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Extremum principle for a nonlinear problem in magneto-elasticity

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Abstract. A minimum principle associated with a class of magneto-elastic boundary-value problems is presented. The principle depends on a result on integral inequalities which appears to be new. An accurate variational solution is obtained for an illustrative one-dimensional problem.

1. *Introduction.* The nonlinear boundary-value problem

$$\phi^{\mathrm{IV}} = \frac{1}{(1-\phi)^2} \quad (0 \leqslant x \leqslant l), \tag{1}$$

with

$$\phi(0) = \phi(l) = \phi'(0) = \phi'(l) = 0, \tag{2}$$

arises in the theory of the bending of a ferromagnetic plate, where ϕ $(0 \leqslant \phi < 1)$ denotes the deflection of the plate due to an applied electromagnetic field (1). Since an analytical solution of this problem is not known, it is necessary to look for approximate methods of solution. One approach to this is to study the associated variational problem, and in this paper we derive a minimum principle which can be used to generate approximate analytic solutions. This principle depends on a result on integral inequalities which appears to be new (see Appendix). To illustrate the theory we perform calculations for the case $l = 1$ and the resulting variational solution is found to be very accurate.

2. *Extremum principle.* Applying the variational theory developed in (2), we are led to introduce the functional

$$J(\Phi) = \int_0^l \left\{ \tfrac{1}{2}(\Phi'')^2 - \frac{1}{1-\Phi} \right\} dx, \tag{3}$$

where Φ belongs to the class of functions having

$$\Phi(0) = \Phi(l) = \Phi'(0) = \Phi'(l) = 0. \tag{4}$$

Then it is readily seen that $J(\Phi)$ is stationary at ϕ, where ϕ is the solution of (1) and (2).

To test $J(\Phi)$ for an extremum property we examine

$$J(\Phi) - J(\phi) = \frac{1}{2}\int_0^l \left\{ (y'')^2 - \frac{2y^2}{(1-\psi)^3} \right\} dx \tag{5}$$

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by (3), where $y = \Phi - \phi$ and $\psi = \phi + \epsilon y$ $(0 \leqslant \epsilon \leqslant 1)$. Since y satisfies the boundary conditions (2), we can use the inequality (A 10) in the Appendix and obtain from (5)

$$J(\Phi) - J(\phi) \geqslant \frac{1}{2}\int_0^l \left\{\frac{K\pi^4}{l^4} - \frac{2}{(1-\psi)^3}\right\} y^2\, dx, \tag{6}$$

where $K \simeq 81/16$. Hence, if the function ψ satisfies

$$\frac{K\pi^4}{l^4} \geqslant \frac{2}{(1-\psi)^3} \quad (0 \leqslant x \leqslant l), \tag{7}$$

we have the minimum principle $\quad J(\phi) \leqslant J(\Phi).$ (8)

For $l = 1$, condition (7) becomes

$$0 \leqslant \psi < 0{\cdot}84, \tag{9}$$

which is clearly satisfied for deformations ϕ that are small compared with unity and reasonably accurate trial functions Φ.

3. *Calculations.* To illustrate the theory we take the case $l = 1$. For the trial function Φ we choose

$$\Phi = \frac{\alpha}{24} x^2(1-x)^2 \tag{10}$$

which satisfies the boundary conditions (2). The variational parameter α is determined by minimizing the functional $J(\Phi)$ and the results obtained are

$$\alpha = 1{\cdot}00295, \quad J = -1{\cdot}000697. \tag{11}$$

The accuracy of the variational solution Φ in (10) and (11) can be measured in terms of an error bound due to Ciarlet, Schultz and Varga (3). For our problem described in (1) and (2) with $l = 1$, this error bound is given by

$$\sup_{x\in[0,1]} |\Phi - \phi| \leqslant \frac{k}{\sqrt{\Lambda}} \|\tau(\Phi)\|_{L^2[0,1]} = E(\Phi), \tag{12}$$

where $$\tau(\Phi) = \Phi^{\mathrm{IV}} - (1-\Phi)^{-2}, \tag{13}$$

Λ is a lower bound for the eigenvalues of the associated eigenvalue problem $\phi^{\mathrm{IV}} = \lambda\phi$, $0 < x < 1$, subject to the boundary conditions (2), and k is a real constant such that

$$\sup_{x\in[0,1]} |y(x)| \leqslant k\left\{\int_0^l (y'')^2\, dx\right\}^{\frac{1}{2}} \tag{14}$$

for all functions $y(x)$ satisfying the conditions

$$y(0) = y(1) = y'(0) = y'(1) = 0, \tag{15}$$

$$y' \text{ continuous in } (0,1), y'' \in L^2[0,1].$$

It can be shown that $$k = \sqrt{(3)}/24, \quad \Lambda = 500. \tag{16}$$

Using the results (12)–(16), we estimate the error in the variational solution (10) to be

$$E(\Phi) = 0{\cdot}593 \times 10^{-5}. \tag{17}$$

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This indicates that Φ is quite an accurate solution of the problem. Since Φ, and hence ϕ, lie between 0 and 0·00261 in this example, we note that the sufficient condition (9) for a minimum is easily satisfied.

