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Two-point correlation in wall turbulence according to the attached-eddy hypothesis

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For the constant-stress layer of wall turbulence, two-point correlations of velocity fluctuations are studied theoretically by using the attached-eddy hypothesis, i.e. a phenomenological model of a random superposition of energy-containing eddies that are attached to the wall. While previous studies had invoked additional assumptions, we focus on the minimum assumptions of the hypothesis to derive the most general forms of the correlation functions. They would allow us to use or assess the hypothesis without any effect of those additional assumptions. We also study the energy spectra and the two-point correlations of the rate of momentum transfer and of the rate of energy dissipation.

Key words: turbulent flows

1. Introduction

This is a theoretical study about wall turbulence that is formed in a pipe, in a channel or over a plate. As shown in figure 1, we set a smooth wall at the x–y plane, set the mean stream along the x direction and use $u(z)$, $v(z)$ and $w(z)$ to denote velocity fluctuations in the streamwise, spanwise and wall-normal directions at a height z from the wall. The mean streamwise velocity itself is not studied here. If the turbulence is stationary and at a high Reynolds number, it has a layer with some constant value of ρu_τ^2 for the mean rate of momentum transfer, i.e. for the Reynolds stress $\rho\langle -uw\rangle$. Here ρ is the mass density, u_τ is the friction velocity and $\langle\cdot\rangle$ denotes the ensemble average.

For this constant-stress layer, there is a phenomenological model of a random superposition of energy-containing eddies that are attached to the wall, i.e. the attached-eddy hypothesis (Townsend 1976). The velocity fields of the eddies are assumed to have an identical shape with a common characteristic velocity u_τ, while their sizes are distributed without any characteristic size. Actually, the constant-stress layer has a characteristic constant u_τ in units of velocity but has no characteristic constant in units of length. If the size distribution of the attached eddies is given so as to reproduce the constant nature of $\langle -uw\rangle$, a logarithmic law is predicted for the variance of the streamwise velocity fluctuations $u(z)$ as

$$\frac{\langle u^2(z_1)\rangle - \langle u^2(z_2)\rangle}{u_\tau^2} \propto \ln\left(\frac{z_1}{z_2}\right). \tag{1.1}$$

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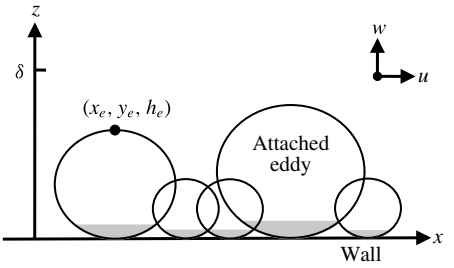


FIGURE 1. Schematic of attached eddies in wall turbulence, where grey areas denote the undermost layers of the eddies.

Since this prediction has been confirmed in experiments of a variety of wall turbulence (Hultmark *et al.* 2012; Marusic *et al.* 2013), the attached-eddy hypothesis has turned out to be a reliable model.

An important application of such a hypothesis would be to the two-point correlations or equivalently the energy spectra of the velocity fluctuations. Besides numerical works based on a particular model of the eddies (e.g. Marusic 2001), there are some theoretical studies (e.g. Perry & Chong 1982; Davidson *et al.* 2006). They have invoked, however, additional assumptions, such as for the similarity (Perry & Abell 1977). We rather focus on the minimum assumptions of the attached-eddy hypothesis that have led to the law of (1.1). These assumptions are yet sufficient to constrain the correlation functions and to clarify their most general forms, which would offer an opportunity to use or assess the hypothesis without any effect of the additional assumptions. We also study the energy spectra and the two-point correlations of some other quantities. They are to be compared with the existing models of the wall turbulence.

2. Basic setting of the hypothesis

Here is a summary of the attached-eddy hypothesis (Townsend 1976). The turbulence is taken to be homogeneous in the streamwise and spanwise directions. For the case of a boundary layer, we assume that it has been well developed and hence is negligibly dependent on the streamwise position x. Since this hypothesis is for the constant-stress layer, the value of the kinematic viscosity ν is not essential. The limit $\nu \to 0$ is taken so as to ignore the viscous length ν/u_τ with respect to the height z. Then, a free-slip condition, i.e. $u \neq 0$ and $v \neq 0$, is imposed on the wall at $z = 0$.

The attached eddies extend from the wall into the flow but are bounded in each of the directions. Regardless of the size of the eddies, they have an identical shape with a common characteristic velocity u_τ. That is, if $\boldsymbol{x}_e = (x_e, y_e, h_e)$ lies at the highest position of an eddy (see figure 1), its velocity field $\boldsymbol{v}_e$ is given for a position $\boldsymbol{x} = (x, y, z)$ as

$$\frac{\boldsymbol{v}_e(\boldsymbol{x})}{u_\tau} = \boldsymbol{f}\left(\frac{\boldsymbol{x} - \boldsymbol{x}_e}{h_e}\right) \quad \text{with } \boldsymbol{f} = (f_u, f_v, f_w). \tag{2.1a}$$

We regard h_e as the size of this eddy. The existence of the wall imposes $f_w = 0$ at $z = 0$. As for the undermost layer at $z \ll h_e$ of such an eddy, it is enough to assume $f_w \propto z/h_e$.

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Furthermore, the free-slip wall condition imposes $f_u \neq 0$ and $f_v \neq 0$ at $z = 0$. These are summarized by using $\boldsymbol{g}$ as some function of $(x - x_e)/h_e$ and $(y - y_e)/h_e$,

$$f_u \to g_u, \quad f_v \to g_v \quad \text{and} \quad f_w \to \frac{z}{h_e} g_w \quad \text{as} \quad \frac{z}{h_e} \to 0. \tag{2.1b–d}$$

The condition at $z > h_e$ is $f_u = f_v = f_w = 0$. No more assumptions are required about the functional form of $\boldsymbol{f}$.

The sizes of the attached eddies are distributed continuously from $h_e = \nu/u_\tau \to 0$ to $h_e = \delta$. Here δ corresponds to the height of the wall turbulence, i.e. pipe radius, channel half-width or boundary layer thickness. The asymptotic laws for $z/\delta \to 0$ are regarded as those for the constant-stress layer. On the wall, the eddies are distributed randomly and independently. They could even overlap one another. We do not assume any more about the distribution of the eddies, although some previous studies have assumed a particular hierarchy in that distribution (e.g. Perry & Chong 1982).

From the random and independent distribution of the attached eddies, it follows that the entire velocity field is a superposition of those of the individual eddies. Any two-point correlation over a streamwise distance $\boldsymbol{r}_l = \boldsymbol{r}_x = (r, 0, 0)$ at a height z is the sum of those within the individual eddies of various sizes from $h_e = z$ to $h_e = \delta$,

$$\frac{\langle v_i(\boldsymbol{x} + \boldsymbol{r}_x) v_j(\boldsymbol{x}) \rangle}{u_\tau^2} = \int_z^\delta \frac{\mathrm{d}h_e}{h_e} \left[h_e^3 n_e(h_e) \int \frac{\mathrm{d}x_e}{h_e} \int \frac{\mathrm{d}y_e}{h_e} f_{v_i}\left(\frac{\boldsymbol{x} + \boldsymbol{r}_x - \boldsymbol{x}_e}{h_e} \right) f_{v_j}\left(\frac{\boldsymbol{x} - \boldsymbol{x}_e}{h_e} \right) \right]. \tag{2.2}$$

