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STOCHASTIC VERSION OF POLTEROVICH'S MODEL: EXPONENTIAL TURNPIKE THEOREMS FOR EQUILIBRIUM PATHS

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The paper examines a stochastic analogue of the Polterovich model of economic dynamics. The study focuses on the asymptotic properties of equilibrium paths. The main results are stochastic turnpike theorems with exponential estimates of the convergence rate.

Keywords: Equilibrium Dynamics, Polterovich's model, Stochastic Equilibrium Paths, Exponential Turnpike Property, Stationary Equilibrium

1. INTRODUCTION

The basis for this study is the work of Polterovich (1978) on equilibrium economic dynamics. The goal of Polterovich's work was to propose an integration of optimal growth theory, including the issues of stability, turnpikes, optimality over an infinite time horizon, etc. [see Gale (1967), McKenzie (1986)], and the ideas of general economic equilibrium. Similar goals were pursued in Bewley's (1982) study. For the above purpose, a dynamic multisector model with monotone demand functions [cf. Wald (1936) and Hildenbrand (1994)] was constructed. Polterovich's model admits a simple description and, at the same time, is rich in content and leads to an interesting theory.

It is a challenging task to develop a stochastic variant of the Polterovich theory. Stochastic models reflect a broader spectrum of economic phenomena and provide a framework for a deeper analysis. The problem of constructing a probabilistic analogue of the Polterovich model has been considered in a series of papers by Evstigneev and Katyshev (1982, 1985) and Evstigneev and Taksar (1995).

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However, a number of important questions still remain unresolved. The present work contributes to the solution of this problem.

We deal with both a stationary and a nonstationary stochastic version of the model. In the more general, nonstationary case, we obtain several turnpike results. These results show, in particular, that the distances in the L_2-norm between two equilibrium paths defined on some time intervals $J = \{0, 1, \ldots, N\}$ and $J' = \{0, 1, \ldots, N'\}$ $(N \leq N' < \infty)$ are small at those moments of time $t \in J$ that are far enough from the end points 0 and N of the interval J. In contrast to the majority of known statements of this kind [for surveys, see, e.g., McKenzie (1986, 1998)], we not only establish the smallness of the above-mentioned distances, but also provide exponential estimates for them.

The exponential estimates add some important quantitative content to the well-known qualitative statements on comparative economic dynamics. They are in the spirit of the modern quantitative stability theory [e.g., Attouch and Wets (1991)], having numerous applications. In particular, results of this kind are often of fundamental interest for the development of new computational methods for economic dynamics models. Having sharper bounds on the convergence of equilibrium trajectories can help to design better computational algorithms, such as value iteration and policy improvement techniques. In some contexts, these quantitative bounds may turn out to be crucial for computations; convergence of the equilibrium paths alone generally sheds no light on the error bounds for a given truncation of the algorithm.

Some facts regarding the exponential convergence of infinite optimal or equilibrium trajectories have been mentioned previously in the mathematical economics literature [e.g., Araujo and Scheinkman (1977), Bewley (1982)]. We concentrate attention on the exponential turnpike property and undertake a systematic investigation of it. It is worth emphasizing that we deal not only with infinite, but also with finite paths, which is the classical context of turnpike theory, in contrast to the conventional stability theory [see Nikaido (1968, pp. 207–208)].

Another specific source of motivation for our study lies in the following. It turns out that the exponential turnpike property plays a key role in proving the asymptotic normality of random additive functionals of equilibrium paths, such as sums of random payoffs or utilities accumulated over time. This was demonstrated in the work of Evstigneev and Flåm (1997), where some asymptotic normality results—central limit theorems—were established. The latter paper focused on a simpler model, dealing with optimal, rather than equilibrium, paths. In our next paper (Amir and Evstigneev, 1998), strongly relying upon the present one, we obtain central limit theorems for the stochastic analogue of the Polterovich model. Such results make sense only in a stochastic world and have no deterministic analogues. To the best of our knowledge, these results are completely new to the literature on the asymptotic properties of economic dynamics models. One of the goals of the present article is to develop technical tools needed for the above purpose.

Another important application of our turnpike results is a proof of existence of a stationary equilibrium path in the stationary version of the model under

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consideration. We construct the path by passing to the limit from equilibrium trajectories defined on finite time intervals. The construction is given in this paper. Because many equilibrium models in economic dynamics are stationary, we believe that the new techniques described in the present paper will turn out to be useful in many related contexts.

The paper is organized as follows. Section 2 describes the basic model. Section 3 states the main results. Sections 4 and 5 contain the proofs. An appendix provides formulations of some results obtained in previous work and used in this paper.

2. BASIC MODEL, DEFINITIONS, AND ASSUMPTIONS

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\dots, s_{-1}(\omega), s_0(\omega), s_1(\omega), \dots$ $(\omega \in \Omega)$ a stochastic process with values in a measurable space $(S, \mathcal{S})$, where S is a set and $\mathcal{S}$ is a countably generated σ-algebra of its subsets (s_t represents a random state of the world at time t). For each $t \in \{0, \pm 1, \pm 2, \dots\}$, we denote by $\mathcal{F}_t$ the σ-subalgebra of the σ-algebra $\mathcal{F}$ generated by the history $s^t = (\dots, s_{t-1}, s_t)$ of the process s_t up to time t. For any $t \in \{0, \pm 1, \pm 2, \dots\}$, we consider the space $L_\infty(t) = L_\infty(\Omega, \mathcal{F}_t, P, R^n)$ of essentially bounded $\mathcal{F}_t$-measurable vector functions $x(\omega)$, $\omega \in \Omega$, taking values in the n-dimensional Euclidean space R^n. The analogous space of integrable functions is denoted by $L_1(t) = L_1(\Omega, \mathcal{F}_t, P, R^n)$. We define

$$\mathcal{X}_t = \{x \in L_\infty(t) : x \geq 0\}, \qquad \mathcal{P}_t = \{p \in L_1(t) : p \geq 0\}.$$

Here and in what follows, equalities and inequalities between vector functions of ω are understood coordinatewise and almost surely (a.s.). As a rule, we identify random variables coinciding a.s.

In the model under consideration, we are given the *technology sets*

$$Z_t \subseteq \mathcal{X}_{t-1} \times \mathcal{X}_t, \qquad t \in \{1, 2, \dots\},$$

and the *demand correspondences*

$$p \mapsto C_t(p), \qquad p \in \mathcal{P}_t,\ C_t(p) \subseteq \mathcal{X}_t, \qquad t \in \{0, 1, \dots\}.$$

The set Z_t specifies the class of input/output pairs (x, y)—*technological processes*—that can be realized during the time period $[t-1, t]$. The input x at time $t-1$ is influenced by random events prior to $t-1$, whereas the output y depends on events prior to t. Vectors in $C_t(p)$ are interpreted as the most preferred bundles of consumption goods given the price vector $p \in \mathcal{P}_t$.

A sequence $\xi = \{(x_t, y_t, c_t, p_t)\}_{t=0}^N$ $(N \leq \infty)$ is called an *equilibrium trajectory* (or an *equilibrium path*) if the following relations hold:

$$(x_{t-1}, y_t) \in Z_t, \qquad t \geq 1; \tag{1}$$

$$Ep_t y_t - Ep_{t-1} x_{t-1} \geq Ep_t y - Ep_{t-1} x, \qquad (x, y) \in Z_t, \qquad t \geq 1; \tag{2}$$

$$c_t \in C_t(p_t), \qquad t \geq 0; \tag{3}$$

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$$y_t - x_t - c_t \geq 0, \qquad t \geq 0; \tag{4}$$

$$p_t(y_t - x_t - c_t) = 0, \qquad t \geq 0. \tag{5}$$

We include in the above definition the following additional requirements:

(a) if $N < \infty$, then

$$x_N = 0; \tag{6}$$

(b) if $N = \infty$, then the sequence $r_t = Ep_t x_t$ satisfies the *moderate growth condition*: For any $\lambda \in (0, 1)$, there exists a sequence of integers $0 < N_1 < N_2 < \cdots$ such that $\lambda^{N_k} r_{N_k} \to 0$.

