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On quantum plasma kinetic equations with a Bohmian force

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Abstract. The dispersion relation arising from a Vlasov–Poisson system with a Bohmian force term is examined and compared to the more fundamental Bohm and Pines dispersion relation for quantum plasmas. Discrepancies are found already when considering the leading order thermal effects. The time-averaged energy densities for longitudinal modes are also shown to be noticeably different.

1. Introduction

The basic linear dispersion relation for quantum plasmas was derived by Bohm and Pines [1], after a series of canonical transformations applied to the N-body Hamiltonian operator for a dense electron gas. Another convenient way to obtain the Bohm and Pines dispersion relation is through a mean field theory formulated in terms of a Wigner–Poisson system [2]. In the context of the plasma physics community, the Wigner–Poisson and Wigner–Maxwell systems have the advantage of a direct correspondence with the Vlasov–Poisson and Vlasov–Maxwell kinetic formalisms. However, the Wigner equation contains a complicated integral term, reflecting the non-locality of quantum interactions. This motivated some recent studies [3–6] where the non-local force term in the Wigner equation is replaced by an effective force, due to the so-called Bohm potential. This Bohmian force is still non-local since it involves the particle density which is the zeroth-order moment of the Wigner function. However, it is comparatively simpler than the original integral term in Wigner equation. The Bohm potential is widely used in quantum fluid models, where only macroscopic variables such as particle density, quantum fluid velocity field and pressure are involved [7, 8]. However, the insertion of the effective quantum force directly as the momentum change into the Vlasov equation has been criticized [9] since the resulting permittivity is different from the Bohm and Pines one.

As a matter of fact, the kinetic equations with a Bohmian force assume that the N-body quantum statistical ensemble is representable as a pure state, see, e.g., the wave function (3) in [3]. In contrast, the Wigner function solving the Wigner–Poisson system is the Weyl transform of the single particle density matrix, hence by definition not restricted to pure states. This can be viewed as the basic reason why the Bohm and Pines dispersion relation has a privileged rôle. On the other hand, both models don't include collisions, since correlations are neglected. A quantum Born–Bogoliubov–Kirkwood–Green–Yvon hierarchy can be taken as a starting point to describe correlations [2].

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It should be recognized that the dispersion relation arising from the Vlasov–Poisson system with a Bohmian force is quite similar to the Bohm and Pines one [10]. Nevertheless, the purpose of the present work is to stress that there are differences, which become bigger as the joint thermal and quantum effects are taken into account. Moreover, even if these discrepancies can be small in certain limits, from a viewpoint of methodology it is preferable to start with kinetic equations and then proceed to fluid-like equations, which are by definition less accurate models. A possible route in this context is via moments of the Wigner (or Vlasov) equation, for instance. The reverse path, from fluid to kinetic theories, is questionable even if resonant wave–particle effects such as Landau damping are not considered. This point is explicitly shown in the following. The fluid models, however, remain valuable in view of the ability to describe the nonlinear aspects of quantum plasma dynamics, in an easier way in comparison to kinetic models (see [11] for a recent review).

The Bohm and Pines permittivity is revisited and compared to the permittivity arising from the Vlasov–Poisson system with a Bohm potential term in Section 2. Considering the leading thermal effects, the corresponding dispersion relations are shown to disagree. In addition, the consequences on the time-averaged energy density for electrostatic oscillations are studied. Final remarks are outlined in Section 3.

2. Dispersion relations

For simplicity, consider a one-dimensional electrostatic plasma with a fixed ionic background of particle density n_0. Our starting point is the dispersion relation derived by Bohm and Pines (1953),

$$\varepsilon = 1 - \frac{m\,\omega_p^2}{\hbar\,k^2}\int dv\,\frac{F_0[v+\hbar\,k/(2\,m)] - F_0[v-\hbar\,k/(2\,m)]}{k\,v-\omega} = 0, \tag{2.1}$$

for a frequency ω and a wavenumber k, with the equilibrium Wigner function $F_0(v)$ normalized according to $\int F_0(v)\,dv = n_0$. In (2.1), ε is the permittivity, m the electron mass, ω_p the plasma frequency and $\hbar$ Planck's constant over 2π. This work is not concerned with Landau damping or instabilities, so that the integrals are understood in the principal value sense. Moreover, for definiteness $F_0(v)$ is assumed to be an even, C^∞ and decaying function.

