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Square functions and variational estimates for Ritt operators on L^1

JENNIFER HULTS† and KARIN REINHOLD LARSSON‡

† *Mathematics and Statistics, University at Albany, USA*
(e-mail: jhults@albany.edu)
‡ *Mathematics and Statistics, University at Albany, State University of New York, USA*
(e-mail: reinhold@albany.edu)

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Abstract. A power-bounded operator T satisfying $\sup_n n\|T^n - T^{n+1}\| < \infty$ is a Ritt operator. For such operators, we study the generalized square function $Q^T_{\alpha,s,r} f = (\sum_n n^\alpha |T^n(I-T)^r f|^s)^{1/s}$. It is known that when T is a positive contraction and a Ritt operator on L^p, $1 < p < \infty$, then for any integer $r \geq 1$, the square function $Q^T_{2r-1,2,r} f$ defines a bounded operator [17] on L^p. In this work, we extend the theory to the endpoint case $p = 1$. We show that if T is a Ritt operator on L^1, then the generalized square function $Q^T_{\alpha,s,r} f$ is bounded on L^1 whenever $\alpha + 1 < sr$. In the particular setting where T is a convolution operator of the form $T_\mu = \sum_k \mu(k) U^k f$, with μ a probability measure on $\mathbb{Z}$ and U the composition operator induced by an invertible measure-preserving transformation, we provide sufficient conditions on μ under which $Q^{T_\mu}_{2r-1,2,r} f$ is of weak type $(1, 1)$, for $r > 0$. We also establish bounds for variational and oscillation norms, $\|n^\beta T^n(1-T)^r\|_{v(s)}$ and $\|n^\beta T^n(1-T)^r\|_{o(s)}$, for Ritt operators, highlighting endpoint behavior.

Key words: Ritt operators, square functions, convolution operators, variational estimates, weak type (1, 1)
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1. *Introduction*

Ritt operators have a long history, tracing back to the foundational work of Ritt [25] and later developed by Lyubich [21], Nevanlinna and Zemanek [22, 23], and others [4]. In recent years, they have been the focus of renewed interest [1, 2, 4, 5, 9–11, 16–18] due to the development of their H^∞-functional calculus which made possible estimates for square functions in L^p for $1 < p < \infty$. Square functions and variational inequalities have been

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fruitful in martingale theory and harmonic analysis, with applications to Littlewood–Paley theory and in ergodic theory. They were instrumental in bounding associated maximal functions, establishing convergence of sequences of operators and controlling their rate of convergence.

Definition 1.1. Let T be a linear operator on a Banach space such that it is (doubly) power-bounded. If $\mathrm{supp}(\nu) \subset \mathbb{N}_0$, assume that T is power-bounded, that is, $\sup_{n\geq 0}\|T^n\| < \infty$. Otherwise assume T is invertible and doubly power-bounded: $\sup_{n\in\mathbb{Z}}\|T^n\| < \infty$. Let ν be a finite signed measure on $\mathbb{Z}$. Define the operator induced by ν as

$$T_\nu f = \sum_k \nu(k) T^k f. \tag{1}$$

Given $0 < a < 1$, the series expansion of $(1-x)^a = 1 - \sum_{k\geq 1} g(a,k)x^k$, $|x| \leq 1$, yields a probability measure ν_a on $\mathbb{N}$ with $\nu_a(1) = a$ and

$$\nu_a(k) = g(a,k) = \frac{a|a-1|\cdots|a-k+1|}{k!} \geq 0, \quad k > 1. \tag{2}$$

Definition 1.2. Let T be a power-bounded linear operator on a Banach space. For $0 < a < 1$, define $(I-T)^a$ as

$$(I-T)^a = I - T_{\nu_a} = I - \sum_{k\geq 1} g(a,k)T^k.$$

The operators T_{ν_a} have been considered by many authors. The first systematic study of their properties is due to Derrienic and Lin [8].

Decomposing any real $r > 0$ into its integer and fractional components allows us to extend the definition of $(I-T)^r$ for all real $r > 0$. In particular, if μ is a finite measure on $\mathbb{Z}$, $T_\mu^n (I-T_\mu)^r f = T_{\nu_{n,r}} f$ where $\nu_{n,r}$ is a signed measure on $\mathbb{Z}$ satisfying $\hat{\nu}_{n,r}(t) = \hat{\mu}^n(t)(1-\hat{\mu}(t))^r$.

Definition 1.3. For real numbers $s \geq 1, r \geq 0$, define generalized square functions as follows:

$$\mathbf{Q}_{\alpha,s,r} f = \mathbf{Q}^T_{\alpha,s,r} f = \left(\sum_{n=1}^{\infty} n^\alpha |T^n (I-T)^r f|^s \right)^{1/s}.$$

Definition 1.4. Let T be a linear operator on a Banach space. T is a Ritt operator if T is power-bounded and $\sup_n n\|T^n(1-T)\| < \infty$.

For any sequence of complex numbers $\{x_n\}$, the s-variational norm is defined as

$$\|\{x_n\}\|_{v(s)} = \sup \left(\sum_k |x_{n_k} - x_{n_{k+1}}|^s \right)^{1/s},$$

where the sup is taken over all possible increasing sequences $\{n_k\}$.

Given any fixed sequence of increasing integers $\{n_k\}$, the s-oscillation norm is defined as

$$\|\{x_n\}\|_{o(s)} = \left(\sum_k \max_{n_k \leq n,m \leq n_{k+1}} |x_n - x_m|^s \right)^{1/s}.$$

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Neither of these are proper norms, but they are seminorms. We refer the reader to [13, 15] for discussion of their properties.

The works of Le Merdy and Xu [17], Arhancet and Le Merdy [2], and Cuny, Cohen and Lin [5], established the following results for Ritt operators on L^p, $1 < p < \infty$.

THEOREM 1.5. *Let $(X, \mathcal{B}, m)$ be a σ-finite measure space, $1 < p < \infty$, and T a positive contraction of $L^p(X)$. If $\sup_n n\|T^n - T^{n+1}\| < \infty$, then, for any fixed real number $r > 0$:*

(a) *for $s \geq 2$, $\mathbf{Q}_{2r-1,s,r} f$ is bounded on L^p;*
(b) *for $s > 2$, $\|\{T^n f\}\|_{v(s)}$ and, for any increasing sequence $\{n_k\}$, $\|\{T^n f\}\|_{o(2)}$, are bounded on L^p; and*
(c) *for $s \geq 2$, $\|\{n^r T^n (I - T)^r f\}\|_{v(s)}$ is bounded on L^p.*
(d) *Let $\{n_k\} \subset \mathbb{N}$ be an increasing sequence with $n_{k+1} - n_k \sim n_k$. Then $(\sum_k \max_{n_k \leq n \leq n_{k+1}} n^{2r} |T^n (I - T)^r|^2)^{1/2}$ is bounded on L^p.*

Note that Le Merdy and Xu [17] proved part (a) for integer r and $s = 2$, but since $(\sum_k |a_k|^s)^{1/s} \leq (\sum_k |a_k|^2)^{1/2}$ if $s \geq 2$, (a) also holds for $s > 2$. Arhancet and Le Merdy [2] subsequently showed that any pair of norms of the form $\|\mathbf{Q}_{2r-1,2,r} f\|_p$ (with $(r > 0)$ are equivalent in L^p, $1 < p < \infty$. Hence, (a) holds for any $r > 0$. Additionally, Le Merdy and Xu [17] proved (b) and that, for $r \geq 1$ integer, $\|\{n^r T^n (I - T)^r f\}\|_{v(s)}$ for $s > 2$; and $\|\{n^r T^n (I - T)^r f\}\|_{o(2)}$ are bounded on L^p. Cuny, Cohen and Lin [5] strengthened these results by establishing that $\|\{n^r T^n (I - T)^r f\}\|_{v(2)}$ is bounded on L^p, $r > 0$. Their theorem focused on $0 < r < 1$, but their methods apply to any $r > 0$. They also showed item (d) for $n_k = 2^k$, but their proof applies to sequences with $n_{k+1} - n_k \sim n_k$ with minor modifications.

In martingale theory, the equivalence between the square function and associated maximal functions is a classical result. This naturally raises the question of whether such an equivalence holds in other contexts. Cohen, Cuny and Lin [5] established that a similar equivalence holds in the context of Ritt operators.

THEOREM 1.6. *Let $(X, \mathcal{B}, m)$ be a σ-finite measure space, $1 < p < \infty$, and T a positive contraction on $L^p(X, m)$. Then the following assertions are equivalent:*

(a) $\sup_n n\|T^n - T^{n+1}\| < \infty$;
(b) *there exists a constant $C_p > 0$ such that $\|\mathbf{Q}_{1,2,1} f\|_p \leq C_p \|f\|_p$.*

Let $X = (X, \beta, m)$ be a σ-finite measure space. Throughout these notes, Y denotes a Banach function space, that is, a Banach space of measurable complex-valued functions on X. In particular, this class includes $L^1(X)$ and Orlicz spaces.

THEOREM 1.7. *Let T be a Ritt operator on a Banach function space Y. Let $s \geq 1$ and $r > 0$. Then the following assertions hold.*

(a) *If $sr > \alpha + 1$, then $\mathbf{Q}^T_{\alpha,s,r} f$ is bounded on Y.*
(b) *If $\beta < r$ and $\{n_k\} \subset \mathbb{N}$ is an increasing sequence with $n_{k+1} - n_k \sim n_k$, then $\sup_n n^\beta |T^n (I - T)^r f|$ and $(\sum_k \sup_{n_k < n < n_{k+1}} n^{\beta s} |T^n (I - T)^r f|^s)^{1/s}$ are bounded on Y, $\lim_{n\to\infty} n^r \|T^n (I - T)^r f\| = 0$ and $\lim_{n\to\infty} n^\beta |T^n (I - T)^r f| = 0$ almost everywhere.*

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(c) *If $\beta < r$ and $s \geq 1$ then both $\|n^\beta T^n (I-T)^r f\|_{v(s)}$ and $\|n^\beta T^n (I-T)^r f\|_{o(s)}$ are bounded on Y.*

(d) *Let $\{n_k\}$ be any increasing sequence with $n_{k+1} - n_k \sim n_k^\gamma$, for some $\gamma \in (0,1]$. If $\beta < r + (1-\gamma)(1-1/s)$ then $(\sum_k n_k^{\beta s} \max_{n_k \leq n,m \leq n_{k+1}} |(T^n - T^m)(I-T)^r f|^s)^{1/s}$ and $(\sum_k n_k^{\beta s} |(T^{n_k} - T^{n_{k+1}})(I-T)^r f|^s)^{1/s}$ are bounded on Y. In particular, these results hold for $\beta < r$ and $s \geq 1$ or $\beta \leq r > 0$ and $s > 1 > \gamma$.*

Remark. By Theorem 1.6, $\mathbf{Q}_{1,2,1} f$ is bounded on L^p for any $p > 1$, but on L^1, $\mathbf{Q}_{1,s,1} f$ is bounded for $s > 2$, and in general, $\mathbf{Q}_{r,s,r} f$ is bounded on L^1 for $s > 1 + 1/r$. More generally, by Theorem 1.5, $\mathbf{Q}_{2r-1,2,r} f$ is bounded on L^p for any $p > 1$, but on L^1, $\mathbf{Q}_{2r-1,s,r} f$ bounded for any $s > 2$, and $\mathbf{Q}_{\alpha,2,r} f$ is bounded for any $\alpha < 2r - 1$.