The vector y_0 is called the *initial vector* (i.v.) of the trajectory ξ.

Conditions (1) and (2) state that the technological process (x_{t-1}, y_t) is feasible and maximizes the expected profit $Ep_t y - Ep_{t-1}x$ among all of the feasible technological processes (x, y). According to (3), c_t is in the set of the most preferred consumption vectors, given the price vector p_t. Inequality (4) expresses a balance between production and consumption. If $N < \infty$, conditions (5), together with (6), imply a financial balance (the Walras law) over the time interval $[0, N]$:

$$\sum_{t=0}^{N} p_t c_t = p_0 y_0 + \sum_{t=1}^{N} (p_t y_t - p_{t-1} x_{t-1}).$$

Assumption (a) is a usual terminal condition saying that the input at the end of a finite equilibrium path is zero. Hypothesis (b) is needed for ruling out pathological infinite paths that may exist among those satisfying conditions (1)–(5) [see Polterovich (1978, Section 5)]. If S consists of a single point (the deterministic case), the model described above reduces to Polterovich's model.

Denote by e the vector $(1, 1, \ldots, 1) \in R^n$. If $a = (a^i)$ is a finite-dimensional vector, then we write $|a| = \sum |a^i|$ and $|a|_2 = (\sum |a^i|^2)^{1/2}$. We impose the following conditions on the technology sets Z_t, $t \in \{1, 2, \ldots\}$.

(Z.1) For each $t \geq 1$, the set Z_t is non-empty, convex, and closed in $\mathcal{X}_{t-1} \times \mathcal{X}_t$ under convergence a.s.

(Z.2) There is a constant $A > 0$ such that

$$x \leq Ae, \qquad y \leq Ae \tag{7}$$

for any $(x, y) \in Z_t$ and any $t \geq 1$.

(Z.3) If $(x, y) \in Z_t$, $(x', y') \in Z_t$, and $\theta(\omega)$ is an $\mathcal{F}_{t-1}$-measurable function $\theta\colon \Omega \to [0, 1]$, then $\theta(x, y) + (1 - \theta)(x', y') \in Z_t$.

(Z.4) There exist vectors $\mathring{x}_t$, $\mathring{y}_t \in \mathcal{X}_t$, $t \in \{0, 1, \ldots\}$, and a constant $\delta > 0$ such that

$$(\mathring{x}_{t-1}, \mathring{y}_t) \in Z_t, \qquad t \geq 1, \tag{8}$$

$$\mathring{y}_t - \mathring{x}_t \geq \delta e, \qquad t \geq 0, \tag{9}$$

$$y \geq \mathring{y}_t \quad \text{for any} \quad (x, y) \in Z_t, \qquad t \geq 1, \tag{10}$$

and $\mathring{y}_0 \leq Ae$.

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(Z.5) There is a constant $\kappa \in (0, 1)$ such that, for any $(x, y) \in Z_t$ and $(x', y') \in Z_t$, there exist $x'' \in \mathcal{X}_{t-1}$, $y'' \in \mathcal{X}_t$ satisfying

$$E(|x''| + |y''|) \geq \kappa E\left(|x - x'|_2^2 + |y - y'|_2^2\right) \tag{11}$$

and

$$\left[\tfrac{1}{2}(x + x') - x'', \tfrac{1}{2}(y + y') + y''\right] \in Z_t. \tag{12}$$

Hypotheses (Z.1) and (Z.2) contain standard assumptions of convexity, closedness and boundedness of the technology sets. Condition (Z.3) describes the possibility of mixing two admissible technological processes with coefficients θ and $1 - \theta$, where θ may depend on random events prior to time $t - 1$. In (Z.4), we postulate the existence of minimal output vectors, $\hat{y}_t$, which can be reproduced "with excess" [conditions of this type are discussed by Evstigneev and Taksar (1994, 1995)]. Assumption (Z.5) expresses a property of strong convexity of the technology sets Z_t. This assumption says that if we combine two different admissible technological processes, (x, y) and (x', y'), then we can save input while gaining output. The effects of this combination are described in terms of the vectors x'' and y''. Requirement (11) implies the inequality

$$\|x''\|_2 + \|y''\|_2 \geq \frac{\kappa}{2\sqrt{n}}(\|x - x'\|_2 + \|y - y'\|_2)^2, \tag{13}$$

where $\|x\|_2$ stands for the L_2-norm $(E|x|_2^2)^{1/2}$. The L_2-norms of x'' and y'' may converge to zero as $\|x-x'\|_2+\|y-y'\|_2 \to 0$; however, the rate of this convergence cannot be too fast: it should not be faster than quadratic, as inequality (13) shows. Examples of technology sets satisfying condition (Z.5) are examined in detail by Evstigneev and Flåm (1997).

We denote by $\|x\|_1$ the L_1-norm $E|x|$ of a random vector x. A sequence $\{x^1, x^2, \ldots\} \subseteq L_\infty(t)$ is said to converge to $x \in L_\infty(t)$ in the topology $\sigma[L_\infty(t), L_1(t)]$ if $Eqx^k \to Eqx$ for any $q \in L_1(t)$.

We assume that, for each $t \in \{0, 1, \ldots\}$, the demand correspondence $p \mapsto C_t(p)$ satisfies conditions (C.1)–(C.5) listed below. All constants involved in these conditions do not depend on t.

(C.1) For any $p \in \mathcal{P}_t$, the set $C_t(p)$ is non-empty and convex. We have

$$c \leq A_1 e \quad \text{for all} \quad c \in C_t(p), \qquad p \in \mathcal{P}_t, \tag{14}$$

where A_1 is some constant.

(C.2) If $p^k \in \mathcal{P}_t$ $(k = 1, 2, \ldots)$, $p \in \mathcal{P}_t$, $\|p^k - p\|_1 \to 0$,

$$c^k \in C_t(p^k), \qquad \text{and } c^k \to c \in \mathcal{X}_t, \sigma[L_\infty(t), L_1(t)], \tag{15}$$

then $c \in C_t(p)$.

(C.3) There is a constant A_0 for which we have

$$pc \leq A_0, \qquad p \in \mathcal{P}_t, \qquad c \in C_t(p). \tag{16}$$

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(C.4) There exists a number $\gamma > 0$ such that the relations

$$P\left\{\min_i(p^i) < \gamma\right\} > 0, \qquad p = (p^1, \ldots, p^n) \in \mathcal{P}_t$$

imply

$$P\left\{\max_j(c^j) > A\right\} > 0$$

for any $c = (c^1, \ldots, c^n) \in C_t(p)$.

