \mathbb{Z}$ in Proposition 9. It is unique. Suppose that we have another $t' \in \mathbb{Z}$ such that Proposition 9 holds. Then we have a modular matrix P such that

$$T_t P = P T_{t'}.$$

Let (x, y) be the first row of P and (u, v) be the second row of P. Then we have that

$$tu = 0, \quad tv = t'x, \quad t'u = 0.$$

If $u \neq 0$, then $t = t' = 0$. If $u = 0$, then both x and v are non-zero since $\det P = 1$. This implies that t and t' are either both 0 or both non-zero. When both t and t' are zero, we have nothing to prove. Suppose that both t and t' are non-zero. Then we get $v = (t'/t)x$. Since $u = 0$, we get $\det P = xv = 1$. This implies that $x^2 = t/t'$ and $x = \pm 1$. Thus, we get $t = t'$. For example,

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

are not linearly conjugate flows on $\mathbb{T}^2$.

Furthermore, let us give an example to which Proposition 9 applies: consider the modular matrix

$$M = \begin{pmatrix} -5 & 6 \\ -6 & 7 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad T_6 = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}.$$

Then we have

$$P^{-1} M P = T_6.$$

When A is not diagonalizable, we show in the following example that there are an oscillating sequence $\mathbf{c} = (c_n)$ and a flow $\mathcal{X}_{A,b}$, $\mathbf{b} \neq \mathbf{0}$, such that $\mathbf{c}$ is not linearly disjoint from $\mathcal{X}_{A,b}$.

Example 7. Let

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix},$$

where $0 < \alpha < 1$ is an irrational number. The sequence $\mathbf{c} = (e^{-\pi i n^2 \alpha})$ is an oscillating sequence (refer to Example 3). But it is not linearly disjoint from the flow $\mathcal{X}_{A,b}$ whose topological entropy is zero.

Proof. Let

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{T}^2.$$

Observe that

$$T^n \mathbf{x} = \begin{pmatrix} x + n\alpha \\ y + nx + \frac{1}{2} n(n-1)\alpha \end{pmatrix}.$$

If we take the continuous function $f(x, y) = e^{2\pi i y}$, then we have

$$\frac{1}{N} \sum_{n=1}^{N} e^{-\pi i n^2 \alpha} f(T^n(\alpha/2, 0)) = \frac{1}{N} \sum_{n=1}^{N} e^{-\pi i n^2 \alpha} e^{\pi i n^2 \alpha} = 1,$$

which does not tend to zero. □

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Remark 7. As a skew product of an irrational rotation, the map $T_{A,b}$ on $\mathbb{T}^2$ is uniquely ergodic with respect to the Lebesque measure, which is the $T_{A,b}$-invariant measure having full support. This implies that $T_{A,b}$ is minimal and $\mathcal{X}_{A,b}$ is MMA. But $\mathcal{X}_{A,b}$ is not MMLS (that is, MLS in this case) because of Theorem 1.

Remark 8. In [**35**], Liu and Sarnak proved that the Möbius sequence $\mathbf{u} = (\mu(n))$ is linearly disjoint from all affine flows $\mathcal{X}_{A,b}$ whose topological entropy is zero, where $A \in \mathrm{SL}(\mathbb{Z}, 2)$ and $\mathbf{b} \in \mathbb{T}^2$ (the result holds in higher dimensions). This result and the above example indicate that in order for an oscillating sequence $\mathbf{c} = (c_n)$ to be linearly disjoint from all zero-entropy flows, it should be oscillating not only in the first order but also in any higher order, that is,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n e^{2\pi i P(n)} = 0$$

for all real polynomials P. Both the Möbius sequence $\mathbf{u} = (\mu(n))$ and the von Mangoldt sequence $\mathbf{m} = (\Lambda(n))$ share the above oscillating property for any higher order, as shown in [**26**]. Recall that the von Mangoldt function is, by definition, $\Lambda(n) = \log p$ if $n = p^k$ for some prime number and some integer $k \geq 1$ and 0 otherwise.

7. *Feigenbaum maps*

A particularly famous type of map with topological entropy zero, called a Feigenbaum map, has been studied extensively for the last 40 years. It is nonlinear and has no mean-equicontinuity property on the whole space. In this section, we will prove that the flow defined by a Feigenbaum map is MMA and MMLS. Thus, Sarnak's conjecture is true for such a flow. We use $\mathrm{ent}(T)$ to denote the topological entropy of a dynamical system $T : X \to X$. For the definition of topological entropy, the reader may refer to [**4**, **44**, **49**].

Let $I = [-1, 1]$. Suppose that

$$T_t(x) = t - (1 + t)x^2 : I \to I.$$

It is a family of quadratic polynomials with parameter $-1/2 \leq t \leq 1$. Then -1 is a fixed point of T_t and T_t maps 1 to -1. The map T_t has a unique critical point 0 and it is non-degenerate, that is, $T_t'(0) = 0$ and $T_t''(0) \neq 0$.

When $t_0 = -1/2$, T_{t_0} has -1 as a parabolic fixed point, that is, $T_{t_0}'(-1) = 1$ (see Figure 1). And, T_{t_0} has no other fixed point. One can see easily that

$$T_{t_0}^n(x) \to -1 \quad \text{as } n \to \infty \text{ for all } x \in I.$$

So, $\mathrm{ent}(T_{t_0}) = 0$ and, in this case, $K_0 = \{-1\}$ is the only minimal set and $\mathrm{Basin}(K_0) = [-1, 1]$. Thus, the flow $\mathcal{X} = ([-1, 1], T_{t_0})$ is MMA and MMLS.

