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PAPER

Cahn–Hilliard equations with singular potential, reaction term and pure phase initial datum

Maurizio Grasselli, Luca Scarpa and Andrea Signori

Dipartimento di Matematica, Politecnico di Milano, Milano, Italy
Corresponding author: Andrea Signori; Email: andrea.signori@polimi.it

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We consider local and nonlocal Cahn–Hilliard equations with constant mobility and singular potentials including, e.g., the Flory–Huggins potential, subject to no-flux (or periodic) boundary conditions. The main goal is to show that the presence of a suitable class of reaction terms allows to establish the existence of a weak solution to the corresponding initial and boundary value problem even though the initial condition is a pure state. This fact was already observed by the authors in a previous contribution devoted to a specific biological model. In this context, we examine the essential assumptions required for the reaction term to ensure the existence of a weak solution. Also, we explore the scenario involving the nonlocal Cahn–Hilliard equation and provide some illustrative examples that contextualize within our abstract framework.

1. Introduction

Phase separation is a relevant phenomenon which appears in a number of different contexts (see, for instance, [20, 22] and references therein). In particular, it has recently become a paradigm in Cell Biology (see, e.g., [7, 19, 44]). Phase separation takes place whenever two (or more) species interact in a way that can be described as a competition between mixing entropy and a demixing effect due to the attraction of particles of the same species. As a prototypical example, one may think of oil and vinegar. From the mathematical viewpoint, a fairly good description of such phenomenon is given by the so-called Cahn–Hilliard equation (see [8, 9], see also [52] and its references). In the case of two species, this equation assumes the following form

$$\partial_t \varphi = \mathrm{div}(m(\varphi)\nabla\mu)$$

in a given bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$ and in a time interval $(0, T)$ for some given $T > 0$. Here, φ represents the (relative) difference between the concentration of the two species, $m(\cdot)$ is a nonnegative mobility function, and μ is the so-called chemical potential. The latter corresponds to the functional derivative of the (local) free energy

$$\mathcal{E}^{\mathrm{loc}}(\varphi(\cdot)) = \int_\Omega |\nabla\varphi(\cdot, x)|^2 \mathrm{d}x + \int_\Omega F(\varphi(\cdot, x))\,\mathrm{d}x, \tag{1.1}$$

where a physically relevant example of potential density F is given by the logarithmic (a.k.a. the Flory–Huggins) potential (see, e.g., [11] and references therein)

$$F_{\log}(r) := \frac{\vartheta}{2}\left[(1+r)\ln(1+r) + (1-r)\ln(1-r)\right] - \frac{\vartheta_0}{2}r^2, \quad r \in (-1, 1), \tag{1.2}$$

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where $0 < \vartheta < \vartheta_0$. An alternative choice is a nonlocal free energy of the form (see [34, 35], see also [1, 14–16, 27, 28] and their references)

$$\mathcal{E}^{\text{nonloc}}(\varphi(\cdot)) = \frac{1}{4}\int_{\Omega\times\Omega} K(x-y)|\varphi(\cdot,x)-\varphi(\cdot,y)|^2 \mathrm{dxdy} + \int_\Omega F(\varphi(\cdot,x))\,\mathrm{dx}, \tag{1.3}$$

where $K : \mathbb{R}^d \to \mathbb{R}$ is a sufficiently smooth interaction kernel such that $K(x) = K(-x)$.

Taking constant mobility equal to one, we have the following local Cahn–Hilliard equation

$$\partial_t \varphi = \Delta(-\Delta\varphi + F'(\varphi)), \tag{1.4}$$

and its nonlocal counterpart

$$\partial_t \varphi = \Delta(a\varphi - K * \varphi + F'(\varphi)), \tag{1.5}$$

in $Q := \Omega \times (0, T)$, where

$$(K * v)(x) := \int_\Omega K(x-y)v(y)\,dy, \quad x \in \Omega,$$

for any given measurable function $v : \Omega \to \mathbb{R}$ and $a = K * 1$.

To ensure mass conservation, the above equations are usually equipped with no-flux boundary conditions on $\Sigma := \partial\Omega \times (0, T)$ for the chemical potential μ, even though periodic boundary conditions can also be considered (see, for instance, [35]). In the vast mathematical literature devoted to these equations, in order to prove the existence of a reasonable (say, weak) solution in the case of a singular potential like (1.2), it is always assumed that the spatial average of the initial datum must belong to $(-1, 1)$ (see, for instance, [18], Theorem 2.2 and [47], Definition 6.1, Theorem 6.1]). However, it must be noted that, in the case of degenerate mobility, pure phases are allowed to be solutions by a suitable splitting of the chemical potential (see [21], see also [23] and references therein for the nonlocal case). In the case of equation (1.4), this fact is less obvious (see [[54], Remark 4.2]).

Here, inspired by a biological model on the formation of condensates (see [32, 33] and references therein, see also [50]) which has been analysed in [42], we show that it is indeed possible to establish the existence of a weak solution to Cahn–Hilliard equations (1.4) and (1.5) with F like, e.g., (1.2), equipped with no-flux boundary conditions, and subject to a suitable reaction term, if the initial datum is a pure phase.

More precisely, letting $d \in \{2, 3\}$, we consider the following initial and boundary value problems

$$\partial_t \varphi - \Delta\mu = S(x, \varphi, g) \qquad \text{in } Q, \tag{1.6}$$
$$\mu = -\Delta\varphi + F'(\varphi) + g \qquad \text{in } Q, \tag{1.7}$$
$$\partial_{\mathbf{n}}\varphi = \partial_{\mathbf{n}}\mu = 0 \qquad \text{on } \Sigma, \tag{1.8}$$
$$\varphi(0) = -1 \qquad \text{in } \Omega, \tag{1.9}$$

and

$$\partial_t \varphi - \Delta\mu = S(x, \varphi, g) \qquad \text{in } Q, \tag{1.10}$$
$$\mu = a\varphi - K * \varphi + F'(\varphi) + g \qquad \text{in } Q, \tag{1.11}$$
$$\partial_{\mathbf{n}}\mu = 0 \qquad \text{on } \Sigma, \tag{1.12}$$
$$\varphi(0) = -1 \qquad \text{in } \Omega, \tag{1.13}$$

where $S(x, \varphi, g)$ is a reaction term and g is a given source, possibly depending on time. We incidentally notice that in the above system, we abuse a bit of notation to denote the function $S(x, \varphi, g)$ by explicitly writing the space variable $x \in \Omega$ which is not specified for all the other variables. As no confusion may arise, that convention will be in force throughout the paper.

The main technical difficulty of the entire work derives from the fact that we can no longer prove the usual property that μ belongs to $L^2(0, T; L^2(\Omega))$, like in [49, 53] (see also [45, 51] for the nonlocal case with degenerate mobility), through an inequality like ([54], Proposition A.2] (see also [[36], Section 5]). This is due to the fact that in our scenario, we start with a pure phase. However, here the mass is not

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conserved because of the reaction term. Thus, under appropriate assumptions, we can show that the spatial average of φ instantly detaches from -1 (i.e., the phase becomes mixed) and to explicitly compute the behaviour close to -1 for small times. This is based on *ad hoc* qualitative analysis of evolution of the spatial average of φ given by

$$(\varphi(t))_\Omega := \frac{1}{|\Omega|}\int_\Omega \varphi(x,t)\,\mathrm{d}x,$$

where $|\Omega|$ denotes the d-Lebesgue measure of Ω, which obeys the Cauchy problem

$$y'(t) = (S(\cdot,\varphi(t),g(t)))_\Omega \quad \text{for all } t\in[0,T], \qquad y(0) = -1, \tag{1.14}$$

where $y(t) := (\varphi(t))_\Omega$. To obtain the above relation, we integrate equation (1.6) ((1.10) resp.) over the domain Ω and then divide by $|\Omega|$.

The second step consists in generalizing the above-mentioned inequality by explicitly tracking the behaviour of $F'(\varphi)$ close to the pure phase. As we shall see, this argument leads to a local in time L^p-control in time of μ for $p\in(1,2)$ which is enough to recover the existence of a weak solution. The fact that we start from -1 is actually just a matter of choice. We could also start from 1 by just taking $-S$ in place of S. This essentially means that we are removing mass instead of adding it.