(C.5) For every $H > 0$, there is a number $\zeta^H > 0$ such that

$$E(c - c')(p - p') \leq -\zeta^H E|p - p'|_2^2 \tag{17}$$

for all

$$c \in C_t(p), \qquad c' \in C_t(p'), \qquad p \in \mathcal{P}_t, \qquad p' \in \mathcal{P}_t, \tag{18}$$

satisfying

$$|p| \leq H, \qquad |p'| \leq H. \tag{19}$$

Hypotheses (C.1) and (C.2) list technical assumptions of regularity of the mappings $C_t(\cdot)$, $t \in \{0, 1, \ldots\}$. Note that the demand correspondence is supposed to be uniformly bounded on the set of all price vectors. This contrasts with more traditional settings, where demand goes to infinity as prices go to zero [instead, we postulate (C.4)]. Conditions (14) and (16), expressing the boundedness of demand and income, are similar to those used by Polterovich (1978).

Assumption (C.4) is a version of a hypothesis of insatiability of demand. This assumption states that if a random price of some commodity takes values lower than a certain level γ with positive probability, then the demand for this (or some other) commodity exceeds the level A with positive probability. At the same time, A is an upper bound for producible amounts of any commodity [see (Z.2)]. This implies, in particular, that for any equilibrium path $\{(x_t, y_t, c_t, p_t)\}$ with i.v. $y_0 \leq Ae$, we have

$$p_t \geq \gamma e, \qquad t \geq 0. \tag{20}$$

This, in turn, yields

$$y_t - x_t - c_t = 0, \qquad t \geq 0, \tag{21}$$

that is, constraints (4) hold as equalities.

It follows from (C.5) that, for any $H \in (0, \infty)$, the multivalued mapping $p \mapsto -C_t(p)$ restricted to the set $\mathcal{P}_t^H \equiv \{p \in \mathcal{P}_t : |p| \leq H\}$ represents a strongly monotone operator [see, e.g., Vainberg (1973)]. A monotonicity hypothesis for aggregate demand functions was introduced first by Wald (1936). Important theoretical results and empirical evidence substantiating the hypothesis were obtained by Hildenbrand (1994). It is well known that appropriate conditions of monotonicity guarantee uniqueness and stability of equilibrium. It also was established [Evstigneev and Katyshev (1982), Evstigneev and Taksar (1994, 1995)] that such conditions could be essential for the existence of equilibria in probabilistic economic models. Assumption (C.5) is a somewhat stronger variant of its

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counterparts used in the previous work cited. In (C.5), we have a *quadratic* estimate $\zeta_{C_t}^H(r) \geq \zeta^H r^2$ for the function $\zeta_{C_t}^H(r) = \inf\{-E(c-c')(p-p')\}$, where the infimum is taken over all c, c', p, p' satisfying (18), (19), and $\|p-p'\|_2 \equiv (E|p-p'|_2^2)^{1/2} \geq r$. The preceding quadratic estimate and inequalities (11) play important roles in the proofs of the exponential turnpike theorems in this paper.

3. STATEMENTS OF THE RESULTS

In this section, we formulate the main results. We assume that hypotheses (Z.1)–(Z.5) and (C.1)–(C.5) hold. The vectors $\mathring{x}_t$, $\mathring{y}_t$ $(t=0,1,\ldots)$, the constants $A, A_0, A_1, \delta, \gamma, \kappa$ and the function $H \mapsto \zeta^H$ involved in these hypotheses are supposed to be fixed. We define

$$I = \{y \in \mathcal{X}_0 : \mathring{y}_0 \leq y \leq Ae\}. \tag{22}$$

Because $\mathring{y}_0 \leq Ae$ [see (Z.4)], we have $I \neq \emptyset$.

We begin by stating a result that shows that the assumptions imposed guarantee the existence, as well as the uniqueness, of the equilibrium paths we examine.

THEOREM 1. *For each $y_0 \in I$ and each $1 \leq N \leq \infty$, there exists a unique equilibrium path $\{(x_t, y_t, c_t, p_t)\}_0^N$ with i.v. y_0.*

For finite trajectories, the above theorem is a consequence of a more general assertion (using milder assumptions) proved by Evstigneev and Taksar (1995). This assertion is formulated in the Appendix. Infinite paths are constructed by passing to the limit from finite ones. The proof of Theorem 1 is given in Section 4.

Let Ξ^N $(N \leq \infty)$ denote the set of equilibrium paths $\{(x_t, y_t, c_t, p_t)\}_0^N$ with some i.v. y_0 in I. Define $\Xi = \Xi^\infty \cup \Xi^1 \cup \Xi^2 \cup \cdots$. Consider two equilibrium paths,

$$\xi^1 = \{(x_t^1, y_t^1, c_t^1, p_t^1)\}_0^{N_1} \in \Xi^{N_1} \quad \text{and} \quad \xi^2 = \{(x_t^2, y_t^2, c_t^2, p_t^2)\}_0^{N_2} \in \Xi^{N_2}, \tag{23}$$

and put

$$w_t^1 = (x_t^1, y_t^1, c_t^1, p_t^1), \qquad w_t^2 = (x_t^2, y_t^2, c_t^2, p_t^2), \tag{24}$$

$$\delta_t(\xi^1, \xi^2) = E|w_t^1 - w_t^2|_2^2, \qquad N = \min\{N_1, N_2\}. \tag{25}$$

Central results of the paper are contained in the following theorem.

THEOREM 2. *There exist constants $L_0 > 0$ and $\theta \in (0,1)$ such that, for any two equilibrium paths* (23)*, the following assertions are true*:

(i) *If $N = \infty$, then*

$$\delta_t(\xi^1, \xi^2) + \delta_{t+1}(\xi^1, \xi^2) + \cdots \leq L_0\theta^t, \qquad t \geq 0. \tag{26}$$

(ii) *If $N < \infty$, then*

$$\delta_t(\xi^1, \xi^2) + \cdots + \delta_{N-t}(\xi^1, \xi^2) \leq L_0\theta^t, \qquad 0 \leq t < N/2. \tag{27}$$

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(iii) *If* $N < \infty$ *and* $y_0^1 = y_0^2$, *then*

$$\delta_0(\xi^1, \xi^2) + \cdots + \delta_{N-t}(\xi^1, \xi^2) \le L_0\theta^t, \qquad 0 \le t \le N. \tag{28}$$

By virtue of (26), the distance between the infinite trajectories ξ^1 and ξ^2, measured in terms of the L_2-norm $\|w_t^1 - w_t^2\|_2 = [\delta_t(\xi^1, \xi^2)]^{1/2}$, tends to zero with exponential rate as $t \to \infty$. In case ξ^1 or ξ^2 is finite ($N < \infty$), formula (27) implies that the distances between ξ^1 and ξ^2 are exponentially small at those points in the time interval $[0, N]$ that are far enough from its ends. In view of (28), a similar statement is true for those points in the interval $[0, N]$ that are close to its left end, provided the initial vectors of ξ^1 and ξ^2 coincide and N is large.

The assertions included into Theorem 2 can be regarded as nonstationary variants of turnpike theorems. For a nonstationary stochastic analogue of Polterovich's model, similar statements (without exponential estimates) were proved by Evstigneev and Katyshev (1985). Conditions guaranteeing exponential convergence of optimal or equilibrium infinite paths have been considered in deterministic settings, e.g., by Araujo and Scheinkman (1977) and Bewley (1982). The turnpike results described in parts (ii) and (iii) of Theorem 2 (involving finite paths) are new in the general context of stochastic models of economic dynamics and equilibrium; assertion (i) is new at least in the context of the model at hand.