Open Question 1. Is $\mathbf{Q}_{\alpha,s,r} f$ of weak type $(1,1)$ for $sr = \alpha + 1$?

Open Question 2. For what range of s are $\|T^n f\|_{v(s)}$ and $\|T^n f\|_{o(s)}$ of weak type $(1,1)$? And in general, for what range of s are $\|n^r T^n (I-T)^r f\|_{v(s)}$ and $\|n^r T^n (I-T)^r f\|_{o(s)}$ of weak type $(1,1)$?

Estimates in L^1 present more challenges. For the maximal function $Mf = \sup_{n \geq 1} |T^n f|$ only partial results are available, showing that Mf is a weak $(1,1)$ operator for a restricted class of operators acting on probability spaces [3, 7, 24]. In what follows (and in §5), we provide answers for the above questions under assumptions analogous to those imposed in the study of Mf.

For the rest of this section $X = (X, \mathcal{B}, m)$ denotes a probability space and τ an invertible measure-preserving transformation on X. T is the composition operator $Tf = f \circ \tau$. To distinguish this case from the general setting, we denote

$$\tau_\mu f(x) = T_\mu f(x) = \sum_k \mu(k) f(\tau^k x). \tag{3}$$

We say that μ has *bounded angular ratio* (BA) if there exists a constant $C > 0$ such that $|1 - \hat{\mu}(t)| \leq C(1 - |\hat{\mu}(t)|)$ for all $|t| \leq 1$, where $\hat{\mu}(t) = \sum_k \mu(k) e^{-2\pi i k t}$.

Bellow and Calderón [3] showed that the maximal function $M_\mu f = \sup_{n \geq 1} |\tau_\mu^n f|$ is weak $(1,1)$ for centered measures μ with finite second moment. Such measures have the BA property. Further extensions of this result, relaxing the moment condition, were obtained by Dungey [9] in 2011, Wedrychowicz [29] also in 2011, and Cuny [7] in 2016, while in 1999 and 2001 Losert [19, 20] constructed measures μ without the BA property, for which pointwise convergence of τ_μ^n failed.

In [7], Cuny introduced a modification of the BA property (see properties BA_1 and BA_2 in §2) to obtain the following L^1 results.

THEOREM 1.8. [7] *Let $(X, \mathcal{B}, m)$ be a probability space and $\tau : X \to X$ an invertible measure-preserving transformation. Let μ be a probability measure on $\mathbb{Z}$ with property BA_1. Then:*

(a) *the maximal function $\sup_n |\tau_\mu^n f|$ is of weak type $(1,1)$; and*

(b) *for $r \in \mathbb{N}$, $m(x \in X : \sup_n n^r |(\tau_\mu^n - \tau_\mu^{n+r}) f(x)| > \lambda) \leq (C_r/\lambda)\|f\|_1$ for any $f \in L^1(X)$.*

(c) *If μ satisfies BA_2, then τ_μ is Ritt on L^1.*

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Strengthening the control over the fluctuations of τ_μ^n, we establish weak (1, 1) results for generalized square functions and variation-type operators.

THEOREM 1.9. *Let $(X, \mathcal{B}, m)$ be a standard probability space, $\tau : X \to X$ an invertible measure-preserving transformation. Let μ be a probability measure on $\mathbb{Z}$ with property BA_2. Then, for any $r \geq 0$, the following assertions hold.*

(A) *If $sr \geq \alpha + 1$ and $\mathbf{Q}^{\tau_\mu}_{\alpha,s,r} f$ is bounded on L^2, then it is of weak type (1, 1).*

(B) *Let $s > 1$ and $0 \leq \beta \leq r$. If $\|n^\beta \tau_\mu^n (I - \tau_\mu)^r f\|_{v(s)}$ is bounded on L^2, then it is of weak type (1, 1), and if $\|n^\beta \tau_\mu^n (I - \tau_\mu)^r f\|_{o(s)}$ is bounded on L^2, then it is of weak type (1, 1).*

THEOREM 1.10. *Let τ, μ be as in Theorem 1.9 and $r > 0$. Then the following assertions hold.*

(a) *For all $s \geq 2$, $\mathbf{Q}^{\tau_\mu}_{s(r-1/2),s,r} f$ is of weak type (1, 1).*

(b) *For all $s > 2$, $\tau_\mu^n f\|_{v(s)}$ is of weak type (1, 1); and $\|\tau_\mu^n f\|_{o(2)}$ is of weak type (1, 1).*

(c) *More generally, $\|n^r \tau_\mu^n (I - \tau_\mu)^r f\|_{v(s)}$ is of weak type (1, 1) for all $s \geq 2$; and $\|n^r \tau_\mu^n (I - \tau_\mu)^r f\|_{o(2)}$ is of weak type (1, 1).*

(d) *Let $\{n_k\} \subset \mathbb{N}$ be an increasing sequence with $n_{k+1} - n_k \sim n_k$, then $(\sum_k \max_{n_k \leq n \leq n_{k+1}} n^{sr} |\tau_\mu^n (I - \tau_\mu)^r f|^s)^{1/s}$ is of weak type (1, 1) for $s \geq 2$.*

Items (a), (b) and (c) of Theorem 1.10 are a straightforward application of Theorems 1.5 and 1.9. Item (d) is proven in §4.

This paper is organized as follows. Section 2 introduces the general framework and develops the main results for the special case of T_μ. Section 3 presents a key technical lemma, central to the proofs of the general case. Section 4 provides detailed proofs for operators of the form T_μ, and concludes with the proof of Theorem 1.7. Section 5 establishes weak type (1, 1) bounds in Theorem 1.10.

2. *The case of T_μ*

We begin this section with a brief discussion of sufficient conditions under which T_μ is a Ritt operator on L^1. For an L^1–L^∞ contraction, Blunck [4] showed, via a clever interpolation argument, that if T is Ritt on L^2, it is also Ritt on L^p, $1 < p < \infty$. This interpolation technique does not extend to the case $p = 1$. However, for the special case of T_μ, Dungey [9] showed T_μ is Ritt on L^1 provided μ satisfies a set of stronger regularity conditions than BA.

THEOREM 2.1. (Dungey [9, Theorem 4.1]) *Let μ be a probability measure on $\mathbb{N}_0$ such that there exist constants $0 < \alpha, c < 1$ with (i) $\operatorname{Re} \hat{\mu}(t) \leq 1 - c|t|^\alpha$, and (ii) $|\hat{\mu}'(t)| \lesssim |t|^{\alpha-1}$ for $0 < |t| \leq 1/2$. If $T \in (L^1)$ is power-bounded, then T_μ is Ritt on L^1.*

Inspired by [9] and also [29], Cuny [7] considered broader spectral conditions, referred to below as properties BA_1 and BA_2, for measures supported on $\mathbb{Z}$ and showed these suffice to guarantee that τ_μ is Ritt on L^1 in a broader setting.

Definition 2.2. Let μ be a probability measure on $\mathbb{Z}$ whose Fourier transform $\hat{\mu}(t)$ is twice continuously differentiable on $|t| \in (0, 1)$. We say μ has property BA_1 if there

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exists a continuous function $h(t)$ on $|t| \le 1$ with $h(0) = 0$, $h(-t) = h(t)$, continuously differentiable on $|t| \in (0, 1)$ satisfying the following conditions: there exist some constants $c, C > 0$ such that (i) $|\hat{\mu}(t)| \le 1 - ch(t)$, (ii) $|t\hat{\mu}'(t)| \le Ch(t)$, (iii) $|\hat{\mu}'(t)| \le Ch'(t)$, and (iv) $|t\hat{\mu}''(t)| \le C|\hat{\mu}'(t)|$. If, in addition, $\hat{\mu}$ satisfies (v) $h(t) \le Cth'(t)$ for $0 < t < 1$, then we say μ satisfies condition BA_2.

Note that BA_1 (iii) implies bounded angular ratio

$$|1 - \hat{\mu}(t)| \le \left| \int_0^t \hat{\mu}'(u)\, du \right| \lesssim \int_0^{|t|} h'(u)\, du = h(t) \lesssim 1 - |\hat{\mu}(t)|.$$

Centered measures with finite second moment satisfy property BA_2 with $h(t) = t^2$. The next example exhibits a non-centered measure without finite first moment that still satisfies BA_2.

Example 2.3. For fixed $0 < a < 1$, let ν_a be the probability measure on $\mathbb{Z}$ defined in (2). Then $\hat{\nu}_a(t) = 1 - (1 - e^{2\pi it})^a$. From [9], ν_a satisfies the conditions of Theorem 2.1 and hence $T_{\nu_a} = I - (I - T)^a$ is a Ritt operator on L^1. Additionally, ν_a satisfies property BA_2.

More examples of measures with property BA_2 can be found in Cuny [7].

For general Banach function spaces, including L^1, we obtain the following results.

