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Adapted metrics for dominated splittings

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Abstract. A Riemannian metric is *adapted* to a hyperbolic set of a diffeomorphism if, in this metric, the expansion/contraction of the unstable/stable directions is seen after only one iteration. A *dominated splitting* is a notion of weak hyperbolicity where the tangent bundle of the manifold splits in invariant subbundles such that the vector expansion on one bundle is uniformly smaller than that on the next bundle. The existence of an *adapted metric* for a dominated splitting has been considered by Hirsch, Pugh and Shub (M. Hisch, C. Pugh and M. Shub. *Invariant Manifolds (Lecture Notes in Mathematics, 583)*. Springer, Berlin, 1977). This paper gives a complete answer to this problem, building adapted metrics for dominated splittings and partially hyperbolic splittings in arbitrarily many subbundles of arbitrary dimensions. These results stand for diffeomorphisms and for flows.

1. *Introduction*

The best known and simplest examples of chaotic dynamical systems are uniformly hyperbolic systems, like Anosov diffeomorphisms. A diffeomorphism f on a compact Riemannian manifold M is said to be an *Anosov diffeomorphism* if there exists a splitting of the tangent bundle TM into two supplementary, df-invariant subbundles, called the *stable* and the *unstable* bundles that are uniformly contracted and expanded, respectively, by an iterate of f. If the hyperbolic systems are now well understood, many dynamical systems are (robustly) not hyperbolic, so that several authors have tried to weaken the notion of hyperbolicity, in order to recover some of its properties on a larger class of systems.

In this spirit, Brin and Pesin [**BP74**] and Hirsch *et al* [**HPS77**] have extended the notion of a hyperbolic diffeomorphism to that of a *partially hyperbolic* diffeomorphism, that is, admitting an invariant splitting $TM = E^s \oplus E^c \oplus E^u$, where the stable bundle E^s is uniformly contracted, the unstable one E^u is uniformly expanded, and the central one E^c is uniformly less contracted (respectively less expanded) than E^s (respectively E^u). Hirsch *et al* showed the structural stability of the central bundle of a partially hyperbolic diffeomorphism (under some extra hypothesis, see [**HPS77**, Theorem 7.1]).

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Working on the stability conjecture†, Liao [**Lia80**], Mañé [**Man82**] and Pliss [**Pli72**] were led to the following general notion: a *dominated splitting* for f is a splitting of TM into two supplementary invariant subbundles such that there exists an iterate of df that uniformly contracts more (or expands less) the first subbundle than the second one. This notion is a key tool for understanding non-hyperbolic systems.

- In dimension two, Pujals and Sambarino [**PS00**] proved that a diffeomorphism with a dominated splitting may be C^1-approached by hyperbolic diffeomorphisms, and diffeomorphisms without dominated splitting may be approached by diffeomorphisms exhibiting a homoclinic tangency: as a consequence, any diffeomorphism of a compact surface can be C^1-approximated either by hyperbolic (Axiom A) diffeomorphisms, or by diffeomorphisms that exhibit a homoclinic tangency (this was conjectured by Palis [**PS70**]).
- In any dimension, Bonatti *et al* showed in [**BDP00**] that a robustly transitive generic diffeomorphism in $\mathrm{Diff}^1(M)$ admits a non-trivial dominated splitting defined on the whole M.

As recalled above, for a hyperbolic set K of a diffeomorphism f, the vectors in the stable and unstable bundles are uniformly contracted and expanded, respectively, by the derivative df^n, for some $n > 0$. The hyperbolicity of K does not depend on the metric on the manifold, but the smallest time n where the contraction/expansion phenomena are seen depends on the metric; a Riemannian metric is said to be *adapted* to the hyperbolic set K if one can take $n = 1$. Applying Holmes' theorem (see [**HPS77**, p. 15]), we obtain that any hyperbolic set admits an adapted Riemannian metric. We will adapt this theorem to the case of dominated behaviours, to show Lemma 4.1. It was asked in [**HPS77**, p. 5] if there existed an *adapted metric for a dominated splitting*, that is a metric such that df uniformly contracts (or respectively expands) the first subbundle more (or respectively less) than the second one, at the first iteration.

The aim of this paper is to give a complete positive answer to this question, proving that such an adapted metric exists for any dominated splitting.