The analysis of (1.14) imposes some restrictions on S (see below). Nonetheless, we can handle, for instance, two well-known types of Cahn–Hilliard equations with reaction terms, namely, a Cahn–Hilliard–Oono type equation (see [41], see also [38] for theoretical results) and the inpainting model proposed by Bertozzi et al. (see [5, 6], see also [10] for the case of logarithmic potential and [46] for the nonlocal case). The former can be deduced by taking $g\equiv 0$ and setting

$$S(x,\varphi,g) = S(\varphi) = -m(\varphi - c), \tag{1.15}$$

where $m>0$ and $c\in(-1,1)$ are given. With reference to the (local) model, we recall that c is related to the backward and forward reaction rates of the chemical reaction (cf. [41]). Observe that this is an off-critical case and the mass is not conserved. Regarding the latter, we have $g\equiv 0$ and

$$S(x,\varphi,g) = S(x,\varphi) = -\lambda_0\chi_U(x)(\varphi - c(x)) \tag{1.16}$$

where $\lambda_0>0$, χ_U is the characteristic function of a given measurable set U (the complement of the inpainting region) such that $0<|U|<|\Omega|$ and $c:\Omega\to[-1,1]$ is a given bounded function (i.e., the given image). In this case, taking -1 as initial datum means that one can start with a totally white image. Then, the image φ evolves keeping itself closer to c in the undamaged region U and diffusing into the damaged one. However, as we shall see (cf. Remark 2.2 below), c can only be taken close to ±1 values, provided that $|U|$ is large enough.

Originally, reaction terms like (1.15) and (1.16) have been proposed in the local case only. Here we also consider and analyse their nonlocal counterpart. In addition, in Section 5, we present an application to (solid) tumour growth models, where the presence of the source term g becomes meaningful. This term, in our abstract models, is motivated by the fact that very often the corresponding systems are coupled with other equations (for instance, reaction-diffusion processes). Thus, including g is the first step towards possible generalizations of these results to more refined models.

Here, for a certain class of S, we prove the existence of a weak solution on a given time interval and, in the case (1.15), its uniqueness (see Sections 2 and 3). However, note that, the spatial average of φ instantaneously belongs to $(-1,1)$. Hence, all the known results apply. Further issues and possible generalizations to other models will be discussed in Section 6.

2. Notation, assumptions and main results

2.1 Notation

To begin with, let us introduce some useful notation. For a given (real) Banach space X, we employ $\|\cdot\|_X$, X^* and $\langle\cdot,\cdot\rangle_X$ to indicate, in the order, the norm of X, its topological dual and the related duality pairing

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between X^* and X. Lebesgue and Sobolev spaces defined on Ω, for every $1 \le p \le \infty$ and $k \ge 0$, are denoted by $L^p(\Omega)$ and $W^{k,p}(\Omega)$, with associated norms $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_p$ and $\|\cdot\|_{W^{k,p}(\Omega)}$, respectively. When $p = 2$, these become Hilbert spaces and we use $\|\cdot\| = \|\cdot\|_2$ for the norm of $L^2(\Omega)$ along with $(\cdot,\cdot)$ for the standard inner product, and account for the standard notation $H^k(\Omega) := W^{k,2}(\Omega)$. Moreover, we set

$$H := L^2(\Omega), \qquad V := H^1(\Omega), \qquad W := \{v \in H^2(\Omega) : \partial_{\mathbf{n}} v = 0 \text{ on } \partial\Omega\}.$$

For any given $v \in V^*$, we define its generalized mean value as $v_\Omega := \frac{1}{|\Omega|}\langle v, 1\rangle_V$, and we recall the Poincaré–Wirtinger inequality:

$$\|v - v_\Omega\| \le c_\Omega \|\nabla v\|, \quad v \in V. \tag{2.1}$$

Here $c_\Omega > 0$ depends on Ω and d only. Besides, we denote by V_0, H_0, V_0^* the closed subspaces of V, H and V^*, respectively, consisting of functions (or functionals) with zero spatial average. Then, we define the linear bounded operator $\mathcal{R} : V \to V^*$ as

$$\langle \mathcal{R}u, v\rangle_V := \int_\Omega \nabla u \cdot \nabla v, \quad u, v \in V.$$

It is well-known that the restrictions $\mathcal{R}|_{V_0}$ yields an isomorphism from V_0 to V_0^* with well defined inverse $\mathcal{R}^{-1} =: \mathcal{N} : V_0^* \to V_0$. Furthermore, it holds that

$$\langle \mathcal{R}u, \mathcal{N}v^*\rangle_V = \langle v^*, u\rangle_V, \qquad u \in V,\ v^* \in V_0^*,$$
$$\langle v^*, \mathcal{N}w^*\rangle_V = \int_\Omega \nabla(\mathcal{N}v^*) \cdot \nabla(\mathcal{N}w^*), \qquad v^*, w^* \in V_0^*.$$

Also, we have that

$$\|v^*\|_* := \|\nabla(\mathcal{N}v^*)\| = (\nabla(\mathcal{N}v^*), \nabla(\mathcal{N}v^*))^{1/2} = \langle v^*, \mathcal{N}v^*\rangle_V^{1/2}, \quad v^* \in V_0^*,$$

is a norm in V_0^* with associated inner product denoted by $(\cdot,\cdot)_* := (\nabla(\mathcal{N}\cdot), \nabla(\mathcal{N}\cdot))$. Similarly, setting

$$\|v^*\|_{-1} := \|v^* - (v^*)_\Omega\|_* + |(v^*)_\Omega|, \qquad v^* \in V^*,$$

we get an equivalent norm in V^*. Finally, we recall the identity

$$\langle \partial_t v^*(t), \mathcal{N}v^*(t)\rangle_V = \frac{1}{2}\frac{d}{dt}\|v^*(t)\|_*^2 \quad \text{for a.e. } t \in (0,T), \qquad v^* \in H^1 0,T;V_0^*,$$

and the inequality

$$|v_\Omega| \le c\,\|v\|_{V^*}, \quad v \in V^*. \tag{2.2}$$

Identifying H with its dual H^* by employing the scalar product of H, we obtain the chain of continuous and dense embeddings $V \hookrightarrow H \hookrightarrow V^*$. This implies the following interpolation inequality:

$$\|v\| = \langle v, v\rangle_V^{1/2} = \langle \mathcal{R}v, \mathcal{N}v\rangle_V^{1/2} \le \|v\|_V^{1/2}\,\|\mathcal{N}v\|_V^{1/2} \le \|v\|_V^{1/2}\,\|v\|_*^{1/2}, \quad v \in V_0. \tag{2.3}$$

Moreover, we will also use the following inequality following from Ehrling's lemma and the compact inclusion $V \hookrightarrow H$:

$$\forall\, \delta > 0, \quad \exists\, C_\delta > 0: \quad \|v\| \le \delta\,\|\nabla v\| + C_\delta\,\|v\|_{-1}, \quad v \in V. \tag{2.4}$$

2.2 Assumptions

The following structural assumptions are in order:

A1 The double-well shaped potential F enjoys the splitting $F = \widehat{\beta} + \widehat{\pi}$, where

$$\widehat{\beta} \in C^0([-1,1]) \cap C^2(-1,1) \text{ is strictly convex, nonnegative, and } \widehat{\beta}(0) = 0, \tag{2.5}$$
$$\widehat{\pi} \in C^1(\mathbb{R}), \tag{2.6}$$
$$\pi := \widehat{\pi}' : \mathbb{R} \to \mathbb{R} \text{ is } L_0\text{-Lipschitz continuous for some } L_0 > 0. \tag{2.7}$$