When $t > t_0$ but close to t_0, -1 becomes a repelling fixed point of T_t, that is, $T_t'(-1) > 1$, but T_t has another fixed point p_t, which is an attracting fixed point, that is, $T_t(p_t) = p_t$ and $|T_t'(p_t)| < 1$ (see Figure 2). Furthermore, we have that

$$T_t^n(x) \to p_t \quad \text{as } n \to \infty \text{ for all } x \in (-1, 1).$$

This phenomenon lasts until p_t becomes a parabolic fixed point at t_1, that is, $T_{t_1}'(p_{t_1}) = -1$. So, for $t_0 < t \leq t_1$, we have that $\mathrm{ent}(T_t) = 0$ and T_t has two minimal

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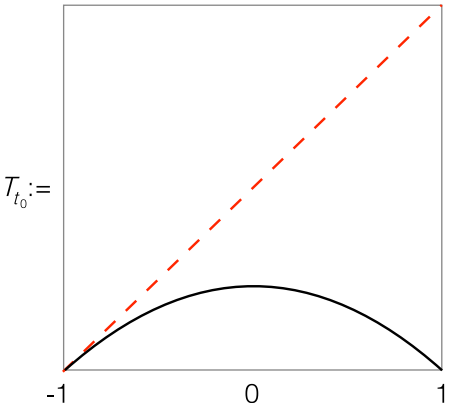


FIGURE 1. One parabolic fixed point.

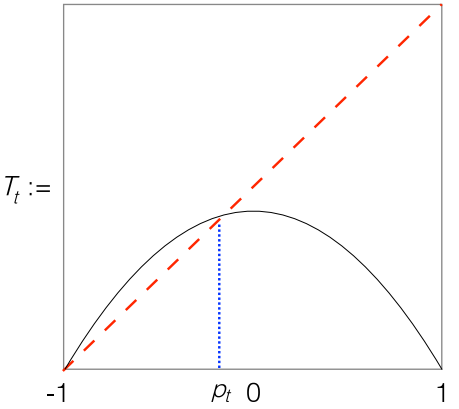


FIGURE 2. One attracting fixed point and one repelling fixed point.

subsets $K_0 = \{-1\}$ and $K_1 = \{p_t\}$ and $\mathrm{Basin}(K_0) = \{-1, 1\}$ and $\mathrm{Basin}(K_1) = (-1, 1)$. This proves that the flow $\mathcal{X} = ([-1, 1], T_t)$ is MMA and MMLS.

When $t > t_1$ but close to t_1, p_t becomes a repelling fixed point of T_t, that is, $T_t(p_t) = p_t$ and $|T_t'(p_t)| > 1$, but there is a periodic cycle of period 2 appearing near p_t, which we denote as $\{p_{t,2,0}, p_{t,2,1}\}$. This cycle is attractive, that is,

$$T_t(p_{t,2,0}) = p_{t,2,1}, \quad T_t(p_{t,2,1}) = p_{t,2,0}$$

and

$$|(T_t^2)'(p_{t,2,0})| = |(T_t^2)'(p_{t,2,1})| < 1$$

(see Figure 3). Moreover,

$$T_t^n(x) \to \{p_{t,2,0}, p_{t,2,1}\} \quad \text{as } n \to \infty \text{ for all } x \in I \setminus \Big(\bigcup_{m=1}^{\infty} T_t^{-m}(p_t) \cup \{-1, 1\}\Big).$$

This phenomenon lasts until the cycle $\{p_{t,2,0}, p_{t,2,1}\}$ becomes a parabolic periodic cycle of period 2 at t_2, that is,

$$|(T_{t_1}^2)'(p_{t_2,2,0})| = |(T_{t_1}^2)'(p_{t_2,2,1})| = 1.$$

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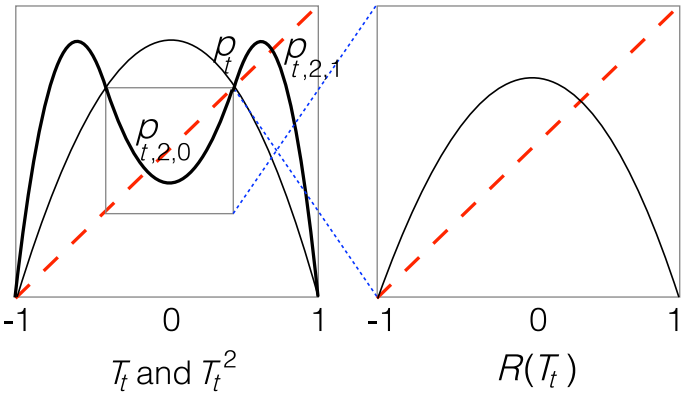


FIGURE 3. Renormalization.

So, for $t_1 < t \le t_2$, we have that $\mathrm{ent}(T_t) = 0$ and T_t has three minimal subsets $K_0 = \{-1\}$, $K_1 = \{p_t\}$, and $K_2 = \{p_{t,2,0},\ p_{t,2,1}\}$ with

$$\mathrm{Basin}(K_0) = \{-1,\ 1\}, \quad \mathrm{Basin}(K_1) = \bigcup_{n=0}^{\infty} T^{-n}(p_t),$$

and

$$\mathrm{Basin}(K_2) = (-1,\ 1)\backslash \mathrm{Basin}(K_1).$$

We have proved that the flow $\mathcal{X} = ([-1,\ 1],\ T_t)$ is MMA and MMLS for $t_1 < t \le t_2$.