The subscripts i and j denote either x, y or z such that v_i and v_j denote either u, v or w. As justified later, the number density of the attached eddies at size h_e per unit area of the wall is $n_e(h_e) = N_e h_e^{-3}$, where N_e is a constant. By using $\langle v_i(x + r, z) v_j(x, z) \rangle$ in place of $\langle v_i(\boldsymbol{x} + \boldsymbol{r}_x) v_j(\boldsymbol{x}) \rangle$,

$$\frac{\langle v_i(x + r, z) v_j(x, z) \rangle}{u_\tau^2} = N_e \int_z^\delta \frac{\mathrm{d}h_e}{h_e} I_{v_i v_j}\left(\frac{r}{h_e}, \frac{z}{h_e} \right), \tag{2.3a}$$

with the correlation for eddies of size h_e,

$$I_{v_i v_j}\left(\frac{r}{h_e}, \frac{z}{h_e} \right) = \int \frac{\mathrm{d}x_e}{h_e} \int \frac{\mathrm{d}y_e}{h_e} f_{v_i}\left(\frac{\boldsymbol{x} + \boldsymbol{r}_x - \boldsymbol{x}_e}{h_e} \right) f_{v_j}\left(\frac{\boldsymbol{x} - \boldsymbol{x}_e}{h_e} \right). \tag{2.3b}$$

Since the streamwise and spanwise sizes of each eddy have been taken as finite, the integral is always finite. Together with the wall condition of (2.1*b–d*), which yields a condition for $I_{v_i v_j}(r/h_e, z/h_e)$ in the case of $r/h_e = 0$, the form of (2.3) serves as the basis of the attached-eddy hypothesis.

To justify the distribution of the eddy size $n_e(h_e) = N_e h_e^{-3}$, the constant nature of $\langle -uw \rangle$ is reproduced in the limit $z/\delta \to 0$. That is, for $r = 0$ in (2.3*a*),

$$\frac{\langle -uw(z) \rangle}{u_\tau^2} = -N_e \int_{z/\delta}^1 \frac{\mathrm{d}z_*}{z_*} I_{uw}(0, z_*) \to \text{const.} \tag{2.4}$$

We have used $z_* = z/h_e$ and $\mathrm{d}z_*/z_* = -\mathrm{d}h_e/h_e$. The wall condition of (2.1*b–d*) yields $I_{uw}(0, z_*) = a_e z_*$ with a constant a_e at $z_* \ll 1$. If z/δ lies in this range of z_* as shown in figure 2(*b*), the integral is equal to $b_e - a_e z/\delta$. The constant $b_e = \int_0^1 (\mathrm{d}z_*/z_*) I_{uw}(0, z_*)$ is dominant in the limit $z/\delta \to 0$. To fix the value as $\langle -uw \rangle / u_\tau^2 = 1$, we require $N_e = -1/b_e$, i.e. $N_e \propto 1/|\boldsymbol{f}|^2$ in (2.3) for a given shape of $\boldsymbol{f}$.

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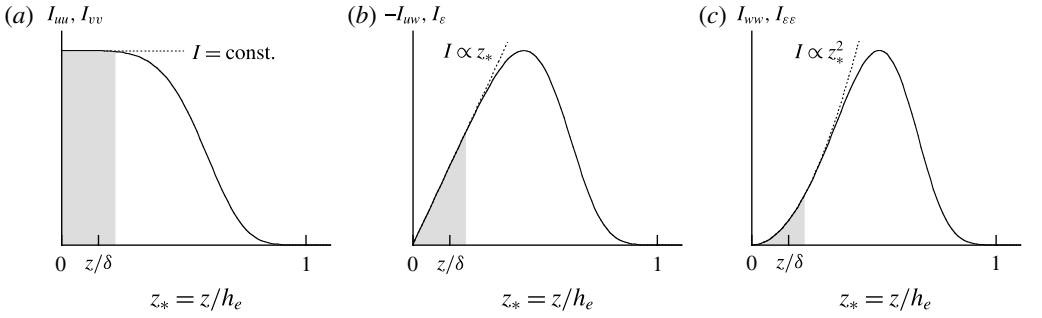


FIGURE 2. Schematic of I_{uu}, I_{vv} (a), $-I_{uw}$, I_ε (b) and I_{ww}, $I_{\varepsilon\varepsilon}$ (c) for $r_* = r/h_e = 0$ as a function of $z_* = z/h_e$, where grey areas correspond to the undermost layers of the eddies.

The velocity variances $\langle v_i^2(z)\rangle$ such as that in (1.1) are obtained in the same manner. As shown in figures 2(a) and 2(c), the wall condition of (2.1b–d) yields

$$I_{uu}(0, z_*) \to \frac{d_{uu}}{N_e}, \quad I_{vv}(0, z_*) \to \frac{d_{vv}}{N_e} \quad \text{and} \quad I_{ww}(0, z_*) \propto z_*^2 \quad \text{as} \quad z_* \to 0. \tag{2.5a–c}$$

Here d_{uu} and d_{vv} are constants. The result for the constant-stress layer $z/\delta \to 0$ is

$$\frac{\langle u^2(z)\rangle}{u_\tau^2} = N_e \int_{z/\delta}^1 \frac{\mathrm{d}z_*}{z_*} I_{uu}(0, z_*) \to c_{uu} + d_{uu} \ln\left(\frac{\delta}{z}\right), \tag{2.6a}$$

$$\frac{\langle v^2(z)\rangle}{u_\tau^2} = N_e \int_{z/\delta}^1 \frac{\mathrm{d}z_*}{z_*} I_{vv}(0, z_*) \to c_{vv} + d_{vv} \ln\left(\frac{\delta}{z}\right), \tag{2.6b}$$

$$\frac{\langle w^2(z)\rangle}{u_\tau^2} = N_e \int_{z/\delta}^1 \frac{\mathrm{d}z_*}{z_*} I_{ww}(0, z_*) \to c_{ww}. \tag{2.6c}$$

Here c_{uu}, c_{vv} and c_{ww} are constants. The law for $\langle u^2(z)\rangle$ has been confirmed with $c_{uu} \simeq$ 1.5–2.5 and $d_{uu} \simeq 1.3$ in a variety of wall turbulence (Hultmark *et al.* 2012; Marusic *et al.* 2013). We rely on the other laws as well. They do hold in the wall turbulence, although we are not yet certain of the values of c_{vv}, c_{ww} and d_{vv} (Sillero *et al.* 2013; Lee & Moser 2015).

3. Velocity correlation along a wall-parallel line

Two-point velocity correlations along a wall-parallel line $\langle v_i(x + r, z)v_i(x, z)\rangle$ are studied at a height z in the constant-stress layer. Although the line has been set streamwise, the same discussion is applicable to the case of a spanwise line. We make use of the correlation length

$$L_{x;v_iv_i}(z) = \frac{\displaystyle\int_0^\infty \mathrm{d}r\, \langle v_i(x + r, z)v_i(x, z)\rangle}{\langle v_i^2(z)\rangle}. \tag{3.1}$$

This is expected to be well defined in the attached-eddy hypothesis, where any correlation is determined by the individual eddies that are not dependent on one

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another and are of finite size. From (2.3*a*) with $r_* = r/h_e$ and $z_* = z/h_e$,

$$L_{x:v_iv_i}(z) = \frac{\displaystyle\int_0^\infty \mathrm{d}r \int_z^\delta (\mathrm{d}h_e/h_e)\, I_{v_iv_i}(r/h_e, z/h_e)}{\displaystyle\int_z^\delta (\mathrm{d}h_e/h_e)\, I_{v_iv_i}(0, z/h_e)} = z\frac{\displaystyle\int_{z/\delta}^1 (\mathrm{d}z_*/z_*^2) \int_0^\infty \mathrm{d}r_*\, I_{v_iv_i}(r_*, z_*)}{\displaystyle\int_{z/\delta}^1 (\mathrm{d}z_*/z_*)\, I_{v_iv_i}(0, z_*)}. \tag{3.2a}$$

The correlation length is also defined in an eddy of a particular size h_e,

$$\Lambda_{x:v_iv_i}(z, h_e) = \frac{\displaystyle\int_0^\infty \mathrm{d}r\, I_{v_iv_i}(r/h_e, z/h_e)}{I_{v_iv_i}(0, z/h_e)} = h_e\frac{\displaystyle\int_0^\infty \mathrm{d}r_*\, I_{v_iv_i}(r_*, z/h_e)}{I_{v_iv_i}(0, z/h_e)}. \tag{3.2b}$$

While $L_{x:v_iv_i}/z$ is dependent on z/δ alone, $\Lambda_{x:v_iv_i}/h_e$ is dependent on z/h_e alone. That is, $L_{x:v_iv_i}$ and $\Lambda_{x:v_iv_i}$ are to be determined respectively by z/δ and z/h_e through $L_{x:v_iv_i}/z$ and $\Lambda_{x:v_iv_i}/h_e$. Although (3.2*a*) could be rewritten so that $L_{x:v_iv_i}/\delta$ is another function of z/δ, it is not considered here because δ is not a characteristic of the constant-stress layer.