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On account of conditions (2.5)–(2.7), using convex analysis we can extend $\widehat{\beta}$ to $+\infty$ outside $[-1, 1]$. Thus, the subdifferential $\partial\widehat{\beta} =: \beta$ yields a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with corresponding domain $D(\beta) := \{r \in \mathbb{R} : \beta(r) \neq \emptyset\}$. In addition, since $\widehat{\beta} \in C^1(-1, 1)$, it readily follows that β is single-valued with $\beta(0) = 0$ and $F' = \beta + \pi$ in $(-1, 1)$. Finally, we suppose that:

$$\lim_{r \searrow (-1)^+} \beta(r) = -\infty, \quad \lim_{r \nearrow 1^-} \beta(r) = +\infty, \tag{2.8}$$

$$\exists\, \rho \in (0, 1): \quad \beta \in L^2(-1, -1 + \rho). \tag{2.9}$$

A2 We assume that g can be written as $g = g_0 + \tilde{g}$, where $g_0 \in V$, $\tilde{g} := g_1 + g_2$, $g_1 \in L^\infty(0, T; H) \cap L^2(0, T; V)$, and $g_2 \in H^1(0, T; H)$. Moreover, we require in the local case that

$$\forall\, \alpha \in (0, 1), \quad \exists\, C_g > 0: \quad \|\tilde{g}\|_{L^\infty(0,t;H)} \leq C_g t^\alpha \quad \forall\, t \in [0, T], \tag{2.10}$$

whereas in the nonlocal case:

$$\begin{cases} g_0 \in W, \\ \tilde{g} \in L^\infty(0, T; V), \\ \forall\, \alpha \in (0, 1),\ \exists\, C_g > 0:\ \|\nabla \tilde{g}\|_{L^\infty(0,t;H)} \leq C_g t^\alpha \quad \forall\, t \in [0, T]. \end{cases} \tag{2.11}$$

A3 Let the source S be defined as

$$S(x, \varphi, g) := -m(x)\varphi + h(x, \varphi, g), \tag{2.12}$$

where

$$m \in L^\infty(\Omega), \qquad m \geq 0 \quad \text{a.e.in } \Omega, \qquad m_\Omega > 0, \tag{2.13}$$

$$h : \Omega \times [-1, 1] \times \mathbb{R} \to \mathbb{R} \quad \text{is a Carathéodory function such that} \tag{2.14}$$

$$h(\cdot, u, v) \in L^\infty(\Omega), \quad \forall\, (u, v) \in [-1, 1] \times \mathbb{R},$$

$$h(x, \cdot, \cdot) \in C^0([-1, 1] \times \mathbb{R}), \quad \text{for a.a. } x \in \Omega, \tag{2.15}$$

and the following compatibility condition between h and m holds:

$$\exists\, \overline{h} > 0: |h(x, u, v)| \leq \overline{h} < m_\Omega, \quad \text{for a.a. } x \in \Omega,\ \forall\, (u, v) \in [-1, 1] \times \mathbb{R}. \tag{2.16}$$

We also set for convenience $\overline{m} := \|m\|_{L^\infty(\Omega)}$ and note that $\overline{m} > 0$ by (2.13).

Remark 2.1. *Assumption (2.16) means that the magnitude of h must be controlled by the space average of m. Note that this allows in particular the degenerate choice*

$$m(x) = \begin{cases} \overline{m} & \text{if } x \in U, \\ 0 & \text{if } x \in \Omega \setminus U, \end{cases} \tag{2.17}$$

where $\overline{m} > 0$ is a fixed constant and $U \subset \Omega$ is a measurable subset such that $0 < |U| < |\Omega|$. Thus, we have $\overline{h} := \operatorname{ess\,sup}_{x,u,v} |h(x, u, v)| < m_\Omega = \frac{|U|}{|\Omega|}\,\overline{m}$.

Remark 2.2. *Let us set $g = 0$ in equation (1.7) ((1.11) resp.). Then, we observe that the Cahn–Hilliard–Oono equation (see (1.15)) can be recovered by taking h and m constants, that is,*

$$h(x, u, v) = mc, \quad x \in \Omega,$$

where $c \in (-1, 1)$ is given. Another meaningful example is provided by the equation proposed for the inpainting (see (1.16)). In this case, we have

$$h(x, u, v) = m(x)c(x)$$

where m is given by (2.17), $c \in L^\infty(\Omega)$ is such that $c(x) \in [-1, 1]$ for almost every $x \in \Omega$, and $\|c\|_{L^\infty(\Omega)} < \frac{|U|}{|\Omega|}$. Observe that the latter condition is a bit restrictive since c cannot take ± 1 values. However, we can say that the smaller the damaged region is, the closer to ± 1 c can be taken. On the other hand, if we consider, for instance, F given by (1.2), then φ cannot take the values ± 1 either. However, in this case, we can take ϑ small enough in order to have minima as close as possible to

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± 1. *For further details concerning this issue, see [[49], Subsection 2.1] and the references provided therein.*

Remark 2.3. *We observe that the Cahn–Hilliard equation with a proliferation term of the form (see [48]) $S(\varphi) := \alpha(1-\varphi^2)$, $\alpha > 0$, is not covered by our assumptions.*

A4 The spatial convolution kernel $K \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ is such that

$$K(x) = K(-x), \quad \text{for a.a. } x \in \mathbb{R}^d.$$

Furthermore, we postulate that

$$a_* := \inf_{x\in\Omega} \int_\Omega K(x-y)\,dy = \inf_{x\in\Omega} a(x) \geq 0,$$

$$a^* := \sup_{x\in\Omega} \int_\Omega |K(x-y)|\,dy < +\infty, \qquad b^* := \sup_{x\in\Omega} \int_\Omega |\nabla K(x-y)|\,dy < +\infty,$$

and set $c_a := \max\{a^* - a_*, 1\} > 0$. Finally, we suppose that

$$\exists\, C_0 > 0 : \quad a_* + F''(r) \geq C_0, \qquad \forall\, r \in (-1,1).$$

In order to handle the fact that the initial datum is a pure phase, we will strongly exploit the following qualitative result, which is one of the main technical novelties of this work.

Proposition 2.4. *Assume **A1**. Then, the following properties hold.*

(P1) For every compact subset $[r_*, r^*] \subset (-1,1)$, $-1 < r_* \leq r^* < 1$, there exist constants $c_*, C_* > 0$ such that, for every $r \in (-1,1)$ and for every $r_0 \in [r_*, r^*]$ it holds that

$$c_*|\beta(r)| \leq \beta(r)(r - r_0) + C_*. \tag{2.18}$$

(P2) For every $\delta \in (0,1)$, there exist constants $c_\beta, C_\beta > 0$ such that, for every $r \in (-1,1)$ and for every $r_0 \in (-1, -1+\delta)$ it holds that

$$c_\beta(r_0+1)|\beta(r)| \leq \beta(r)(r-r_0) + \big[(r+1) + C_\beta(r_0+1)\big]\,|\beta(-1+(r_0+1)/2)|. \tag{2.19}$$

In particular, this implies that, for every $\delta \in (0,1)$, there exist $c_\beta, C_\beta > 0$ such that, for every measurable $\phi : \Omega \to (-1,1)$ with $\phi_\Omega \in (-1, -1+\delta)$, we have

$$c_\beta(\phi_\Omega + 1) \int_\Omega |\beta(\phi)| \leq \int_\Omega \beta(\phi)(\phi - \phi_\Omega) + C_\beta(\phi_\Omega + 1)|\beta(-1+(\phi_\Omega+1)/2)|. \tag{2.20}$$

Note that the first property (P1) is a classical result in the literature of Cahn–Hilliard problems and dates back to the ideas in [[36], Section 5] and [54]: this is typically used to control the L^1-norm in space of $\beta(\varphi)$ whenever the mass of the initial datum, represented by r_0 in the above estimates, is confined in the interior of the physical domain $(-1,1)$. Nevertheless, in our case the initial datum is a pure phase so, unfortunately, the first property (P1) cannot be employed as the constants c_* and C_* strongly depend on r_* and r^*. The main technical novelty that we present is then the refinement of such estimate, i.e., property (P2): here, the qualitative behaviour of the constants appearing in (2.18) is explicitly tracked when r_0 approaches the pure phase -1. Note that the constants c_β, C_β appearing in (2.19) depend only on the threshold value δ, but are *independent of r and r_0*. The role of the parameter δ is only technical as it represents the closeness of r_0 to the pure phase: for example, one can think to fix $\delta = \frac{1}{3}$. For the corresponding proof, we refer the reader to the Appendix.