To see how the periodic cycle of period 2 was born, we can use the period doubling operator $\mathcal{R}$ as follows. When $t_1 \le t \le t_2$, let $\beta = p_t > 0$. Consider the map

$$\mathcal{R}(T_t)(x) = -\beta^{-1} T_t^2(-\beta x) : I \to I.$$

It is not a quadratic polynomial but has a graph similar to T_t for $t_0 \le t \le t_1$. So, when $t = t_1$, $\mathcal{R}(T_t)$ has a parabolic fixed point -1 and when $t_1 < t < t_2$, $\mathcal{R}(T_t)$ has -1 as a repelling fixed point and there is an attractive fixed point q_t. When $t = t_2$, $\mathcal{R}(T_t)$ has -1 as a repelling fixed point and q_t becomes a parabolic fixed point. Note that

$$p_{t,2,0} = -\beta q_t \quad \text{and} \quad p_{t,2,1} = T_t(p_{t,2,0})$$

become a cycle of periodic points of period 2 for T_t.

In general, the above process gives the following dichotomy. There is an increasing sequence

$$t_0 < t_1 < \cdots < t_n < t_{n+1} < \cdots < 1$$

for any $t_{n-1} < t \le t_n$ and T_t has a repelling periodic cycle

$$\mathrm{Orb}(i) = \{p_{t,2^i,k}\}_{k=0}^{2^i-1}$$

of period 2^i for any $0 \le i < n$ and an attractive (or parabolic but semi-attractive) periodic cycle

$$\mathrm{Orb}(n) = \{p_{t,2^n,k}\}_{k=0}^{2^n-1}$$

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such that

$$T_t^n(x) \to \mathrm{Orb}(n) \quad \text{as } n \to \infty \text{ for all } x \in I \setminus P_t,$$

where $P_t = \bigcup_{m=0}^{\infty} T_t^{-m}(\bigcup_{i=0}^{n-1} \mathrm{Orb}(i))$. So, we have that $\mathrm{ent}(T_t) = 0$ for $t_0 \le t < t_\infty$, where

$$t_\infty = \lim_{n\to\infty} t_n.$$

In the case $t_{n-1} < t \le t_n$, T_t has $n+1$ minimal subsets, $K_0 = \{-1\}$, $K_1 = \{p_t\}$, and

$$K_{i+1} = \{p_{t,2^i,k}\}_{k=0}^{2^i-1}, \quad i = 1, 2, \ldots, n-1,$$

with $\mathrm{Basin}(K_0) = \{-1, 1\}$ and

$$\mathrm{Basin}(K_i) = \bigcup_{n=0}^{\infty} T^{-n}(K_i), \quad i = 1, 2, \ldots, n-1$$

and

$$\mathrm{Basin}(K_n) = [-1, 1] \setminus \bigcup_{i=0}^{\infty} \mathrm{Basin}(K_i).$$

Thus, we have proved that the flow $\mathcal{X} = ([-1, 1], T_t)$ is MMA and MMLS.

Concluding from the above, we have the following result.

THEOREM 3. *For any $-1/2 \le t < t_\infty$, the flow*

$$\mathcal{X} = (I, T_t(x) = t - (1+t)x^2)$$

is MMA and MMLS.

COROLLARY 9. *Any oscillating sequence $\mathbf{c} = (c_n)$ is linearly disjoint from*

$$\mathcal{X} = (I, T_t(x) = t - (1+t)x^2)$$

for any $-1/2 \le t < t_\infty$.

A special case in Corollary 9 is the Möbius sequence $\mathbf{u} = (\mu(n))$.

The quadratic polynomial T_{t_∞} is called the *Feigenbaum quadratic polynomial*. This polynomial has been studied for the past 40 years since it was discovered by Feigenbaum [**18**, **19**] and, independently, by Coullet and Tresser [**8**] in the 1970s. The following theorem can be found in the literature [**7**, **27**, **39**, **41**].

THEOREM 4. *The quadratic polynomial T_{t_∞} on I as a dynamical system has two repelling fixed points -1 and p_{t_∞} and a repelling periodic cycle*

$$\mathrm{Orb}(n) = \{p_{t,n,k}\}_{k=0}^{2^n-1}$$

of period 2^n for every $n \ge 1$. There is a Cantor set Λ such that

$$T_{t_\infty}^n(x) \to \Lambda \quad \text{as } n \to \infty \text{ for all } x \in I \setminus P_{t_\infty},$$

where

$$P_{t_\infty} = \bigcup_{m=0}^{\infty} T^{-m}\left(\{-1, p_{t_\infty}\} \cup \left(\bigcup_{n=0}^{\infty} \mathrm{Orb}(n)\right)\right).$$

Moreover, the topological entropy

$$\mathrm{ent}(T_{t_\infty}) = 0.$$

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In the rest of this section, we will give a rigorous proof that the flow defined by T_{t_∞} on $I \setminus P_{t_\infty}$ is MLS (that is, mean-equicontinuous). Actually, we will prove that all flows defined by Feigenbaum maps restricted to their corresponding sets are MLS.

Our strategy in the proof is as follows. First we will give a rigorous proof that the restriction of a Feigenbaum map to its Cantor set attractor is topologically conjugate to the adding machine on the 2-adic symbolic space (refer to §5). This fact was first observed by Feigenbaum in [**18**, **19**] when he proposed his famous universality theory in chaos (and, independently, by Coullet and Tresser [**8**]). Then it was emphasized by Sullivan in his many lectures attended by the second author in 1986–1989 and in our paper and book [**27**, **28**].

