The probability certain random quadratics have real roots

CHRIS BOUCHER

Early in high school algebra, quadratics chosen as examples by teachers and textbooks alike tend to have integer coefficients and to factorise over the integers. This can give the misleading impression that such quadratics are the norm. As students progress into calculus and begin regularly seeing quadratics that are not as 'nice', we hope they become disabused of this notion. Indeed, even if the coefficients of the quadratic are integers, the probability that the quadratic factorises over the integers tends to zero as the range from which the integers are drawn grows (see [1]). But what if we ask about a behaviour less restrictive than factorising, say merely having real roots? This is the problem that concerns this Article.

For now, we restrict ourselves to the case where the coefficients are positive integers. As we will see at the end, once this case is resolved, addressing the question for integer coefficients is straightforward. Our question, then, is this: if A_n, B_n and C_n are independent, discrete uniform random variables taking values in the set $\{1, \ldots, n\}$ and P_n is the probability that $A_n x^2 + B_n x + C_n = 0$ has real roots, what is $\lim_{n \to \infty} P_n$?

The study of polynomials with random coefficients is not a new one. The systematic study of random polynomials began with the work of Bloch and Pólya [2]. Littlewood, Offord [3], and Kac [4, 5] also made seminal contributions with Bharuca-Reid [6] providing a comprehensive work on the subject. Our question is far less ambitious than the issues these works address. Much of the work cited above seeks to understand the asymptotic behaviour of the number of zeroes of a polynomial with independent, identically distributed coefficients as the degree of the polynomial approaches infinity. Kac (see [6] pp. 12-13) gives an exact formula for the expected number of zeroes of a random polynomial in the case where the coefficients are independent, identically distributed Gaussian random variables.

For fixed n, our question is a combinatorial one. If we let $\Omega_n \doteq \{1, 2, \ldots, n\}^3$ and $D_n = \{(a, b, c) \in \Omega_n : b^2 - 4ac \geqslant 0\}$, then we seek

$$\lim_{n \to \infty} \frac{|D_n|}{|\Omega_n|} = \lim_{n \to \infty} \frac{|D_n|}{n^3}.$$

We estimate $|D_n|$ in two ways, first using an integral and then more directly.

Integral estimate:

Throughout the paper we write $a \vee b$ to mean the maximum of a and b and $a \wedge b$ to mean the minimum of a and b. Noting that D_n is the collection

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of lattice points in the first octant of $\mathbb{R}^3$ between the surfaces $z = n$ and $z = 2\sqrt{xy} \wedge n$, we can estimate $|D_n|/n^3$ by the fraction of the volume of the cube $[1, n]^3$ between these surfaces.

$$\frac{|D_n|}{n^3} \approx \frac{1}{(n-1)^3}\int_1^n\int_1^n \left(n - 2\sqrt{xy} \wedge n\right) dx\, dy \to \frac{5 + 6\log 2}{36}.$$

Direct estimate:

A second approach is to estimate the size of D_n more directly by using a partition $\{D_{n,1}, D_{n,2}, \ldots, D_{n,n}\}$, where $D_{n,i} \doteq \{(a, b, c) \in D_n : b = i\}$.

Theorem 1: For all positive integers n and all $i \in \{1, 2, \ldots, n\}$

$$|D_{n,i}| = \sum_{a=1}^{n \wedge \left\lfloor \frac{i^2}{4} \right\rfloor} \left\lfloor \frac{i^2}{4a} \right\rfloor \wedge n \tag{1}$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Proof: Fix $n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, n\}$. If $a, c \in \{1, 2, \ldots, n\}$ satisfy $4ac \leqslant i^2$, then $a \leqslant n \wedge \lfloor i^2/4 \rfloor$. For each value of a that satisfies $1 \leqslant a \leqslant n \wedge \lfloor i^2/4 \rfloor$, any value of c that satisfies $1 \leqslant c \leqslant n \wedge \lfloor i^2/4a \rfloor$ will produce a pair (a, c) that satisfies $ac \leqslant i^2/4$. Thus the number of (a, c) pairs that satisfy $ac \leqslant i^2/4$ is

$$\sum_{a=1}^{n \wedge \left\lfloor \frac{i^2}{4} \right\rfloor} \left\lfloor \frac{i^2}{4a} \right\rfloor \wedge n.$$

This completes the proof.