We point out that the Flory–Huggins potential (1.2) satisfies A1. Thus, it also enjoys the two properties (P1)–(P2). Let us note that Proposition 2.4 also holds in the setting of a three-phase Flory–Huggins potential, for which we refer to [[42], Appendix A].

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2.3 Main results

The existence results for systems (1.6) – (1.9) and (1.10) – (1.13) read as follows.

Theorem 2.5 (local case). *Suppose that **A1–A3** hold and let the initial datum be the pure phase*

$$\varphi_0 \equiv -1.$$

Then, problem (1.6)–(1.9) admits a weak solution (φ, μ). *Namely, it holds that*

$$\begin{aligned}
&\varphi \in H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W),\\
&\varphi \in L^4(\sigma,T;W) \cap L^2(\sigma,T;W^{2,\gamma}(\Omega)) \quad \forall\, \sigma \in (0,T),\\
&\varphi \in L^{2p}W \cap L^p(0,T;W^{2,\gamma}(\Omega)) \quad \forall\, p \in (1,\overline{p}),\\
&\varphi \in L^\infty(Q), \quad |\varphi(x,t)| < 1 \quad \textit{for a.e. } (x,t) \in Q,\\
&\mu \in L^p(0,T;V) \cap L^2(\sigma,T;V) \quad \forall\, p \in (1,\overline{p}), \quad \forall\, \sigma \in (0,T),\\
&\nabla\mu \in L^2(0,T;H),
\end{aligned}$$

where

$$\gamma \in \begin{cases} [1,6] & \textit{if } d=3,\\ [1,+\infty) & \textit{if } d=2, \end{cases} \qquad \overline{p} := \begin{cases} \frac{4}{3} & \textit{if } g_0 \in V \setminus W,\\ 2 & \textit{if } g_0 \in W, \end{cases} \tag{2.21}$$

such that

$$\mu = -\Delta\varphi + F'(\varphi) + g, \quad \textit{a.e. in } Q,$$

and the variational equality

$$\langle \partial_t\varphi, v\rangle_V + \int_\Omega \nabla\mu \cdot \nabla v = \int_\Omega S(\cdot,\varphi,g)v$$

is satisfied for every test function $v \in V$, *and almost everywhere in* $(0,T)$. *Moreover, it holds that*

$$\varphi(0) = -1 \quad \textit{a.e. in } \Omega.$$

Theorem 2.6 (nonlocal case). *Suppose that **A1–A4** hold and let the initial datum be the pure phase*

$$\varphi_0 \equiv -1.$$

Then, problem (1.10)–(1.13) admits a weak solution (φ, μ). *Namely, it holds that*

$$\begin{aligned}
&\varphi \in H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V),\\
&\varphi \in L^\infty(Q), \quad |\varphi(x,t)| < 1 \quad \textit{for a.e. } (x,t) \in Q,\\
&\mu \in L^p(0,T;V) \cap L^2(\sigma,T;V) \quad \forall\, p \in (1,2), \quad \forall\, \sigma \in (0,T),\\
&\nabla\mu \in L^2(0,T;H),
\end{aligned}$$

such that

$$\mu = a\varphi - K * \varphi + F'(\varphi) + g, \quad \textit{a.e. in } Q,$$

and the variational equality

$$\langle \partial_t\varphi, v\rangle_V + \int_\Omega \nabla\mu \cdot \nabla v = \int_\Omega S(\cdot,\varphi,g)v$$

is satisfied for every test function $v \in V$, *and almost everywhere in* $(0,T)$. *Moreover, it holds that*

$$\varphi(0) = -1 \quad \textit{a.e. in } \Omega.$$

Remark 2.7. *It is worth mentioning that the standard assumptions* $\varphi_0 \in V$, $F(\varphi_0) \in L^1(\Omega)$ *and* $(\varphi_0)_\Omega \in (-1,1)$ *entail that all regularities in the above theorems also hold for* $\sigma = 0$ *as the usual approach can*

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be applied. The same goes if the initial condition $\varphi(0) \equiv -1$ *is approximated by* $\varphi(0) = \varphi_{0,\delta} := -1 + \delta$ *for some* $\delta \in (0, 1)$.

Finally, assuming that h is constant, we establish a continuous dependence estimate on the initial datum, which may assume the value -1. Extending this result to more general reaction terms remains an open issue. Under this additional assumption, we obtain the following result.

Theorem 2.8 (local case). *Suppose that* ***A1*** *–* ***A3*** *hold and* h *is independent of the phase variable* φ. *Let* $\varphi_{0j} \in V$ *be such that* $(\varphi_{0j})_\Omega \in [-1, 1)$, $j = 1, 2$ *and denote by* φ_j *a weak solution originating from* φ_{0j}. *Then, there exists* $C > 0$, *depending on* T, Ω, F, m, *and* h, *such that*

$$\begin{aligned} &\|(\varphi_2 - \varphi_1)(t)\|_{-1}^2 + |(\varphi_2 - \varphi_1)_\Omega(t)| + \int_0^t \|(\varphi_2 - \varphi_1)(s)\|_V^2 \, ds \\ &\quad \leq C\|\varphi_{02} - \varphi_{01}\|_{-1}^2 + C|(\varphi_{02})_\Omega - (\varphi_{01})_\Omega|, \quad \forall\, t \in [0, T]. \end{aligned} \tag{2.22}$$

Theorem 2.9 (nonlocal case). *Suppose that* ***A1–A4*** *hold and* h *is independent of the phase variable* φ. *Let* $\varphi_{0j} \in H$ *be such that* $(\varphi_{0j})_\Omega \in [-1, 1)$, $j = 1, 2$ *and denote by* φ_j *a weak solution originating from* φ_{0j}. *Then, there exists* $C > 0$, *depending on* T, Ω, K, F, m, *and* h, *such that*

$$\begin{aligned} &\|(\varphi_2 - \varphi_1)(t)\|_{-1}^2 + |(\varphi_2 - \varphi_1)_\Omega(t)| + \int_0^t \|(\varphi_2 - \varphi_1)(s)\|^2 \, ds \\ &\quad \leq C\|\varphi_{02} - \varphi_{01}\|_{-1}^2 + C|(\varphi_{02})_\Omega - (\varphi_{01})_\Omega|, \quad \forall\, t \in [0, T]. \end{aligned} \tag{2.23}$$

Thus, we can conclude that our problems are well posed, provided that the function h does not depend on the phase variable φ. This holds, in particular, for the Cahn–Hilliard–Oono equation (see (1.15)), where h is a constant, as well as for the inpainting equation (see (1.16)).

3. Existence of weak solutions

The strategy of the proofs of Theorem 2.5 and Theorem 2.6 is outlined here below. The local and nonlocal cases slightly differ in the type of estimates and regularities we can obtain. Indeed, the fourth-order nature of the local Cahn–Hilliard equation allows higher regularity with respect to the nonlocal one.

(I) Approximation. The first step consists in considering suitable approximate problems. Both in the local and the nonlocal case, the approximation only consists in shifting to the right the pure phase initial datum -1, so that we can recover the classical assumption that the initial mass is strictly confined in the physical domain $(-1, 1)$. More precisely, for every approximation parameter $\lambda \in (0, 1)$ one considers the family of problems starting from the initial datum $\varphi_{0,\lambda} := -1 + \lambda$ instead, and with respect to the same singular potential F.

(II) Preliminary estimate on the phase variable. By exploiting the Cahn–Hilliard structure, a suitable preliminary, uniform with respect to λ, estimate on the phase variable φ_λ can be derived without involving the chemical potential μ_λ. This is the first information to evaluate the rate at which φ_λ reaches its initial value for small times.

(III) Qualitative analysis of the mean value. Before proceeding with further uniform estimates, a control of the spatial mean of φ_λ is necessary. In particular, in order to exploit (P1) and (P2), one needs to prove that $(\varphi_\lambda)_\Omega$ remains detached from -1 for positive times, uniformly in λ. This is not obvious as the initial datum only satisfies $(\varphi_\lambda(0))_\Omega = -1 + \lambda$. The qualitative behaviour of $(\varphi_\lambda)_\Omega$ is obtained via an analysis of the ODE in (1.14). This allows to show that the mean values instantly detach from -1 at a linear rate, uniformly in λ.