THEOREM 1. *Let f be a diffeomorphism of a Riemannian manifold M, and K a compact invariant subset of M, such that the restriction of f to K admits a dominated splitting $TM|_K = E^1 \oplus E^2 \oplus \cdots \oplus E^d$, where the vectors in E^i are uniformly less expanded than those in E^{i+1} by df^n for some $n > 0$. Then there exists a Riemannian metric $\|\cdot\|$ on M (necessarily equivalent to the first metric) and adapted to the dominated splitting: there exists a constant $0 < \mu < 1$ such that for any $x \in K$, any $i \in \{1, \ldots, d-1\}$, and any unit vectors $u \in E^i_x$, $v \in E^{i+1}_x$, one has $\|df(u)\| < \mu.\|df(v)\|$.*

This result was already known by Hisch *et al* [**HPS77**] for a dominated splitting in two bundles, $TM|_K = E_1 \oplus E_2$, such that $\dim(E_1) = 1$ or $\dim(E_2) = 1$. In addition, they showed that any absolutely normally hyperbolic system admits an adapted metric, but it was not known whether it was true for a relatively normally hyperbolic system, which is, with our definitions, a partially hyperbolic system. We answer by showing (see Theorem 3)

† The stability conjecture, proved by Mañé in [**Man82**] for diffeomorphisms and then by Hayashi for flows in [**Hay97**], asserts that any C^1-structurally stable system is hyperbolic, i.e. satisfies the Axiom A and the strong transversality condition.

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that an adapted metric exists for any partially hyperbolic splitting, that is a metric adapted to the corresponding dominated splitting, and such that the stable/unstable bundles are uniformly expanded/contracted at the first iterate. Finally, we show in §5 how to transpose these results from diffeomorphisms to flows.

In order to present more clearly the idea of the proof, we will first focus, in §3, on dominated splittings into two subbundles, over an invariant compact set of a diffeomorphism. Then, we will show that there exists an adapted Finsler for any dominated splitting into d bundles for a Banach bundle automorphism (see Theorem 2).

2. *Definition and notations*

For a morphism A of normed vector spaces, define the *norm* and the *minimum norm* of A:

$$\|A\| = \sup_{\|u\|=1} \|A(u)\|, \quad \mathfrak{m}(A) = \inf_{\|u\|=1} \|A(u)\|.$$

When A is invertible, $\mathfrak{m}(A) = \|A^{-1}\|^{-1}$. For a Banach bundle E, we denote by E_x the fibre of E above a point x of the base. If $\mathcal{A}$ is an automorphism of a Banach bundle E with compact base K, then, for any point x of K, we denote by $\mathcal{A}_x$ the restriction of $\mathcal{A}$ to the fibre E_x. We refer the reader to [**HPS77**] for definitions.

We say that a sequence of functions $g_n(x)\colon K \to \mathbb{R}$ *converges exponentially to zero* if there exists positive constants C and $\mu < 1$ such that, for all x and n,

$$|g_n(x)| \leq C\mu^n.$$

Given an automorphism $\mathcal{A}$ of a Banach bundle E with compact base K

$$\begin{array}{ccc} E & \xrightarrow{\mathcal{A}} & E \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ K & \xrightarrow{f} & K \end{array}$$

and a positive continuous function $r\colon K \to \mathbb{R}$, we denote by $R_n(x)$ the product

$$R_n(x) = \prod_{i=0}^{n-1} r[f^i(x)] = r(x)r[f(x)]\cdots r[f^{n-1}(x)].$$

Definition 2.1. A positive continuous function $r\colon K \to \mathbb{R}$ *dominates* $\mathcal{A}$, if the sequence of ratios $x \to \|\mathcal{A}_x^n\|/R_n(x)$ converges exponentially to zero as $n \to \infty$, where $\mathcal{A}_x^n = \mathcal{A}^n_{|E_x}$.

In this case we write $df_{|E} \prec r$. Symmetrically, we say that r is *dominated* by $\mathcal{A}$ and we write $r \prec \mathcal{A}$ if and only if the ratio $R_n/\mathfrak{m}(\mathcal{A}^n)$ goes exponentially to zero as $n \to \infty$. Note that $r \prec \mathcal{A}$ is equivalent to $\mathcal{A}^{-1} \prec 1/r$.

Definition 2.2. Let E be a Banach bundle over a compact base K, and $E = E^1 \oplus \cdots \oplus E^d$ be an invariant splitting for an automorphism $\mathcal{A}$, where the E^i are vector subbundles with constant dimension. Then we say that it is a *dominated splitting* if, for each integer $0 < i < d$, the ratio $\|\mathcal{A}^n_{|E^i}\|/\mathfrak{m}(\mathcal{A}^n_{|E^{i+1}})$ tends exponentially to zero, as n goes to infinity.