THEOREM 2.4. *Let μ be as in Theorem 1.8 and Y be a Banach function space. Let T be a (doubly) power-bounded linear operator on Y. If $r > 0$ and $sr > \alpha + 1$, then $\mathbf{Q}^{T_\mu}_{\alpha,s,r} f$ is bounded on Y.*

COROLLARY 2.5. *Let μ, T and Y be as in Theorem 2.4 and $\{n_k\} \subset \mathbb{N}$ an increasing sequence with $n_{k+1} - n_k \sim n_k$. If $0 \le \beta < r$ and $s \ge 1$, $\sup_n n^\beta |T_\mu^n (I - T_\mu)^r f|$ and $(\sum_k \max_{n_k \le n \le n_{k+1}} n^{\beta s} |T_\mu^n (I - T_\mu)^r f|^s)^{1/s}$ are both bounded on Y, $\lim_{n\to\infty} n^r \|T_\mu^n (I - T_\mu)^r f\| = 0$ and $\lim_{n\to\infty} n^\beta |T_\mu^n (I - T_\mu)^r f| = 0$ almost everywhere.*

PROPOSITION 2.6. *Let μ, T and Y be as in Theorem 2.4. Let $\{n_k\}$ be an increasing sequence with $n_{k+1} - n_k \sim n_k^\gamma$ for some $0 < \gamma \le 1$. Let $s \ge 1$, $r \ge 0$. If $\beta < r + (1 - \gamma)(1 - 1/s)$, then $(\sum_k n_k^{\beta s} \max_{n_k \le n, m \le n_{k+1}} |(T_\mu^n - T_\mu^m)(I - T_\mu)^r f|^s)^{1/s}$ and $(\sum_k n_k^{\beta s} |(T_\mu^{n_k} - T_\mu^{n_{k+1}})(I - T_\mu)^r f|^s)^{1/s}$ are bounded on Y.*

PROPOSITION 2.7. *Let μ, T and Y be as in Theorem 2.4. Let $r > 0$. If $\beta < r$ and $s \ge 1$, then both $\|n^\beta T_\mu^n (I - T_\mu)^r f\|_{v(s)}$ and $\|n^\beta T_\mu^n (I - T_\mu)^r f\|_{o(s)}$ are bounded on Y.*

3. *Auxiliary lemma*

Across these notes, c and C denote constants whose values may change from one instance to the next, and $e(x) = e^{2\pi ix}$. For $0 \le x, y$, we say $x \lesssim y$ if there exists a constant $c > 0$ such that $x \le cy$, and $x \sim y$ if $x \lesssim y$ and $y \lesssim x$.

For completeness, we recall the Stein–Minkowski integral inequality [27]. Suppose that (X_1, ν_1) and (X_2, ν_2) are two σ-finite measure spaces and $g : X_1 \times X_2 \to \mathbb{R}$ is measurable. Then, for any $1 \le s < \infty$,

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$$\left[\int_{X_2}\left|\int_{X_1} g(x_1,x_2)\,d\nu_1(x_1)\right|^s d\nu_2(x_2)\right]^{1/s} \le \int_{X_1}\left(\int_{X_2}|g(x_1,x_2)|^s\,d\nu_2(x_2)\right)^{1/s} d\nu_1(x_1).$$

LEMMA 3.1. *Let $S=\{\Delta_n\}$ be a sequence of (finite) signed measures on $\mathbb{Z}$ whose Fourier transforms $\hat{\Delta}_n$ are twice continuously differentiable on $|t|\in(0,1)$. Let T be a linear (doubly) power-bounded operator on a Banach function space, and let*

$$A=\int_{|t|<1/2}\frac{1}{|t|}\Big(\sum_n|\hat{\Delta}_n(t)|^s\Big)^{1/s}dt, \qquad B=\int_{|t|<1/2}|t|\Big(\sum_n|\hat{\Delta}_n''(t)|^s\Big)^{1/s}dt,$$

$$C=\sum_{k\neq0}\frac{1}{|k|}\Big(\sum_n|\hat{\Delta}_n(1/|2k|)|^s\Big)^{1/s}, \qquad D=\sum_{k\neq0}\frac{1}{|k|^2}\Big(\sum_n|\hat{\Delta}_n'(1/|2k|)|^s\Big)^{1/s}.$$

If A, B, C and D are all finite, then, for any $f\in Y$, $\|(\sum_n|T_{\Delta_n}f|^s)^{1/s}\|\lesssim\|f\|$.

Proof. Without loss of generality, we assume $\sup_{n\ge0}\|T^n\|=1$ if $\bigcup_n\operatorname{supp}(\Delta_n)\subset\mathbb{N}_0$ and assume $\sup_{n\in\mathbb{Z}}\|T^n\|=1$ otherwise. We have

$$\begin{aligned}\Big\|\Big(\sum_n|T_{\Delta_n}f|^s\Big)^{1/s}\Big\| &= \Big\|\Big(\sum_n\Big|\sum_k\Delta_n(k)T^kf\Big|^s\Big)^{1/s}\Big\|\\ &\lesssim \Big\|\Big(\sum_n\Big|\sum_{k\neq0}\int_{|t|<1/|2k|}\hat{\Delta}_n(t)e(kt)\,dt\;T^kf\Big|^s\Big)^{1/s}\Big\|\\ &\quad+\Big\|\Big(\sum_n\Big|\sum_{k\neq0}\int_{1/|2k|<|t|<1/2}\hat{\Delta}_n(t)e(kt)\,dt\;T^kf\Big|^s\Big)^{1/s}\Big\|\\ &\quad+\Big(\sum_n|\Delta_n(0)|^s\Big)^{1/s}\|f\|\\ &= \mathrm{I}+\mathrm{II}+\Big(\sum_n|\Delta_n(0)|^s\Big)^{1/s}\|f\|.\end{aligned}$$

Applying the Stein–Minkowski integral inequality, we obtain the following estimate for the first term:

$$\begin{aligned}\mathrm{I} &= \Big\|\Big(\sum_n\Big|\sum_{k\neq0}\int_{|t|<1/|2k|}\hat{\Delta}_n(t)e(kt)\,dt\;T^kf\Big|^s\Big)^{1/s}\Big\|\\ &\le\|f\|\sum_{k\neq0}\Big(\sum_n\Big|\int_{|t|<1/|2k|}\hat{\Delta}_n(t)e(kt)\,dt\Big|^s\Big)^{1/s}\\ &\le\|f\|\int_{|t|<1/2}\frac{1}{|t|}\Big(\sum_n|\hat{\Delta}_n(t)|^s\Big)^{1/s}dt=A\|f\|.\end{aligned}$$

Also

$$\Big(\sum_n|\Delta_n(0)|^s\Big)^{1/s}\le\int_{|t|<1/2}\Big(\sum_n|\hat{\Delta}_n(t)|^s\Big)^{1/s}dt\le A.$$

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For the second term, by the Stein–Minkowski integral inequality, we have

$$\begin{aligned}\mathrm{II} &\le \left\| \left(\sum_n \left| \sum_{k\neq 0} \int_{1/|2k|<|t|<1/2} \hat{\Delta}_n(t) e(kt)\, dt\, T^k f \right|^s \right)^{1/s} \right\| \\ &\le \left\| \sum_{k\neq 0} \left(\sum_n \left| \int_{1/|2k|<|t|<1/2} \hat{\Delta}_n(t) e(kt)\, dt \right|^s \right)^{1/s} |T^k f| \right\| \\ &\le \|f\| \sum_{k\neq 0} \left(\sum_n \left| \int_{1/|2k|<|t|<1/2} \hat{\Delta}_n(t) e(kt)\, dt \right|^s \right)^{1/s}.\end{aligned}$$

Since $\Delta_n(t)$, $\Delta_n'(t)$ and $e(kt)$ are 1-periodic, $\Delta_n(-t) = \overline{\Delta_n(t)}$, and $\Delta_n'(-t) = \overline{\Delta_n'(t)}$, we estimate

$$\begin{aligned}\left| \int_{1/|2k|<|t|<1/2} \hat{\Delta}_n(t) e(kt)\, dt \right| &\le \left| \int_{1/|2k|<|t|<1/2} \hat{\Delta}_n'(t) \frac{e(kt)}{2\pi k}\, dt \right| \\ &\quad + \left| \frac{\hat{\Delta}_n(1/|2k|) e(k/|2k|)}{2\pi k} - \frac{\hat{\Delta}_n(-1/|2k|) e(-k/|2k|)}{2\pi k} \right| \\ &\lesssim \left| \int_{1/|2k|<|t|<1/2} \hat{\Delta}_n''(t) \frac{e(kt)}{4\pi^2 k^2}\, dt \right| + \left| \frac{\hat{\Delta}_n(1/|2k|)}{2\pi k} \right| + \left| \frac{\hat{\Delta}_n'(1/|2k|)}{4\pi^2 k^2} \right|. \qquad (4)\end{aligned}$$

By the Stein–Minkowski integral inequality,

$$\begin{aligned}\sum_{k\neq 0} \left(\sum_n \left| \int_{1/|2k|<|t|<1/2} \hat{\Delta}_n''(t) \frac{e(kt)}{4\pi^2 k^2}\, dt \right|^s \right)^{1/s} &\lesssim \sum_{k\neq 0} \frac{1}{k^2} \int_{1/|2k|<|t|<1/2} \left(\sum_n \left| \hat{\Delta}_n''(t) \right|^s \right)^{1/s} dt \\ &\lesssim \int_{0<|t|<1/2} |t| \left(\sum_n \left| \hat{\Delta}_n''(t) \right|^s \right)^{1/s} dt = B.\end{aligned}$$

Thus, $\mathrm{II} \lesssim (B + C + D)\|f\|$. □

The proof of this lemma can be adapted to the following setting.

LEMMA 3.2. *Let Δ_n and T be as in Lemma 3.1, $\{n_k\}$ an increasing sequence, and $J_k = [n_k, n_{k+1})$. Let*

$$A = \int_{|t|<1/2} \frac{1}{|t|} \left(\sum_k n_k^\beta \max_{n\in J_k} |\hat{\Delta}_n(t)|^s \right)^{1/s} dt, \quad B = \int_{|t|<1/2} |t| \left(\sum_k n_k^\beta \max_{n\in J_k} |\hat{\Delta}_n''(t)|^s \right)^{1/s} dt,$$

$$C = \sum_{l\neq 0} \frac{1}{|l|} \left(\sum_k n_k^\beta \max_{n\in J_k} |\hat{\Delta}_n(1/|2l|)|^s \right)^{1/s}, \qquad D = \sum_{l\neq 0} \frac{1}{|l|^2} \left(\sum_k n_k^\beta \max_{n\in J_k} |\hat{\Delta}_n'(1/|2l|)|^s \right)^{1/s}.$$

If A, B, C and D are all finite, then, for any $f \in Y$, $\|(\sum_k n_k^\beta \max_{n\in I_k} |T_{\Delta_n} f|^s)^{1/s}\| \lesssim \|f\|$.

4. *Proofs of results for T_μ and Theorem 1.7*

Proof of Theorem 2.4. Let Δ_n be the measure on the integers defined by

$$\hat{\Delta}_n = n^{\alpha/s} \hat{\mu}^n (1 - \hat{\mu})^r,$$

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that is, $T_{\Delta_n} = n^{\alpha/s} T_\mu^n (I - T_\mu)^r$. Using Lemma 3.1 with Δ_n, it suffices to show that the corresponding terms defined in the lemma, A, B, C, D, are bounded.

We assume $\alpha \geq -1$ because, if $\alpha < -1$, $\mathbf{Q}_{\alpha,s,r} f \leq \mathbf{Q}_{-1,s,r} f$.