In order to access the rôle of quantum effects, it is convenient to approximate (2.1) in the vicinity of the classical permittivity. On integrating by parts, after Taylor expanding F_0 and retaining only the leading quantum corrections, we get

$$\varepsilon = 1 - \frac{\omega_p^2}{n_0}\int dv\,\frac{F_0(v)}{(kv-\omega)^2}\left[1+\frac{\omega_q^2}{(kv-\omega)^2}+\frac{\omega_q^4}{(kv-\omega)^4}\right] + O\left(\left(\frac{\omega_q}{\omega_p}\right)^6\right), \tag{2.2}$$

where $\omega_q = \hbar\,k^2/(2\,m)$.

The structure of (2.2) suggests the rescaling

$$F = \frac{\omega_p\,F_0}{n_0\,k}, \quad u = \frac{k\,v}{\omega_p}, \quad \Omega = \frac{\omega}{\omega_p}, \quad \Omega_q = \frac{\omega_q}{\omega_p}. \tag{2.3}$$

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In the new notation, (2.2) gives

$$\varepsilon = 1 - \frac{1}{\Omega^2} \int du \, \frac{F(u)}{(1-u/\Omega)^2} \left[1 + \frac{\Omega_q^2}{\Omega^2 \, (1-u/\Omega)^2} + \frac{\Omega_q^4}{\Omega^4 \, (1-u/\Omega)^4} \right], \qquad (2.4)$$

with the normalization condition $\int F(u)\, du = 1$ and where the $O\left(\Omega_q^6\right)$ contribution was neglected.

To deal with (2.4), it is convenient to expand like

$$\frac{1}{(1-u/\Omega)^i} = \sum_{j=0}^{\infty} C(i+j-1, i-1) \left(\frac{u}{\Omega}\right)^j , \qquad (2.5)$$

where $C(i,j)$ is a binomial coefficient. The above series diverges for $|u/\Omega| \geqslant 1$ but we suppose that $F(u)$ is such that the contribution from this range is negligible. In other words, as usual, a high-frequency (or, alternatively, a long wave–length) assumption is implicit. Moreover, admitting that $\langle u^4 \rangle / \Omega^4 \sim \langle u^2 \rangle^2 / \Omega^4 \ll \langle u^2 \rangle / \Omega^2 \ll 1$, where $\langle u^2 \rangle = \int du\, F(u)\, u^2$ and $\langle u^4 \rangle = \int du\, F(u)\, u^4$, the dispersion relation (2.4) gives

$$\Omega^2 = 1 + \frac{\Omega_q^2}{\Omega^2} + \frac{\Omega_q^4}{\Omega^4} + \left(3 + \frac{10\,\Omega_q^2}{\Omega^2} + \frac{10\,\Omega_q^4}{\Omega^4} \right) \frac{\langle u^2 \rangle}{\Omega^2} , \qquad (2.6)$$

which is valid up to the first order in $\langle u^2 \rangle / \Omega^2$ and to second order in the quantum parameter Ω_q^2. In addition, $\langle u^n \rangle = 0$ for odd n was used to get (2.6), since $F(u)$ is taken as an even function.

Solving (2.6) by successive approximations, the result is

$$\Omega^2 = 1 + \Omega_q^2 + \left(3 + 4\,\Omega_q^2 - 11\,\Omega_q^4 \right) \langle u^2 \rangle , \qquad (2.7)$$

which is correct if the $O(\langle u^4 \rangle)$ and $O(\Omega_q^6)$ contributions are negligible. As expected, the $O(\Omega_q^4)$ terms show up only when taking into account the thermal spread. Indeed, in the case of zero velocity dispersion ($F(u) = \delta(u)$), one has $\Omega^2 = 1 + \Omega_q^2$ up to all orders in the quantum parameter (as shown in [2]). Here it is referred to 'zero velocity dispersion' rather than to 'zero-temperature' to not confound with, for instance, a zero-temperature Fermi gas, where $\langle u^2 \rangle \neq 0$ in consequence of the exclusion principle. In addition, it is curious that the thermal effects already present in (2.7) disappear for the particular value $\Omega_q^2 = 0.74$, which still can be considered as a moderate number, with some optimism.