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We have written $\|\mathcal{A}^n_{|E^i}\|/\mathrm{m}(\mathcal{A}^n_{|E^{i+1}})$ for the function $x \mapsto \|\mathcal{A}^n_{|E^i_x}\|/\mathrm{m}(\mathcal{A}^n_{|E^{i+1}_x})$. In this case, we say that $\mathcal{A}_{|E^i}$ is *dominated* by $\mathcal{A}_{|E^{i+1}}$, and we write $\mathcal{A}_{|E^i} \prec \mathcal{A}_{|E^{i+1}}$. We recall that the subbundles E^i are necessarily continuous (see [**BDV04**, Appendix B] for a proof).

Remark 2.3. Since the bundles and automorphisms are continuous, and the base K is compact, the definitions of domination and dominated splitting are independent of the Finsler. Thus we will be allowed to change to equivalent metrics; in finite dimension, all Finslers can be replaced by smooth Riemannian metrics.

A Finsler $\|\cdot\|_*$ is *adapted* to the dominated splitting if and only if, for all $i < d$, we have

$$\frac{\|\mathcal{A}_{|E^i}\|_*}{\mathrm{m}_*(\mathcal{A}_{|E^{i+1}})} < 1$$

where $\|\cdot\|_*$ and m_* are the norm and the minimum norm, with respect to the Finsler $\|\cdot\|_*$. Equivalently, by compactness of the base, there exists a real number $0 < C < 1$ such that, for any $x \in K$, for any non-zero unit vectors $u \in E^i_x$, $v \in E^{i+1}_x$, we have $\|\mathcal{A}(u)\|_* < C\|\mathcal{A}(v)\|_*$.

3. *Two-bundle splittings*

Let M be a compact smooth manifold endowed with a Riemannian metric $\|\cdot\|$, let f be a diffeomorphism of M and let K be an invariant compact set in M. We will show the following theorem.

THEOREM 3.1. *If $T_K M = E \oplus F$ is a dominated splitting for the diffeomorphism f on the compact K, then there exists a smooth Riemannian metric on M that is adapted to that dominated splitting.*

Proof. The proof consists in first building a *separator* for the dominated splitting, that is, a positive function $r\colon K \to \mathbb{R}$ such that we have $df_{|E} \prec r \prec df_{|F}$. Then, by a dominated version of Holmes' theorem, we will build two metrics $\|\cdot\|_E$ and $\|\cdot\|_F$ on the bundles E and F, such that, for any $x \in K$, for any unit vectors $u \in E_x$ and $v \in F_x$, we have

$$\|df(u)\|_E < r(x) < \|df(v)\|_F.$$

These metrics will induce, up to perturbation, an adapted Riemannian metric on M.

LEMMA 3.1. *A two-bundle dominated splitting has a separator.*

Proof. In the following, we fix a dominated splitting $T_K M = E \oplus F$ for the diffeomorphism f. For simplicity, call $df_{|E} = \mathcal{A}$ and $df_{|F} = \mathcal{B}$. By hypothesis, the ratio $\|\mathcal{A}^n\|/\mathrm{m}(\mathcal{B}^n)$ tends exponentially to zero. In particular, for N large enough, the function $x \mapsto \|\mathcal{A}^N_x\|/\mathrm{m}(\mathcal{B}^N_x)$ is smaller than $1/2$. Therefore, for $a > 1 > b$ close enough to one, we have for all x in K:

$$a\|\mathcal{A}^N_x\|^{1/N} < b\mathrm{m}(\mathcal{B}^N_x)^{1/N}.$$

Hence, Lemma 3.1 comes from the following claim. □

CLAIM 1. *Any continuous function $r\colon K \to \mathbb{R}$ such that $a.\|\mathcal{A}^N_x\|^{1/N} \leq r(x) \leq b.\mathrm{m}(\mathcal{B}^N_x)^{1/N}$ separates the splitting.*

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Proof. For any integer $n > N$, and each $0 \le k \le N-1$, we can write the iterate $\mathcal{A}_x^n$ as the composition

$$\mathcal{A}^l_{f^{k+mN}(x)} \circ \mathcal{A}^{mN}_{f^k(x)} \circ \mathcal{A}^k_x$$

for some integers $0 \le l \le N-1$ and $m \ge 0$. More precisely, take the integer part of $(n-k)/N$ for m, and $l = n - nM - k$. Denote by c the upper bound of the norms of the ith forward or backward iterates of A, for $i \le N$,

$$c = \sup_{|i| \le N, y \in K} \|A^i_y\|.$$

It is finite, as K is compact. We then have

$$\begin{aligned} \|\mathcal{A}^n_x\| &\le \|\mathcal{A}^l_{f^{k+nm}(x)}\| \left(\prod_{i=0}^{m-1} \|\mathcal{A}^N_{f^{k+iN}(x)}\| \right) \|\mathcal{A}^k_x\| \\ \|\mathcal{A}^n_x\| &\le c^2 \prod_{j \in J_k} \|\mathcal{A}^N_{f^j(x)}\| \end{aligned} \tag{1_k}$$