If $\alpha > -1$, dominating the sum by an integral and using property BA_1 (i), we obtain

$$\sum_n n^\alpha |\hat{\mu}(t)|^{ns} \leq \sum_n n^\alpha (1 - ch(t))^{ns} \lesssim \frac{1}{h(t)^{\alpha+1}}.$$

If $\alpha = -1$,

$$\sum_n \frac{1}{n} |\hat{\mu}(t)|^{ns} \leq \sum_n \frac{1}{n} (1 - ch(t))^{ns} = |\ln(1 - (1 - ch(t))^s)| \lesssim \frac{1}{h(t)^\gamma}$$

for any $\gamma > 0$.

For term A, using properties BA_2, we estimate

$$\begin{aligned} A &= \int_{|t|<1/2} \frac{1}{|t|} \Big(\sum_n \Big| \hat{\Delta}_n(t) \Big|^s \Big)^{1/s} dt \\ &= \int_{|t|<1/2} \Big(\sum_n n^\alpha |\hat{\mu}(t)|^{ns} \Big)^{1/s} \frac{|1 - \hat{\mu}(t)|^r}{|t|} \, dt \\ &\lesssim \begin{cases} \displaystyle\int_{0<t<1/2} h(t)^{r-(\alpha+1)/s-1} \, h'(t) \, dt & \text{for } \alpha > -1, \\ \displaystyle\int_{0<t<1/2} h(t)^{r-\gamma/s-1} \, h'(t) \, dt & \text{for } \alpha = -1. \end{cases} \end{aligned} \tag{5}$$

When $\alpha > -1$ the integral is finite for $sr > \alpha + 1$, and when $\alpha = -1$ the integral is finite for $r > 0$ since we can choose γ arbitrarily small, say $\gamma = sr/2$.

With $B_n = T^n (I - T)^r$,

$$|\hat{B}_n'(t)| = |n\hat{\mu}^{n-1}(t)(\hat{\mu}'(t))(1 - \hat{\mu}(t))^r - \hat{\mu}^n(t)\hat{\mu}'(t)(1 - \hat{\mu}(t))^{r-1}|, \tag{6}$$

and by BA_1,

$$\begin{aligned} |\hat{B}_n''(t)| &= |n(n-1)\hat{\mu}^{n-2}(t)(\hat{\mu}'(t))^2(1 - \hat{\mu}(t))^r \\ &\quad + n\hat{\mu}^{n-1}(t)\hat{\mu}''(t)(1 - \hat{\mu}(t))^r - 2nr\hat{\mu}^{n-1}(t)(\hat{\mu}'(t))^2(1 - \hat{\mu}(t))^{r-1} \\ &\quad + r(r-1)\hat{\mu}^n(t)(1 - \hat{\mu}(t))^{r-2}(\hat{\mu}'(t))^2 - r\hat{\mu}^n(t)(1 - \hat{\mu}(t))^{r-1}\hat{\mu}''(t)| \\ &\lesssim [n^2|\hat{\mu}^{n-2}(t)|h^2(t) + n|\hat{\mu}^{n-1}(t)|h(t) + |\hat{\mu}^n(t)|] \frac{h^{r-1}(t)|\hat{\mu}'(t)|}{|t|}. \end{aligned}$$

Thus, for $\alpha > -1$,

$$\Big(\sum_n |\hat{\Delta}_n''(t)|^s \Big)^{1/s} = \Big(\sum_n n^\alpha |\hat{B}_n''(t)|^s \Big)^{1/s} \lesssim h(t)^{(r-1)-(\alpha+1)/s} \frac{|\hat{\mu}'(t)|}{|t|}. \tag{7}$$

Since $(1 + \alpha) < sr$,

$$B = \int_{|t|<1/2} |t| \Big(\sum_n |\hat{\Delta}_n''(t)|^s \Big)^{1/s} dt \lesssim \int_{0<t<1/2} h(t)^{(r-1)-(\alpha+1)/s} \, h'(t) \, dt < \infty.$$

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When $\alpha = -1$, choosing $0 < \gamma < r$, the estimate is

$$\int_{|t|<1/2} |t| \left(\sum_n |\hat{\Delta}_n''(t)|^s \right)^{1/s} dt \lesssim \int_{0<t<1/2} h(t)^{r-\gamma-1} h'(t)\, dt < \infty.$$

For the remaining terms, we address the case $\alpha > -1$ since the estimates for the case $\alpha = -1$ follow similar arguments.

For $(1+\alpha) < sr$, by BA_2 and dominating the kth term by an integral over intervals $(1/k+1, 1/k)$, we have

$$\begin{aligned} C &= \sum_{k\neq 0} \frac{1}{|k|} \left(\sum_n |\hat{\Delta}_n(1/|2k|)|^s \right)^{1/s} \\ &\lesssim \sum_{k\neq 0} \frac{1}{|k|} \left(\sum_n n^\alpha |\hat{\mu}(1/|2k|)|^{ns} |1-\hat{\mu}(1/|2k|)|^{sr} \right)^{1/s} \\ &\lesssim \sum_{k>0} \frac{1}{k} h(1/2k)^{r-(\alpha+1)/s} \lesssim c + \int_0^{1/2} \frac{h(t)^{r-(\alpha+1)/s}}{t}\, dt \\ &< c + \int_0^{1/2} h(t)^{r-(\alpha+1)/s-1} h'(t)\, dt < \infty. \end{aligned}$$

From (6) and BA_1(ii),

$$\left(\sum_n |\hat{\Delta}_n'(t)|^s \right)^{1/s} = \left(\sum_n n^\alpha |\hat{B}_n'(t)|^s \right)^{1/s} \lesssim h(t)^{r-1-(\alpha+1)/s} |\mu'(t)| \lesssim \frac{h(t)^{r-(\alpha+1)/s}}{|t|}. \tag{8}$$

Thus,

$$D = \sum_{k\neq 0} \frac{1}{|k|^2} \left(\sum_n |\hat{\Delta}_n'(1/|2k|)|^s \right)^{1/s} \lesssim \sum_{k\neq 0} \frac{1}{|k|} h(1/2k)^{r-(\alpha+1)/s} < \infty. \qquad \square$$

Proof of Corollary 2.5. Let $\{n_k\}$ be an increasing sequence in $\mathbb{N}$ such that $n_{k+1} - n_k \sim n_k$. Then we have $n_{k+1} = n_{k+1} - n_k + n_k \lesssim 2n_k$.

Applying Abel's summation,

$$\begin{aligned} \sum_{j=1}^u j^\beta T_\mu^{j-1}(1-T_\mu) f &= \sum_{j=1}^u j^\beta T_\mu^{j-1} - \sum_{j=1}^u j^\beta T_\mu^j f = \sum_{j=0}^{u-1} (j+1)^\beta T_\mu^j f - \sum_{j=1}^u j^\beta T_\mu^j f \\ &= \sum_{j=0}^{u-1} ((j+1)^\beta - j^\beta) T_\mu^j f - u^\beta T_\mu^u f \\ &= \sum_{j=1}^{u} (j^\beta - (j-1)^\beta) T_\mu^{j-1} f - u^\beta T_\mu^u f. \end{aligned}$$

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Thus,

$$u^\beta |T_\mu^u f| \le \sum_{j=1}^{u} (j^\beta - (j-1)^\beta) |T_\mu^{j-1} f| + \sum_{j=1}^{u} j^\beta |T_\mu^{j-1}(1-T_\mu) f| \\ \lesssim \sum_{j=1}^{u} j^{\beta-1} |T_\mu^{j-1} f| + \sum_{j=1}^{u} j^\beta |T_\mu^{j-1}(1-T_\mu) f|. \tag{9}$$

Decomposing $n \in (n_k, n_{k+1}]$ as $n = n_{k-1} + l$ with $l \in (n_k - n_{k-1}, n_{k+1} - n_{k-1}]$, and applying the above to the function $T_\mu^{n_{k-1}}(1-T_\mu)^r f$ with $u = l$,

$$n^\beta |T_\mu^n (1-T_\mu)^r f| \lesssim \frac{n^\beta}{l^\beta} \left[\sum_{j=1}^{l} j^{\beta-1} |T_\mu^{j+n_{k-1}-1}(1-T_\mu)^r f| + \sum_{j=1}^{l} j^\beta |T_\mu^{j+n_{k-1}-1}(1-T_\mu)^{r+1} f| \right] \\ \lesssim \sum_{j=n_{k-1}}^{n_{k+1}-1} j^{\beta-1} |T_\mu^{j}(1-T_\mu)^r f| + \sum_{j=n_{k-1}}^{n_{k+1}-1} j^\beta |T_\mu^{j}(1-T_\mu)^{r+1} f|$$

because $n/l \le (n_{k+1}/n_k - n_{k-1}) \lesssim (2n_k/n_{k-1}) \lesssim 4$. Thus,

$$\left(\sum_k \max_{n_k < n \le n_{k+1}} n^{\beta s} |T_\mu^n (I-T_\mu)^r f|^s \right)^{1/s} \le \sum_k \max_{n_k < n \le n_{k+1}} n^\beta |T_\mu^n (I-T_\mu)^r f| \\ \lesssim \sum_k \sum_{j=n_{k-1}}^{n_{k+1}-1} j^{\beta-1} |T_\mu^j (1-T_\mu)^r f| + \sum_k \sum_{j=n_{k-1}}^{n_{k+1}-1} j^\beta |T_\mu^j (1-T_\mu)^{r+1} f| \\ \lesssim \mathbf{Q}_{\beta-1,1,r} f + \mathbf{Q}_{\beta,1,r+1} f.$$

By Theorem 2.4, both generalized square functions are bounded in Y when $0 \le \beta < r$. Therefore $\sup_n n^\beta |T_\mu^n (I-T_\mu)^r f|$ is also bounded in Y.

By [5], $\lim_{n\to\infty} n^r \|T_\mu^n (I-T_\mu)^r f\| = 0$.