(IV) Energy estimate. By using the above information, the standard Cahn–Hilliard weak energy estimate can be now performed on the λ-approximation level. Namely, we can test the (approximated version of the) first equation (1.6) ((1.10) resp.) by μ_λ and the second (1.7) ((1.11) resp.) by $\partial_t \varphi_\lambda$. As is well known for Cahn–Hilliard-type systems, this procedure produces just a L^2-control

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of $\nabla\mu_\lambda$ and no information is available for μ_λ itself due to the homogeneous Neumann boundary conditions. This issue is usually overcome by the Poincaré–Wirtinger inequality provided to show that $(\mu_\lambda)_\Omega$ is bounded in $L^2(0,T)$. Note that, in the present case, we also need to handle the inner product $(S(\cdot,\varphi_\lambda,g),\mu_\lambda)$. To this end, comparison in the second equation (1.7) ((1.11) resp.) reveals that this, roughly speaking, equals to controlling the singular term $\beta(\varphi_\lambda)$ in $L^2(0,T;L^1(\Omega))$. However, the singular nature of β and the pathological initial condition $\varphi_\lambda(0)=-1$ prevent us to get such a bound through the usual approach based on (P2). Nevertheless, hinging on (P2), we are able to suitably combine the qualitative estimates on the behaviour of $(\varphi_\lambda)_\Omega$ with the blow-up of the nonlinearity and to close the energy estimate. This yields directly a first L^2-control of $\nabla\mu_\lambda$. In order to gain a full control of the H^1-norm of μ_λ we are only left to show specific integrability properties of $(\mu_\lambda)_\Omega$. These follow again by a sharp qualitative analysis of the behaviour of $(\varphi_\lambda)_\Omega$, making use of (P1)–(P2). The only drawback in having a pure phase initial datum results in a slight loss of integrability of μ_λ close to the initial time. Notice that such a loss of information is almost negligible and seems reasonable to us in view of the initial pure phase. Indeed, by comparing the classical scenario of Cahn–Hilliard equations with ours, assuming g to fulfil either (2.10) or (2.11), it turns out that the correct spaces where one has to bound the chemical potentials μ_λ are

$$\underbrace{L^2(0,T;H^1(\Omega))}_{\text{in the classical setting}} \qquad \text{vs} \qquad \underbrace{\bigcap_{\sigma\in(0,T),\,p\in(1,\bar p)} L^2(\sigma,T;H^1(\Omega))\cap L^p(0,\sigma;H^1(\Omega))}_{\text{in our setting}}.$$

As noted before, the loss of integrability on the chemical potential caused by the pure phase initial datum is extremely light and absolutely reasonable if taking into account the singular nature of the problem.

(V) Passing to the limit. The final step consists of realizing that all the previous estimates are independent of λ so that weak and weak star compactness arguments can be exploited to rigorously justify the passage to the limit along with a suitable sequence $\lambda_k \searrow 0$. This procedure provides a limit pair (φ,μ) which fulfils suitable regularities and, in a suitable sense, yields a solution to the original problem (1.6)–(1.9) ((1.10)–(1.13) resp.). We highlight that, as expected from the above lines, the time integrability of μ becomes the usual one, as soon as we detach from $t=0$, i.e., on every time interval of the form (σ,T) for all $\sigma\in(0,T)$.

Since the above plan works for both the proofs of Theorem 2.5 and Theorem 2.6, we are going to present them in a unified fashion, just pointing out the main differences between the local and nonlocal cases. The proof will be divided into the aforementioned steps.

3.1 Step (I)

In what follows, we are going to indicate with C a generic positive constant, independent of λ, whose value may change from line to line. We now fix $\lambda\in(0,1)$ and introduce the λ-approximation of problems (1.6)–(1.9) and (1.10)–(1.13), respectively.

3.1.1 Local case

We consider the approximating problem

$$\partial_t\varphi_\lambda - \Delta\mu_\lambda = S(x,\varphi_\lambda,g) \qquad \text{in } Q, \tag{3.1}$$

$$\mu_\lambda = -\Delta\varphi_\lambda + F'(\varphi_\lambda) + g \qquad \text{in } Q, \tag{3.2}$$

$$\partial_{\mathbf{n}}\varphi_\lambda = \partial_{\mathbf{n}}\mu_\lambda = 0 \qquad \text{on } \Sigma, \tag{3.3}$$

$$\varphi_\lambda(0) = \varphi_{0,\lambda} := -1+\lambda \qquad \text{in } \Omega. \tag{3.4}$$

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Thanks to previous results (see, e.g., [53]), the above problem has a weak solution $(\varphi_\lambda, \mu_\lambda)$ such that

$$\varphi_\lambda \in H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), \quad \mu_\lambda \in L^2(0,T;V).$$

Moreover, due to the singularity of the potential F, we have $\varphi_\lambda \in L^\infty(Q)$ with

$$-1 \le \varphi_\lambda(x,t) \le 1 \quad \text{for a.e. } (x,t) \in Q$$

whence also the corresponding boundedness of S using the structural assumptions in A3.

3.1.2 Nonlocal case

In this case, we consider the following approximating problem

$$\partial_t \varphi_\lambda - \Delta \mu_\lambda = S(x, \varphi_\lambda, g) \quad \text{in } Q, \tag{3.5}$$

$$\mu_\lambda = a\varphi_\lambda - K * \varphi_\lambda + F'(\varphi_\lambda) + g \quad \text{in } Q, \tag{3.6}$$

$$\partial_{\mathbf{n}} \mu_\lambda = 0 \quad \text{on } \Sigma, \tag{3.7}$$

$$\varphi_\lambda(0) := \varphi_{0,\lambda} = -1 + \lambda \quad \text{in } \Omega. \tag{3.8}$$

We can adapt the argument in [[24], Section 3] for the existence (see also references therein). Summing up, there exists a weak solution $(\varphi_\lambda, \mu_\lambda)$ to the above problem, such that

$$\varphi_\lambda \in H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V), \quad \mu_\lambda \in L^2(0,T;V).$$

Moreover, arguing as above, it holds that $\varphi_\lambda \in L^\infty(Q)$ with

$$-1 \le \varphi_\lambda(x,t) \le 1 \quad \text{for a.e. } (x,t) \in Q,$$

and the corresponding boundedness of S.

3.2 Step (II)

Let us now establish some uniform a priori estimates for the phase field variable φ_λ.

3.2.1 Local case

Recalling (2.10), A2, and that $\varphi_{0,\lambda} = -1 + \lambda$, we test equation (3.1) by $\varphi_\lambda - \varphi_{0,\lambda} = \varphi_\lambda + 1 - \lambda$, equation (3.2) by $-\Delta(\varphi_\lambda - \varphi_{0,\lambda}) = -\Delta\varphi_\lambda$, and add the resulting identities together to obtain that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|\varphi_\lambda - \varphi_{0,\lambda}\right\|^2 + \|\Delta\varphi_\lambda\|^2 - \int_\Omega \beta(\varphi_\lambda)\Delta\varphi_\lambda \\ &\quad = \int_\Omega S(\cdot, \varphi_\lambda, g)(\varphi_\lambda - \varphi_{0,\lambda}) + \int_\Omega \pi(\varphi_\lambda)\Delta\varphi_\lambda + \int_\Omega g_0 \Delta\varphi_\lambda + \int_\Omega \tilde{g}\Delta\varphi_\lambda. \end{aligned}$$

As for the third term on the left-hand side, integrating by parts and using the convexity of $\widehat{\beta}$, we get

$$-\int_\Omega \beta(\varphi_\lambda)\Delta\varphi_\lambda = \int_\Omega \beta'(\varphi_\lambda)|\nabla\varphi_\lambda|^2 \ge 0.$$