where J_k is the set of integers $\{k + iN, i = 0, \dots, m-1\}$, that is, the set of integers of the form $k + iN$ and comprised between 0 and $n - N$. Obviously, the sets J_k for $k = 0, \dots, N-1$ are pairwise disjoint and their union is the interval $\{0, \dots, n-N\}$. Hence, taking the product of inequalities (1_k), for $k = 0, \dots, N-1$ we obtain

$$\|\mathcal{A}^n_x\|^N \le c^{2N} \prod_{j \in \{0,\dots,n-N\}} \|\mathcal{A}^N_{f^j(x)}\|.$$

Since $\|\mathcal{A}^N_{f^j(x)}\|^{-1} \le \|\mathcal{A}^{-N}_{f^{j+N}(x)}\| \le c$, we get

$$\|\mathcal{A}^n_x\|^N \le c^{2N} c^N \prod_{j \in \{0,\dots,n\}} \|\mathcal{A}^N_{f^j(x)}\|.$$

Thus, as $a\|\mathcal{A}^N_{f^j(x)}\|^{1/N} \le r[f^j(x)]$, we get that, for any $x \in K$,

$$\|\mathcal{A}^n_x\| \le c^3 a^{-n} R_n(x),$$

which proves that $\mathcal{A} \prec r$. Note that $(1/b).\|\mathcal{B}^{-N}_x\|^{1/N} \le 1/r[F^N(x)]$, for all x. Thus, we have in the same way $\mathcal{B}^{-1} \prec 1/r$ and then $r \prec \mathcal{B}$. This ends the proof of the claim, and that of Lemma 3.1. □

We now show the following lemma (which can actually be seen as a particular case of Lemma 4.1 stated below).

LEMMA 3.2. *Let $r\colon K \to \mathbb{R}$ be a positive function that separates the continuous splitting $E \oplus F$, that is, $df_{|E} \prec r \prec df_{|F}$. Then there exists a Riemannian metric $\|\cdot\|_*$ on M that is* adapted *to the domination; namely, for all $x \in K$, for all unit vectors $u \in E_x$, $v \in F_x$ we have*

$$\|df(u)\|_* < r(x) < \|df(v)\|_*.$$

Proof. We define on E a metric $\|\cdot\|_E$ by

$$\|u\|_E^2 = \sum_{n=0}^{\infty} \frac{\|df^n(u)\|^2}{[R_n(x)]^2}$$

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for any $u \in E_x$, where $R_n(x) = r(x) \cdots r[f^{n-1}(x)]$ as above. By domination, this is a sum of a normally convergent series of continuous functions; therefore $\|\cdot\|_E$ is well defined and continuous. As a sum of quadratic forms, $\|\cdot\|_E^2$ is a quadratic form, and thus $\|\cdot\|_E$ is a Hilbertian metric (it arises from an inner product). Moreover, we have

$$\|df(u)\|_E^2 = \sum_{n=0}^{\infty} \frac{\|df^{n+1}(u)\|^2}{[R_n(x)]^2} = \sum_{n=1}^{\infty} \frac{\|df^n(u)\|^2}{[R_{n-1}(x)]^2} = r(x)^2 \sum_{n=1}^{\infty} \frac{\|df^n(u)\|^2}{[R_n(x)]^2}$$

since $R_{n-1}(x) = R_n(x)/r(x)$. We obtain $\|df(u)\|_E^2 = r(x)^2[\|u\|_E^2 - \|u\|^2]$ where $\|u\|^2$ is the first term of the series defining $\|u\|_E^2$. Hence, for any non-zero u, $\|df(u)\|_E < r(x)\|u\|_E$. Up to change of f into f^{-1} and r into $1/r$, we find the same way a Hilbertian metric $\|\cdot\|_F$ on F such that, for all non-zero v in F, $r(x)\|v\|_F < \|df(v)\|_F$. Consider now the Hilbertian metric $\|\cdot\|_*$ on $T_K M$ that extends $\|\cdot\|_E$ and $\|\cdot\|_F$ and that makes E and F orthogonal. It is continuous, since $\|\cdot\|_E$ and $\|\cdot\|_F$ are continuous.

The inequality $\|df(u)\|_* < r(x) < \|df(v)\|_*$ holds for all unit vectors $u \in E$, $v \in F$ above each point x of the base K. We extend the metric $\|\cdot\|_*$ to the whole M, and smooth it into a Riemannian metric by a small perturbation, so that, by compactness of K, the inequality is preserved. □

This together with the existence of a separator (Lemma 3.1) ends the proof of Theorem 3.1. □

4. *Multiple bundles splittings*

We will show the most general result of our paper in this section.

