We now divide Q into two triangles by the diagonal BD and apply the cosine rule twice; this gives

$$b^2 + c^2 - 2bc\cos\gamma = a^2 + d^2 - 2ad\cos\alpha. \qquad (2)$$

Because of the (algebraic) symmetry here, we may assume that $b^2 + c^2 \geqslant a^2 + d^2$; thus

$$0 \leqslant b^2 + c^2 - a^2 - d^2 = 2(ad + bc)\cos\gamma - 2ad\big(\cos\alpha + \cos\gamma\big). \qquad (3)$$

As some configuration of Q (with the given side lengths) is assumed to exist, we have

$$(a + d)^2 \geqslant BD^2 = b^2 + c^2 - 2bc\cos\gamma > (b - c)^2$$

or, equivalently,

$$0 \leqslant b^2 + c^2 - a^2 - d^2 < 2(ad + bc).$$

This means that if arrange the sides BC and CD by choosing γ such that

$$2(ad + bc)\cos\gamma = b^2 + c^2 - a^2 - d^2, \qquad 0 < \gamma \leqslant \tfrac{1}{2}\pi;$$

then, from (2) we can reconstruct Q, and from (1), $\cos\alpha + \cos\gamma = 0$, so that $\alpha + \gamma = \pi$ and the reconstructed Q is cyclic.

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101.35 Some properties of the Kepler triangle

The Kepler triangle is a right-angled triangle whose side lengths are in geometric progression. If the common ratio is $\sqrt{x}$, it follows that $1 + x = x^2$, and so x is the golden ratio ϕ, where $\phi = \frac{1}{2}(1 + \sqrt{5})$. In this Note, we will show some interesting geometric properties hidden in it.

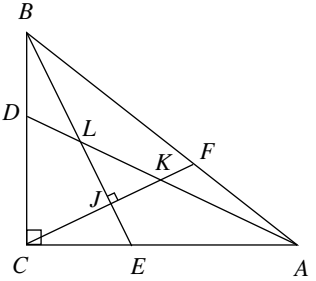


FIGURE 1

In Figure 1, let ΔBCA be a Kepler triangle with $BC = 1$, $CA = \sqrt{\phi}$ and $AB = \phi$. Let D, E, F be points of BC, CA, AB respectively so that

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = \frac{1}{\phi}.$$

Let J, K, L be the intersection points of BE, CF; CF, AD; AD, BE respectively.

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Theorem: $\triangle JKL \sim \triangle CBA$.

Proof: We see that BCF is an isosceles triangle with $BC = BF = 1$. Since $\frac{BC}{BA} = \frac{CE}{EA} = \frac{1}{\phi}$, we get $\angle CBE = \angle EBA$ by the angle bisector theorem. Thus, $BE \perp CF$, which means LJK is a right-angled triangle. Then set $\angle CBA = \theta$, $\angle CDA = \theta_1$. We have $\tan\theta = \frac{CA}{CB} = \sqrt{\phi}$ and $\tan\theta_1 = \frac{CA}{CD} = \phi\sqrt{\phi}$, then we get

$$\tan 2\theta_1 = \frac{2\tan\theta_1}{1 - \tan^2\theta_1} = -\sqrt{\phi}.$$

Now $\tan\theta + \tan 2\theta_1 = 0$, and so $\theta + 2\theta_1 = \pi$, and

$$\angle JLK = \angle DLB = \angle CDA - \angle CBE = \theta_1 - \frac{\theta}{2} = \frac{\pi}{2} - \angle CBA = \angle CAB.$$

Hence the result follows.

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101.36 On the area of a tangential quadrilateral

A tangential quadrilateral is a quadrilateral with an incircle, that is, a circle inside the quadrilateral that is tangent to all four sides. These quadrilaterals are related to, but unfortunately not so well-known as, the more famous cyclic quadrilateral. There are numerous beautiful characterisations of tangential quadrilaterals (see [1]) as well as interesting metric relations in these quadrilaterals (see [2] and [3]).

Several formulas that give the area K of tangential quadrilaterals $ABCD$ are known, for instance

$$K = \sqrt{abcd}\,\sin\frac{A + C}{2}$$

and

$$K = \sqrt{(e + f + g + h)(efg + fgh + ghe + hef)}.$$

Here a, b, c, d are the lengths of the sides of $ABCD$, and e, f, g, h are the lengths of the tangents from the vertices to the incircle as in Figure 1. These and similar formulas can be found at [4].

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