From these results, it follows that $n^\beta |T_\mu^n (I-T_\mu)^r f| \to 0$ almost everywhere as well. □

Note that in [5, Proposition 6.4], it was shown that

$$\left(\sum_k \max_{n_k < n \le n_{k+1}} n^{2\beta} |T_\mu^n (I-T_\mu)^r f|^2 \right)^{1/2} \lesssim \mathbf{Q}_{2\beta-1,2,r} f + \mathbf{Q}_{2(\beta+1)-1,2,r+1} f.$$

This result holds for general $s > 1$. Let $s' = s/(s-1)$. From equation (9) with $\beta = 1$,

$$|T_\mu^u f| \lesssim \frac{1}{u} \sum_{j=1}^{u} |T_\mu^{j-1} f| + \frac{1}{u} \sum_{j=1}^{u} j |T_\mu^{j-1}(1-T_\mu) f| \\ \le \left(\frac{1}{u} \sum_{j=1}^{u} |T_\mu^{j-1} f|^s \right)^{1/s} + \left(\sum_{j=1}^{u} j^{s/s'} |T_\mu^{j-1}(1-T_\mu) f|^s \right)^{1/s} \frac{1}{u} \left(\sum_{j=1}^{u} j^{s'/s} \right)^{1/s'}.$$

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Since $s/s' = s - 1$ and $\sum_{j=1}^{u} j^{1/(s-1)} \lesssim u^{1+1/(s-1)} = u^{s'}$,

$$|T_\mu^u f| \lesssim \left(\frac{1}{u}\sum_{j=1}^{u} |T_\mu^{j-1} f|^s\right)^{1/s} + \left(\sum_{j=1}^{u} j^{s-1} |T_\mu^{j-1}(1-T_\mu) f|^s\right)^{1/s}.$$

Following arguments similar to those in the corollary, it follows that

$$\left(\sum_k \max_{n_k < n \le n_{k+1}} n^{s\beta} |T_\mu^n (I - T_\mu)^r f|^s\right)^{1/s} \lesssim \mathbf{Q}_{\beta s-1,s,r} f + \mathbf{Q}_{(\beta+1)s-1,s,r+1} f. \tag{10}$$

Proof of Proposition 2.6. We prove only the result for the convolution measures with $\hat{\Delta}_{k,r} = n_k^\beta (\hat{\mu}^{n_k} - \hat{\mu}^{n_{k+1}})(1-\hat{\mu})^r$ because the other case follows similar arguments. It suffices to verify that $\{\Delta_{k,r}\}_k$ satisfy the conditions of Lemma 3.1.

If $\delta \ge 0$,

$$\sum_k n_k^{s\delta} (n_{k+1} - n_k) |\hat{\mu}(t)|^{n_k s} \lesssim \frac{1}{h(t)^{s\delta+1}}.$$

We also have

$$\begin{aligned}
\left(\sum_k |\hat{\Delta}_{k,r}(t)|^s\right)^{1/s} &\le \left(\sum_k n_k^{\beta s} |\hat{\mu}(t)|^{s n_k} \, |1 - \hat{\mu}(t)^{(n_{k+1}-n_k)}|^s\right)^{1/s} |1-\hat{\mu}(t)|^r \\
&\le \left(\sum_k n_k^{\beta s} (n_{k+1}-n_k)^s |\hat{\mu}(t)|^{s n_k}\right)^{1/s} |1-\hat{\mu}(t)|^{r+1} \\
&\lesssim \left(\sum_k n_k^{\beta s + \gamma(s-1)} (n_{k+1}-n_k) |\hat{\mu}(t)|^{s n_k}\right)^{1/s} |h(t)|^{r+1} \\
&\lesssim \frac{h(t)^{r+1}}{h(t)^{\beta+\gamma+(1-\gamma)/s}}.
\end{aligned} \tag{11}$$

Under the assumptions for γ, β and s we have $\beta + \gamma + (1-\gamma)/s < 1 + r$.

Then, since $|h(t)| \lesssim |t| h'(t)$,

$$A = \int_{|t|<1/2} \frac{1}{|t|}\left(\sum_k |\hat{\Delta}_{k,r}(t)|^s\right)^{1/s} dt \lesssim \int_{|t|<1/2} \frac{h(t)^r \, h'(t)}{h(t)^{\beta+\gamma+(1-\gamma)/s}} \, dt = K < \infty,$$

and dominating the terms of the sum by an integral over intervals of length l^{-2},

$$C = \sum_{l \ne 0} \frac{1}{|l|}\left(\sum_k |\hat{\Delta}_{k,r}(1/|2l|)|^s\right)^{1/s} \lesssim \sum_{l \ne 0} \frac{h(1/|2l|)^{1+r-(\gamma+\beta)-(1-\gamma)/s}}{|l|} \lesssim K < \infty.$$

Let $B_k = \hat{\mu}^{n_k}(t) - \hat{\mu}^{n_{k+1}}(t)$. Then

$$\begin{aligned}
|B_k'(t)| &\le n_k |\hat{\mu}^{n_k-1}(t)| \, |1 - \hat{\mu}(t)^{n_{k+1}-n_k}| \, |\hat{\mu}'(t)| + (n_{k+1}-n_k) |\hat{\mu}(t)|^{n_{k+1}-1} \, |\hat{\mu}'(t)| \\
&\lesssim n_k^{1+\gamma} |\hat{\mu}^{n_k-1}(t)| \, |\hat{\mu}'(t)| |1-\hat{\mu}(t)| + n_k^\gamma |\hat{\mu}^{n_{k+1}-1}(t)| \, |\hat{\mu}'(t)|
\end{aligned}$$

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and

$$\begin{aligned}B_k'' &= n_k(n_k-1)\hat{\mu}^{n_k-2}(t)(\hat{\mu}'(t))^2 - n_{k+1}(n_{k+1}-1)\hat{\mu}^{n_{k+1}-2}(t)(\hat{\mu}'(t))^2\\ &\quad + n_k\hat{\mu}^{n_k-1}(t)\hat{\mu}''(t) - n_{k+1}\hat{\mu}^{n_{k+1}-1}(t)\hat{\mu}''(t)\\ &= n_k(n_k-1)\hat{\mu}^{n_k-2}(t)(1-\hat{\mu}^{n_{k+1}-n_k}(t))(\hat{\mu}'(t))^2\\ &\quad + [n_k(n_k-1) - n_{k+1}(n_{k+1}-1)]\hat{\mu}^{n_{k+1}-2}(\hat{\mu}'(t))^2\\ &\quad + n_k\hat{\mu}^{n_k-1}(t)(1-\hat{\mu}^{n_{k+1}-n_k}(t))\hat{\mu}''(t)\\ &\quad + (n_k-n_{k+1})\hat{\mu}^{n_{k+1}-1}(t)\hat{\mu}''(t).\end{aligned}$$

Since

$$\hat{\Delta}_{k,r}' = n_k^{\beta}[B_k'(t)(1-\hat{\mu}(t))^r + B_k(t)r(1-\hat{\mu}(t))^{r-1}\hat{\mu}'(t)],$$

we have

$$\left(\sum_k |\hat{\Delta}_{k,r}'|^s\right)^{1/s} \lesssim \frac{h^r(t)|\hat{\mu}'(t)|}{h(t)^{(\gamma+\beta)+(1-\gamma)/s}}. \tag{12}$$

Estimating

$$|n_k(n_k-1) - n_{k+1}(n_{k+1}-1)| \lesssim n_k^{\gamma}(n_{k+1}+n_k+1) \lesssim n_k^{1+\gamma},$$

we have

$$\begin{aligned}|n^{\beta}B_k''(t)(1-\hat{\mu}(t))^r| &\lesssim [n_k^{2+\gamma+\beta}\hat{\mu}^{n_k-2}(t)h(t)^2 + n_k^{1+\gamma+\beta}|\hat{\mu}(t)|^{n_{k+1}-1}h(t)\\ &\quad + n_k^{1+\gamma+\beta}|\hat{\mu}(t)|^{n_{k+1}-2}h(t) + n_k^{\gamma+\beta}|\hat{\mu}(t)|^{n_{k+1}-1}]\frac{h(t)^r|\hat{\mu}'(t)|}{|t|}.\end{aligned}$$

Since

$$\begin{aligned}|\hat{\Delta}_{k,r}''| \sim n_k^{\beta}[&|B_k''(t)||1-\hat{\mu}(t)|^r + |B_k'(t)||1-\hat{\mu}(t)|^{r-1}|\hat{\mu}(t)|\\ &+ |B_k(t)||1-\hat{\mu}(t)|^{r-2}|\hat{\mu}'(t)|^2 + |B_k(t)||1-\hat{\mu}(t)|^{r-1}|\hat{\mu}''(t)|],\end{aligned}$$

we have

$$\left(\sum_k |\hat{\Delta}_{k,r}''(t)|^s\right)^{1/s} \lesssim \frac{h(t)^r}{h(t)^{(\gamma+\beta)+(1-\gamma)/s}}\frac{|\hat{\mu}'(t)|}{|t|}, \tag{13}$$

and

$$B = \int_{|t|<1/2} |t|\left(\sum_k |\hat{\Delta}_{k,r}''(t)|^s\right)^{1/s} dt \lesssim \int_{0<t<1/2} \frac{h(t)^r\, h'(t)}{h(t)^{(\gamma+\beta)+(1-\gamma)/s}}\, dt < \infty.$$

For the last term, from equation (12) and $|t\hat{\mu}'(t)| \lesssim h(t)$, we have

$$D \lesssim \sum_{l\neq 0}\frac{1}{l^2}\frac{h(1/|2l|)^r|\hat{\mu}'(1/|2l|)|}{h(1/|2l|)^{(\gamma+\beta)+(1-\gamma)/s}} \lesssim \sum_{l\neq 0}\frac{h(1/|2l|)^{r+1-(\gamma+\beta)-(1-\gamma)/s}}{|l|} \lesssim K < \infty. \qquad \square$$

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Proof of Proposition 2.7. Let $\Delta_{n,r} f = T_\mu^n (I - T_\mu)^r f$, and $\{n_k\}$ be any increasing sequence. By Abel summation,

$$\sum_{u=n}^{m} u^\beta T_\mu^u (I - T_\mu) f = \sum_{u=n+1}^{m+1} (u^\beta - (u-1)^\beta) T_\mu^u f + n^\beta T_\mu^n - (m+1)^\beta T_\mu^{m+1} f.$$

Hence,

$$|n_k^\beta T_\mu^{n_k} f - n_{k+1}^\beta T_\mu^{n_{k+1}} f| \lesssim \left| \sum_{u=n_k}^{n_{k+1}-1} u^\beta T_\mu^u (I - T_\mu) f \right| + \left| \sum_{u=n_k+1}^{n_{k+1}} (u^\beta - (u-1)^\beta) T_\mu^u f \right|.$$

Applied to the function $(I - T_\mu)^r f$, we obtain

$$|D_{k,\beta} f| = |n_k^\beta \Delta_{n_k,r} f - n_{k+1}^\beta \Delta_{n_{k+1},r} f| \lesssim \sum_{u=n_k+1}^{n_{k+1}} u^{\beta-1} |\Delta_{u,r} f| + \sum_{u=n_k}^{n_{k+1}-1} u^\beta |\Delta_{u,r+1} f|. \tag{14}$$

Since $s \geq 1$,

$$\left(\sum_k |D_{k,\beta} f|^s \right)^{1/s} \leq \sum_k |D_{k,\beta} f| \lesssim \sum_u u^{\beta-1} |\Delta_{u,r} f| + \sum_u u^\beta |\Delta_{u,r+1} f|.$$

Then

$$\| n^\beta \Delta_{n,r} f \|_{v(s)} \lesssim \mathbf{Q}_{\beta-1,1,r} f + \mathbf{Q}_{\beta,1,r+1} f,$$

which, by Theorem 2.4, are bounded on Y because $\beta < r$.