THEOREM 2. *Let E be a finite-dimensional Banach bundle on a compact base, and let $\mathcal{A}$ be an automorphism of E. If $E = E^1 \oplus \cdots \oplus E^d$ is a dominated splitting for $\mathcal{A}$, then there is a Finsler $\|\cdot\|_*$ on E adapted to the domination, that is, for each $i = 1 \ldots d-1$, for any $x \in K$, we have*

$$\|\mathcal{A}_{|E_x^i}\|_* < \mathfrak{m}_*(\mathcal{A}_{|E_x^{i+1}}).$$

Furthermore, if the original metric on E is Hilbertian, then the adapted metric can be chosen to be also Hilbertian.

Let F be a Banach bundle with compact base K, and $\mathcal{B}$ be an automorphism

$$\begin{array}{ccc} F & \xrightarrow{\mathcal{B}} & F \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ K & \xrightarrow{f} & K \end{array}$$

Then we have this dominated version of Holmes' theorem (see [**HPS77**, p. 15]).

LEMMA 4.1. *Let $r, s\colon K \to \mathbb{R}$ be two positive continuous functions such that the domination $r \prec \mathcal{B} \prec s$ and the inequality $r < s$ hold on K. Then there is a Finsler $\|\cdot\|_*$ on F that is adapted to the domination, namely, for any $x \in K$, for any $u \in F_x \setminus \{0\}$, we have*

$$r(x)\|u\|_* < \|\mathcal{B}(u)\|_* < s(x)\|u\|_*.$$

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Proof. For all u in F, we define

$$\|u\|_*^2 = \sum_{n=1}^{\infty} R_n^2[f^{-n}(x)]\|\mathcal{B}^{-n}(u)\|^2 + \sum_{n=0}^{\infty} \frac{\|\mathcal{B}^n(u)\|^2}{[S_n(x)]^2}$$

where $\|\cdot\|$ is the original metric on F, and where $R_n[f^{-n}(y)] = r(f^{-n}(y)) \cdots r(f^{-1}(y))$, $S_n(y) = s(y)s(f(y)) \cdots s(f^{n-1}(y))$ as before. Still by domination, the series normally converges to a continuous function. Thus $\|\cdot\|_*$ is a well-defined Finsler. We have

$$\begin{aligned}\|\mathcal{B}(u)\|_*^2 &= \sum_{n=1}^{\infty} R_n^2[f^{-n+1}(x)]\|\mathcal{B}^{-n+1}(u)\|^2 + \sum_{n=0}^{\infty} \frac{\|\mathcal{B}^{n+1}(u)\|^2}{[S_n(f(x))]^2} \\ &= \sum_{n=0}^{\infty} R_{n+1}^2[f^{-n}(x)]\|\mathcal{B}^{-n}(u)\|^2 + \sum_{n=1}^{\infty} \frac{\|\mathcal{B}^n(u)\|^2}{[S_{n-1}(f(x))]^2}.\end{aligned}$$

As we have $R_{n+1}[f^{-n}(x)] = R_n[f^{-n}(x)].r(x)$ and $S_{n-1}[f(x)] = S_n(x)/s(x)$, we get

$$\|\mathcal{B}(u)\|_*^2 = [r(x)]^2 \sum_{n=0}^{\infty} R_n^2[f^{-n}(x)]\|\mathcal{B}^{-n}(u)\|^2 + [s(x)]^2 \sum_{n=1}^{\infty} \frac{\|\mathcal{B}^n(u)\|^2}{[S_n(x)]^2}.$$

On the other hand, we have $R_0(x)\|\mathcal{B}^{-0}(v)\| = \|v\| = \|\mathcal{B}^0(v)\|/S_0(x)$ since R_0 and S_0 are empty products equal to one. So, from the definition of $\|u\|_*^2$, we have

$$\|u\|_*^2 = \sum_{n=0}^{\infty} R_n^2[f^{-n}(x)]\|\mathcal{B}^{-n}(u)\|^2 + \sum_{n=1}^{\infty} \frac{\|\mathcal{B}^n(u)\|^2}{[S_n(x)]^2}.$$

Finally, since $r < s$, we obtain, for any non-zero vector u, the inequality

$$[r(x)]^2\|u\|_*^2 < \|\mathcal{B}(u)\|_*^2 < [s(x)]^2\|u\|_*^2$$

the square root of which concludes the proof. □

Remark 4.2. Having chosen this quadratic construction, if $\|\cdot\|$ is a Hilbertian metric, then $\|\cdot\|_*$ is still a Hilbertian metric.