Suppose that a one-to-one mapping $T\colon \Omega \to \Omega$—a *shift transformation*—is given such that T and T^{-1} are measurable and preserve the measure P. The transformation T may be thought of as a shift of the timescale which establishes a correspondence between events occurring at some time t and the analogous events occurring at time $t+1$. The model under consideration is said to be *stationary* if the following conditions hold:

$$s_{t+1}(\omega) = s_t(T\omega), \qquad t \in \{0, \pm 1, \pm 2, \ldots\}, \qquad \omega \in \Omega; \tag{29}$$

$$Z_{t+1} = TZ_t, \qquad t \ge 1; \qquad C_{t+1}(Tp) = TC_t(p), \qquad p \in \mathcal{P}_t, \qquad t \ge 0. \tag{30}$$

By a standard abuse of notation, the letter T denotes both the given mapping $\Omega \to \Omega$ and the transformation of functions of $\omega \in \Omega$, which acts according to the formula $Tx(\omega) = x(T\omega)$. Condition (29) implies that the process $\{s_t\}$ is stationary and $\mathcal{F}_{t+1} = \{T^{-1}\Gamma\colon \Gamma \in \mathcal{F}_t\}$. According to (30), technology and demand are time invariant (in the stochastic sense specified). When dealing with stationary models, we assume that the random vectors $\mathring{x}_t$, $\mathring{y}_t$ involved in (Z.4) satisfy

$$\mathring{x}_{t+1} = T\mathring{x}_t, \qquad \mathring{y}_{t+1} = T\mathring{y}_t, \qquad t \ge 0. \tag{31}$$

An infinite path $\xi = \{w_t\}_0^\infty$, $w_t = (x_t, y_t, c_t, p_t)$ is called *stationary* if $w_{t+1} = Tw_t$ for each $t \ge 0$.

THEOREM 3. *In a stationary model, there exists a unique stationary equilibrium path.*

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A similar result in a different context was obtained by Bewley (1981).

Denote the path described in Theorem 3 by $\bar{\xi} = \{(\bar{x}_t, \bar{y}_t, \bar{c}_t, \bar{p}_t)\}_0^\infty$. From (7), (10), and (31), it is easily seen that $\bar{\xi} \in \Xi$. Let ξ be any equilibrium trajectory in Ξ. By Theorem 2, if $\xi \in \Xi$ is infinite, then ξ converges in L_2 to the stationary equilibrium path $\bar{\xi}$. If ξ is finite, then ξ is most of the time close to $\bar{\xi}$, significant deviations between ξ and $\bar{\xi}$ being possible only at the beginning or at the end of the time interval of ξ. In accord with the well-known interpretation of results of this kind, the stationary equilibrium path $\bar{\xi}$ plays here the role of a turnpike. For the literature where similar concepts are discussed, see Samuelson (1960), Nikaido (1968), Bewley (1982), McKenzie (1986, 1998), Arkin and Evstigneev (1987), and Majumdar and Zilcha (1987).

We note that, in this article, the existence theorem for a stationary equilibrium path is obtained as a byproduct of the turnpike theory developed (see Section 5). The questions of existence are not our main concern in the present paper. We examine them only to the extent that is needed to show that the classes of paths directly related to the turnpike results are non-empty in the context we deal with. It is possible that Theorem 3 holds under milder assumptions than those under which Theorem 2 holds. The analysis of this issue, which is beyond the scope of the paper, represents an interesting topic for further study.

4. EXPONENTIAL TURNPIKE PROPERTY

The main goal of this section is to prove Theorem 2. We derive this theorem from a more general statement, Theorem 4. The latter, in turn, is obtained as a result of a chain of elementary steps, Propositions 1–4.

Let $\xi^k = \{(x_t^k, y_t^k, c_t^k, p_t^k)\}_0^{N_k}$, $0 < N_k \le \infty$, $k \in \{1, 2\}$, be two sequences of random vectors $x_t^k, y_t^k, c_t^k \in \mathcal{X}_t$, $p_t^k \in \mathcal{P}_t$. Suppose that

$$y_t^k - x_t^k - c_t^k = 0 \tag{32}$$

for any $k \in \{1, 2\}$ and all $t \in \mathbf{N}$, $t \le N$, where $\mathbf{N} = \{0, 1, 2, \ldots\}$, $N = \min(N_1, N_2)$. Define

$$q_t^k = \left(p_{t-1}^k, p_t^k\right), \qquad v_t^k = \left(-x_{t-1}^k - c_{t-1}^k, y_t^k\right), \qquad k \in \{1, 2\}. \tag{33}$$

Consider the scalar product

$$\begin{aligned}\left(q_t^1 - q_t^2\right)\left(v_t^1 - v_t^2\right) &= \left(p_t^1 - p_t^2\right)\left(y_t^1 - y_t^2\right) - \left(p_{t-1}^1 - p_{t-1}^2\right)\left(x_{t-1}^1 - x_{t-1}^2\right) \\ &\quad - \left(p_{t-1}^1 - p_{t-1}^2\right)\left(c_{t-1}^1 - c_{t-1}^2\right).\end{aligned} \tag{34}$$

PROPOSITION 1. *For all $i, j \in \mathbf{N}$, $1 \le i \le j \le N$, we have*

$$\sum_{t=i}^{j} \left(q_t^1 - q_t^2\right)\left(v_t^1 - v_t^2\right) = \left(p_j^1 - p_j^2\right)\left(y_j^1 - y_j^2\right) - \left(p_{i-1}^1 - p_{i-1}^2\right)\left(y_{i-1}^1 - y_{i-1}^2\right). \tag{35}$$

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Proof. Equality (35) holds because

$$\sum_{t=i}^{j} q_t^k v_t^m = \sum_{t=i}^{j} \left[p_t^k y_t^m - p_{t-1}^k \left(x_{t-1}^m + c_{t-1}^m\right)\right] = \sum_{t=i}^{j} \left(p_t^k y_t^m - p_{t-1}^k y_{t-1}^m\right)$$

$$= p_j^k y_j^m - p_{i-1}^k y_{i-1}^m \tag{36}$$

for any $k, m = 1, 2$ [see (32)]. ∎

For the two given sequences ξ^k, $k \in \{1, 2\}$, define

$$\Delta_t = E\left\{\left[\left(p_{t-1}^1, p_t^1\right) - \left(p_{t-1}^2, p_t^2\right)\right]\left[\left(-x_{t-1}^1, y_t^1\right) - \left(-x_{t-1}^2, y_t^2\right)\right]\right\}; \tag{37}$$

$$z_t^k = \left(x_{t-1}^k, y_t^k\right), \qquad d_t = E\left|z_t^1 - z_t^2\right|_2^2, \qquad d_0 = E\left|y_0^1 - y_0^2\right|_2^2 \tag{38}$$

($t \in \mathbf{N}$, $1 \le t \le N$). Fix some constant $\kappa_1 \in (0, 1)$ and suppose that the sequences ξ^k, $k \in \{1, 2\}$, satisfy [in addition to (32)] the following conditions:

$$E\left(p_t^1 - p_t^2\right)\left(c_t^1 - c_t^2\right) \le 0, \qquad t \in \mathbf{N}, \qquad t \le N; \tag{39}$$

$$\Delta_t \ge \kappa_1 d_t, \qquad t \in \mathbf{N}, \qquad 1 \le t \le N. \tag{40}$$

PROPOSITION 2. *For all* $i, j \in \mathbf{N}$, $1 \le i \le j \le N$,

$$\sum_{t=i}^{j} d_t \le \frac{1}{\kappa_1}\left[E\left(p_j^1 - p_j^2\right)\left(y_j^1 - y_j^2\right) - E\left(p_{i-1}^1 - p_{i-1}^2\right)\left(y_{i-1}^1 - y_{i-1}^2\right)\right]. \tag{41}$$