Since $\| n^\beta \Delta_{n,r} f \|_{o(s)} \leq \| n^\beta \Delta_{n,r} f \|_{v(s)}$ the result also holds for oscillation norms. □

Proof of Theorem 1.7. By [9], there exist a power-bounded operator R and $a \in (0, 1)$ such that $T = I - (I - R)^a = R_{\nu_a}$, where ν_a is the probability measure on $\mathbb{Z}$ defined in equation (2). Therefore Theorem 2.4, Corollary 2.5 and Propositions 2.6 and 2.7 apply to T. □

5. *The case of τ_μ*

We now turn to the special case of the operators τ_μ induced by a measure-preserving transformation on a probability space.

By the ergodic decomposition of probability measures, it suffices to work with ergodic systems.

LEMMA 5.1. *Let $X = (X, \mathcal{B}, m)$ be a standard probability space and $\tau : X \to X$ a measure-preserving transformation. Let U_τ be a sublinear operator defined on measurable functions on X. Suppose that there exists a constant $C > 0$ such that for any τ-invariant ergodic probability measure ν on $(X, \mathcal{B})$, $\nu(|U_\tau f| > \lambda) \leq (C/\lambda) \int |f| \, d\nu$. Then*

$$m(|U_\tau f| > \lambda) \leq \frac{C}{\lambda} \int |f| \, dm.$$

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Proof. There are several versions of the ergodic decomposition of measures [6, 28]. We use the formulation based on conditional expectation on a standard probability space [26]. (The argument works for any other formulation.)

Let $\mathrm{Inv}(\tau) = \{E \in \mathcal{B} : E = \tau^{-1}E\}$. Let m_y be the conditional probabilities with respect to $\mathrm{Inv}(\tau)$. (Refer to [26] to see how the $\{m_y\}$ are constructed.) Then, for m-almost every $y \in X$, m_y is τ-invariant and ergodic and $\int_X f\,dm = E(E(f|\mathrm{Inv}(\tau)) = \int_X \int_X f\,dm_y\,dm(y)$. Then

$$\begin{aligned} m(|U_\tau f| > \lambda) &= \int_X m_y(|U_\tau f| > \lambda)\,dm(y) \\ &\le \int_X \frac{C}{\lambda} \int_X |f|\,dm_y\,dm(y) = \frac{C}{\lambda}\int_X |f|\,dm. \end{aligned}$$ □

For operators τ_μ, the estimates developed in §§4 can be improved with the aid of the ergodic Calderón–Zygmund decomposition introduced by Jones [12], because the cancelation provided by the 'bad' function gives improved control on Fourier coefficients.

PROPOSITION 5.2. (Ergodic Calderón–Zygmund decomposition [12]) *Let (X, m) be a probability space and $\tau : X \to X$ and ergodic, invertible measure-preserving transformation. Let $f \in L^1(X)$ and $f^*(x) = \sup_{n,m}(1/n+m+1)|\sum_{k=-m}^{n} f(\tau^k x)|$, the maximal function for averages. Then f can be decomposed as $f = g + b$, with $g \in L^\infty$ and $b \in L^1(X)$, called the 'bad' function, with the following properties:*

(a) *the set $B = \{x : f^*(x) \le \lambda\}$ decomposes as $B = \bigcup_i B_i$ where the B_i are pairwise disjoint;*
(b) *$b = \sum_i b_i$ where each b_i is supported on a set E_i of the form $E_i = \bigcup_{j=1}^{i} \tau^j B_i$, union of pairwise disjoint sets;*
(c) *$\sum_{k=1}^{i} b_i(\tau^k x) = 0$ if $x \in B_i$;*
(d) *$\sum_{k=1}^{i} |b_i(\tau^k x)| \le 2\lambda$ if $x \in B_i$;*
(e) *$\sum_i m(E_i) \le 2\|f\|_1/\lambda$;*
(f) *$\|g\|_\infty \le 2\lambda$ and $\|g\|_1 \le \|f\|_1$.*

This decomposition reduces the problem on L^1 to working with simpler building blocks called 'atoms', which are defined below.

Definition 5.3. Let $\tau : X \to X$ be an invertible, measure-preserving transformation on a standard probability space $(X, \mathcal{B}, m)$. Given $\lambda > 0$, a λ-*atom* is a function $a \in L^1(X)$ for which there exist a measurable set $B \in X$ and an integer $d > 0$ such that:

(a) $\{\tau^k B\}_{k=1}^{n}$ are pairwise disjoint sets;
(b) a is supported on the set $E = \bigcup_{k=1}^{d} \tau^k B$;
(c) $\sum_{k=1}^{d} a(\tau^k x) = 0$ for all $x \in B$;
(d) $\sum_{k=1}^{d} |a(\tau^k x)| \le 2\lambda$ if $x \in B$.

We call B the generator set (of length d) and define

$$E^* = E \cup \tau^d E \cup \tau^{2d} E \cup \tau^{-d} E \cup \tau^{-2d} E.$$

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Notice that, as a consequence of the properties of atoms,

$$\|a\|_1 = \int_E |a|\,dm = \sum_{k=1}^{d} \int_{\tau^k B} |a|\,dm = \sum_{k=1}^{d} \int_B |a \circ \tau^k|\,dm \le 2\lambda\, m(B). \tag{15}$$

PROPOSITION 5.4. *Let X and τ as in Definition 5.3. Suppose Q is a sublinear operator defined on measurable functions on X satisfying the following properties:*

(a) *Q is bounded on $L^2(X)$;*

(b) *there exists a constant $C > 0$ such that, for every λ-atom a,*

$$\int_{(E^*)^c} |Q(a)|\,dm \le C\,\lambda m(E),$$

where E and E^* are defined as in Definition 5.3.

Then Q is of weak type (1, 1).

Proof. By Lemma 5.1, it suffices to consider the case when $(X, \mathcal{B}, m, \tau)$ is an ergodic measure-preserving system.

Given $\lambda > 0$ and $f \in L^1$, decompose f via the ergodic Calderón–Zygmund decomposition, with the sets B, B_i, E_i and the associated good and bad functions g and b defined as in Proposition 5.2.

Since Q is sublinear, we have

$$m(x : |Qf(x)| > 2\lambda) \le m(x : |Qg(x)| > \lambda) + m(x : |Qb(x)| > \lambda).$$

By the L^2-boundedness of Q, $m(x : |Qg(x)| > \lambda) \lesssim \|g\|_2^2/\lambda^2 \le 2\|g\|_1/\lambda \le 2\|f\|_1/\lambda$.

Let $E_i^* = E_i \cup \tau^i E_i \cup \tau^{2i} E_i \cup \tau^{-i} E_i \cup \tau^{-2i} E_i$. By property (e) of the ergodic Calderón–Zygmund decomposition,

$$\begin{aligned} m(x : |Qb(x)| > \lambda) \le& m\left(\bigcup_i E_i^*\right) + m\left(x \in \left(\bigcup_i E_i^*\right)^c : |Qb(x)| > \lambda\right) \\ \le& 5 \sum_i m(E_i) + \frac{1}{\lambda} \int_{(\bigcup_i E_i^*)^{*c}} |Qb|\,dm \\ \le& 10 \frac{\|f\|_1}{\lambda} + \frac{1}{\lambda} \sum_i \int_{(E_i^*)^{*c}} |Qb_i|\,dm. \end{aligned}$$

Noting that each b_i is a λ-atom, apply assumption (b) to obtain

$$\int_{(\bigcup_i E^*)^c} |Qb(x)|\,dm \le \sum_i \int_{E_i^{*c}} |Qb_i(x)|\,dm \le C\lambda \sum_i m(E_i) \le 2C\|f\|_1,$$

completing the proof. □

THEOREM 5.5. *Let $\tau : X \to X$ be an invertible, measure-preserving transformation on a standard probability space $(X, \mathcal{B}, m)$. Let $s \ge 1$, $\{n_k\}$ be an increasing sequence of positive integers, $J_k = [n_k, n_{k+1})$, and $\{\Delta_n\}$ a collection of finite, signed measures on $\mathbb{Z}$ such that $\{\hat{\Delta}_n\}$ are twice continuously differentiable on $|t| \in (0, 1)$. Suppose $\{\Delta_n\}$ satisfy the following conditions:*

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- $\int_{|t|<0.5}(\sum_k \max_{n\in J_k} |\hat{\Delta}_n(t)|^s)^{1/s}\, dt < \infty$;
- $\sum_{|j|>1}(1/j^2)(\sum_k \max_{n\in J_k} |\hat{\Delta}_n(1/|j|)|^s)^{1/s} < \infty$;
- $\int_{|t|<0.5}(\sum_k \max_{n\in J_k} |\hat{\Delta}_n''(t)|^s)^{1/s}|t|^2\, dt < \infty$;
- $\sum_{|j|>1}(1/|j|^3)(\sum_k \max_{n\in J_k} |\hat{\Delta}_n'(1/|j|)|^s)^{1/s} < \infty$.

Let

$$Qf = \left(\sum_k \max_{n\in J_k} |\tau_{\Delta_n} f|^s\right)^{1/s}.$$

Then the following assertions hold:

(a) *for every λ-atom a (λ > 0),*

$$\int_{(E^*)^c} Qa\, dm \le C\, \lambda m(E),$$

where E and E^* are as in Definition 5.3;

(b) *if Qf is bounded on L^2 then Qf is weak* (1, 1).

Proof. Let a be a λ-atom and let B, d, E and E^* as in Definition 5.3. Since a is supported on E, and using the cancelation property (c) of Definition 5.3,

$$\tau_{\Delta_n} a(x) = \sum_j \Delta_n(j) a(\tau^j x) = \sum_{j:\tau^j x\in B} \sum_{l=1}^{d} (\Delta_n(j+l) - \Delta_n(j)) a(\tau^{j+l} x).$$

Then

$$\begin{aligned}(Qa(x))^s &= \sum_{k=1}^{\infty} \max_{n\in J_k} \left| \sum_{j:\tau^j x\in B} \sum_{l=1}^{d} (\Delta_n(j+l) - \Delta_n(j)) a(\tau^{j+l} x)\right|^s \\ &= \sum_{k=1}^{\infty} \max_{n\in J_k} \left| \sum_{j:\tau^j x\in B} \sum_{l=1}^{d} a(\tau^{j+l} x) \int_{|t|<0.5} \hat{\Delta}_n(t)(e((j+l)t) - e(jt))\, dt\right|^s.\end{aligned}$$

Let $I(x)$ and $II(x)$ be the expressions obtained by replacing the integral over $\{|t| < 0.5\}$ with integrals over $\{|t| < 1/|j|\}$ and $\{1/|j| < |t| < 0.5\}$, respectively. We have

$$I(x) = \left(\sum_{k=1}^{\infty} \max_{n\in J_k} \left| \sum_{j:\tau^j x\in B} \sum_{l=1}^{d} a(\tau^{j+l} x) \int_{|t|<1/|j|} \hat{\Delta}_n(t)(e((j+l)t) - e(jt))\, dt\right|^s\right)^{1/s},$$

$$II(x) = \left(\sum_{k=1}^{\infty} \max_{n\in J_k} \left| \sum_{j:\tau^j x\in B} \sum_{l=1}^{d} a(\tau^{j+l} x) \int_{1/|j|<|t|<0.5} \hat{\Delta}_n(t)(e((j+l)t) - e(jt))\, dt\right|^s\right)^{1/s}.$$

By the triangular inequality of the ℓ^s-norm,

$$Qa(x) \le I(x) + II(x).$$

From the construction of E^*, if $x \in (E^*)^c$ and $\tau^j x \in B$, we must have $|j| \ge 2d > 1$.