Proof of Theorem 2. By definition, the ratios $\|\mathcal{A}^n_{|E^i}\|/\mathfrak{m}(\mathcal{A}^n_{|E^{i+1}})$ converge exponentially to zero, for each i, as n goes to infinity. Thus we can find an integer N such that, for each i, the ratio $\|\mathcal{A}^N_{|E^i}\|/\mathfrak{m}(\mathcal{A}^N_{|E^{i+1}})$ is smaller than $1/4$. The proof of Lemma 3.1 still works when $TM_{|K} = E \oplus F$ is replaced by the Banach bundle $E^i \oplus E^{i+1}$, and when $df_{|K}$ is replaced by the automorphism $\mathcal{A}_{|E^i \oplus E^{i+1}}$.

Then choose a family $(r_i)_{0<i<d}$ of continuous functions such that $2^{1/N}\|\mathcal{A}^N_{|E^i}\|^{1/N} < r_i < 2^{-1/N}\mathfrak{m}(\mathcal{A}^N_{|E^{i+1}})^{1/N}$. We then have $r_1(x) < \cdots < r_{d-1}(x)$, for all $x \in K$; furthermore, by Claim 1, we have

$$E_1 \prec r_1 \prec E_2 \prec \cdots \prec r_{d-1} \prec E_d.$$

In order to have two-sided dominations for the extremal bundles, we may add two functions $r_0 \prec E_1$ and $E_d \prec r_d$, with $0 < r_0 < r_1$ and $r_{d-1} < r_d$. We now apply Lemma 4.1 to find

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a Finsler $\|\cdot\|_i$ on each E_i that is adapted to the domination $r_i \prec E_{i+1} \prec r_{i+1}$. Define the new metric

$$\|u\|_* = \sqrt{\sum_{i=1,\dots,d} \|p_i(u)\|_i^2},$$

for all $u \in E$, where p_i is the projection on E_i along $E_1 \oplus \cdots \oplus E^{i-1} \oplus E^{i+1} \cdots \oplus E^d$. It is a Finsler that is clearly adapted to the dominated splitting: for any unit vectors $u \in E^i_x$, $v \in E^{i+1}_x$ we have $\|u\|_* = \|u\|_i < r_i(x) < \|v\|_{i+1} = \|v\|_*$.

Remark 4.3. Obviously by the previous remark, if the original metric $\|\cdot\|$ was a Hilbertian metric, then the metrics $\|\cdot\|_i$ are so, and the metric $\|\cdot\|_*$ we have built is still a Hilbertian metric.

After smoothing the adapted Hilbertian metric, we obtain the following, which is a reformulation of Theorem 1.

COROLLARY 4.4. *Let M be a Riemannian manifold, K a compact invariant set for a diffeomorphism f, and $T_K M = E^1 \oplus \cdots \oplus E^d$ a dominated splitting for f above K. Then there exists a smooth Riemannian metric on M that is adapted to it.*

The existence of an adapted metric was shown for *absolute*- and not *relative*-normally hyperbolic systems (see [**HPS77**] for proofs and definitions). The bases of all Banach bundles are still compact. A dominated splitting $E = E^1 \oplus \cdots \oplus E^d$ for an automorphism $\mathcal{A}$ is *partially hyperbolic* if and only if, for some $1 \le k < k+1 < l \le d$, the bundles $E^s = E^1 \oplus \cdots \oplus E^k$ and $E^u = E^l \oplus \cdots \oplus E^d$ are respectively stable and unstable, that is $\|\mathcal{A}^n_{|E^s}\|$ and $\|\mathcal{A}^{-n}_{|E^u}\|$ converge exponentially to zero as n goes to infinity. We say that a metric $\|\cdot\|_*$ is *adapted* to such a partially hyperbolic splitting if it is adapted to the dominated splitting, and if $\|\mathcal{A}_{|E^s}\| < 1$ and $\mathfrak{m}(\mathcal{A}_{|E^u}) > 1$.