We require $L_{x:v_iv_i}/z$ to be finite. If $L_{x:v_iv_i}/z$ were not finite, it would follow that $L_{x:v_iv_i}$ is not determined and is not well defined in the attached-eddy hypothesis. The finiteness is also required for $\Lambda_{x:v_iv_i}/h_e$ because the streamwise size of an eddy is finite compared with its wall-normal size. For the undermost layer of such an eddy $z_* = z/h_e \ll 1$, there is a constant $\alpha \geqslant 0$ in relation to (3.2*b*),

$$\frac{\Lambda_{x:v_iv_i}(z, h_e)}{h_e} \propto z_*^\alpha \quad \text{and hence} \quad \int_0^\infty \mathrm{d}r_*\, I_{v_iv_i}(r_*, z_*) \propto z_*^\alpha I_{v_iv_i}(0, z_*) \quad \text{as } z_* \to 0. \tag{3.3}$$

The asymptotic form of $I_{v_iv_i}(0, z_*)$ is given in (2.5). By substituting the resultant form of $\int_0^\infty \mathrm{d}r_*\, I_{v_iv_i}(r_*, z_*)$ into (3.2*a*), along with $\int_{z/\delta}^1 (\mathrm{d}z_*/z_*)\, I_{v_iv_i}(0, z_*)$ given in (2.6), we obtain $L_{x:v_iv_i}/z$ in the limit $z/\delta \to 0$. The minimum α value for $L_{x:v_iv_i}/z$ to be finite is used as the leading order to determine the asymptotic form of $I_{v_iv_i}(r_*, z_*)$ in (3.3). It determines the asymptotic form of $\langle v_i(x + r, z)v_i(x, z)\rangle$ in the constant-stress layer $z/\delta \to 0$ in a manner similar to that for the variance $\langle v_i^2(z)\rangle$ in (2.6).

The individual cases of $v_i = u$, v and w are as follows. Only leading-order behaviour for $z/\delta \to 0$ is studied about $\int_{z/\delta}^1 (\mathrm{d}z_*/z_*^2) \int_0^\infty \mathrm{d}r_*\, I_{v_iv_i}(r_*, z_*)$ and $\int_{z/\delta}^1 (\mathrm{d}z_*/z_*)\, I_{v_iv_i}(0, z_*)$, e.g. $\ln(\delta/z)$ about $c_{uu} + d_{uu}\ln(\delta/z)$ in (2.6*a*).

For the streamwise velocity u, we take $\alpha = 1$ in (3.3) and use (2.5) to obtain $\int_0^\infty \mathrm{d}r_*\, I_{uu}(r_*, z_*) \propto z_*$ in the limit $z_* \to 0$. This is applied for $z/\delta \to 0$ as

$$\int_{z/\delta}^1 \frac{\mathrm{d}z_*}{z_*^2} \int_0^\infty \mathrm{d}r_*\, I_{uu}(r_*, z_*) \propto \ln\left(\frac{\delta}{z}\right). \tag{3.4a}$$

On the other hand, from (2.6*a*),

$$\int_{z/\delta}^1 \frac{\mathrm{d}z_*}{z_*} I_{uu}(0, z_*) \propto \ln\left(\frac{\delta}{z}\right). \tag{3.4b}$$

By substituting these into (3.2*a*), we confirm that $L_{x:uu}/z$ is finite. If $\alpha < 1$, it would diverge as $(\delta/z)^{1-\alpha}/\ln(\delta/z)$. The above asymptotic form of $\int_0^\infty \mathrm{d}r_*\, I_{uu}(r_*, z_*) \propto z_*$ is equivalent through $r_* = r/h_e$ and $z_* = z/h_e$ to

$$\int_0^\infty \mathrm{d}r\, I_{uu}\left(\frac{r}{h_e}, \frac{z}{h_e}\right) \propto z \quad \text{as } \frac{z}{h_e} \to 0. \tag{3.5}$$

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Thus, $I_{uu}(r/h_e,\ z/h_e)$ in this limit $z/h_e \to 0$ is independent of h_e but is only a function of r/z, say, $D_{x:uu}(r/z)/N_e$. It is independent of $z_* = z/h_e$ if r/z is fixed as $I_{uu}(r/h_e,\ z/h_e) = I_{uu}(rz_*/z,\ z_*)$. The correlation function in the constant-stress layer $z/\delta \to 0$ is thereby derived from

$$\frac{\langle u(x+r,\ z)u(x,\ z)\rangle}{u_\tau^2} = N_e \int_{z/\delta}^1 \frac{\mathrm{d}z_*}{z_*}\, I_{uu}\left(\frac{r}{z}z_*,\ z_*\right). \tag{3.6a}$$

The result is

$$\frac{\langle u(x+r,\ z)u(x,\ z)\rangle}{u_\tau^2} \to C_{x:uu}\left(\frac{r}{z}\right) + \ln\left(\frac{\delta}{z}\right) D_{x:uu}\left(\frac{r}{z}\right). \tag{3.6b}$$

Here $C_{x:uu}(r/z)$ is a function of r/z. The correlation function of the spanwise velocity v is derived in the same manner,

$$\frac{\langle v(x+r,\ z)v(x,\ z)\rangle}{u_\tau^2} \to C_{x:vv}\left(\frac{r}{z}\right) + \ln\left(\frac{\delta}{z}\right) D_{x:vv}\left(\frac{r}{z}\right). \tag{3.6c}$$

For the wall-normal velocity w, we take $\alpha = 0$ in (3.3) and use (2.5) to obtain $\int_0^\infty \mathrm{d}r_*\, I_{ww}(r_*,\ z_*) \propto z_*^2$ in the limit $z_* \to 0$. This is applied for $z/\delta \to 0$ as

$$\int_{z/\delta}^1 \frac{\mathrm{d}z_*}{z_*^2} \int_0^\infty \mathrm{d}r_*\, I_{ww}(r_*,\ z_*) \to \text{const.} \tag{3.7a}$$

On the other hand, from (2.6*c*),

$$\int_{z/\delta}^1 \frac{\mathrm{d}z_*}{z_*}\, I_{ww}(0,\ z_*) \to \text{const.} \tag{3.7b}$$