We now determine an upper and a lower bound for $|D_{n,i}|$. We begin determining the upper bound by noting that

$$|D_{n,i}| \leqslant \sum_{k=1}^{n} \left\lfloor \frac{i^2}{4k} \right\rfloor \wedge n = \sum_{k=1}^{\left\lfloor \frac{i^2}{4n} \right\rfloor} n + \sum_{k=\left\lfloor \frac{i^2}{4n} \right\rfloor + 1}^{n} \left\lfloor \frac{i^2}{4k} \right\rfloor \leqslant \frac{i^2}{4} + \frac{i^2}{4} \sum_{k=\left\lfloor \frac{i^2}{4n} \right\rfloor + 1}^{n} \frac{1}{k}. \tag{2}$$

Let $p = \left\lfloor \frac{i^2}{4n} \right\rfloor + 1$ and observe that

$$\sum_{k=p}^{n} \frac{1}{k} \leqslant \log \frac{n}{p} + \frac{1}{p} \leqslant \log \frac{n}{i^2/4n} + \frac{1}{i^2/4n} = \log \frac{4n^2}{i^2} + \frac{4n}{i^2}. \tag{3}$$

The first inequality in (3) is evident from Figure 1, where the area of the rectangles is the left-hand side and the total area shown is the right-hand side.

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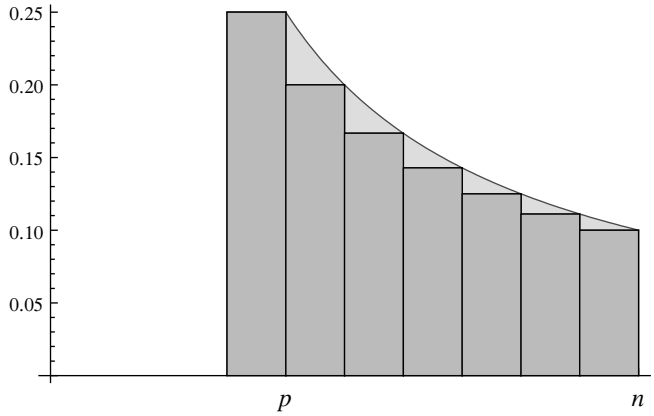


FIGURE 1

Putting (2) and (3) together,

$$|D_{n,i}| \leqslant \frac{i^2}{4} + \frac{i^2}{4}\log\frac{4n^2}{i^2} + n. \tag{4}$$

Thus,

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}|D_{n,i}| &\leqslant \lim_{n\to\infty}\frac{1}{4n^3}\sum_{i=1}^{n}i^2 + \limsup_{n\to\infty}\frac{1}{4n^3}\sum_{i=1}^{n}i^2\log\left(\frac{2n}{i}\right)^2\\ &= \frac{1}{12} + \frac{\log 2}{6} + \frac{1}{2}\limsup_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}i^2\log\left(\frac{n}{i}\right). \qquad (5)\end{aligned}$$

The second and third lines make use of the fact that $\lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}i^2 = \frac{1}{3}$.

The only tools used in arriving at the upper bound in (4) were easy estimates of ceiling and floor functions. These tools can be used to produce a lower bound analogous to that in (4) given by

$$|D_{n,i}| \geqslant \frac{i^2}{4}\left(1 + \frac{1}{n} - \frac{1}{n+1} + \log\left(\frac{4n^2}{i^2}\right)\right) - \left(\frac{3n}{2} + 2\right) \doteq B_{n,i}. \tag{6}$$

Thus, calculating much as we did in (5),

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}|D_{n,i}| &\geqslant \liminf_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}B_{n,i}\\ &\geqslant \liminf_{n\to\infty}\frac{1}{4n^3}\sum_{i=1}^{n}i^2 + \liminf_{n\to\infty}\frac{1}{4n^3}\sum_{i=1}^{n}i^2\log\left(\frac{2n}{i}\right)^2\\ &= \frac{1}{12} + \frac{\log 2}{6} + \frac{1}{2}\liminf_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}i^2\log\left(\frac{n}{i}\right). \qquad (7)\end{aligned}$$

Remarkably, the limit that occurs in (5) and (7) is an integral.

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Theorem 2: $\lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 \log\left(\frac{n}{i}\right) = \frac{1}{9}$.

Proof:

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 \log\left(\frac{n}{i}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \log\left(\frac{1}{i/n}\right)$$

$$= \int_0^1 x^2 \log\left(\frac{1}{x}\right) dx$$

$$= \frac{1}{9}.$$

The integral can be calculated using integration by parts.

Letting $\alpha_n = \frac{1}{n^3} \sum_{i=1}^{n} \left(\frac{i^2}{4} + \frac{i^2}{4} \log \frac{4n^2}{i^2}\right)$, (5), (7) and Theorem 2 show that

$$\alpha_n \to \frac{1}{12} + \frac{\log 2}{6} + \frac{1}{18} = \frac{6 \log 2 + 5}{36},$$

which must be the value of $\lim_{n\to\infty} \frac{|D_n|}{n^3}$. In fact, the bounds (4) and (6) give somewhat more detailed information about the asymptotics of $\frac{|D_n|}{n^3}$. The upper bound (4) reads

$$\frac{|D_n|}{n^3} \leqslant \alpha_n + \frac{1}{n}$$

and the lower bound (6) reads

$$\frac{|D_n|}{n^3} \geqslant \alpha_n + \frac{1}{n^3} \sum_{i=1}^{n} \left(\frac{1}{n} - \frac{1}{n+1} - \frac{3n}{2} - 2\right) = \alpha_n + \frac{3n^3 - 7n^2 - 4n + 2}{2n^3(n+1)}.$$