Proof. Using (40) and (39), we obtain

$$\sum_{t=i}^{j} d_t \le \kappa_1^{-1} \sum_{t=i}^{j} \Delta_t = \kappa_1^{-1} E \sum_{t=i}^{j} \left[\left(q_t^1 - q_t^2\right)\left(v_t^1 - v_t^2\right)\right.$$

$$\left. + \left(p_{t-1}^1 - p_{t-1}^2\right)\left(c_{t-1}^1 - c_{t-1}^2\right)\right] \le \kappa_1^{-1} E \sum_{t=i}^{j} \left(q_t^1 - q_t^2\right)\left(v_t^1 - v_t^2\right).$$

This yields (41) in view of (35). ∎

Let us impose a further assumption on ξ^k, $k \in \{1, 2\}$. Let $K_1 > 0$ be a constant. Assume that

$$E\left|p_t^1 - p_t^2\right|_2^2 \le K_1 \cdot E\left|c_t^1 - c_t^2\right|_2^2 \tag{42}$$

for any $t \in \mathbf{N}$, $t \le N$. In accord with (25), define $\delta_t = \delta_t(\xi^1, \xi^2) = E|w_t^1 - w_t^2|_2^2$. Clearly

$$\delta_t = E\left|x_t^1 - x_t^2\right|_2^2 + E\left|y_t^1 - y_t^2\right|_2^2 + E\left|c_t^1 - c_t^2\right|_2^2 + E\left|p_t^1 - p_t^2\right|_2^2. \tag{43}$$

PROPOSITION 3. *For any* $t \in \mathbf{N}$, $t < N$,

$$\delta_t \le (3 + 2K_1)(d_t + d_{t+1}). \tag{44}$$

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Proof. Taking into account (32), we write

$$E\left|c_t^1 - c_t^2\right|_2^2 = E\left|\left(y_t^1 - y_t^2\right) - \left(x_t^1 - x_t^2\right)\right|_2^2 \le 2E\left|y_t^1 - y_t^2\right|_2^2 + 2E\left|x_t^1 - x_t^2\right|_2^2, \quad t \in \mathbf{N}, \quad t \le N. \quad (45)$$

If $t \in [1, N)$, then (45) yields

$$E\left|c_t^1 - c_t^2\right|_2^2 \le 2E\left|z_t^1 - z_t^2\right|_2^2 + 2E\left|z_{t+1}^1 - z_{t+1}^2\right|_2^2 = 2(d_t + d_{t+1}). \quad (46)$$

For $t = 0$, we have

$$E\left|c_0^1 - c_0^2\right|_2^2 \le 2E\left|y_0^1 - y_0^2\right|_2^2 + 2d_1 = 2(d_0 + d_1). \quad (47)$$

Thus, by using (42), we find

$$E\left|p_t^1 - p_t^2\right|_2^2 \le 2K_1(d_t + d_{t+1}), \quad t \in \mathbf{N}, \quad t < N. \quad (48)$$

By combining (46), (47), and (48) and using the inequality

$$E\left|y_t^1 - y_t^2\right|_2^2 + E\left|x_t^1 - x_t^2\right|_2^2 \le d_t + d_{t+1}, \quad t \in \mathbf{N}, \quad t < N,$$

we obtain (44). ∎

Remark 1. Suppose the sequences ξ^1 and ξ^2 satisfy

$$-E\left(p_t^1 - p_t^2\right)\left(c_t^1 - c_t^2\right) \ge \kappa_2 E\left|p_t^1 - p_t^2\right|_2^2, \quad t \in \mathbf{N}, \quad t \le N, \quad (49)$$

for some constant $\kappa_2 > 0$. Then, conditions (39) and (42) are fulfilled with $K_1 = (\kappa_2)^{-2}$. Indeed, inequality (39) is a direct consequence of (49). To derive (42), we write

$$\kappa_2\left\|p_t^1 - p_t^2\right\|_2^2 \le -E\left(p_t^1 - p_t^2\right)\left(c_t^1 - c_t^2\right) \le \left\|p_t^1 - p_t^2\right\|_2\left\|c_t^1 - c_t^2\right\|_2.$$

This yields $\kappa_2\|p_t^1 - p_t^2\|_2 \le \|c_t^1 - c_t^2\|_2$, and so we obtain (42) with $K_1 = (\kappa_2)^{-2}$.

PROPOSITION 4. *For any two sequences ξ^1 and ξ^2 satisfying* (32), (39), (40), *and* (42), *the following inequalities hold*:

$$\delta_i + \cdots + \delta_j \le (6 + 4K_1)(\kappa_1)^{-1} \cdot (\delta_i + \delta_j), \quad (50)$$

where $i, j \in \mathbf{N}$, $i < j \le N$. If, in addition, $y_0^1 = y_0^2$, then

$$\delta_0 + \cdots + \delta_j \le (6 + 4K_1)(\kappa_1)^{-1} \cdot \delta_j \quad (51)$$

for any $j \in \mathbf{N}$, $j \le N$.

The constants $\kappa_1 \in (0, 1)$ and $K_1 > 0$ involved in (50) and (51) are described in (40) and (42).

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Proof of Proposition 4. Clearly, (50) is true if $i+1=j$ (recall that $0<\kappa_1<1$ and $K_1>0$). Suppose $i+1<j$. By using (41) and the inequality $ab\le\frac{1}{2}(|a|_2^2+|b|_2^2)$ $(a,b\in R^n)$, we get

$$\sum_{t=i+1}^{j} d_t \le \frac{1}{2\kappa_1}\{E|p_j^1-p_j^2|_2^2+E|y_j^1-y_j^2|_2^2+E|p_i^1-p_i^2|_2^2+E|y_i^1-y_i^2|_2^2\}$$

$$\le \frac{1}{2\kappa_1}(\delta_j+\delta_i). \tag{52}$$

For shortness, denote $K_2=3+2K_1$. In view of (44), we can write

$$\delta_{i+1}+\cdots+\delta_{j-1}\le K_2[(d_{i+1}+d_{i+2})+(d_{i+2}+d_{i+3})+\cdots+(d_{j-1}+d_j)]$$

$$\le 2K_2\sum_{t=i+1}^{j} d_t.$$

This, combined with (52), gives

$$\delta_{i+1}+\cdots+\delta_{j-1}\le(K_2/\kappa_1)(\delta_i+\delta_j).$$

By adding $\delta_i+\delta_j$ to the both sides of this inequality, we obtain (50), because

$$K_2/\kappa_1+1\le 2K_2/\kappa_1=(6+4K_1)/\kappa_1. \tag{53}$$

Suppose $y_0^1=y_0^2$. If $j=0$, inequality (51) holds trivially. Let $j>0$. By employing (44), we find

$$\delta_0+\cdots+\delta_{j-1}\le K_2[(d_0+d_1)+\cdots+(d_{j-1}+d_j)]\le 2K_2\sum_{t=1}^{j} d_t$$

(the last inequality is true because $d_0=E|y_0^1-y_0^2|_2^2=0$). By virtue of Proposition 2,

$$\sum_{t=1}^{j} d_t \le (\kappa_1)^{-1}E\left(p_j^1-p_j^2\right)\left(y_j^1-y_j^2\right)\le(2\kappa_1)^{-1}\delta_j,$$

and so, $\delta_0+\cdots+\delta_j\le(K_2/\kappa_1+1)\delta_j$, which yields (51). ∎

We need the following simple general fact.