By substituting these into (3.2*a*), we confirm that $L_{x:ww}/z$ is finite. The above form of $\int_0^\infty \mathrm{d}r_*\, I_{ww}(r_*,\ z_*) \propto z_*^2$ in the limit $z_* \to 0$ leads to $\beta \geqslant 2$ for $I_{ww}(r_*,\ z_*) \propto z_*^\beta$ at each r_* in this limit. Also, as for $I_{ww}(r_*,\ z_*) \propto r_*^\gamma$ at each z_* in the limit $r_* \to 0$, we require $\gamma \geqslant 0$ because $I_{ww}(0,\ z_*)$ is finite. With use of $I_{ww}(rz_*/z,\ z_*) \propto z_*^{\beta+\gamma}$ in the case of $z_* \ll 1$,

$$\frac{\langle w(x+r,\ z)w(x,\ z)\rangle}{u_\tau^2} = N_e \int_{z/\delta}^1 \frac{\mathrm{d}z_*}{z_*}\, I_{ww}\left(\frac{r}{z}z_*,\ z_*\right), \tag{3.8a}$$

and hence

$$\frac{\langle w(x+r,\ z)w(x,\ z)\rangle}{u_\tau^2} \to C_{x:ww}\left(\frac{r}{z}\right). \tag{3.8b}$$

To relate these correlations continuously to the variances $\langle v_i^2(z)\rangle$ in (2.6), we require $C_{x:v_iv_i}(0) = c_{v_iv_i}$ and $D_{x:v_iv_i}(0) = d_{v_iv_i}$. Especially if $d_{v_iv_i}$ is existent and $\langle v_i^2(z)\rangle$ is dependent on $\ln(\delta/z)$, then $D_{x:v_iv_i}(r/z)$ is existent and $\langle v_i(x+r,\ z)v_i(x,\ z)\rangle$ is dependent on $\ln(\delta/z)$.

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The functions $C_{x:v_iv_i}(r/z)$ and $D_{x:v_iv_i}(r/z)$ are to reflect the internal structures of the individual eddies. Since $D_{x:v_iv_i}(r/z)$ is due to eddies of sizes h_e up to the height δ of the turbulence, it is multiplied by $\ln(\delta/z)$ and exists only for the wall-parallel velocities u and v that are not blocked by the wall. There is also $C_{x:v_iv_i}(r/z)$ due to eddies of sizes h_e comparable to the height z, existent for all the velocities u, v and w.

The functional forms of (3.6*b*), (3.6*c*) and (3.8*b*) hold for distances r down to those in the inertial range. Below this range lies the dissipation range where the kinematic viscosity ν is important. Since we have taken the limit $\nu \to 0$, those forms do not hold. The velocity fluctuations in the dissipation range are regarded to have been coarse grained at a length scale in the inertial range (see also § 6.2).

Lastly, we reconsider the correlation lengths in the constant-stress layer $z/\delta \to 0$, by substituting (3.6*b*), (3.6*c*) and (3.8*b*) into (3.1) with $r_\star = r/z$,

$$L_{x:uu}(z) \;\to\; z\frac{\displaystyle\int_0^\infty \mathrm{d}r_\star\, C_{x:uu}(r_\star) + \ln(\delta/z)\int_0^\infty \mathrm{d}r_\star\, D_{x:uu}(r_\star)}{C_{x:uu}(0) + \ln(\delta/z)D_{x:uu}(0)}, \tag{3.9a}$$

$$L_{x:vv}(z) \;\to\; z\frac{\displaystyle\int_0^\infty \mathrm{d}r_\star\, C_{x:vv}(r_\star) + \ln(\delta/z)\int_0^\infty \mathrm{d}r_\star\, D_{x:vv}(r_\star)}{C_{x:vv}(0) + \ln(\delta/z)D_{x:vv}(0)}, \tag{3.9b}$$

$$L_{x:ww}(z) \;\to\; z\frac{\displaystyle\int_0^\infty \mathrm{d}r_\star\, C_{x:ww}(r_\star)}{C_{x:ww}(0)}. \tag{3.9c}$$

These tend to increase with an increase in the height z. However, $L_{x:uu}(z)$ and $L_{x:vv}(z)$ are not proportional to the height z because of the factor $\ln(\delta/z)$. This is not the case for $L_{x:ww}(z)$, which is exactly proportional to the height z.

4. Velocity correlation along a wall-normal line

Two-point velocity correlations are also studied over a wall-normal distance $\boldsymbol{r}_l = \boldsymbol{r}_z = (0, 0, -r)$ in the range of $0 \leqslant r < z$. According to the attached-eddy hypothesis (see § 2), the correlation function is given by

$$\frac{\langle v_i(z-r)v_j(z)\rangle}{u_\tau^2} = N_e\int_z^\delta \frac{\mathrm{d}h_e}{h_e}\, J_{v_iv_j}\left(\frac{r}{h_e}, \frac{z}{h_e}\right), \tag{4.1a}$$

with the correlation for eddies of size h_e,

$$J_{v_iv_j}\left(\frac{r}{h_e}, \frac{z}{h_e}\right) = \int \frac{\mathrm{d}x_e}{h_e}\int \frac{\mathrm{d}y_e}{h_e} f_{v_i}\left(\frac{\boldsymbol{x}+\boldsymbol{r}_z-\boldsymbol{x}_e}{h_e}\right) f_{v_j}\left(\frac{\boldsymbol{x}-\boldsymbol{x}_e}{h_e}\right). \tag{4.1b}$$

Here $J_{v_iv_j}(0, z/h_e)$ is identical to $I_{v_iv_j}(0, z/h_e)$. The correlation length is defined by using an integration from $r = 0$ to $r = z$ as

$$L_{z:v_iv_i}(z) = \frac{\displaystyle\int_0^z \mathrm{d}r\, \langle v_i(z-r)v_i(z)\rangle}{\langle v_i^2(z)\rangle}. \tag{4.2}$$

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Correlation	Functional form	Equation
$\dfrac{\langle u(\boldsymbol{x}+\boldsymbol{r}_l)u(\boldsymbol{x})\rangle}{u_\tau^2}$	$C_{l:uu}\left(\dfrac{r}{z}\right)+\ln\left(\dfrac{\delta}{z}\right)D_{l:uu}\left(\dfrac{r}{z}\right)$	(3.6*b*) and (4.4*b*)
$\dfrac{\langle v(\boldsymbol{x}+\boldsymbol{r}_l)v(\boldsymbol{x})\rangle}{u_\tau^2}$	$C_{l:vv}\left(\dfrac{r}{z}\right)+\ln\left(\dfrac{\delta}{z}\right)D_{l:vv}\left(\dfrac{r}{z}\right)$	(3.6*c*) and (4.4*c*)
$\dfrac{\langle w(\boldsymbol{x}+\boldsymbol{r}_l)w(\boldsymbol{x})\rangle}{u_\tau^2}$	$C_{l:ww}\left(\dfrac{r}{z}\right)$	(3.8*b*) and (4.4*d*)

TABLE 1. Correlations over the distance $\boldsymbol{r}_l=\boldsymbol{r}_x=(r,0,0)$, $\boldsymbol{r}_y=(0,r,0)$ or $\boldsymbol{r}_z=(0,0,-r)$ according to the attached-eddy hypothesis.