Our rigorous proof is based on two facts. The first one is the existence of the fixed point of the periodic doubling operator, which was a hot topic during the past 40 years in the research of complex dynamics and chaos. The existence of the fixed point was first observed by Feigenbaum in [**18**, **19**] and used to explain his universality theory in chaos. Then it was proved by Lanford in [**33**] with computer assistance (but still a legitimate mathematical proof). Furthermore, a conceptual understanding and a more rigorous mathematical proof, by using Teichmüller theory and following the idea of the Moscow rigidity theorem, has been given very recently, thanks to the work of many mathematicians (refer to [**27**, **37**, **40**, **50**]). The other one is the connection between the Cantor set attractor of the Feigenbaum fixed point and the non-escaping set of an induced expanding map, as studied in our previous paper [**28**].

To state our main result, we first define the space of all Feigenbaum quadratic-like maps (for simplicity, we call them Feigenbaum maps). Suppose that $T : I \to I$ is a C^3-map. We call it a *quadratic-like map* if:

(1) $T(-1) = T(1) = -1$;
(2) $T'(0) = 0$ and $T''(0) \neq 0$;
(3) $T'(x) > 0$ for $x \in [-1, 0)$ and $T'(x) < 0$ for $x \in (0, 1]$;
(4) the Schwarzian derivative

$$S(T)(x) := \frac{T'''(x)}{T'(x)} - \frac{3}{2}\left(\frac{T''(x)}{T'(x)}\right)^2 < 0 \quad \text{for all } x \in I.$$

Let $\mathcal{Q}$ be the space of all quadratic-like maps with C^3-topology. Then the topological entropy

$$\mathrm{ent}(T) : \mathcal{Q} \to \mathbb{R}^+ = \{x \geq 0\}$$

defines a functional. Using the theory of kneading sequences of Milnor and Thurston [**7**, **42**], the space $\mathcal{Q}$ is divided into two parts,

$$\mathcal{Q}_0 = \{T \in \mathcal{Q} \mid \mathrm{ent}(T) = 0\}$$

and

$$\mathcal{Q}_+ = \{T \in \mathcal{Q} \mid \mathrm{ent}(T) > 0\}.$$

The boundary of both spaces

$$\mathcal{F}_\infty = \partial\mathcal{Q}_0 = \partial\mathcal{Q}_+$$

is called the *space of Feigenbaum maps* (or the boundary of chaos (see [**27**])).

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Remark 9. The space of Feigenbaum maps can be defined much more generally. For example, we can assume that the type of critical point is $|x|^\alpha$ for some $\alpha > 1$ and that the smoothness is C^{1+Z}, where $1 + Z$ means that the first derivative belongs to the Zygmund class of functions. Then we can assign the C^{1+Z}-topology to this space of functions. See [**27**] for a more general definition of Feigenbaum folding maps. To avoid distracting the reader from our main purpose, we use a simpler definition of a Feigenbaum map. What we will prove in Theorem 6 works for any Feigenbaum folding map, but the general argument is more complicated.

We would like to note that the space $\mathcal{F}_\infty$ consists of all quadratic-like maps with the same Feigenbaum kneading sequence, which is the kneading sequence of T_{t_∞}. Moreover, we have the following result from the literature [**27**, **39**, **41**].

THEOREM 5. *Any $T \in \mathcal{F}_\infty$ has two fixed points -1 and β and a repelling periodic cycle*

$$\mathrm{Orb}(n) = \{p_{T,2^n,k}\}_{k=0}^{2^n-1}$$

of period 2^n for every $n \geq 1$. There is a Cantor set Λ_T such that

$$T^n(x) \to \Lambda_T \quad \text{as } n \to \infty \text{ for all } x \in I \backslash P_T,$$

where

$$P_T = \bigcup_{m=0}^{\infty} T^{-m}\left(\{-1, \beta\} \cup \left(\bigcup_{n=1}^{\infty} \mathrm{Orb}(n)\right)\right).$$

Furthermore, there is a homeomorphism h of I such that

$$h \circ T = T_{t_\infty} \circ h \quad \text{on } I.$$

Thus, the topological entropy

$$\mathrm{ent}(T) = 0.$$

We call $\mathcal{X} = (I, T)$ for $T \in \mathcal{F}_\infty$ a Feigenbaum flow.

THEOREM 6. (MMA/MMLS and Feigenbaum flow) *Any Feigenbaum flow $\mathcal{X}$ is MMA and MMLS.*

Proof. We first define the period doubling operator on $\mathcal{F}_\infty$ as follows. Any $T \in \mathcal{F}_\infty$ has two fixed points $\alpha = -1$ and $\beta \in (0, 1)$. Define

$$\mathcal{R}(T) = -\beta^{-1} T^2(-\beta x), \quad x \in I.$$

Then $\mathcal{R}(T) \in \mathcal{F}_\infty$. Thus,

$$\mathcal{R} : \mathcal{F}_\infty \to \mathcal{F}_\infty$$

defines an operator, which is called the period doubling operator.

As we remarked in the third paragraph after Theorem 4, the period doubling operator has a unique fixed point $G \in \mathcal{F}_\infty$, that is,

$$\mathcal{R}(G) = G.$$

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With the existence of this fixed point, we continue our proof as follows. Consider the critical value $c_1 = G(0)$ and let $c_2 = G^2(0)$, $c_3 = G^3(0)$, $c_4 = G^4(0)$. Then we have

$$c_2 < 0 < c_4 < c_3 < c_1.$$

Let $J = [c_2, c_1]$ and let $J_0 = [c_2, c_4]$ and $J_1 = [c_3, c_1]$.