PROPOSITION 5. *Let L be a number greater than* 1 *and $\{r_t\}_{t=0}^{M}$ $(1\le M\le\infty)$ a sequence of real numbers satisfying $0\le r_t\le L$. Let $\theta=1-L^{-1}$.*

Suppose that $r_i+\cdots+r_j\le L(r_i+r_j)$ for all $i,j\in\mathbf{N}$, $i<j\le M$. Then the following assertions are true: (*a*) *if $M=\infty$, then $r_t+r_{t+1}+\cdots\le 2L^2\theta^t$, $t\in\mathbf{N}$*; (*b*) *if $M<\infty$, then $r_t+\cdots+r_{M-t}\le 2L^2\theta^t$, $t\in\mathbf{N}$, $t<M/2$.*

Assume that $M<\infty$ and $r_0+\cdots+r_j\le Lr_j$ for all $j\in\mathbf{N}$, $j\le M$. Then $r_0+\cdots+r_{M-t}\le L^2\theta^t$, $t\in\mathbf{N}$, $t\le M$.

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Proof. To prove (b), put $\sigma_t = r_t + \cdots + r_{M-t}$, where $0 \le t < M/2$. Then $\sigma_0 = r_0 + \cdots + r_M \le L(r_0 + r_M) \le 2L^2$. Furthermore, $\sigma_t \le L(r_t + r_{M-t}) = L(\sigma_t - \sigma_{t+1})$ for $0 \le t < M/2 - 1$. Thus, $\sigma_{t+1} \le \theta\sigma_t$ for all $0 \le t < M/2 - 1$, and so $\sigma_t \le \theta^t\sigma_0 \le \theta^t \cdot 2L^2$, which yields (b). To prove the other inequalities in Proposition 5, one has to apply analogous considerations to $\sigma_t = r_t + r_{t+1} + \cdots$ and $\sigma_t = r_0 + \cdots + r_{M-t}$. ∎

Let us apply the above proposition to the sequence $\{r_t\}_{t=0}^M = \{\delta_t\}_{t=0}^N$ $(M = N)$ defined in terms of ξ^1 and ξ^2 [see (43)]. Fix some constant D and assume

$$\delta_t \le D, \qquad t \in \mathbf{N}, \qquad t \le N. \tag{54}$$

Define

$$L = \max\{D, (6 + 4K_1)(\kappa_1)^{-1}\}, \qquad L_0 = 2L^2, \qquad \theta = 1 - L^{-1}. \tag{55}$$

As a consequence of Propositions 4 and 5, we obtain

THEOREM 4. *Let the sequences ξ^1 and ξ^2 satisfy conditions* (32), (39), (40), (42), *and* (54). *Then, assertions* (i)–(iii) *of Theorem 2 hold with L_0 and θ defined by* (55).

To deduce Theorem 2 from Theorem 4, it is sufficient to verify the following:

PROPOSITION 6. *Let ξ^1 and ξ^2 be equilibrium trajectories belonging to the class Ξ. Then conditions listed in Theorem 4 are fulfilled with $\kappa_1 = 4\gamma\kappa$, $K_1 = (\zeta^{B_0})^{-2}$, and $D = 3n(A^2 + B_0^2)$, where $B_0 = 2A_0\delta^{-1}(2A + \delta)$.*

Proof. We have already noticed in Section 2 that, for any equilibrium path in Ξ, constraints (4) hold as equalities [see (21)], which yields (32). By virtue of Theorem A.3 (see Appendix), the equilibrium price vectors p_t^k satisfy $|p_t^k| \le B_0$. By using this fact and hypothesis (C.5), we obtain (49), where $\kappa_2 = \zeta^{B_0}$. According to Remark 1, (49) implies (39) and (42). Let us verify (40). By virtue of (Z.5), there exist $a_{t-1} \in \mathcal{X}_{t-1}$ and $b_t \in \mathcal{X}_t$ such that

$$\left[\left(x_{t-1}^1 + x_{t-1}^2\right)/2 - a_{t-1}, \left(y_t^1 + y_t^2\right)/2 + b_t\right] \in Z_t, \qquad t \in \mathbf{N}, \qquad 1 \le t \le N,$$

and

$$E(|a_{t-1}| + |b_t|) \ge \kappa E\left(\left|x_{t-1}^1 - x_{t-1}^2\right|_2^2 + \left|y_t^1 - y_t^2\right|_2^2\right).$$

In view of (2), we have

$$Ep_t^1 y_t^1 - Ep_{t-1}^1 x_{t-1}^1 \ge Ep_t^1\left[\left(y_t^1 + y_t^2\right)/2 + b_t\right] - Ep_{t-1}^1\left[\left(x_{t-1}^1 + x_{t-1}^2\right)/2 - a_{t-1}\right].$$

Consequently,

$$Ep_t^1\left(y_t^1 - y_t^2\right) - Ep_{t-1}^1\left(x_{t-1}^1 - x_{t-1}^2\right) \ge 2Ep_t^1 b_t + 2Ep_{t-1}^1 a_{t-1}$$
$$\ge 2\gamma E(|b_t| + |a_{t-1}|) \ge 2\gamma\kappa E\left(\left|y_t^1 - y_t^2\right|_2^2 + \left|x_{t-1}^1 - x_{t-1}^2\right|_2^2\right),$$

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because $p_t^k \geq \gamma e,\ k \in \{1, 2\}$ [see (20)]. Thus, we find

$$Ep_t^1 \left(y_t^1 - y_t^2\right) - Ep_{t-1}^1 \left(x_{t-1}^1 - x_{t-1}^2\right) \geq 2\gamma\kappa E\left(\left|z_t^1 - z_t^2\right|_2^2\right).$$

The above inequality remains valid if we modify it by interchanging the superscripts 1 and 2. By adding the original and the modified inequalities, we find

$$E\left(p_t^1 - p_t^2\right)\left(y_t^1 - y_t^2\right) - E\left(p_{t-1}^1 - p_{t-1}^2\right)\left(x_{t-1}^1 - x_{t-1}^2\right) \geq 4\gamma\kappa E\left(\left|z_t^1 - z_t^2\right|_2^2\right),$$

which gives us (40) with $\kappa_1 = 4\gamma\kappa$. From (Z.2), (32), and the inequality $|p_t^k| \leq B_0$, we conclude that (54) holds for $D = 3n(A^2 + B_0^2)$. ■

Proof of Theorem 2. Theorem 2 follows at once from Theorem 4 and Proposition 6. ■

Remark 2. Our results concerning the turnpike properties of equilibrium paths have been obtained under the assumptions of strong convexity of the technology sets and strong monotonicity of the demand correspondences. Such assumptions appear—in one form or another—in many studies dealing with problems of stability and sensitivity of optimal or equilibrium paths in dynamic economic models. In recent years, another approach to such problems, based on entirely different hypotheses, has been developed. The approach makes use of lattice partial orderings on sets of economic alternatives and relies on hypotheses of supermodularity of objective functionals [Topkis (1978), Milgrom and Shannon (1991)]. Techniques of this type have proved to be appropriate in various comparative dynamics problems and have led to a number of new results and insights [Amir et al. (1991), Amir (1996)]. It would be of interest to apply these techniques to the study of turnpike properties of stochastic equilibrium paths.

5. INFINITE EQUILIBRIUM PATHS: EXISTENCE AND UNIQUENESS

We here give proofs of Theorems 1 and 3. Some of the results in the present section rely on those in the preceding one.