From (4.1*a*) with $r_*=r/h_e$ and $z_*=z/h_e$,

$$L_{z:v_iv_i}(z)=\frac{\displaystyle\int_0^z \mathrm{d}r\int_z^\delta (\mathrm{d}h_e/h_e)\,J_{v_iv_i}(r/h_e,z/h_e)}{\displaystyle\int_z^\delta (\mathrm{d}h_e/h_e)\,J_{v_iv_i}(0,z/h_e)}=z\frac{\displaystyle\int_{z/\delta}^1 (\mathrm{d}z_*/z_*^2)\int_0^{z_*}\mathrm{d}r_*\,J_{v_iv_i}(r_*,z_*)}{\displaystyle\int_{z/\delta}^1 (\mathrm{d}z_*/z_*)\,J_{v_iv_i}(0,z_*)}. \tag{4.3a}$$

The correlation length is also defined in an eddy of a particular size h_e,

$$\Lambda_{z:v_iv_i}(z,h_e)=\frac{\displaystyle\int_0^z \mathrm{d}r\,J_{v_iv_i}(r/h_e,z/h_e)}{J_{v_iv_i}(0,z/h_e)}=h_e\frac{\displaystyle\int_0^{z/h_e}\mathrm{d}r_*\,J_{v_iv_i}(r_*,z/h_e)}{J_{v_iv_i}(0,z/h_e)}. \tag{4.3b}$$

We require $L_{z:v_iv_i}/z$ and $\Lambda_{z:v_iv_i}/h_e$ to be finite (see § 3). Then, since $\Lambda_{z:v_iv_i}/h_e$ is finite in the limit $z_*=z/h_e\to 0$, its asymptotic form is identical to that of $\Lambda_{x:v_iv_i}/h_e$ in (3.3). For the constant-stress layer $z/\delta\to 0$, the correlation function is derived in the same manner as for those in (3.6*a*) and (3.8*a*)

$$\frac{\langle v_i(z-r)v_i(z)\rangle}{u_\tau^2}=N_e\int_{z/\delta}^1\frac{\mathrm{d}z_*}{z_*}J_{v_iv_i}\left(\frac{r}{z}z_*,z_*\right). \tag{4.4a}$$

The result is

$$\frac{\langle u(z-r)u(z)\rangle}{u_\tau^2}\to C_{z:uu}\left(\frac{r}{z}\right)+\ln\left(\frac{\delta}{z}\right)D_{z:uu}\left(\frac{r}{z}\right), \tag{4.4b}$$

$$\frac{\langle v(z-r)v(z)\rangle}{u_\tau^2}\to C_{z:vv}\left(\frac{r}{z}\right)+\ln\left(\frac{\delta}{z}\right)D_{z:vv}\left(\frac{r}{z}\right), \tag{4.4c}$$

$$\frac{\langle w(z-r)w(z)\rangle}{u_\tau^2}\to C_{z:ww}\left(\frac{r}{z}\right). \tag{4.4d}$$

Here $C_{z:v_iv_i}$ and $D_{z:v_iv_i}$ are functions of r/z with $C_{z:v_iv_i}(0)=C_{x:v_iv_i}(0)=c_{v_iv_i}$ or with $D_{z:v_iv_i}(0)=D_{x:v_iv_i}(0)=d_{v_iv_i}$. The functional forms of these correlations are summarized in table 1.

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5. Energy spectrum along a wall-parallel line

From the velocity correlation along a wall-parallel line $\langle v_i(x+r,\,z)v_i(x,\,z)\rangle$, the energy spectrum $\Phi_{x:v_iv_i}(k,\,z)$ at a streamwise wavenumber k is obtained as

$$\Phi_{x:v_iv_i}(k,\,z)=\frac{2}{\pi}\int_0^\infty \mathrm{d}r\,\langle v_i(x+r,\,z)v_i(x,\,z)\rangle\cos(kr). \tag{5.1}$$

Substitution of (3.6*b*), (3.6*c*) and (3.8*b*) into (5.1) leads to the energy spectra of the attached-eddy hypothesis in the constant-stress layer $z/\delta\to 0$,

$$\frac{\Phi_{x:uu}(k,\,z)}{zu_\tau^2}\;\to\;\tilde{C}_{x:uu}(kz)+\ln\left(\frac{\delta}{z}\right)\tilde{D}_{x:uu}(kz), \tag{5.2a}$$

$$\frac{\Phi_{x:vv}(k,\,z)}{zu_\tau^2}\;\to\;\tilde{C}_{x:vv}(kz)+\ln\left(\frac{\delta}{z}\right)\tilde{D}_{x:vv}(kz), \tag{5.2b}$$

$$\frac{\Phi_{x:ww}(k,\,z)}{zu_\tau^2}\;\to\;\tilde{C}_{x:ww}(kz). \tag{5.2c}$$

Here, we have defined $\tilde{C}_{x:v_iv_i}(kz)=(2/\pi)\int_0^\infty \mathrm{d}r_\star\,C_{x:v_iv_i}(r_\star)\cos(kzr_\star)$ and $\tilde{D}_{x:v_iv_i}(kz)=(2/\pi)\int_0^\infty \mathrm{d}r_\star\,D_{x:v_iv_i}(r_\star)\cos(kzr_\star)$ with $r_\star=r/z$. The functional forms are the same even if the wavenumber is spanwise.

There is another model for the energy spectrum (Perry & Abell 1977; Perry & Chong 1982). Within a range of wavenumbers k, some similarity is assumed among eddies of streamwise size $1/k$ so that $\Phi_{x:uu}(k,\,z)$ depends only on u_τ and k, i.e. $\Phi_{x:uu}(k,\,z)\propto u_\tau^2/k$. If the range of the similarity is from $k=a_s/\delta$ to $k=b_s/z$, where a_s and b_s are constants, it follows that $\langle u^2(z)\rangle=\int_0^\infty \mathrm{d}k\,\Phi_{x:uu}(k,\,z)$ is dependent on $\ln(\delta/z)$ as in the case of (2.6*a*). The same law of u_τ^2/k is assumed for $\Phi_{x:vv}(k,\,z)$.

Since (2.6) originates in the attached-eddy hypothesis, the energy spectra of the hypothesis have been related to the law of u_τ^2/k (Perry & Chong 1982). This law is also used to constrain the functional forms of the correlations of the hypothesis (Davidson *et al.* 2006). However, the existence of such a law remains controversial (e.g. Vallikivi *et al.* 2015). The law of u_τ^2/k is in fact not consistent with the attached-eddy hypothesis. At any height $z\ll\delta$, we find in (5.2) that $\Phi_{x:uu}(k,\,z)$ and $\Phi_{x:vv}(k,\,z)$ are dominated by $\ln(\delta/z)\tilde{D}_{x:uu}(kz)$ and $\ln(\delta/z)\tilde{D}_{x:vv}(kz)$. Their spectral shapes are not allowed to yield another factor of $\ln(\delta/z)$ for the variances $\langle u^2(z)\rangle$ and $\langle v^2(z)\rangle$. Furthermore, the eddies of streamwise size $1/k$ are not identical to the attached eddies, which have various sizes h_e but could contribute to the same streamwise wavenumber k at each height z. It is coincidental for (2.6) to be derived from these two.

The similarity law of u_τ^2/k could yet approximate the internal structures of the attached eddies, i.e. $\tilde{C}_{x:v_iv_i}(kz)$ or $\tilde{D}_{x:v_iv_i}(kz)$ in (5.2), if the range of the similarity is not from $k=a_s/\delta$ but from $k=a_s/z$. At the height $z\ll\delta$, the premultiplied spectrum $k\Phi_{x:uu}(k,\,z)$ is known to exhibit an energy-containing broad plateau (e.g. Vallikivi *et al.* 2015). This could be approximated by the law of u_τ^2/k through $\tilde{D}_{x:uu}(kz)\propto 1/kz$. From (5.1),

$$\langle v_i(x+r,\,z)v_i(x,\,z)\rangle=\int_0^\infty \mathrm{d}k\,\Phi_{x:v_iv_i}(k,\,z)\cos(kr). \tag{5.3}$$

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Correlation	Functional form	Equation
$\dfrac{\langle v_i^2(\boldsymbol{x}+\boldsymbol{r}_l)v_i^2(\boldsymbol{x})\rangle - \langle v_i^2(\boldsymbol{x}+\boldsymbol{r}_l)\rangle\langle v_i^2(\boldsymbol{x})\rangle}{u_\tau^4}$	$\dfrac{2\langle v_i(\boldsymbol{x}+\boldsymbol{r}_l)v_i(\boldsymbol{x})\rangle^2}{u_\tau^4}$	(6.2)
$\dfrac{\langle uw(\boldsymbol{x}+\boldsymbol{r}_l)uw(\boldsymbol{x})\rangle - \langle -uw\rangle^2}{u_\tau^4}$	$C_{l:uwuw}\left(\dfrac{r}{z}\right) + \ln\left(\dfrac{\delta}{z}\right) D_{l:uwuw}\left(\dfrac{r}{z}\right)$	(6.3)
$\dfrac{\langle \varepsilon(\boldsymbol{x}+\boldsymbol{r}_l)\varepsilon(\boldsymbol{x})\rangle - \langle \varepsilon(\boldsymbol{x}+\boldsymbol{r}_l)\rangle\langle \varepsilon(\boldsymbol{x})\rangle}{u_\tau^6/z^2}$	$C_{l:\varepsilon\varepsilon}\left(\dfrac{r}{z}\right)$	(6.9)

TABLE 2. Correlations over the distance $\boldsymbol{r}_l = \boldsymbol{r}_x = (r, 0, 0)$, $\boldsymbol{r}_y = (0, r, 0)$ or $\boldsymbol{r}_z = (0, 0, -r)$ according to the attached-eddy hypothesis with some additional assumptions.