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We estimate $I(x)$ as follows:

$$\begin{aligned} I(x) &= \left(\sum_{k=1}^{\infty} \max_{n\in J_k} \left| \sum_{j:\tau^j x\in B} \sum_{l=1}^{d} a(\tau^{j+l}x) \int_{|t|<1/|j|} \hat{\Delta}_n(t)(e((j+l)t)-e(jt))\,dt \right|^s \right)^{1/s} \\ &\le \sum_{j:\tau^j x\in B} \sum_{l=1}^{d} |a(\tau^{j+l}x)| \int_{|t|<1/|j|} \left(\sum_{k=1}^{\infty} \max_{n\in J_k} |\hat{\Delta}_n(t)|^s \right)^{1/s} |e((j+l)t)-e(jt)|\,dt \\ &\le \sum_{j:\tau^j x\in B} \sum_{l=1}^{d} |l|\,|a(\tau^{j+l}x)| \int_{|t|<1/|j|} \left(\sum_{k=1}^{\infty} \max_{n\in J_k} |\hat{\Delta}_n(t)|^s \right)^{1/s} |t|\,dt \\ &\le 2\lambda d \int_{|t|<0.5} \left(\sum_{k=1}^{\infty} \max_{n\in J_k} |\hat{\Delta}_n(t)|^s \right)^{1/s} |t| \sum_{|j|>1} 1_{[x:x\in\tau^{-j}B]}(x) 1_{[j:|j|<1/|t|]}(j)\,dt. \end{aligned}$$

Then

$$\begin{aligned} \int_{(E^*)^c} I(x)\,dm &\le 2\lambda d \int_{|t|<0.5} \left(\sum_{k=1}^{\infty} \max_{n\in J_k} |\hat{\Delta}_n(t)|^s \right)^{1/s} |t| \sum_{1<|j|<1/|t|} m((E^*)^c \cap \tau^{-j}B) \\ &\le 4\lambda d\, m(B) \int_{|t|<0.5} \left(\sum_{k=1}^{\infty} \max_{n\in J_k} |\hat{\Delta}_n(t)|^s \right)^{1/s} dt \le C\lambda m(E). \end{aligned}$$

Handling $II(x)$ requires integration by parts. We will need the following estimates. Because $\Delta_n(t), \Delta_n'(t), e(kt)$ are 1-periodic, the same procedure as (4) yields, for $|j| > 2|l|$,

$$\begin{aligned} &\left| \int_{|t|>1/|j|} \hat{\Delta}_n(t)(e((j+l)t)-e(jt))\,dt \right| \\ &\lesssim \left| \int_{|t|>1/|j|} \hat{\Delta}_n'(t) \left(\frac{e((j+l)t)}{j+l} - \frac{e(jt)}{j} \right) dt \right| + \left| \hat{\Delta}_n\left(\frac{1}{|j|}\right) \right| \frac{|l|}{j^2}. \end{aligned}$$

From [3], since $2|l| < 2d < |j|$, we have

$$\left| \frac{e((j+l)t)-1}{(j+l)^2} - \frac{e(jt)-1}{j^2} \right| \lesssim \frac{|l||t|}{|j|^2}.$$

Thus,

$$\begin{aligned} &\int_{|t|>1/|j|} \hat{\Delta}_n'(t) \left(\frac{e((j+l)t)}{(j+l)} - \frac{e(jt)}{j} \right) dt \\ &\lesssim \left| \int_{|t|>1/|j|} \hat{\Delta}_n''(t) \left(\frac{e((j+l)t)-1}{(j+l)^2} - \frac{e(jt)-1}{j^2} \right) dt \right| \\ &\quad + \left| \hat{\Delta}_n'\left(\frac{1}{|j|}\right) \right| \frac{|l|}{|j|^3} \\ &\le \int_{|t|>1/|j|} |\hat{\Delta}_n''(t)||t| \frac{|l|}{|j|^2}\,dt + \left| \hat{\Delta}_n'\left(\frac{1}{|j|}\right) \right| \frac{|l|}{|j|^3}. \end{aligned}$$

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Accordingly, $II(x)$ is dominated by three parts II_1, II_2 and II_3 estimated below.

$$\begin{aligned} II_1(x) &= \left(\sum_{k=1}^{\infty} \max_{n\in J_k} \left| \sum_{j:\tau^j x\in B} \sum_{l=1}^{d} a(\tau^{j+l}x) \int_{1/|j|<|t|<0.5} |\hat{\Delta}_n''(t)| \frac{l|t|}{|j|^2}\, dt \right|^s \right)^{1/s} \\ &\le d \sum_{j:\tau^j x\in B} \sum_{l=1}^{d} |a(\tau^{j+l}x)| \int_{1/|j|<|t|<0.5} \left(\sum_{k=1}^{\infty} \max_{n\in J_k} |\hat{\Delta}_n''(t)|^s \right)^{1/s} \frac{|t|}{|j|^2}\, dt \\ &\le 2\lambda d \int_{|t|<0.5} \left(\sum_{k=1}^{\infty} \max_{n\in J_k} |\hat{\Delta}_n''(t)|^s \right)^{1/s} |t| \sum_{|j|>1/|t|} 1_{[\tau^j x\in B]}(x) \frac{1}{|j|^2}\, dt. \end{aligned}$$

Integrating over $(E^*)^c$,

$$\begin{aligned} \int_{(E^*)^c} II_1(x)\, dm &\le 2\lambda d \int_{|t|<0.5} \left(\sum_{k=1}^{\infty} \max_{n\in J_k} |\hat{\Delta}_n''(t)|^s \right)^{1/s} |t| \sum_{|j|>1/|t|} \frac{1}{|j|^2} m(B) \\ &\lesssim \lambda m(E) \int_{|t|<0.5} \left(\sum_{k=1}^{\infty} \max_{n\in J_k} |\hat{\Delta}_n''(t)|^s \right)^{1/s} |t|^2\, dt = C\lambda m(E). \end{aligned}$$

Lastly, we need to handle two remaining pieces:

$$\begin{aligned} II_2(x) &= \int_{(E^*)^c} \left(\sum_{k} \max_{n\in J_k} \left| \sum_{j:\tau^j x\in B} \sum_{l=1}^{d} |a(\tau^{j+l}x)| \left| \hat{\Delta}_n\left(\frac{1}{|j|}\right) \right| \frac{|l|}{|j+l|^2} \right|^s \right)^{1/s} dm \\ &\lesssim \int_{(E^*)^c} d \sum_{j:\tau^j x\in B} \sum_{l=1}^{d} |a(\tau^{j+l}x)| \left(\sum_{k} \max_{n\in J_k} \left| \hat{\Delta}_n\left(\frac{1}{|j|}\right) \right|^s \right)^{1/s} \frac{1}{j^2}\, dm \\ &\lesssim \lambda m(E) \sum_{|j|>1} \frac{1}{j^2} \left(\sum_{k} \max_{n\in J_k} \left| \hat{\Delta}_n\left(\frac{1}{|j|}\right) \right|^s \right)^{1/s} \le C\lambda m(E) \end{aligned}$$

and

$$\begin{aligned} II_3(x) &= \int_{(E^*)^c} \left(\sum_{k} \max_{n\in J_k} \left| \sum_{j:\tau^j x\in B} \sum_{l=1}^{d} |a(\tau^{j+l}x)| \left| \hat{\Delta}_n'\left(\frac{1}{|j|}\right) \right| \frac{|l|}{|j+l|^3} \right|^s \right)^{1/s} dm \\ &\lesssim \int_{(E^*)^c} d \sum_{j:\tau^j x\in B} \sum_{l=1}^{d} |a(\tau^{j+l}x)| \left(\sum_{k} \max_{n\in J_k} \left| \hat{\Delta}_n'\left(\frac{1}{|j|}\right) \right|^s \right)^{1/s} \frac{1}{|j|^3}\, dm \\ &\lesssim \lambda m(B) d \sum_{|j|>1} \left(\sum_{k} \max_{n\in J_k} \left| \hat{\Delta}_n'\left(\frac{1}{|j|}\right) \right|^s \right)^{1/s} \frac{1}{|j|^3} \le C\lambda m(E). \end{aligned}$$

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Combining the estimates for $I(x)$, $II_1(x)$, $II_2(x)$ and $II_3(x)$, we obtain the desired result:

$$\int_{(\bigcup_i E^*)^c} Qa(x)\,dm \le C\lambda\, m(E). \qquad \square$$

Proof of Theorem 1.9.

Proof of Part A. Assuming that $\mathbf{Q}_{\alpha,s,r}$ is bounded on L^2, we simply need to check the requirements in Theorem 5.5 for $\hat{\Delta}_n = n^{\alpha/s}\hat{\mu}^n(1-\hat{\mu})^r$ and the sequence $n_k = k$, $r \ge 0$.