APPENDIX

To minimize the expression

$$T(y) = \int_0^\pi \{y''(x)\}^2\,dx \Big/ \int_0^\pi \{y(x)\}^2\,dx$$

subject to the conditions

$$y(0) = y'(0) = y(\pi) = y'(\pi) = 0 \tag{A 1}$$

we set

$$y(x) = \sum_{n=1}^{\infty} a_n \sin nx \tag{A 2}$$

so that

$$T(y) = \sum_{n=1}^{\infty} n^4 a_n^2 \Big/ \sum_{n=1}^{\infty} a_n^2. \tag{A 3}$$

The function $\sin x - \frac{1}{3}\sin 3x$ gives $T = 9$; since any non-vanishing a_n for which n is even gives a factor $\geqslant 16$ in (A 3), the extremal function will have $a_{2r} = 0$ identically. Note that (A 1) is equivalent to

$$a_1 + 3a_3 + 5a_5 + \ldots = 0 \tag{A 4}$$

and so by Cauchy's inequality:

$$a_1^2 \leqslant \left(\frac{1}{k_3^2} + \frac{1}{k_5^2} + \frac{1}{k_7^2} + \ldots\right)(3^2 k_3^2 a_3^2 + 5^2 k_5^2 a_5^2 + \ldots)$$

with equality if

$$nk_n^2 a_n = \text{constant}, \quad \text{for odd } n \geqslant 3. \tag{A 5}$$

Hence, for $\alpha \geqslant 0$,

$$\sum_{n=1}^{\infty}{}' a_n^2 \leqslant (1-\alpha)a_1^2 + \sum_{n=3}^{\infty}{}' (1 + Cn^2 k_n^2 \alpha) a_n^2 \tag{A 6}$$

where $'$ denotes summation restricted to odd n, and

$$C = \sum_{n=3}^{\infty}{}' \frac{1}{k_n^2}. \tag{A 7}$$

There is equality in (A 6) if (A 5) holds. Let

$$1 - \alpha = \beta, \quad 1 + Cn^2 k_n^2 \alpha = \beta n^4 \quad \text{for odd} \quad n \geqslant 3. \tag{A 8}$$

Then $T(y) \geqslant 1/\beta$ for all functions y satisfying (A 1); moreover there is equality if (A 5) holds. Thus

$$K = \min_y T(y) = 1/\beta$$

where (A 7) and (A 8) give

$$\sum_{n=1}^{\infty}{}' \frac{(K-1)n^2}{n^4 - K} = 0$$

or

$$\tanh\frac{\pi K^{\frac{1}{4}}}{2} + \tan\frac{\pi K^{\frac{1}{4}}}{2} = 0 \quad (1 \leqslant K \leqslant 9).$$

If we set

$$\frac{\pi K^{\frac{1}{4}}}{2} = \frac{3\pi}{4} + \epsilon$$

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we find that $$e^{2\epsilon}\tan\epsilon = e^{-\frac{3}{2}\pi}$$

so that K is very slightly larger than $(3/2)^4 = 81/16$. From (A 4), (A 5) and (A 8) we see that the extremal function is (any scalar multiple of)

$$y_0(x) = \sum_{n=1}^{\infty}{}' \frac{n\sin nx}{n^4 - K}.$$

One method of summation is by Laplace transforms. We find that

$$y_0(x) = \frac{\pi}{8\sqrt{K}}[(\sinh K^{\frac{1}{4}}x - \sin K^{\frac{1}{4}}x)\tanh \tfrac{1}{2}(\pi K^{\frac{1}{4}}) - (\cosh K^{\frac{1}{4}}x - \cos K^{\frac{1}{4}}x)].$$

We therefore have the following result: if

$$y(0) = y'(0) = y(L) = y'(L) = 0 \tag{A 9}$$

then

$$\int_0^L \{y''(x)\}^2\,dx \geqslant \frac{\pi^4 K}{L^4}\int_0^L \{y(x)\}^2\,dx \tag{A 10}$$

where

$$\tanh\frac{\pi K^{\frac{1}{4}}}{2} + \tan\frac{\pi K^{\frac{1}{4}}}{2} = 0 \quad (K \doteqdot 81/16). \tag{A 11}$$

The inequality is best possible.

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