We now define a C^3 expanding map $E(x)$. Let $\beta \in (0, 1)$ be the fixed point of G. Define

$$E(x) = \begin{cases} -\beta^{-1}x, & x \in J_0, \\ -\beta^{-1}g(x), & x \in J_1. \end{cases}$$

It has been shown in [**28**] that $E(x)$ is expanding, that is, $|E'(x)| > 1$. Therefore, the non-escaping set

$$NE = \bigcap_{n=0}^{\infty} E^{-n}(J)$$

is a Cantor set (see [**27**, Ch. 1] for the detailed proof of this fact).

Let

$$PC = \{c_n = G^n(0)\}_{n=1}^{\infty}$$

be the set of all post-critical points for G. Then the attracting Cantor set for G satisfies

$$\Lambda_G = \overline{PC}.$$

Since $\mathcal{R}(G) = G$, we can see that end points of all intervals in $\bigcap_{n=0}^{N} E^{-n}(J)$ are contained in PC. This implies that

$$NE = \Lambda_G.$$

Since $E(x)$ is an expanding map of degree 2, we can conjugate $E|_{NE}$ to the shift map σ of the symbolic space (the 2-adic space in §5)

$$\mathbb{Z}_2 = \prod_{n=0}^{\infty} \{0, 1\} = \{w = i_0 i_1 \cdots i_{n-2} i_{n-1} \cdots \mid i_{n-1} \in \{0, 1\}, n = 1, 2, \ldots\}$$

as follows. Consider the initial partition

$$\eta_0 = \{J_0, J_1\}.$$

We can pull this back to get the nth partition

$$\eta_n = E^{-n}(\eta_0) = \{J_{w_n}\}, \tag{26}$$

where $w_n = i_0 i_1 \cdots i_{n-2} i_{n-1}$. We label the intervals in η_n inductively by

$$E(J_{w_n}) = J_{\sigma(w_n)} \in \eta_{n-1},$$

where $\sigma(w_n) = i_1 \cdots i_{n-2} i_{n-1}$. From this labelling, we have

$$\cdots \subset J_{i_0 i_1 \cdots i_{n-2} i_{n-1}} \subset J_{i_0 i_1 \cdots i_{n-2}} \subset \cdots \subset J_{i_0}.$$

Thus, for any $w = i_0 i_1 \cdots i_{n-2} i_{n-1} \cdots \in \mathbb{Z}_2$, we have that the set

$$\bigcap_{n=1}^{\infty} J_{i_0 i_1 \cdots i_{n-1}} = \{x_w\}$$

contains one point. Set $\pi(w) = x_w$.

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Suppose that the Cantor set NE is a metric space with the metric induced from the Lebesgue metric on I and suppose that $\mathbb{Z}_2$ is a metric space with the metric defined as

$$d(w, w') = \frac{1}{2^n}$$

if $w = i_0 i_1 \cdots i_{n-1} i_n \cdots$ and $w' = i_0 i_1 \cdots i_{n-1} i'_n \cdots$ and if $i_n \neq i'_n$. Then we have that

$$\pi : \mathbb{Z}_2 \to NE$$

is a homeomorphism such that

$$\pi \circ E = \sigma \circ \pi.$$

Now we define the adding machine $\mathrm{add}(w) = w + 1$ on $\mathbb{Z}_2$ as follows. For any $w = i_0 i_1 \cdots i_{n-1} \cdots$, if $i_0 = 0$, then $\mathrm{add}(w) = 1 i_1 \cdots i_{n-1} \cdots$ and, if $i_0 = 1$, then the first digit of $\mathrm{add}(w)$ is 0 and consider $i_1 + 1$, etc. From the construction of $NE = \Lambda_G$, we have that

$$G(x_w) = x_{\mathrm{add}(w)}.$$

This implies that

$$G(x) = \pi \circ \mathrm{add} \circ \pi^{-1}(x)$$

and, furthermore, $G^n(x) = \pi \circ \mathrm{add}^n \circ \pi^{-1}(x)$ for $x \in NE$. Since the adding machine add is isometric, it is equicontinuous. Since π is a conjugacy, we have that $\mathcal{G} = (NE, G)$ is equicontinuous.

Now, for any $T \in \mathcal{F}_\infty$, we have a homeomorphism $h : I \to I$ such that

$$T^n = h \circ G^n \circ h^{-1} \quad \text{on } I \text{ for all } n \geq 0.$$

Both $h|_{\Lambda_G}$ and $h^{-1}|_{\Lambda_T}$ are uniformly continuous. So, we have that $\mathcal{T} = (\Lambda_T, T)$ is equicontinuous.

Now what we have proved is that T has countably many minimal subsets $K_0 = \{-1\}$, $K_1 = \{p\}$, $K_{n+1} = \mathrm{Orb}(n)$ for all $n \geq 1$, and $K_\infty = \Lambda_T$.

Since K_0 or K_1 contains only one repelling fixed point, it is easy to see that

$$\mathrm{Basin}(K_0) = \{-1, 1\}, \quad \mathrm{Basin}(K_1) = \bigcup_{m=0}^{\infty} T^{-n}(K_1).$$

Similarly, since K_{n+1} contains only one repelling periodic cycle of period 2^n for $n \geq 1$,

$$\mathrm{Basin}(K_{n+1}) = \bigcup_{m=0}^{\infty} T^{-n}(K_{n+1}), \quad n = 1, 2, \ldots.$$

Now we claim that

$$\mathrm{Basin}(K_\infty) = [-1, 1] \Big\backslash \bigcup_{n=0}^{\infty} \mathrm{Basin}(K_n).$$

To prove this claim, we consider G and the induced expanding map E. Let η_n be the nth partition in (26). Since E is expanding, we have a constant $0 < \tau < 1$ such that

$$\tau_n = \max_{J \in \eta_n} |J| \leq \tau^n \quad \text{for all } n \geq 1.$$

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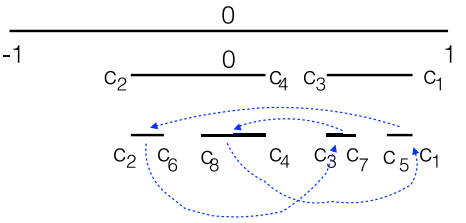


FIGURE 4. Permutation.