Theorem 3: If A_n, B_n and C_n are independent, discrete uniform random variables taking values in the set $\{1, \ldots, n\}$ and P_n is the probability that $A_n x^2 + B_n x + C_n$ has real roots, then

$$\lim_{n\to\infty} P_n = \frac{6 \log 2 + 5}{36} \approx 0.254413.$$

It follows that if E_n is the expected number of real roots of $A_n x^2 + B_n x + C_n$, then

$$\lim_{n\to\infty} E_n = \frac{6 \log 2 + 5}{18} \approx 0.508827. \tag{8}$$

The one loose end in Theorem 3 is the calculation of the asymptotic probability that $A_n x^2 + B_n x + C_n = 0$ has a double real root. Let $F_n = \{(a, b, c) : b^2 = 4ac\}$. Since a specification of the values or any

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two of the variables in the equation $b^2 = 4ac$ completely determines the value of the third, $|F_n| \leqslant n^2$ (in fact, n^2 is a gross overestimate). Thus, denoting by Q_n the probability that the quadratic has a double real root,

$$Q_n = \frac{|F_n|}{n^3} \leqslant \frac{1}{n}.$$

Equation (8) follows.

We can extend these results to cover the case where the coefficients B_n and C_n are discrete uniform random variables taking values in the set $\{-n, \ldots, n\}$ and A_n is a discrete uniform random variable taking values in the set $\{-n, \ldots, n\} \setminus \{0\}$. Let

$$\Delta_n = \{(a, b, c) : a, b, c \in \{-n, \ldots, n\}, a \neq 0, b^2 - 4ac \geqslant 0\}.$$

So $|\Delta_n| (2n)(2n + 1)^2$. We partition Δ_n based on the signs of a, b and c. Let

$$+ + + = \{(a, b, c) \in \Delta_n : a > 0, b > 0, c > 0\}$$

$$- + - = \{(a, b, c) \in \Delta_n : a < 0, b > 0, c < 0\}$$

$$+ 0 - = \{(a, b, c) \in \Delta_n : a > 0, b > 0, c < 0\}$$

$$\vdots$$

and so forth. There are one-to-one correspondences between the set $+ + +$, which we analysed above, and each of the sets $+ - +$, $- + -$, and $- - -$; they are $(a, b, c) \mapsto (a, -b, c)$, $(a, b, c) \mapsto (-a, b, -c)$, $(a, b, c) \mapsto (-a, -b, -c)$, respectively. Thus each of these four cells of the partition has size $|D_n|$. All the other cells in the partition are such that either every triple in the cell satisfies $b^2 - 4ac \geqslant 0$ or none of them does. Table 1 gives a complete summary of the partition. We have $|\Delta_n| = 4|D_n| + 4n^3 + 6n^2 + 2n$ and

$$\lim_{n \to \infty} \frac{|\Delta_n|}{2n(2n + 1)} = \lim_{n \to \infty} \frac{1}{2} \cdot \frac{6 \log 2 + 5}{36} + \frac{1}{2} \approx 0.627207.$$

cell	size of cell	cell	size of cell
$+ + +$	$\lvert D_n \rvert$	$+ + 0$	n^2
$+ + -$	n^3	$+ 0 0$	n
$+ - +$	$\lvert D_n \rvert$	$+ 0 -$	n^2
$- + +$	n^3	$+ - 0$	n^2
$+ - -$	n^3	$- 0 +$	n^2
$- + -$	$\lvert D_n \rvert$	$- + 0$	n^2
$- - +$	n^3	$- 0 0$	n
$- - -$	$\lvert D_n \rvert$	$- 0 -$	0
$+ 0 +$	0	$- - 0$	n^2

TABLE 1: Partition of Δ_n

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CHRIS BOUCHER

Mathematics Department, Salem State University

352 Lafayette Street, Salem, MA 01970 USA

e-mail: *cboucher@salemstate.edu*

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Quotations for *Nemo* (continued from page 396)

2. Let Clemison, house of Clemison, rejoice with Helix a kind of Ivy.
   God be praised for the vision of the Redcap and packet.

3. The helix revolves like a timeless thought,
   Instantaneous from apex to rim
   Like a dance whose figure is limpet or murex,
   cowrie or golden winkle.

4. Is there no way out of the mind?
   Steps at my back spiral into a well.

5. You were gone up as a rocket,
   Bending your passages from right to left and from left to right
   In the flat projection of a spiral.

6. concentric geometries of transparency slightly
   joggled sink through algebras of proud
   inwardlyness to collide spirally with iron arithmetics

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