Moreover, thanks to the boundedness of S, the first term on the right-hand side can be bounded as

$$\int_\Omega S(\cdot, \varphi_\lambda, g)(\varphi_\lambda - \varphi_{0,\lambda}) \le C\left\|\varphi_\lambda - \varphi_{0,\lambda}\right\|,$$

while using A1 and integration by parts, we get

$$\int_\Omega \pi(\varphi_\lambda)\Delta\varphi_\lambda \le L_0 \|\nabla\varphi_\lambda\|^2 = L_0\left\|\nabla(\varphi_\lambda - \varphi_{0,\lambda})\right\|^2 \le \frac{1}{4}\|\Delta\varphi_\lambda\|^2 + C\left\|\varphi_\lambda - \varphi_{0,\lambda}\right\|^2.$$

Furthermore, recalling A2 and integrating by parts yield

$$\int_\Omega g_0\Delta\varphi_\lambda + \int_\Omega \tilde{g}\Delta\varphi_\lambda \le C\left(\|\nabla\varphi_\lambda\| + t^\alpha \|\Delta\varphi_\lambda\|\right) \le \frac{1}{4}\|\Delta\varphi_\lambda\|^2 + C\left(t^{2\alpha} + \|\nabla\varphi_\lambda\|\right). \tag{3.9}$$

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Upon rearranging the terms, we obtain that

$$\begin{aligned}
&\frac{1}{2}\sup_{r\in[0,t]}\left\|\varphi_\lambda-\varphi_{0,\lambda}\right\|^2+\frac{1}{2}\int_0^t\|\Delta\varphi_\lambda(s)\|^2\,\mathrm{ds}\\
&\quad\le Ct^{2\alpha+1}+C\,\|\nabla\varphi_\lambda\|_{L^1(0,t;H)}+C\int_0^t\left(\left\|\varphi_\lambda(s)-\varphi_{0,\lambda}\right\|+\left\|\varphi_\lambda(s)-\varphi_{0,\lambda}\right\|^2\right)\mathrm{ds}\\
&\quad\le Ct^{2\alpha+1}+Ct^{\frac{3}{4}}\,\|\nabla\varphi_\lambda\|_{L^4(0,t;H)}+C\int_0^t\left(\left\|\varphi_\lambda(s)-\varphi_{0,\lambda}\right\|+\left\|\varphi_\lambda(s)-\varphi_{0,\lambda}\right\|^2\right)\mathrm{ds}.
\end{aligned}$$

By interpolation and the Young inequality, it readily follows that

$$\begin{aligned}
Ct^{\frac{3}{4}}\,\|\nabla\varphi_\lambda\|_{L^4(0,t;H)}&=Ct^{\frac{3}{4}}\left\|\nabla(\varphi_\lambda-\varphi_{0,\lambda})\right\|_{L^4(0,t;H)}\\
&\le Ct^{\frac{3}{4}}\,\|\Delta\varphi_\lambda\|^{\frac{1}{2}}_{L^2(0,t;H)}\left\|\varphi_\lambda-\varphi_{0,\lambda}\right\|^{\frac{1}{2}}_{L^\infty(0,t;H)}\\
&\le Ct^{\frac{3}{2}}+\frac{1}{2}\,\|\Delta\varphi_\lambda\|_{L^2(0,t;H)}\left\|\varphi_\lambda-\varphi_{0,\lambda}\right\|_{L^\infty(0,t;H)}\\
&\le Ct^{\frac{3}{2}}+\frac{1}{4}\,\|\Delta\varphi_\lambda\|^2_{L^2(0,t;H)}+\frac{1}{4}\left\|\varphi_\lambda-\varphi_{0,\lambda}\right\|^2_{L^\infty(0,t;H)},
\end{aligned}$$

from which, collecting all the terms, we deduce that

$$\begin{aligned}
&\frac{1}{4}\sup_{r\in[0,t]}\left\|\varphi_\lambda(r)-\varphi_{0,\lambda}\right\|^2+\frac{1}{4}\int_0^t\|\Delta\varphi_\lambda(s)\|^2\,\mathrm{ds}\\
&\quad\le C(t^{2\alpha+1}+t^{\frac{3}{2}})+C\int_0^t\left(\left\|\varphi_\lambda(s)-\varphi_{0,\lambda}\right\|+\left\|\varphi_\lambda(s)-\varphi_{0,\lambda}\right\|^2\right)\mathrm{ds}.
\end{aligned}\tag{3.10}$$

Thus, Gronwall's lemma and elliptic regularity theory yield

$$\|\varphi_\lambda\|_{L^\infty(0,T;H)\cap L^2(0,T;W)}\le C.\tag{3.11}$$

Using this information, we can go back to (3.10) and bound the right-hand side and infer that

$$\left\|\varphi_\lambda-\varphi_{0,\lambda}\right\|_{L^\infty(0,t;H)\cap L^2(0,t;W)}\le Ct^{1/2},\quad\forall t\in[0,T].$$

Noting that $2\alpha+1\ge\frac{3}{2}$ for some $\alpha\in(0,1)$, iterating the argument on the right-hand side of (3.10) yields

$$\forall\alpha\in[0,3/4],\quad\exists\,C_\alpha>0:\quad\left\|\varphi_\lambda-\varphi_{0,\lambda}\right\|_{L^\infty(0,t;H)\cap L^2(0,t;W)}\le C_\alpha t^\alpha,\quad\forall\,t\in[0,T],\tag{3.12}$$

where the constants C_α are independent of λ. Let us note that if, besides (2.10), it also holds that $g_0\in W$, one can see that (cf. (3.9))

$$\int_\Omega g_0\Delta\varphi_\lambda=\int_\Omega g_0\Delta(\varphi_\lambda-\varphi_{0,\lambda})=\int_\Omega\Delta g_0(\varphi_\lambda-\varphi_{0,\lambda})\le C\left\|\varphi_\lambda-\varphi_{0,\lambda}\right\|.\tag{3.13}$$

Hence, the same argument above allows us to infer (3.12), for every $\alpha\in[0,1)$.

3.2.2 Nonlocal case

We can argue as above. Namely, we test (3.5) by $\varphi_\lambda-\varphi_{0,\lambda}$, the gradient of equation (3.6) by $-\nabla(\varphi_\lambda-\varphi_{0,\lambda})=-\nabla\varphi_\lambda$, and add the resulting identities. This gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\varphi_\lambda-\varphi_{0,\lambda}\right\|^2+\int_\Omega F''(\varphi_\lambda)|\nabla\varphi_\lambda|^2+\int_\Omega\nabla(a\varphi_\lambda-K*\varphi_\lambda)\cdot\nabla\varphi_\lambda\\
&\quad=\int_\Omega S(\cdot,\varphi_\lambda,g)(\varphi_\lambda-\varphi_{0,\lambda})-\int_\Omega\nabla g_0\cdot\nabla\varphi_\lambda-\int_\Omega\nabla\tilde{g}\cdot\nabla\varphi_\lambda.
\end{aligned}$$

We note straightaway that, being $\varphi_{0,\lambda}=-1+\lambda$, by definition of a, one has that

$$\int_\Omega\nabla(a\varphi_{0,\lambda}-K*\varphi_{0,\lambda})\cdot\nabla\varphi_\lambda=\int_\Omega\nabla(-a+\lambda a+a-\lambda a)\cdot\nabla\varphi_\lambda=0.$$

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Taking this information into account, the second and third terms on the left-hand side can be bounded by recalling A4 along with Young's inequality as

$$\begin{aligned}
&\int_\Omega F''(\varphi_\lambda)|\nabla\varphi_\lambda|^2 + \int_\Omega \nabla(a\varphi_\lambda - K*\varphi_\lambda)\cdot\nabla\varphi_\lambda \\
&\quad= \int_\Omega F''(\varphi_\lambda)|\nabla\varphi_\lambda|^2 + \int_\Omega \nabla\big(a(\varphi_\lambda-\varphi_{0,\lambda}) - K*(\varphi_\lambda-\varphi_{0,\lambda})\big)\cdot\nabla\varphi_\lambda \\
&\quad= \int_\Omega (F''(\varphi_\lambda)+a)|\nabla\varphi_\lambda|^2 + \int_\Omega \big((\nabla a)(\varphi_\lambda-\varphi_{0,\lambda}) - (\nabla K)*(\varphi_\lambda-\varphi_{0,\lambda})\big)\cdot\nabla\varphi_\lambda \\
&\quad\geq C_0\,\|\nabla\varphi_\lambda\|^2 - 2b^*\,\|\varphi_\lambda-\varphi_{0,\lambda}\|\,\|\nabla\varphi_\lambda\| \\
&\quad\geq \frac{C_0}{2}\,\|\nabla\varphi_\lambda\|^2 - C\,\|\varphi_\lambda-\varphi_{0,\lambda}\|^2 .
\end{aligned}$$