There is an asymptotic relation for the cosine integral function,

$$\int_T^\infty \mathrm{d}t\,\frac{\cos(t)}{t} \to -\ln|T| - \gamma_E \quad \text{with Euler's constant } \gamma_E = 0.57721\ldots \text{ as } T \to 0. \tag{5.4a}$$

If the distance r is comparable to the height z, our law of u_τ^2/k yields a factor of $\ln(r/z)$ as

$$\int_{a_s/z}^{b_s/z} \mathrm{d}k\,\frac{\cos(kr)}{k} \to -\ln\left(\frac{a_s r}{z}\right) - \gamma_E \quad \text{as } a_s \to 0 \text{ and } b_s \to \infty. \tag{5.4b}$$

Thus, the correlation function $\langle u(x+r, z)u(x, z)\rangle$ could exhibit approximate dependence on $\ln(r/z)$ in a range of r/z, which is actually seen in the wall turbulence (Davidson *et al.* 2006; Chung *et al.* 2015, see also Davidson & Krogstad 2014).

6. Other correlations

Having studied the velocity correlations in table 1 by using the minimum assumptions of the attached-eddy hypothesis, some other correlations are studied by adding assumptions that are consistent with the hypothesis. Their functional forms are summarized in table 2.

6.1. *Kinetic energy and momentum transfer rate*

To obtain a moment $\langle v_i^m(\boldsymbol{x}+\boldsymbol{r}_l)v_j^m(\boldsymbol{x})\rangle$ of some order m, the corresponding cumulants for eddies of each size h_e are integrated from $h_e = z$ to $h_e = \delta$. Any cumulant of a sum of random variables is equal to the sum of cumulants of the variables if they are independent of one another (e.g. Monin & Yaglom 1971). The moment of the sum is not equal to the sum of the moments, except for low-order central moments such as $I_{v_iv_j}$ and $J_{v_iv_j}$, defined for $\langle v_i(\boldsymbol{x}+\boldsymbol{r}_l)v_j(\boldsymbol{x})\rangle$ in (2.3) and (4.1), which serve also as cumulants. Although a combination of those integrals could yield $\langle v_i^m(\boldsymbol{x}+\boldsymbol{r}_l)v_j^m(\boldsymbol{x})\rangle$, it tends to be a complicated combination and tends to depend on an uncertain value of N_e. There is a more practical approach.

We set many eddies at each position on the wall, by increasing the number of the eddies N_e. According to the central limit theorem (e.g. Monin & Yaglom 1971), the resultant velocity field is approximated to be Gaussian and to be determined by $\langle v_i(\boldsymbol{x}+\boldsymbol{r}_l)v_j(\boldsymbol{x})\rangle$ alone. The other cumulants are all negligible.

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This is a good approximation for the constant-stress layer of the wall turbulence (Meneveau & Marusic 2013). The boundary layer actually exhibits $\langle u^4\rangle/\langle u^2\rangle^2 \simeq 2.8$ and $\langle v^4\rangle/\langle v^2\rangle^2 \simeq \langle w^4\rangle/\langle w^2\rangle^2 \simeq 3.4$ (Fernholz & Finley 1996), for which the Gaussian value is 3.

The odd-order moments are all equal to 0 in such a Gaussian field, while any even-order moment is equal to a sum of products of the above correlations. In particular,

$$\begin{aligned}\langle v_{i_1}(\boldsymbol{x}_1)v_{i_2}(\boldsymbol{x}_2)v_{i_3}(\boldsymbol{x}_3)v_{i_4}(\boldsymbol{x}_4)\rangle \;=\; &\langle v_{i_1}(\boldsymbol{x}_1)v_{i_2}(\boldsymbol{x}_2)\rangle\langle v_{i_3}(\boldsymbol{x}_3)v_{i_4}(\boldsymbol{x}_4)\rangle\\ &+\langle v_{i_1}(\boldsymbol{x}_1)v_{i_3}(\boldsymbol{x}_3)\rangle\langle v_{i_2}(\boldsymbol{x}_2)v_{i_4}(\boldsymbol{x}_4)\rangle\\ &+\langle v_{i_1}(\boldsymbol{x}_1)v_{i_4}(\boldsymbol{x}_4)\rangle\langle v_{i_2}(\boldsymbol{x}_2)v_{i_3}(\boldsymbol{x}_3)\rangle. \end{aligned}\tag{6.1}$$

The correlation of the kinetic energy per unit mass $v_i^2 = u^2$, v^2 or w^2 in the direction $l = x$, y or z is thereby related to that of the velocity v_i in (3.6*b*), (3.6*c*) and (3.8*b*) or (4.4*b–d*),

$$\langle v_i^2(\boldsymbol{x}+\boldsymbol{r}_l)v_i^2(\boldsymbol{x})\rangle - \langle v_i^2(\boldsymbol{x}+\boldsymbol{r}_l)\rangle\langle v_i^2(\boldsymbol{x})\rangle = 2\langle v_i(\boldsymbol{x}+\boldsymbol{r}_l)v_i(\boldsymbol{x})\rangle^2. \tag{6.2}$$

From (6.1), we also obtain

$$\begin{aligned}&\langle uw(\boldsymbol{x}+\boldsymbol{r}_l)uw(\boldsymbol{x})\rangle - \langle uw(\boldsymbol{x}+\boldsymbol{r}_l)\rangle\langle uw(\boldsymbol{x})\rangle\\ &\quad= \langle u(\boldsymbol{x}+\boldsymbol{r}_l)u(\boldsymbol{x})\rangle\langle w(\boldsymbol{x}+\boldsymbol{r}_l)w(\boldsymbol{x})\rangle + \langle u(\boldsymbol{x}+\boldsymbol{r}_l)w(\boldsymbol{x})\rangle\langle w(\boldsymbol{x}+\boldsymbol{r}_l)u(\boldsymbol{x})\rangle. \end{aligned}\tag{6.3a}$$

Since the wall condition of (2.1*b–d*) yields $I_{uw}(0, z_*) \propto z_*$ for $z_* \to 0$ (see § 2), the correlations $\langle u(\boldsymbol{x}+\boldsymbol{r}_l)w(\boldsymbol{x})\rangle$ and $\langle w(\boldsymbol{x}+\boldsymbol{r}_l)u(\boldsymbol{x})\rangle$ depend only on r/z in the limit $z/\delta \to 0$ (see § 3). By also using (3.6*b*) and (3.8*b*) or (4.4*b*) and (4.4*d*) for $\langle u(\boldsymbol{x}+\boldsymbol{r}_l)u(\boldsymbol{x})\rangle$ and $\langle w(\boldsymbol{x}+\boldsymbol{r}_l)w(\boldsymbol{x})\rangle$, the correlation function of the rate of momentum transfer per unit mass $-uw$ is derived for the constant-stress layer $z/\delta \to 0$ as

$$\frac{\langle uw(x+r,z)uw(x,z)\rangle - \langle -uw\rangle^2}{u_\tau^4} \to C_{x:uwuw}\left(\frac{r}{z}\right) + \ln\left(\frac{\delta}{z}\right) D_{x:uwuw}\left(\frac{r}{z}\right), \tag{6.3b}$$

$$\frac{\langle uw(z-r)uw(z)\rangle - \langle -uw\rangle^2}{u_\tau^4} \to C_{z:uwuw}\left(\frac{r}{z}\right) + \ln\left(\frac{\delta}{z}\right) D_{z:uwuw}\left(\frac{r}{z}\right). \tag{6.3c}$$