Proof of Theorem 1. Let $y_0 \in I$. The existence of an equilibrium path with i.v. y_0 over a finite time interval $[0, N]$ $(1 \leq N < \infty)$ follows from Theorem A.1 stated in Appendix. Let us show that this path is unique. Suppose there are two such paths, $\xi^k = \{(x_t^k, y_t^k, c_t^k, p_t^k)\}_{t=0}^N$, $k \in \{1, 2\}$. Because $x_N^k = 0$, we have $y_N^k = c_N^k$ [see (21)], and so,

$$E\left(p_N^1 - p_N^2\right)\left(y_N^1 - y_N^2\right) = E\left(p_N^1 - p_N^2\right)\left(c_N^1 - c_N^2\right) \leq 0, \tag{56}$$

where the last inequality follows from (C.5) in view of the relations $|p_N^1| \leq B_0$ and $|p_N^2| \leq B_0$. By using (56) and (41), we find

$$\sum_{t=1}^{N} d_t \leq -(\kappa_1)^{-1} E\left(p_0^1 - p_0^2\right)\left(y_0^1 - y_0^2\right) = 0,$$

where $y_0^1 = y_0^2 = y_0$. Therefore, $d_1 = \cdots = d_N = 0$. By virtue of (44), this implies $\delta_1 = \delta_2 = \cdots = \delta_{N-1} = 0$. Because $x_0^1 = x_0^2$ and $y_0^1 = y_0^2$, we obtain $c_0^1 = c_0^2$ [see

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(21)], and $p_0^1 = p_0^2$ [see (C.5)]. Analogously, because $y_N^1 = y_N^2$ and $x_N^1 = x_N^2$ (=0), we conclude that $c_N^1 = c_N^2$ and $p_N^1 = p_N^2$. Thus, $\delta_0 = \delta_N = 0$, and so, $\xi^1 = \xi^2$.

Let us prove that, for each $y_0 \in I$, there exists an infinite equilibrium path with i.v. y_0. For any $1 \le N < \infty$, consider the finite equilibrium path $\{w_0^N, \dots, w_N^N\}$, $w_t^N = (x_t^N, y_t^N, c_t^N, p_t^N)$, with i.v. $y_0^N = y_0$. As we have already proved, this path exists, is unique, and the price vectors p_t^N satisfy $|p_t^N| \le B_0$. Furthermore, we have $0 \le x_t^N, y_t^N, c_t^N \le Ae$. By virtue of (28),

$$\left\| w_t^N - w_t^{N+m} \right\|_2^2 \le L_0 \theta^{N-t}, \qquad N \ge t, \qquad m \ge 0,$$

where $\theta \in (0, 1)$. Thus, the sequence w_t^N, $N = t, t+1, \dots$, is fundamental with respect to the L_2-norm, and so, there exists a $w_t = (x_t, y_t, c_t, p_t)$ in $\mathcal{X}_t^4 \equiv \mathcal{X}_t \times \mathcal{X}_t \times \mathcal{X}_t \times \mathcal{X}_t$ such that $\|w_t^N - w_t\|_2 \to 0$ as $N \to \infty$. Any sequence converging in the L_2-norm contains a subsequence converging a.s. Using this and applying Cantor's diagonal process, we construct a sequence of natural numbers $N_1 < N_2 < \cdots$ such that $w_t^{N_m} \to w_t$ a.s. ($m \ge t$) for every $t \ge 0$. Clearly, $y_t = x_t + c_t$ and $|p_t| \le B_0$. In view of (Z.1), $(x_{t-1}, y_t) \in Z_t$. By passing to the limit in the inequality

$$E\left[p_t^{N_m} y_t^{N_m} - p_{t-1}^{N_m} x_{t-1}^{N_m}\right] \ge E\left[p_t^{N_m} y - p_{t-1}^{N_m} x\right], \qquad (x, y) \in Z_t,$$

we find $E[p_t y_t - p_{t-1} x_{t-1}] \ge E[p_t y - p_{t-1} x]$. Finally, we have $c_t^{N_m} \in C_t(p_t^{N_m})$, where $\|p_t^{N_m} - p_t\|_1 \to 0$ (because $\|p_t^{N_m} - p_t\|_2 \to 0$) and $c_t^{N_m} \to c_t$ $\{\sigma[L_\infty(t), L_1(t)]\}$ because $c_t^{N_m} \to c_t$ a.s. and $c_t^{N_m}$ is uniformly bounded in $L_\infty(t)$. Thanks to (C.2) we conclude that $c_t \in C_t(p_t)$. Thus, $\{w_t\}_{t=0}^\infty$ is an infinite equilibrium trajectory with i.v. y_0.

Suppose there is another infinite equilibrium trajectory with i.v. $y_0' = y_0$. Denote it by $\{w_t'\}_{t=0}^\infty$. By virtue of Theorem 2, $\|w_t - w_t'\|_2 \to 0$ as $t \to \infty$. By virtue of Proposition 6, the trajectories ξ and ξ' satisfy (32), (39), (40), and (42). Therefore, as a consequence of Proposition 4, we find

$$\sum_{t=0}^{j} \|w_t - w_t'\|_2^2 \le K \cdot \|w_j - w_j'\|_2^2, \qquad j \ge 0, \qquad K = \text{const}.$$

Hence $\|w_t - w_t'\|_2 = 0$ for all $t \ge 0$. ∎

Proof of Theorem 3. Suppose that the model under consideration is stationary. Fix some $y_0 \in I$, and consider the infinite equilibrium path $\xi = \{w_t\}_{t=0}^\infty = \{(x_t, y_t, c_t, p_t)\}_{t=0}^\infty$ with i.v. y_0. Define $\xi^k = \{w_t^k\}_{t=0}^\infty$, where

$$w_t^k = \left(x_t^k, y_t^k, c_t^k, p_t^k\right) = T^{-k} w_{t+k}, \qquad k \in \{0, 1, \dots\}.$$

Let us show that ξ^k is an infinite equilibrium path belonging to the class Ξ. Because $s_{t+1}(\omega) = s_t(T\omega)$, we have $\mathcal{F}_{t+k} = T^{-k}\mathcal{F}_t (= \{T^{-k}\Gamma\colon\ \Gamma \in \mathcal{F}_t\})$. Consequently, $T^{-k}\mathcal{X}_{t+k} = \mathcal{X}_t$ and $T^{-k}\mathcal{P}_{t+k} = \mathcal{P}_t$ ($k \ge 0$). Therefore, $x_t^k, y_t^k, c_t^k \in \mathcal{X}_t$, and $p_t^k \in \mathcal{P}_t$. It follows from (30) that

$$T^{-k} Z_{t+k} = Z_t (t \ge 1), \qquad T^{-k} C_{t+k}(p) = C_t(T^{-k} p)\ (p \in \mathcal{P}_{t+k},\ t \ge 0), \tag{57}$$

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for each $k \geq 0$. In view of (57), we have $(x_{t-1}^k, y_t^k) \in Z_t$ because $(x_{t+k-1}, y_{t+k}) \in Z_{t+k}$ $(t \geq 1)$, and $c_t^k \in C_t(p_t^k)$ because $c_{t+k} \in C_{t+k}(p_{t+k})$. By using the fact that the transformation T^{-k} preserves the measure P, we find that (x_{t-1}^k, y_t^k) maximizes $Ep_t^k y - Ep_{t-1}^k x$ over $(x, y) \in Z_t$; the equalities $y_t^k = x_t^k + c_t^k$ $(t \geq 0, k \geq 0)$ hold a.s. The i.v. $y_0^k = T^{-k} y_k$ of ξ^k satisfies $\mathring{y}_0 \leq y_0^k \leq Ae$ by virtue of (7), (10), and (31). Thus $\xi^k \in \Xi$.