On the other hand, recalling A2, (2.11), and using the boundedness of S, integration by parts and the Young inequality, we obtain

$$\begin{aligned}
&\int_\Omega S(\cdot,\varphi_\lambda,g)(\varphi_\lambda-\varphi_{0,\lambda}) - \int_\Omega \nabla g_0\cdot\nabla\varphi_\lambda - \int_\Omega \nabla\tilde{g}\cdot\nabla\varphi_\lambda \\
&\quad= \int_\Omega S(\cdot,\varphi_\lambda,g)(\varphi_\lambda-\varphi_{0,\lambda}) + \int_\Omega \Delta g_0(\varphi_\lambda-\varphi_{0,\lambda}) - \int_\Omega \nabla\tilde{g}\cdot\nabla\varphi_\lambda \\
&\quad\leq C\,\|\varphi_\lambda-\varphi_{0,\lambda}\| + Ct^\alpha\,\|\nabla\varphi_\lambda\| \\
&\quad\leq C\,\|\varphi_\lambda-\varphi_{0,\lambda}\| + \frac{C_0}{4}\,\|\nabla\varphi_\lambda\|^2 + Ct^{\alpha+1}.
\end{aligned}$$

Thus, upon rearranging terms, we find

$$\begin{aligned}
&\frac{1}{2}\,\|\varphi_\lambda(t)-\varphi_{0,\lambda}\|^2 + \frac{C_0}{4}\int_0^t \|\nabla\varphi_\lambda(s)\|^2\,\mathrm{d}s \\
&\qquad\leq Ct^{\alpha+1} + C\int_0^t \big(\,\|\varphi_\lambda(s)-\varphi_{0,\lambda}\| + \|\varphi_\lambda(s)-\varphi_{0,\lambda}\|^2\,\big)\,\mathrm{d}s
\end{aligned} \tag{3.14}$$

and Gronwall's lemma entails

$$\|\varphi_\lambda\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq C. \tag{3.15}$$

Then, noting that $\alpha+1\in(1,2)$ for all $\alpha\in(0,1)$, we argue as before by iteration to infer that

$$\forall\,\alpha\in[0,1),\quad \exists\,C_\alpha>0:\quad \|\varphi_\lambda-\varphi_{0,\lambda}\|_{L^\infty(0,t;H)\cap L^2(0,t;V)} \leq C_\alpha t^\alpha,\quad \forall\,t\in[0,T], \tag{3.16}$$

where the constants C_α are independent of λ.

3.3 Step (III)

Unlike the previous steps, this one is independent of the choice of the local or nonlocal model as the mass evolution of φ is always ruled by the first equation in (1.14) with initial condition $\varphi_{0,\lambda}$. More precisely, the aim of this step is to prove the following result.

Lemma 3.1. *In the current setting, there exists a constant $c_0\in(0,1)$, depending only on m and h, such that, for all $\lambda\in(0,1)$ and $t\in[0,T]$,*

$$-1\leq -1+c_0(1-e^{-\overline{m}t}) \leq (\varphi_\lambda(t))_\Omega \leq -1+(2-c_0)(1-e^{-\overline{m}t})+\lambda e^{-\overline{m}t}. \tag{3.17}$$

In particular, there are constants $c_1,c_2>0$ and $\lambda_0\in(0,1)$, depending only on m and h, such that, for all $\lambda\in(0,\lambda_0)$,

$$-1+2c_1t\leq(\varphi_\lambda(t))_\Omega\leq -1+\lambda+c_2(1-e^{-\overline{m}t})\leq 1-\lambda_0<1\qquad\forall\,t\in[0,T]. \tag{3.18}$$

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Proof of Lemma 3.1. Integrating (3.1) ((3.5) resp.) over Ω and multiplying by $1/|\Omega|$ lead to a Cauchy problem for the mean of the phase variable $y_\lambda := (\varphi_\lambda)_\Omega$ (cf. (1.14))

$$y_\lambda' = (S(\cdot, \varphi_\lambda, g))_\Omega, \quad y_\lambda(0) = -1 + \lambda,$$

Thus, recalling the boundedness of S, the associated Cauchy problem for y_λ reads as

$$\begin{cases} y_\lambda' + (m(\cdot)\varphi_\lambda)_\Omega = (h(\cdot, \varphi_\lambda, g))_\Omega \quad \text{in } (0, T), \\ y_\lambda(0) = -1 + \lambda, \end{cases}$$

which can be equivalently reformulated as

$$\begin{cases} y_\lambda' + \overline{m} y_\lambda = (h(\cdot, \varphi_\lambda, g))_\Omega + \overline{m} y_\lambda - (m(\cdot)\varphi_\lambda)_\Omega \quad \text{in}(0, T), \\ y_\lambda(0) = -1 + \lambda. \end{cases} \tag{3.19}$$

Recalling that $\overline{m} = \|m\|_{L^\infty(\Omega)} > 0$ and noting that

$$\overline{m} y_\lambda - (m(\cdot)\varphi_\lambda)_\Omega = \big((\overline{m} - m(\cdot))\varphi_\lambda\big)_\Omega,$$

we have the solution formula

$$y_\lambda(t) = (-1 + \lambda)e^{-\overline{m}t} + \int_0^t e^{-\overline{m}(t-s)} \Big((h(\cdot, \varphi_\lambda(s), g(s)))_\Omega + ((\overline{m} - m(\cdot))\varphi_\lambda(s))_\Omega \Big)\, \mathrm{d}s.$$

Let us focus on the estimate from below. Recalling that $|h| \le \overline{h}$, we deduce that

$$y_\lambda(t) + 1 \ge (1 - e^{-\overline{m}t}) + \lambda e^{-\overline{m}t} - \frac{\overline{h}}{\overline{m}}(1 - e^{-\overline{m}t}) + \int_0^t e^{-\overline{m}(t-s)} ((\overline{m} - m(\cdot))\varphi_\lambda(s))_\Omega \, \mathrm{d}s.$$

Now, since $|\varphi_\lambda| \le 1$ and $0 \le m \le \overline{m}$ in Ω, we have

$$((\overline{m} - m(\cdot))\varphi_\lambda)_\Omega = \frac{1}{|\Omega|} \int_\Omega (\overline{m} - m(\cdot))\varphi_\lambda \ge -\frac{1}{|\Omega|} \int_\Omega (\overline{m} - m(\cdot)) = -\overline{m} + m_\Omega.$$

Putting everything together, we infer that

$$y_\lambda(t) + 1 \ge \Big(1 - \frac{\overline{h}}{\overline{m}} - 1 + \frac{m_\Omega}{\overline{m}}\Big)(1 - e^{-\overline{m}t}) = \frac{1}{\overline{m}}\Big(m_\Omega - \overline{h}\Big)(1 - e^{-\overline{m}t}).$$

Since $\overline{h} < m_\Omega \le \overline{m}$, the estimate from below in (3.17) follows with the choice

$$c_0 := \frac{m_\Omega - \overline{h}}{\overline{m}} \in (0, 1).$$

Let us focus now on the estimate from above. Recalling that $|h| \le \overline{h}$, $|\varphi_\lambda| \le 1$, and $0 \le m \le \overline{m}$ in Ω, we infer that

$$\begin{aligned} y_\lambda(t) + 1 &\le (1 - e^{-\overline{m}t}) + \lambda e^{-\overline{m}t} + \frac{\overline{h}}{\overline{m}}(1 - e^{-\overline{m}t}) + \Big(\frac{\overline{m} - m_\Omega}{\overline{m}}\Big)(1 - e^{-\overline{m}t}) \\ &= \left(2 - \frac{m_\Omega - \overline{h}}{\overline{m}}\right)(1 - e^{-\overline{m}t}) + \lambda e^{-\overline{m}t} \\ &= (2 - c_0)(1 - e^{-\overline{m}t}) + \lambda e^{-\overline{m}t}, \end{aligned}$$

and (3.17) is proved. The second inequality in (3.17) is then a direct consequence of (3.17) by taking, for instance, $\lambda_0 := \frac{c_0}{2}$. □