First note that, under the assumptions for h, if $(1+\alpha) \le sr$, $\int_{|t|<0.5} h^{r-(1+\alpha)/s}(t)\,dt < \infty$. Using estimates for $\Delta_n(t)$ in equation (5), for $(1+\alpha) \le sr$ we have

$$\int_{|t|<0.5} \left(\sum_{n=1}^{\infty} |\hat{\Delta}_n(t)|^s \right)^{1/s} dt \le \int_{|t|<0.5} h^{r-(1+\alpha)/s}(t)\,dt < \infty$$

and

$$\sum_{|j|>1} \frac{1}{j^2} \left(\sum_n \left| \hat{\Delta}_n\left(\frac{1}{|j|}\right) \right|^s \right)^{1/s} \le \sum_{|j|>1} \frac{1}{j^2} h\left(\frac{1}{j}\right)^{r-(1+\alpha)/s} \lesssim \int_0^1 h(t)^{r-(\alpha+1)/s}\,dt < \infty.$$

From property BA_1 (ii) and equation (8),

$$\sum_{j>1} \frac{1}{|j|^3} \left(\sum_n \left| \hat{\Delta}_n'\left(\frac{1}{|j|}\right) \right|^s \right)^{1/s} \le \sum_{j>1} \frac{1}{j^2} h\left(\frac{1}{j}\right)^{r-(\alpha+1)/s} < \infty.$$

By the estimates for $|\hat{\Delta}_n''(t)|$ in equation (7) and BA_1 (ii),

$$\begin{aligned} \int_{|t|<0.5} \left(\sum_{n=1}^{\infty} |\hat{\Delta}_n''(t)|^s \right)^{1/s} |t|^2\,dt &\le \int_{|t|<0.5} h(t)^{(r-1)-(\alpha+1)/s} |\hat{\mu}'(t)||t|\,dt \\ &\le \int_{|t|<0.5} h(t)^{r-(\alpha+1)/s}\,dt < \infty. \end{aligned}$$

Proof of Part B. Since $\|\{n^\beta \tau_\mu^n (I-\tau_\mu)^r f\}\|_{v(s)}$ is bounded on L^2, by Proposition 5.4 it suffices to check the conditions of Theorem 5.5 for the corresponding sublinear operator Q defined by $\hat{\Delta}_n = n^\beta(\hat{\mu}^n(1-\hat{\mu})^r)$.

Because $s > 1$ we can choose $0 < \delta < \min(1, (s-1)/(s+1))$. Starting with the sequence $\{2^k\}$, enlarge it by adding $N_k = 2^{k(1-\delta)}$ equally spaced points in between; call these $\{r_{k,j}\}_{j=0,N_k-1}$. These points have the properties $r_{k,0} = 2^k$ and $r_{k,j+1} - r_{k,j} \sim 2^k/N_k = 2^{\delta k} \sim r_{k,j}^\delta$. Denote the full collection of these by $\{r_j\}$ with the indices rearranged appropriately.

Let $B_{n,r} = n^\beta \tau_\mu^n (I-\tau_\mu)^r$ and $K_n = B_{r_j,r}$ for $r_j \le n < r_{j+1}$. Also let $I_j = [r_j, r_{j+1}]$. We have

$$\|\{B_{n,r}a\}\|_{v(s)} \le \|\{(B_{n,r} - K_n)a\}\|_{v(s)} + \|\{K_n a\}\|_{v(s)}$$

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By properties of variational norms [13–15],

$$\|\{(B_{n,r} - K_n)a\}\|_{v(s)} \lesssim \Bigg(\sum_j \|\{(B_{n,r} - B_{r_j,r})a : n \in I_j\}\|_{v(s)}^s \Bigg)^{1/s}.$$

Using Abel summation as in equation (14), bound the last term by

$$\begin{aligned}
\|\{(B_{n,r} - B_{r_j,r})a : n \in I_j\}\|_{v(s)}^s &\lesssim \sum_{n \in I_j} |(B_{n,r} - B_{r_j,r})a|^s \\
&\lesssim r_j^{\delta} \Bigg(\sum_{n=r_j+1}^{r_{j+1}} n^{\beta-1} |\tau_\mu^n (1-\tau_\mu)^r a| + \sum_{n=r_j}^{r_{j+1}-1} n^{\beta} |\tau_\mu^n (1-\tau_\mu)^{r+1} a| \Bigg)^s \\
&\lesssim r_j^{\delta(1+s)} \sum_{n=r_j}^{r_{j+1}} n^{(\beta-1)s} |\tau_\mu^n (1-\tau_\mu)^r a|^s + r_j^{\delta(1+s)} \sum_{n=r_j}^{r_{j+1}-1} n^{\beta s} |\tau_\mu^n (1-\tau_\mu)^{r+1} a|^s.
\end{aligned}$$

Thus,

$$\|\{(B_{n,r} - K_n)a\}\|_{v(s)} \lesssim \mathbf{Q}_{(\beta-1)s+\delta(1+s),s,r} a + \mathbf{Q}_{\beta s+\delta(1+s),s,r+1} a. \tag{16}$$

Since $\beta \le r$ and $\delta(1+s) + 1 < s$, these are bounded on L^1.

When $r > 0$,

$$\begin{aligned}
\|\{K_n a\}\|_{v(s)} &\lesssim \sum_j |(r_j^\beta \tau_\mu^{r_j} - r_{j+1}^\beta \tau_\mu^{r_{j+1}})(1-\tau_\mu)^r a| \\
&\lesssim \sum_j r_j^\beta |(\tau_\mu^{r_j} - \tau_\mu^{r_{j+1}})(1-\tau_\mu)^r a| + \sum_j (r_{j+1}^\beta - r_j^\beta) |\tau_\mu^{r_{j+1}} (1-\tau_\mu)^r a| \\
&\lesssim \sum_j r_j^\beta |(\tau_\mu^{r_j} - \tau_\mu^{r_{j+1}})(1-\tau_\mu)^r a| + \sum_j r_{j+1}^{\beta-1+\delta} |\tau_\mu^{r_{j+1}} (1-\tau_\mu)^r a|.
\end{aligned}$$

Since $\beta \le r$, Lemma 5.6(a) applies to the first term with $s = 1$, and Lemma 5.6(b) applies to the second term.

When $r = 0$ and $\beta = 0$, we have

$$\|\{K_n a\}\|_{v(s)} \lesssim \sum_j |(\tau_\mu^{r_j} - \tau_\mu^{r_{j+1}})a|.$$

Thus, Lemma 5.6(a) applies.

In either case, we obtain

$$\int_{(E^*)^c} \|\{K_n a(x)\}\|_{v(s)} \, dm \lesssim \lambda m(E). \tag{17}$$

Finally, from equations (15), (16) and (17),

$$\begin{aligned}
\int_{(E^*)^c} \|\{B_{n,r} a\}\|_{v(s)} \, dm &\le \int_{(E^*)^c} [\|\{(B_{n,r} - K_n)a\}\|_{v(s)} + \|\{K_n a(x)\}\|_{v(s)}] \, dm \\
&\lesssim \lambda m(E).
\end{aligned}$$

The result for the oscillation norm follows from noticing that $\|B_{n,r} a\|_{o(s)} \le \|B_{n,r} a\|_{v(s)}$. □

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Proof of Theorem 1.10(d). By equation (10), if $s \geq 2$, then

$$\left(\sum_k \max_{n_k < n \leq n_{k+1}} n^{rs} |\tau_\mu^n (1-\tau_\mu)^r f|^s\right)^{1/s} \lesssim \mathbf{Q}_{2r-1,2,r} f + \mathbf{Q}_{2(r+1)-1,1,r+1} f.$$

Thus, the result follows from Theorem 1.10(a) with $s = 2$. □

LEMMA 5.6. *Let (X, β, m) be a probability space, $\tau : X \to X$ an ergodic, invertible measure-preserving transformation. Let μ be a probability measure on $\mathbb{Z}$ with property BA_1, and $s \geq 1$. Let $\{r_{n+1}\}$ be a sequence with $r_{n+1} - r_n \sim r_n^\delta$ for some $0 < \delta \leq 1$. Let $\lambda > 0$ and a be a λ-atom, with E and E^* as in definition 5.3.*

(a) *If $\beta \leq r + (1-\delta)(1-1/s)$, then*

$$\int_{(E^*)^c} \left(\sum_n r_n^{\beta s} |(\tau_\mu^{r_n} - \tau_\mu^{r_{n+1}})(I-\tau_\mu)^r a|^s\right)^{1/s} dm \lesssim \lambda m(E).$$

(b) *If $r > 0$ and $\beta \leq r - (1-\delta)/s$, then*

$$\int_{(E^*)^c} \left(\sum_n r_n^{\beta s} |\tau_\mu^{r_n} (I-\tau_\mu)^r a|^s\right)^{1/s} dm \lesssim \lambda m(E).$$

Proof. For simplicity, we only prove case (a) because case (b) follows from similar estimates.

By Theorem 5.5, it suffices to check the assumptions of that theorem for $\hat{\Delta}_n = r_n^\beta (\hat{\mu}^{r_n} - \hat{\mu}^{r_{n+1}})(1-\hat{\mu})^r$ and the sequence $n_k = k$.

From equation (11), and since $\beta + \delta + (1-\delta)/s \leq r+1$,

$$\left(\sum_{n=1}^\infty |\hat{\Delta}_n(t)|^s\right)^{1/s} \lesssim \frac{h(t)^{r+1}}{h(t)^{\beta+\delta+(1-\delta)/s}} \lesssim 1.$$

Hence,

$$\int_{|t|<0.5} \left(\sum_{n=1}^\infty |\hat{\Delta}_n(t)|^s\right)^{1/s} dt \lesssim 1 \quad \text{and} \quad \sum_{|j|>0} \frac{1}{|j|^2} \left(\sum_{n=1}^\infty \left|\hat{\Delta}_n\left(\frac{1}{|j|}\right)\right|^s\right)^{1/s} < \infty.$$

By equation (12),

$$\left(\sum_{n=1}^\infty |\hat{\Delta}_n'(t)|^s\right)^{1/s} \lesssim \frac{h(t)^r}{h(t)^{\beta+\delta+(1-\delta)/s}} |\hat{\mu}'(t)|,$$

and by BA_1(ii), $|t\hat{\mu}'(t)| \lesssim h(t)$,

$$\begin{aligned}\sum_{|j|>0} \frac{1}{|j|^3} \left(\sum_{n=1}^\infty |\hat{\Delta}_n'(1/|j|)|^s\right)^{1/s} &\lesssim \sum_{|j|>0} \frac{1}{|j|^3} \frac{h(1/|j|)^r}{h(1/|j|)^{\beta+\delta+(1-\delta)/s}} |\hat{\mu}'\left(\frac{1}{|j|}\right)| \\ &\lesssim \sum_{|j|>0} \frac{1}{|j|^2} h\left(\frac{1}{|j|}\right)^{r+1-(\beta+\delta+(1-\delta)/s)} \lesssim \sum_{|j|>0} \frac{1}{|j|^2} < \infty.\end{aligned}$$

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By equation (13) and BA_1,

$$\int_{|t|<0.5} \left(\sum_{n=1}^{\infty} |\hat{\Delta}_n''(t)|^s \right)^{1/s} |t|^2 \, dt \lesssim \int_{|t|<0.5} \frac{h(t)^r}{h(t)^{\beta+\delta+(1-\delta)/s}} \frac{|\hat{\mu}'(t)|}{|t|} |t|^2 \, dt \lesssim 1. \qquad \square$$

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