For all $t, k, m \in \{0, 1, \ldots\}$,

$$\left\| w_t^k - w_t^{k+m} \right\|_2^2 = \left\| T^{-k} w_{t+k} - T^{-k} T^{-m} w_{t+k+m} \right\|_2^2 = \left\| w_{t+k} - w_{t+k}^m \right\|_2^2.$$

Hence $\|w_t^k - w_t^{k+m}\|_2^2 \leq L_0 \theta^{t+k} \leq L_0 \theta^k$ by virtue of (26). Thus, $w_t^0, w_t^1, w_t^2, \ldots (t \geq 0)$ is a fundamental sequence with respect to the norm $\|\cdot\|_2$. Furthermore, $|p_t^k| \leq B_0$. Consequently, there exist random vectors $\bar{w}_t = (\bar{x}_t, \bar{y}_t, \bar{c}_t, \bar{p}_t) \in \mathcal{X}_t^4$ $(t \geq 0)$ such that, for any $t \geq 0$, $\|w_t^k - \bar{w}_t\|_2 \to 0$ as $k \to \infty$. By using the same considerations as in the proof of the existence of an infinite equilibrium path (see the proof of Theorem 1), we establish that $\bar{\xi} = \{\bar{w}_t\}_{t=0}^{\infty}$ is an equilibrium trajectory. Because $\mathring{y}_0 \leq y_0^k \leq Ae$, we have $\mathring{y}_0 \leq \bar{y}_0 \leq Ae$, and hence $\bar{\xi} \in \Xi$. For any $t \geq 0$,

$$\|T\bar{w}_t - \bar{w}_{t+1}\|_2 = \lim_{k\to\infty} \left\| T w_t^k - w_{t+1}^k \right\|_2 = \lim_{k\to\infty} \left\| T T^{-k} w_{t+k} - w_{t+1}^k \right\|_2$$

$$= \lim_{k\to\infty} \left\| T^{-(k-1)} w_{(t+1)+(k-1)} - w_{t+1}^k \right\|_2 = \lim_{k\to\infty} \left\| w_{t+1}^{k-1} - w_{t+1}^k \right\|_2 = 0.$$

Thus, $\bar{\xi}$ is a stationary equilibrium path belonging to the class Ξ.

Suppose there is another stationary equilibrium path $\hat{\xi} = \{\hat{w}_t\}_{t=0}^{\infty}$, where $\hat{w}_t = (\hat{x}_t, \hat{y}_t, \hat{c}_t, \hat{p}_t)$. Because $(\hat{x}_0, \hat{y}_1) \in Z_1$, we have $\mathring{y}_1 \leq \hat{y}_1 \leq Ae$ [see (Z.2) and (Z.4)]. Therefore, $\mathring{y}_0 \leq \hat{y}_0 \leq Ae$, and so $\hat{\xi} \in \Xi$. By virtue of Theorem 2, $\|\hat{w}_t - \bar{w}_t\|_2 \to 0$ as $t \to \infty$. On the other hand, $\|\hat{w}_t - \bar{w}_t\|_2$ does not depend on t because $\|\hat{w}_{t+1} - \bar{w}_{t+1}\|_2 = \|T\hat{w}_t - T\bar{w}_t\|_2 = \|\hat{w}_t - \bar{w}_t\|_2$. Thus, $\|\hat{w}_t - \bar{w}_t\|_2 = 0$ for all $t \geq 0$, and hence $\hat{\xi} = \bar{\xi}$. ■

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APPENDIX

We state some results of previous work used in this paper.

THEOREM A.1. *Let hypotheses* (*Z.1*)–(*Z.4*), (*C.1*)–(*C.3*), *and* (*C.5*) *hold. Then for any* $1 \leq N < \infty$ *and any* $y_0 \in I$, *there exists an equilibrium path with i.v.* y_0 *over the time interval* $[0, N]$.

The assertion formulated follows from Theorem 1 of Evstigneev and Taksar (1995).

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THEOREM A.2. *Let* $\{(x_t, y_t)\}_0^N$ *and* $\{(\check{x}_t, \check{y}_t)\}_0^N$ $(N \leq \infty)$ *be two sequences of random vectors such that, for all* $t \geq 0$, *we have* $x_t, y_t, \check{x}_t, \check{y}_t \in \mathcal{X}_t$,

$$y_t - x_t \geq 0, \qquad \check{y}_t - \check{x}_t \geq \delta e, \tag{A.1}$$

$$\check{y}_t \leq y_t \leq Ae, \tag{A.2}$$

where $\delta > 0$ *and* $A > 0$ *are some* (*nonrandom*) *constants. Let* $\{p_t\}_0^N$ *be a sequence of vector functions* $p_t \in \mathcal{P}_t$, *satisfying the following conditions*:

(i) *For any* $t \geq 1$ *and any* $\mathcal{F}_{t-1}$*-measurable function* $\theta\colon \Omega \to [0, 1]$, *we have*

$$Ep_t y_t^\theta - Ep_{t-1} x_{t-1}^\theta \leq Ep_t y_t - Ep_{t-1} x_{t-1}, \tag{A.3}$$

where $(x_{t-1}^\theta, y_t^\theta) = \theta(\check{x}_{t-1}, \check{y}_t) + (1 - \theta)(x_{t-1}, y_t)$.

(ii) *There is a constant* A_0 *such that*

$$p_t(y_t - x_t) \leq A_0 \tag{A.4}$$

for all $t \geq 0$. *Suppose one of the following assumptions holds*: (*a*) $N < \infty$, *and* $Ep_N x_N = 0$; (*b*) $N = \infty$, *and the sequence* $Ep_t x_t$, $t = 0, 1, \ldots$, *satisfies the moderate growth condition. Then for each* t, *we have* $|p_t| \leq B_0$, *where*

$$B_0 = 2A_0\delta^{-1}(2A + \delta). \tag{A.5}$$

The above result is a consequence of Propositions 3.3 and 3.4 of Evstigneev and Flåm (1997). From Theorem A.2, we deduce the following fact:

THEOREM A.3. *If* $\{x_t, y_t, c_t, p_t\}$ *is an equilibrium trajectory in the class* Ξ, *then the price vectors* p_t *satisfy*

$$|p_t| \leq B_0, \ t \geq 0, \tag{A.6}$$

where B_0 *is defined by (A.5).*

According to (A.6) and (20), all of the equilibrium prices are bounded above and bounded away from zero (a.s.) by constants that are determined by the data of the model and do not depend on the equilibrium trajectory.

Proof. To prove Theorem A.3, it suffices to show that the sequences $\{(x_t, y_t)\}_0^N$ and $\{\check{x}_t, \check{y}_t)\}_0^N = \{(\mathring{x}_t, \mathring{y}_t)\}_0^N$ meet the requirements listed in Theorem A.2. We have $y_t - x_t \geq c_t \geq 0$. The inequality $\mathring{y}_t - \mathring{x}_t \geq \delta e$ holds by virtue of (9). By using (Z.2), (Z.4), and the fact that $y_0 \in I$, we find that $0 \leq \mathring{y}_t \leq y_t \leq Ae$. This proves (A.1) and (A.2). In view of (Z.3) and (8), $(x_{t-1}^\theta, y_t^\theta) \in Z_t$, and so equation (A.3) is true by the definition of an equilibrium path. Finally, we conclude that $p_t(y_t - x_t) = p_t c_t \leq A_0$ thanks to (16), which completes the proof. ■

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