Using the same notation, the modified permittivity ϵ and the associated dispersion relation arising from the Vlasov–Poisson system with a Bohmian force is

$$\epsilon = 1 - \frac{(1+\Omega_q^2)}{\Omega^2} \int du \, \frac{F(u)}{(1-u/\Omega)^2} = 0, \qquad (2.8)$$

see [3–6] and also (3) of [9]. Contrary to (2.4), (2.8) is supposed to hold to all orders in Ω_q.

Proceeding as before, (2.8) gives

$$\Omega^2 = \left(1 + \Omega_q^2\right) \left(1 + \frac{3\,\langle u^2 \rangle}{\Omega^2} \right) + O\left(\frac{\langle u^4 \rangle}{\Omega^4} \right) , \qquad (2.9)$$

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to be compared to (2.7). Discarding the $O(\langle u^4\rangle/\Omega^4)$ terms, the pertinent root for (2.9) can be expressed as

$$\Omega^2 = 1 + \Omega_q^2 + \frac{3\,(1+2\,\Omega_q^2)\,\langle u^2\rangle}{1+\Omega_q^2}\,, \tag{2.10}$$

where there is no limitation on the quantum effects in the context of the modified theory. However, if the $O(\Omega_q^6)$ contribution is disregarded, (2.10) gives

$$\Omega^2 = 1 + \Omega_q^2 + 3\,\left(1+\Omega_q^2-\Omega_q^4\right)\,\langle u^2\rangle\,, \tag{2.11}$$

which is in the same level of approximation as (2.7). It is seen that the quantum corrections in the velocity dispersion term are not the same, starting from the Bohm and Pines or the Vlasov–Poisson with effective quantum force theories. In particular, the thermal effects in (2.11) are vanishing for $\Omega_q^2 = 1.62$, well above the Bohm and Pines value.

Finally, even if the resulting dispersion relations are somehow similar, the use of different permittivities has tangible consequences. For instance, restoring physical variables,

$$\langle W\rangle = \frac{\varepsilon_0}{4}\,\frac{\partial\,(\omega\,\varepsilon_h)}{\partial\,\omega}\,|E_1|^2 \tag{2.12}$$

is the time-averaged energy density $\langle W\rangle$ for longitudinal oscillations, where ε_0 is the vacuum permittivity, ε_h is the hermitian part of the dielectric function and E_1 is the amplitude of the perturbation electric field. For dissipation-free models, ε_h is the same as ε or ϵ, as given by (2.2) or (2.8) respectively. Relation (2.12) is a consequence of just the Maxwell equations so that it applies equally well to classical and quantum plasmas. It includes not only the electrostatic energy, but also the 'acoustic energy', which is the component of the particles kinetic energy due to the coherent wave motion [12].

The results are

$$\langle W\rangle = \left(1+\Omega_q^2+\left[3+8\,\Omega_q^2-29\,\Omega_q^4\right]\,\langle u^2\rangle\right)\,\frac{\varepsilon_0}{2}\,|E_1|^2 \tag{2.13}$$

from the Bohm–Pines permittivity ε in (2.2) and

$$\langle W\rangle = \left(1+3\left[1-2\,\Omega_q^2+3\,\Omega_q^4\right]\,\langle u^2\rangle\right)\,\frac{\varepsilon_0}{2}\,|E_1|^2 \tag{2.14}$$

from the modified permittivity ϵ in (2.8). As before, the calculations neglect the $O(\Omega_q^6)$ and $O(\langle u^4\rangle)$ terms. At this instance, comparison between (2.13) and (2.14) shows discrepancies even for zero thermal effects. Moreover, the energy density following from ϵ does not include quantum effects at all, for vanishing thermal spread.

3. Conclusions

It is evident that different permittivities would produce distinct dispersion relations, except in very particular limits. Nevertheless, it is relevant to explicitly determine what are the discrepancies between the Bohm and Pines theory and some recently introduced kinetic equations inspired by fluid models. To summarize, it is found that the corresponding wave frequencies disagree already when considering the leading thermal effects. The same apply to the energy density for longitudinal

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waves. Similar disagreement arise if higher order thermal effects are taken into account.

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