As for the functions $C_{l:uwuw}(r/z)$ and $D_{l:uwuw}(r/z)$, we require $C_{x:uwuw}(0) = C_{z:uwuw}(0)$ and $D_{x:uwuw}(0) = D_{z:uwuw}(0)$. The fluctuations of $-uw$ are dependent on those of u, which are in turn dependent on the factor $\ln(\delta/z)$. This is also the case for the correlation along a spanwise line.

6.2. *Energy dissipation rate*

Since the attached-eddy hypothesis is a model for the energy-containing eddies, it has not been applied to the rate of energy dissipation per unit mass $\varepsilon(\boldsymbol{x})$. There is nevertheless a possibility for this application if we add an assumption and if we ignore fluctuations at the smallest length scales.

The dissipation fields of the individual eddies $\varepsilon_e(\boldsymbol{x})$ are assumed to have some identical shape with a common characteristic rate u_τ^3/h_e,

$$\varepsilon_e(\boldsymbol{x}) = \frac{u_\tau^3}{h_e} f_\varepsilon\left(\frac{\boldsymbol{x}-\boldsymbol{x}_e}{h_e}\right). \tag{6.4}$$

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We have the corresponding Kolmogorov length $\eta_e = (\nu^3/\varepsilon_e)^{1/4} \propto (\nu/u_\tau)^{3/4} h_e^{1/4}$, which is not proportional to the eddy size h_e. Thus, (6.4) does not hold for fluctuations in the dissipation range around that scale η_e. Such fluctuations are regarded to have been coarse grained at a length scale in the inertial range.

We set the condition at $z = 0$ to be $f_\varepsilon \neq 0$, although the last result is the same even in the case of $f_\varepsilon = 0$. The dissipation rate $\varepsilon(\boldsymbol{x})$ is averaged at a height z as

$$\langle \varepsilon(\boldsymbol{x}) \rangle = \int_z^\delta \frac{\mathrm{d}h_e}{h_e} \left[h_e^3 \, n_e(h_e) \int \frac{\mathrm{d}x_e}{h_e} \int \frac{\mathrm{d}y_e}{h_e} \frac{u_\tau^3}{h_e} f_\varepsilon \left(\frac{\boldsymbol{x} - \boldsymbol{x}_e}{h_e} \right) \right]. \tag{6.5}$$

By using $n_e(h_e) = N_e h_e^{-3}$ with $N_e \propto 1/f_\varepsilon$ for a given shape of f_ε (see § 2) and also by using $\langle \varepsilon(z) \rangle$ in place of $\langle \varepsilon(\boldsymbol{x}) \rangle$,

$$\frac{\langle \varepsilon(z) \rangle}{u_\tau^3/z} = N_e \int_z^\delta \frac{\mathrm{d}h_e}{h_e} I_\varepsilon \left(\frac{z}{h_e} \right), \tag{6.6a}$$

with the average for eddies of size h_e,

$$I_\varepsilon \left(\frac{z}{h_e} \right) = \frac{z}{h_e} \int \frac{\mathrm{d}x_e}{h_e} \int \frac{\mathrm{d}y_e}{h_e} f_\varepsilon \left(\frac{\boldsymbol{x} - \boldsymbol{x}_e}{h_e} \right). \tag{6.6b}$$

For $z_* = z/h_e \to 0$, we have $I_\varepsilon(z_*) \propto z_*$ (see figure 2). The mean rate of energy dissipation in the constant-stress layer $z/\delta \to 0$ is

$$\frac{\langle \varepsilon(z) \rangle}{u_\tau^3/z} = N_e \int_{z/\delta}^1 \frac{\mathrm{d}z_*}{z_*} I_\varepsilon(z_*) \to \text{const.} \tag{6.7}$$

To determine the coefficient for $f_\varepsilon \propto 1/N_e$, the above constant $N_e \int_0^1 (\mathrm{d}z_*/z_*) \, I_\varepsilon(z_*)$ is set equal to the inverse of the von Kármán constant, $1/\kappa \simeq 2.5$. It corresponds to a result of the local equilibrium approximation (Townsend 1961), where the constant-stress layer is under an equilibrium between local rates of production and dissipation of the turbulence kinetic energy. For the actual constant-stress layer, it is considered that the local rate of energy dissipation is 90 %–100 % of the local rate of energy production (Lee & Moser 2015).

The correlation function over a streamwise distance $\boldsymbol{r}_l = \boldsymbol{r}_x = (r, 0, 0)$ at a height z is given by

$$\frac{\langle \varepsilon(x + r, z) \varepsilon(x, z) \rangle - \langle \varepsilon(z) \rangle^2}{u_\tau^6/z^2} = N_e \int_z^\delta \frac{\mathrm{d}h_e}{h_e} \left[I_{\varepsilon\varepsilon} \left(\frac{r}{h_e}, \frac{z}{h_e} \right) - I_\varepsilon^2 \left(\frac{z}{h_e} \right) \right], \tag{6.8a}$$

with the correlation $I_{\varepsilon\varepsilon} - I_\varepsilon^2$ for eddies of size h_e, which is made up of $I_\varepsilon(z/h_e)$ in (6.6*b*) and also of

$$I_{\varepsilon\varepsilon} \left(\frac{r}{h_e}, \frac{z}{h_e} \right) = \frac{z^2}{h_e^2} \int \frac{\mathrm{d}x_e}{h_e} \int \frac{\mathrm{d}y_e}{h_e} f_\varepsilon \left(\frac{\boldsymbol{x} + \boldsymbol{r}_x - \boldsymbol{x}_e}{h_e} \right) f_\varepsilon \left(\frac{\boldsymbol{x} - \boldsymbol{x}_e}{h_e} \right). \tag{6.8b}$$

For $z_* = z/h_e \to 0$, we have $I_{\varepsilon\varepsilon}(0, z_*) - I_\varepsilon^2(z_*) \propto z_*^2$ (see figure 2). In a manner similar to that for the velocity correlations (see § 3),

$$\frac{\langle \varepsilon(x + r, z) \varepsilon(x, z) \rangle - \langle \varepsilon(z) \rangle^2}{u_\tau^6/z^2} = N_e \int_{z/\delta}^1 \frac{\mathrm{d}z_*}{z_*} \left[I_{\varepsilon\varepsilon} \left(\frac{r}{z} z_*, z_* \right) - I_\varepsilon^2 (z_*) \right], \tag{6.9a}$$

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and hence

$$\frac{\langle\varepsilon(x+r,z)\varepsilon(x,z)\rangle-\langle\varepsilon(z)\rangle^2}{u_\tau^6/z^2}\to C_{x:\varepsilon\varepsilon}\left(\frac{r}{z}\right). \tag{6.9b}$$