The map G permutes intervals in η_n (see Figure 4). For any

$$x \in [-1, 1] \setminus \bigcup_{n=0}^{\infty} \mathrm{Basin}_G(K_n),$$

we know that $G^n(x) \to NE = \Lambda_G$ as $n \to \infty$. Note that

$$\Lambda_G = NE = \bigcap_{n=1}^{\infty} \bigcup_{J \in \eta_n} J.$$

Therefore, for any $\epsilon > 0$, we have an integer $N' > 0$ such that

$$\tau_{N'} \leq \tau^{N'} < \epsilon.$$

We also have an integer $N'' \geq N'$ such that $G^{N''}(x)$ is in some interval $J \in \eta_{N'}$. Take a point $z_0 \in J \cap \Lambda_G$. Then we have that

$$|G^{N''}(x) - z_0| < \epsilon.$$

Since $G : \Lambda_G \to \Lambda_G$ is a homeomorphism, we have a point $z \in J' \cap \Lambda_G$ for $J' \in \eta_{N'}$ such that $z_0 = G^{N''}(z)$ (refer to Figure 4). This implies that

$$|G^{N''}(x) - G^{N''}(z)| < \epsilon.$$

Since G permutes intervals in η'_N, for any $n \geq N''$, we have that $G^n(x)$ and $G^n(z)$ are in the same interval in $\eta_{N'}$. This further implies that

$$|G^n(x) - G^n(z)| < \epsilon \quad \text{for all } n \geq N''.$$

Thus, we have that

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |G^n(x) - G^n(z)| < \epsilon.$$

This implies that

$$\mathrm{Basin}_G(K_\infty) = [-1, 1] \setminus \bigcup_{n=0}^{\infty} \mathrm{Basin}_G(K_n).$$

We have proved the claim for G. Since $T^n = h \circ G^n \circ h^{-1}$ and since both h and h^{-1} are uniformly continuous on I, we have now proved the claim for T.

We already knew that $T : K_\infty \to K_\infty$ is minimal and equicontinuous. For each $n \geq 0$, the map $T : K_n \to K_n$ is minimal and equicontinuous because it is just a periodic cycle. Thus, we have finally proved the MMA and MMLS properties for $\mathcal{X}$. □

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One consequence of Theorems 1 and 6 is the following corollary, with the Möbius sequence $\mathbf{u} = (\mu(n))$ as a special example.

COROLLARY 10. (Feigenbaum disjointness) *Any oscillating sequence is linearly disjoint from all Feigenbaum flows.*

Remark 10. After the first version of this paper was completed, we learned that Karagulyan [**31**] has proved that the Möbius sequence $\mathbf{u} = (\mu(n))$ is linearly disjoint from all continuous interval maps with zero topological entropy. Actually, in his proof only the oscillating property is needed and the special property of the Möbius function is not needed. Therefore, the statement that any oscillating sequence is linearly disjoint from flows defined by all continuous interval maps with zero topological entropy can be proved. We now believe the statement that flows defined by all continuous interval maps with zero topological are MMA and MMLS, and will write down the detailed proof in another place.

8. *Circle homeomorphisms*

In this section, we discuss the MMA and MMLS properties for flows defined by circle homeomorphisms.

Recall that we use $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ to denote the unit circle in the complex plane. Suppose that $T : S^1 \to S^1$ is a continuous map. Then from [**6**, **51**], we know that if $\mathcal{X} = (S^1, T)$ is equicontinuous, then:

(1) $\deg(T) = 0$ and, for every $x \in S^1$, $T^n x \to E$ as $n \to \infty$, where E is the fixed point set of T^2;
(2) $\deg(T) = 1$ and T is topologically conjugate to $R_\rho(z) = e^{2\pi i\rho} z$ for some $0 \le \rho < 1$; or
(3) $\deg(T) = -1$ and T^2 is the identity.

The above three cases are simple and Corollary 2 applies.

We call a homeomorphism of S^1 a circle homeomorphism. If a circle homeomorphism is orientation-preserving, then it has two fixed points, its second iterate is orientation-preserving, and its rotation number is 0. So, we only need to consider orientation-preserving ones. We call $R_\rho(z) = e^{2\pi i\rho} z : S^1 \to S^1$ for some $0 \le \rho < 1$ a rigid rotation.

We know that the flow defined by an orientation-preserving circle homeomorphism is equicontinuous if and only if it is topologically conjugate to a rigid rotation. Therefore, we know from Corollary 2 that any oscillating sequence is linearly disjoint from flows defined by all orientation-preserving circle homeomorphisms which are topologically conjugate to rigid rotations. However, not all orientation-preserving circle homeomorphisms are topologically conjugate to rigid rotations.

If an orientation-preserving circle homeomorphisms is not topologically conjugate to a rigid rotation but its rotation number is rational, then its non-wandering set consists of periodic points of the same period, and all other points are in its wandering set and tend to its non-wandering set under iterations. This implies that the flow defined by this map is MMA and MMLS (refer to Proposition 7). Therefore, we conclude that any oscillating sequence is linearly disjoint from flows defined by all orientation-preserving circle homeomorphisms with rational rotation numbers.