THEOREM 3. *A partially hyperbolic splitting has an adapted metric.*

Proof. We show it in the three-bundle case (it is the same idea for the general case). Consider a partially hyperbolic splitting $E = E^s \oplus E^c \oplus E^u$ with compact base for an automorphism $\mathcal{A}$. Then $\|\mathcal{A}^n_{|E^s}\|$ and $\|\mathcal{A}^{-n}_{|E^u}\|$ tend exponentially to zero as n goes to infinity. From the construction we gave in the proof of Lemma 3.1, we can find two functions $0 < r < 1 < s$ such that

$$\mathcal{A}_{|E^s} \prec r \prec \mathcal{A}_{|E^c} \prec s \prec \mathcal{A}_{|E^u}.$$

With respect to this domination, the metric $\|\cdot\|_*$ produced in the proof of Theorem 2 is adapted to the dominated splitting and satisfies $\|\mathcal{A}_{|E^s}\| < 1 < \mathfrak{m}(\mathcal{A}_{|E^u})$. Hence, it is adapted to the partially hyperbolic splitting. □

5. *Dominated splittings for flows*

In this section, we briefly show that the same results apply for flows. In the following, ϕ is a flow on a compact subset K of a Riemannian manifold M. A Finsler $\|\cdot\|_*$ on M is *adapted* to a dominated splitting $T_K M = E^1 \oplus \cdots \oplus E^d$ for ϕ, if and only if, for any point $x \in K$, for all unit vectors v, w in any pair E^i_x, E^{i+1}_x, we have

$$\text{for all } t > 0, \quad \|d\phi^t(v)\|_* < \|d\phi^t(w)\|_*,$$

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where $d\phi^t$ is the derivative of the time-t map of ϕ. The existence of adapted metrics for flows is not a straightforward consequence of our results on diffeomorphisms. At best, applying the former results would provide, for each $\epsilon > 0$, a metric $\|\cdot\|_*$ such that the inequality above holds for all $t > \epsilon$. To get adapted metrics, we have to transpose the notion of separator to the flow case.

Let E be a subbundle of $T_K M$, invariant by ϕ. Fix two strictly positive, continuous functions $r, s : K \to \mathbb{R}$. Then, for any $x \in K$, for all $t \in \mathbb{R}$, define

$$R_t(x) = \exp\left(\int_0^t \ln[r(\phi^u(x))].du\right),$$
$$S_t(x) = \exp\left(\int_0^t \ln[s(\phi^u(x))].du\right).$$

For any fixed x, the functions $t \mapsto R_t(x)$ and $t \mapsto S_t(x)$ are C^1, and for all real numbers t, k,

$$R_{t+k}(x) = R_t(x).R_k[\phi^t(x)], \tag{1}$$
$$S_{t+k}(x) = R_t(x).S_k[\phi^t(x)]. \tag{2}$$

Assume that $r < s$, and that we have the domination relation $r \prec d_E\phi \prec s$, that is, for all $x \in K$, for any vector $v \in E_x$, the quantities $\|d\phi^t(v)\|/S_t(x)$ and $\|d\phi^t(v)\|/R_t(x)$ go exponentially to zero, respectively, as t goes to $+\infty$, and as t goes to $-\infty$. Then, define the Finsler $\|\cdot\|_*$ on E by

$$\|v\|_*^2 = \int_{-\infty}^0 \frac{\|d\phi^t(v)\|^2}{R_t(x)^2}.dt + \int_0^\infty \frac{\|d\phi^t(v)\|^2}{S_t(x)^2}.dt$$

for any $x \in K$, for all $v \in E_x$.

CLAIM 2. *For any non-zero vector $v \in E$ above any point $x \in K$, the metric $\|\cdot\|_*$ satisfies*

$$\textit{for all } k > 0, \quad R_k(x).\|v\|_* < \|d\phi^k(v)\|_* < S_k(x).\|v\|_*.$$

That is, the metric $\|\cdot\|_*$ is *adapted* to the domination $r \prec d_E\phi \prec s$.

Remark 5.1. This is merely Lemma 4.1 for flows.

Proof. After a change of variable, we get

$$\begin{aligned}\|d\phi^k(v)\|_*^2 &= \int_{-\infty}^k \frac{\|d\phi^t(v)\|^2}{R_{t-k}^2 \circ \phi^k(x)}.dt + \int_k^{+\infty} \frac{\|d\phi^t(v)\|^2}{S_{t-k}^2 \circ \phi^k(x)}.dt \\ &= R_k^2(x) \int_{-\infty}^k \frac{\|d\phi^t(v)\|^2}{R_t^2(x)}.dt + S_k^2(x) \int_k^{+\infty} \frac{\|d\phi^t(v)\|^2}{S_t^2(x)}.dt,\end{aligned}$$

by formulae (1) and (2). Let $\theta(k)$ be the quotient $S_k^2(x)/R_k^2(x)$, and define the function

$$f : k \mapsto \frac{\|d\phi^k(v)\|_*^2}{R_k^2(x)} = \int_{-\infty}^k \frac{\|d\phi^t(v)\|^2}{R_t^2(x)}.dt + \theta(k) \int_k^{+\infty} \frac{\|d\phi^t(v)\|^2}{S_t^2(x)}.dt.$$