The same form is derived for the correlation function over a spanwise distance. For that over a wall-normal distance $\boldsymbol{r}_l=\boldsymbol{r}_z=(0,0,-r)$, we replace $\boldsymbol{r}_x$ with $\boldsymbol{r}_z$ in (6.8*b*) to define $J_{\varepsilon\varepsilon}(r/h_e, z/h_e)$. Then,

$$\frac{\langle\varepsilon(z-r)\varepsilon(z)\rangle-\langle\varepsilon(z-r)\rangle\langle\varepsilon(z)\rangle}{u_\tau^6/z^2}=N_e\int_{z/\delta}^1\frac{\mathrm{d}z_*}{z_*}\left[J_{\varepsilon\varepsilon}\left(\frac{r}{z}z_*,z_*\right)-I_\varepsilon\left(z_*-\frac{r}{z}z_*\right)I_\varepsilon(z_*)\right]. \tag{6.9c}$$

The result is

$$\frac{\langle\varepsilon(z-r)\varepsilon(z)\rangle-\langle\varepsilon(z-r)\rangle\langle\varepsilon(z)\rangle}{u_\tau^6/z^2}\to C_{z:\varepsilon\varepsilon}\left(\frac{r}{z}\right). \tag{6.9d}$$

Here $C_{l:\varepsilon\varepsilon}$ is a function of r/z with $C_{x:\varepsilon\varepsilon}(0)=C_{y:\varepsilon\varepsilon}(0)=C_{z:\varepsilon\varepsilon}(0)$. Since the rate ε_e of an eddy is inversely proportional to its size h_e, the entire rate ε is independent of $\ln(\delta/z)$ that originates in eddies of sizes h_e much larger than the height z. On the other hand, from $\varepsilon_e\propto f_\varepsilon\propto 1/N_e$, we have $C_{l:\varepsilon\varepsilon}(r/z)\propto 1/N_e$. If N_e were too large, the fluctuations of the entire rate ε would be negligible. They are actually not negligible even over large distances r in the wall turbulence (e.g. Mouri *et al.* 2006).

7. Concluding discussion

Two-point velocity correlations have been studied theoretically for the constant-stress layer of wall turbulence on the basis of the attached-eddy hypothesis, i.e. a phenomenological model of a random superposition of energy-containing eddies that are attached to the wall (Townsend 1976). While the shapes of the eddies are taken to be identical to one another with a common characteristic velocity u_τ, their sizes are distributed up to a finite value in each of the directions.

From the random distribution and the finite sizes of the attached eddies, it follows that the correlation lengths defined in (3.2) and (4.3) always exist. We have used this fact to derive the most general forms of the correlation functions in (3.6*b*), (3.6*c*), (3.8*b*) and (4.4*b*–*d*) and of the energy spectra in (5.2). The results are summarized in table 1.

The correlation functions $\langle v_i(\boldsymbol{x}+\boldsymbol{r}_l)v_i(\boldsymbol{x})\rangle$ are made up of $C_{l:v_iv_i}(r/z)$ due to eddies of sizes h_e comparable to the height z and of $D_{l:v_iv_i}(r/z)$ due to eddies of sizes h_e larger than the height z. We have obtained $D_{l:v_iv_i}(r/z)$ only for the wall-parallel velocities u and v that are not blocked by the wall. It is multiplied by $\ln(\delta/z)$ because the size h_e of the large eddies is up to the height δ of the turbulence.

These results are not consistent with the results of previous studies, e.g. the spectra in the form of $\Phi_{x:v_iv_i}(k,z)\propto u_\tau^2/k$ (Perry & Chong 1982). While we have focused on the minimum assumptions of the attached-eddy hypothesis, the previous studies had invoked additional assumptions. They are in fact not consistent with the hypothesis, aside from which is more reliable as a model for the wall turbulence.

Without any confounding effect of such additional assumptions, our results would offer an opportunity to assess the extent to which the attached-eddy hypothesis is consistent with the actual wall turbulence. However, at present, the direct numerical simulations are not at Reynolds numbers that are high enough to attain a layer of the strictly constant stress (Sillero *et al.* 2013; Lee & Moser 2015). The experiments are so far based mostly on Taylor's frozen-eddy hypothesis. Improvements are desirable for the simulations and for the experiments.

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There are nevertheless some experimental implications. It is known that the value of $d_{uu} = D_{l:uu}(0)$ is common to a variety of flows (Marusic *et al.* 2013). This commonness is expected for the entire function $D_{l:uu}(r/z)$, which reflects the undermost layers of the attached eddies that are insensitive to the flow configuration. The value of $c_{uu} = C_{l:uu}(0)$ is not common, and so is not the entire function $C_{l:uu}(r/z)$. We expect these implications to apply further to the other velocities v and w.

The two-point correlation of the wall-normal velocity w is determined by the attached eddies of sizes h_e comparable to the height z of the two points. If such eddies are actually dominant in the wall turbulence (Adrian 2007), their internal structures would be clarified in that correlation, especially as to whether the shape of the eddy is independent of its size. This is not the case for the correlations of the wall-parallel velocities u and v, which are affected by the larger eddies.

By adding assumptions that are consistent with the attached-eddy hypothesis, we have derived the correlations of some other quantities. That is, if the velocity field is Gaussian, the correlation functions of the kinetic energies u^2, v^2 and w^2 and that of the momentum transfer rate $-uw$ are given in (6.2) and (6.3). If the energy dissipation rate of an attached eddy ε_e is proportional to u_τ^3/h_e, the correlation function of the entire rate ε is given in (6.9). The results are summarized in table 2.

A logarithmic law such as that in (1.1) has been found also for the variance of the pressure fluctuations p (Jiménez & Hoyas 2008; Sillero *et al.* 2013). If this law is due to the attached eddies, the two-point correlation $\langle p(\boldsymbol{x}+\boldsymbol{r}_l)p(\boldsymbol{x})\rangle$ is expected to have a functional form that is identical to those of the wall-parallel velocities u and v given in (3.6*b*) and (3.6*c*) or (4.4*b*) and (4.4*c*).

Lastly, we note that these results would allow us to use the attached-eddy hypothesis in its most general form. An example is the coarse graining to characterize or model the wall turbulence, say, the atmospheric boundary layer in a particular class of numerical simulations (Wyngaard 2004). Over a streamwise length R, we coarse grain a quantity q as

$$q_R(x, y, z) = \frac{1}{R}\int_0^R \mathrm{d}r\, q(x+r, y, z). \tag{7.1}$$

Through a relation (e.g. Monin & Yaglom 1971), the variance of q_R is dependent on the two-point correlation of q,

$$\langle q_R^2(z)\rangle - \langle q_R(z)\rangle^2 = \frac{2}{R^2}\int_0^R \mathrm{d}r_1 \int_0^{r_1} \mathrm{d}r_2 \left[\langle q(x+r_2, z)q(x, z)\rangle - \langle q(z)\rangle^2\right], \tag{7.2a}$$

and hence

$$\langle q_R^2(z)\rangle - \langle q_R(z)\rangle^2 = \frac{2}{R^2}\int_0^R \mathrm{d}r\,(R-r)\left[\langle q(x+r, z)q(x, z)\rangle - \langle q(z)\rangle^2\right]. \tag{7.2b}$$

By using (3.1) to define the correlation length $L_{x:qq}(z)$,

$$\langle q_R^2(z)\rangle - \langle q_R(z)\rangle^2 \to \frac{2L_{x:qq}(z)}{R}\left[\langle q^2(z)\rangle - \langle q(z)\rangle^2\right] \quad \text{as } R\to\infty. \tag{7.3}$$

If the two-point correlation is a function of r/z alone as for $q = w$, w^2 and ε in tables 1 and 2, the variance is a function of R/z alone. The remaining variances depend also on $\ln(\delta/z)$. Such a constraint would in turn constrain the theory or model on that coarse graining. Thus, for this and various other studies of wall turbulence, we could make use of the general forms of the correlation functions derived here from the minimum assumptions of the attached-eddy hypothesis.

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