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Definition 7. (Denjoy counter-example) We call an orientation-preserving circle homeomorphisms a Denjoy counter-example if it is not topologically conjugate to a rigid rotation and its rotation number is irrational.

We now show that Theorem 1 applies to all flows defined by Denjoy counter-examples. Let us first state the following classical result in one-dimensional dynamical systems.

THEOREM 7. (Poincaré theorem) *Suppose that $T : S^1 \to S^1$ is an orientation-preserving circle homeomorphism with irrational rotation number ρ. Then there is an orientation-preserving continuous surjective map $h : S^1 \to S^1$ such that*

$$h \circ T = R_\rho \circ h \quad \text{on } S^1. \tag{27}$$

Moreover, the ω-limit set $\omega(z)$ for any point $z \in S^1$ is either a Cantor set Λ, which is independent of z, or the whole circle S^1. In the case $\omega(z) = S^1$, h is a homeomorphism. In the case $\omega(z) = \Lambda$, a Cantor set, $S^1 \backslash \Lambda = \cup J$ is the union of all wandering intervals. Here Λ is also called the non-wandering set for T.

Denjoy constructed an example of an orientation-preserving circle homeomorphism with irrational rotation number such that for any $z \in S^1$, $\omega(z) = \Lambda$ is a Cantor set in [**12**]. In this case, $T : \Lambda \to \Lambda$ is minimal and the flow $\mathcal{X} = (S^1, T)$ is not equicontinuous according to [**6**, **51**]. For a Denjoy counter-example, we actually have two flows $\mathcal{X} = (S^1, T)$ and $\mathcal{A} = (\Lambda, T)$. We have the following theorem.

THEOREM 8. (MMA/MMLS and Denjoy counter-example) *Suppose that $T : S^1 \to S^1$ is a Denjoy counter-example. The flow $\mathcal{X} = (S^1, T)$ is MMA but the flow $\mathcal{A} = (\Lambda, T)$ is not equicontinuous. However, the flow $\mathcal{A} = (\Lambda, T)$ is MLS. Thus, we have that the flow $\mathcal{X} = (S^1, T)$ is MMA and MMLS.*

Proof. For any $x, y \in S^1$, let $[x, y]$ be the shortest arc with end points x and y. We use $|x - y|$ to denote the arc length of $[x, y]$. Assume that the total length of S^1 is 1. Since T is a Denjoy counter-example, Λ is a Cantor set and

$$S^1 \backslash \Lambda = \bigcup I$$

is the union of all wandering intervals. We have that

$$\sum_I |I| \le 1.$$

For any $x \in S^1 \backslash \Lambda$, it is in a wandering interval $I = (a, b)$, where $a, b \in \Lambda$. Let $I_n = T^n(I)$, $n = 0, 1, \ldots$. Then we have that

$$\sum_{n=0}^{\infty} |I_n| \le 1.$$

Thus, $|I_n| \to 0$ as $n \to \infty$. Since $T^n(x) \in I_n$ and $T^n(a) \in \overline{I}_n$, we have that

$$\lim_{n \to +\infty} |T^n(a) - T^n(x)| = \lim_{n \to +\infty} |I_n| = 0,$$

which implies that $\mathcal{X} = (S^1, T)$ is MMA because $T : \Lambda \to \Lambda$ is minimal.

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Consider the set $\mathcal{W}$ of all wandering intervals, which is countable. We partition it into full orbits

$$\mathcal{W} = \{I_{m,n} \mid I_{m,n} = T^n(I_{m,0}),\ 0 \le m < L,\ n \in \mathbb{Z}\},$$

where L is the number of full orbits of wandering intervals. Here L is either a positive integer or ∞. Since Λ is a Cantor set, we have

$$S^1 = \overline{\bigcup_{0\le m<L} \bigcup_{n\in\mathbb{Z}} I_{m,n}}.$$

Let h be the semi-conjugacy between T and the rigid rotation R_ρ. See (27). Then the image $h(I_{m,n})$ is a singleton $\{x_{m,n}\}$ and we write $x_{m,n} = h(I_{m,n})$. By (27),

$$\{x_{m,n} = h(I_{m,n}) = R_\rho^n(x_{m,0}) \mid n \in \mathbb{Z}\} \tag{28}$$

is the full orbit of $x_{m,0}$ under iterations of R_ρ for any fixed $0 \le m < L$.

Also notice that h is monotone increasing. So, if x and y are two points in the Cantor set Λ, we have

$$I_{m,n} \subset [x,\ y] \quad \text{if and only if } x_{m,n} \in [h(x),\ h(y)]. \tag{29}$$

The rigid rotation R_ρ is uniquely ergodic with the Lebesgue measure as the unique invariant measure. This fact, together with (28), implies that for any $z,\ w \in S^1$,

$$\lim_{n\to\infty} \frac{\#\{k \mid x_{m,k} \in [z,\ w],\ -n \le k \le 0\}}{n} = |z - w|. \tag{30}$$

This allows us to prove that $\mathcal{A} = (\Lambda,\ T)$ is not equicontinuous. In fact, given any $x \ne y \in \Lambda$, by (29) and (30), we have that

$$\begin{aligned}&\lim_{n\to\infty} \frac{\#\{k \mid I_{0,k} \subset [x,\ y],\ -n \le k \le 0\}}{n}\\ &\quad = \lim_{n\to\infty} \frac{\#\{k \mid x_{0,k} \in [h(x),\ h(y)],\ -n \le k \le 0\}}{n} = |h(x) - h(y)| > 0.\end{aligned}$$

Thus, there are infinitely many negative integers k such that $I_{0,k} \subset [x,\ y]$. For such k, we have

$$I_{0,0} = T^{|k|}(I_{0,k}) \subset [T^{|k|}(x),\ T^{|k|}(y)],$$

so that

$$|T^{|k|}(x) - T^{|k|}(y)| \ge |I_0|$$

for infinitely many positive integers $|k|$. Therefore, $\mathcal{A} = (\Lambda,\ T)$ is not equicontinuous.