Since $\theta(k) = \exp\big(2.\int_0^k \ln[(s/r)(\phi^u(x))].du\big)$, and $s < r$, the derivative θ' is strictly positive. The function f is obviously C^1, and its derivative, after some calculation, is

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$$f'(k) = \theta'(k) \int_k^{+\infty} \frac{\|d\phi^t(v)\|^2}{S_t^2(x)}.dt.$$

Hence f' is strictly positive, and f is strictly increasing. As we have $f(0) = \|v\|_*^2/R_0^2(x) = \|v\|_*^2$, the inequality $f(k) > f(0)$ leads to $R_k(x)^2.\|v\|_*^2 < \|d\phi^k(v)\|_*^2$, for all $k > 0$. The inequality $\|d\phi^k(v)\|_*^2 < S_k^2(x).\|v\|_*^2$ comes the same way, considering this time the function $g : \left(k \mapsto \|d\phi^k(v)\|_*^2/S_k^2(x)\right)$. This concludes the proof of the claim. □

On the other hand, any dominated splitting for a flow has a separator. Let $T_K M = E \oplus F$ be a dominated splitting. For simplicity, Φ will denote the restriction of $d\phi$ to the bundle E. For instance, we will write $\|\Phi_x^t\|$ for the maximum norm of the restriction of $d\phi$ to E_x. More precisely, we assert the following claim.

CLAIM 3. *The function* $r = \left(x \mapsto a.\|\Phi_x^T\|^{1/T}\right)$ *is a separator* $E \prec r \prec F$, *for some* $a > 1$, *and some* T *large enough.*

Remark 5.2. This is Lemma 3.1 for flows; the proof is comparable step by step to that of Claim 1.

Proof. Fix a real $t > T$. Let m be the largest integer such that $T + mT \le t$. For any real $0 \le \kappa \le T$, we decompose Φ^t to obtain, for all $x \in K$,

$$\|\Phi_x^t\| \le \|\Phi^{\lambda}_{\phi^{\kappa+mT}(x)}\| \cdot \|\Phi^T_{\phi^{\kappa+(m-1)T}(x)}\| \cdot \cdots \|\Phi^T_{\phi^{\kappa+T}(x)}\| \cdot \|\Phi^T_{\phi^{\kappa}(x)}\| \cdot \|\Phi_x^{\kappa}\|,$$

where $\lambda > 0$ satisfies $t = \kappa + mT + \lambda$. Denote by c the upper bound $c = \sup_{|\tau|\le 2T, y\in K} \|\Phi_y^{\tau}\|$, and take the logarithm of the inequality,

$$\ln \|\Phi_x^t\| \le 2\ln(c) + \ln \|\Phi^T_{\phi^{\kappa+(m-1)T}(x)}\| + \cdots + \ln \|\Phi^T_{\phi^{\kappa}(x)}\|.$$

This stands for all $0 \le \kappa \le T$. Therefore, we have

$$\begin{aligned}\int_0^T \ln \|\Phi_x^t\|.d\kappa \le \int_0^T 2\ln(c).d\kappa + \int_0^T \ln \|\Phi^T_{\phi^{\kappa+(m-1)T}(x)}\|.d\kappa \\ + \cdots + \int_0^T \ln \|\Phi^T_{\phi^{\kappa}(x)}\|.d\kappa,\end{aligned}$$

$$T \ln \|\Phi_x^t\| \le 2T \ln(c) + \int_0^{mT} \ln \|\Phi^T_{\phi^u(x)}\|.du,$$

$$T \ln \|\Phi_x^t\| \le 4T \ln(c) + \int_0^{t} \ln \|\Phi^T_{\phi^u(x)}\|.du,$$

since $0 \le t - mT \le 2T$, and $-\ln \|\Phi_y^T\| \le \ln \|\Phi_y^{-T}\| \le \ln(c)$. Then, dividing by T, we obtain

$$\ln \|\Phi_x^t\| \le \ln(c^4) + \int_0^t \ln[a^{-1}.r(\phi^u(x))].du.$$

Writing the exponential form, we get $\|\Phi_x^t\| \le c^4 a^{-t} R_t(x)$, which means that $\Phi = d_E\phi \prec r$. We are left to check that $r \prec d_F\phi$ for some $T > 0$ big enough, and some $a > 1$. This is done the same way as in Lemma 3.1. □

Clearly, referring the reader to the proof of Theorem 2, flow versions of Lemmas 3.1 and 4.1 are all the ingredients we need, to transpose the results of §4 to flows.

THEOREM 4. *A dominated (respectively partially hyperbolic) splitting for a flow on a compact subset of a Riemannian manifold admits an adapted metric.*

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