Next we prove that $\mathcal{A} = (\Lambda,\ T)$ is MLS. We first assume that the non-wandering set Λ has zero Lebesgue measure. Under this assumption, we have that

$$\sum_{0\le m<L} \sum_{n\in\mathbb{Z}} |I_{m,n}| = 1.$$

Moreover, for any two points $x \ne y \in \Lambda$,

$$|x - y| = \sum_{I\in\mathcal{W},\, I\subset[x,y]} |I|. \tag{31}$$

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For any $N \geq 1$, consider the rectangle

$$R_N = \{(m, n) \in \mathbb{N} \times \mathbb{Z} \mid 0 \leq m \leq N \wedge (L-1), |n| \leq N\},$$

where $N \wedge (L-1) = \min\{N, L-1\}$. For any $\epsilon > 0$, we have a large integer $N_0 \geq 1$ such that

$$\sum_{(m,n)\in\mathbb{N}\times\mathbb{Z}\setminus R_{N_0}} |I_{m,n}| < \epsilon.$$

Since h is uniformly continuous, we can find a $0 < \delta \leq \epsilon$ such that for any $x \neq y \in \Lambda$ with $|x - y| < \delta$,

$$0 < |h(x) - h(y)| < \frac{\epsilon}{(N_0 \wedge (L-1) + 1)(2N_0 + 1)}.$$

Let

$$E = \bigcup_{(m,l)\in R_{N_0}} E_{m,l},$$

where

$$E_{m,l} = \{0 \leq k < \infty \mid I_{m,-k+l} \subset [x, y]\}.$$

For any fixed $(m, l) \in R_{N_0}$, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{\#(E_{m,l} \cap [1, n])}{n} &= \lim_{n\to\infty} \frac{\#\{k \mid I_{m,k+l} \subset [x, y], -n \leq k \leq 0\}}{n} \\ &= \lim_{n\to\infty} \frac{\#\{k \mid x_{m,k+l} \in [h(x), h(y)], -n \leq k \leq 0\}}{n} \\ &= |h(x) - h(y)| > 0.\end{aligned}$$

Therefore, the density of E exists and is equal to

$$\overline{D}(E) = \limsup_{n\to\infty} \frac{\#(E \cap [1, n])}{n} = (N_0 \wedge (L-1) + 1)(2N_0 + 1)|h(x) - h(y)| < \epsilon.$$

For any $n \notin E$, the interval $[T^n(x), T^n(y)]$ contains only wandering intervals from

$$\mathcal{W}_{N_0} = \{I_{m,n} = T^n(I_{m,0}) \mid (m, n) \in \mathbb{N} \times \mathbb{Z}\setminus R_{N_0}\}.$$

From (31), we have that

$$|T^n(x) - T^n(y)| \leq \sum_{(m,n)\in\mathbb{N}\times\mathbb{Z}\setminus R_{N_0}} |I_{m,n}| < \epsilon.$$

This implies that $\mathcal{A} = (\Lambda, T)$ is MLS (equivalently, mean-equicontinuous). Since the flow $\mathcal{X} = (S^1, T)$ has only one mean minimal attractor Λ, it is MMLS.

When the non-wandering set Λ has a positive Lebesgue measure, there is another Denjoy counter-example $\widetilde{T}$ whose non-wandering set $\widetilde{\Lambda}$ has zero Lebesgue measure such that both of them are semi-conjugate to the same rigidity rotation R_ρ. That is, we have two continuous onto maps $h, \widetilde{h} : \mathbb{T} \to \mathbb{T}$ such that

$$h \circ T = R_\rho \circ h \quad \text{and} \quad \widetilde{h} \circ \widetilde{T} = R_\rho \circ \widetilde{h}.$$

Now we can define

$$\widehat{h} : \mathbb{T} \to \mathbb{T},$$

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which formally can be written as $\widetilde{h}^{-1} \circ h$. It maps every wandering interval to the corresponding wandering interval bijectively and preserves orders of wandering intervals. Furthermore, it maps Λ to $\widetilde{\Lambda}$ bijectively. Thus, it is a homeomorphism of the circle such that

$$\widehat{h} \circ T = \widetilde{T} \circ \widehat{h}.$$

Since $\widetilde{T}$ is MMLS and since T is topologically conjugate to $\widetilde{T}$, we get that T is MMLS. □

One consequence of Theorems 1 and 8 is the following corollary, with the Möbius sequence $\mathbf{u} = (\mu(n))$ as a special example.

COROLLARY 11. (Denjoy disjointness) *Any oscillating sequence is linearly disjoint from flows defined by all Denjoy counter-examples.*

This result combined with the explanation we gave at the beginning of this section finally implies the following theorem.

THEOREM 9. (Circle disjointness) *Any oscillating sequence is linearly disjoint from flows defined by all orientation-preserving circle homeomorphisms.*

In particular, we have Sarnak's conjecture for orientation-preserving circle homeomorphisms.

COROLLARY 12. (Karagulyan [**31**]) *The Möbius sequence* $\mathbf{u} = (\mu(n))$ *is linearly disjoint from flows defined by all orientation-preserving circle homeomorphisms.*

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