Remark 3.2. *We highlight that Lemma 3.1 ensures that*

$$(\varphi_\lambda(t))_\Omega \in [-1, 1), \quad \forall\, t \in [0, T].$$

This will be a crucial property to motivate the interpretation of the chemical potential μ_λ for small times in the limit $\lambda \to 0$. Besides, as a straightforward consequence of the above analysis, we have that

$$\forall\, \sigma \in (0, T), \quad \exists\, c_\sigma \in (0, 1) : \quad -1 + c_\sigma \le (\varphi_\lambda(t))_\Omega < 1 - c_\sigma, \quad \forall\, t \in [\sigma, T], \tag{3.20}$$

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meaning that for every positive time σ, the mass of the phase variable φ_λ is confined in the physical interval $(-1,1)$ independently of λ. This is the standard condition that is usually required in the Cahn–Hilliard literature to infer from inequality (2.18) that $\beta(\varphi_\lambda) \in L^2(\sigma,T;L^1(\Omega))$. Hence, from comparison in (1.7) ((1.11) resp.), it follows that $(\mu_\lambda)_\Omega \in L^2(\sigma,T)$ and, via the Poincaré–Wirtinger inequality and the above energy estimate, one recovers that $\mu_\lambda \in L^2(\sigma,T;V)$.

3.4 Step (IV)

This step is devoted to the derivation of the energy estimates. The idea is to combine the classical energy estimate with the enhanced version (2.19) of the inequality (2.18). This enables to derive insights into the L^1-spatial norm of the nonlinear term $\beta(\varphi_\lambda)$ by the use of the evolution properties of the mean of φ_λ and the preliminary estimate.

3.4.1 Local case

We test (3.1) by μ_λ, (3.2) by $-\partial_t\varphi_\lambda$, add the resulting equalities and integrate in time, for any arbitrary $t \in (0,T]$. This gives

$$\begin{aligned}&\frac{1}{2}\|\nabla\varphi_\lambda(t)\|^2 + \int_\Omega F(\varphi_\lambda(t)) + \int_0^t \|\nabla\mu_\lambda(s)\|^2 \,\mathrm{d}s\\ &\quad = |\Omega|F(-1+\lambda) + \int_0^t\int_\Omega S(\cdot,\varphi_\lambda(s),g(s))\mu_\lambda(s)\,\mathrm{d}s - \int_0^t\int_\Omega g(s)\partial_t\varphi_\lambda(s)\,\mathrm{d}s. \end{aligned} \tag{3.21}$$

It readily follows from the previous estimates that

$$\int_\Omega F(\varphi_\lambda(t)) \geq \int_\Omega \widehat{\beta}(\varphi_\lambda(t)) - C.$$

As for the last term on the right-hand side, we point out that the notation is only formal. More precisely, recalling A2 and using again that $\varphi_\lambda(0) = -1+\lambda$, the formal integral is rigorously defined as follows

$$\begin{aligned}-\int_0^t\int_\Omega g(s)\partial_t\varphi_\lambda(s)\,\mathrm{d}s = &-\int_0^t \langle\partial_t\varphi_\lambda(s), g_0 + g_1(s)\rangle_V \,\mathrm{d}s - \int_\Omega g_2(t)\varphi_\lambda(t)\\ &- \int_\Omega g_2(0)(1-\lambda) + \int_0^t\int_\Omega \partial_t g_2(s)\varphi_\lambda(s)\,\mathrm{d}s.\end{aligned}$$

By comparison in (3.1), using the boundedness of S, A2, and (3.11), thanks to the Young inequality, we infer that

$$-\int_0^t\int_\Omega g(s)\partial_t\varphi_\lambda(s)\,\mathrm{d}s \leq \frac{1}{4}\int_0^t \|\nabla\mu_\lambda(s)\|^2 \,\mathrm{d}s + C.$$

Moreover, using again the boundedness of S, (3.2), and (3.11), we find

$$\begin{aligned}&\int_0^t\int_\Omega S(\cdot,\varphi_\lambda(s),g(s))\mu_\lambda(s)\,\mathrm{d}s\\ &\quad = \int_0^t\int_\Omega S(\cdot,\varphi_\lambda(s),g(s))(\mu_\lambda(s) - (\mu_\lambda(s))_\Omega)\,\mathrm{d}s + \int_0^t\int_\Omega S(\cdot,\varphi_\lambda(s),g(s))(\mu_\lambda(s))_\Omega\,\mathrm{d}s\\ &\quad \leq \frac{1}{4}\int_0^t \|\nabla\mu_\lambda(s)\|^2 \,\mathrm{d}s + C + C\int_0^t |(\mu_\lambda(s))_\Omega|.\end{aligned}$$

Observe now that, by comparison in equation (3.2), thanks to **A2** and (3.11), it holds

$$|\Omega||(\mu_\lambda)_\Omega| \leq \|\beta(\varphi_\lambda)\|_1 + \|\pi(\varphi_\lambda)\|_1 + \|g\|_1 \leq \|\beta(\varphi_\lambda)\|_1 + C.$$

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Putting everything together, we infer that there exists a constant $C_* > 0$, independent of λ, such that, for every $t \in [0, T]$,

$$\|\nabla\varphi_\lambda(t)\|^2 + \|\widehat{\beta}(\varphi_\lambda(t))\|_1 + \int_0^t \|\nabla\mu_\lambda(s)\|^2 \, \mathrm{d}s \le C_* + C_* \int_0^t \|\beta(\varphi_\lambda(s))\|_1 \mathrm{d}s. \tag{3.22}$$

Now, we go back to equation (3.2) tested by $\varphi_\lambda - (\varphi_\lambda)_\Omega$ and get

$$\|\nabla\varphi_\lambda\|^2 + \int_\Omega \beta(\varphi_\lambda)(\varphi_\lambda - (\varphi_\lambda)_\Omega) \le \int_\Omega \mu_\lambda\big(\varphi_\lambda - (\varphi_\lambda)_\Omega\big) - \int_\Omega (\pi(\varphi_\lambda) + g)\big(\varphi_\lambda - (\varphi_\lambda)_\Omega\big).$$

By exploiting the fact that $\varphi_{0,\lambda} = (\varphi_{0,\lambda})_\Omega = -1 + \lambda$, using the Poincaré–Wirtinger inequality, and assumptions **A1–A2**, one has that

$$\begin{aligned}
&\int_\Omega \mu_\lambda\big(\varphi_\lambda - (\varphi_\lambda)_\Omega\big) - \int_\Omega (\pi(\varphi_\lambda) + g)\big(\varphi_\lambda - (\varphi_\lambda)_\Omega\big) \\
&\quad = \int_\Omega (\mu_\lambda - (\mu_\lambda)_\Omega)\big(\varphi_\lambda - (\varphi_\lambda)_\Omega\big) - \int_\Omega (\pi(\varphi_\lambda) + g)\big(\varphi_\lambda - (\varphi_\lambda)_\Omega\big) \\
&\quad = \int_\Omega (\mu_\lambda - (\mu_\lambda)_\Omega)\big((\varphi_\lambda - \varphi_{0,\lambda}) - (\varphi_\lambda - (\varphi_{0,\lambda}))_\Omega\big) \\
&\qquad - \int_\Omega (\pi(\varphi_\lambda) + g)\big((\varphi_\lambda - \varphi_{0,\lambda}) - (\varphi_\lambda - (\varphi_{0,\lambda}))_\Omega\big) \\
&\quad \le C\big\|\nabla\mu_\lambda\big\|\big\|\varphi_\lambda - \varphi_{0,\lambda}\big\| + C(\big\|\varphi_\lambda\big\| + 1)\big\|\varphi_\lambda - \varphi_{0,\lambda}\big\|.
\end{aligned}$$

Here we have used the identity $\int_\Omega (\mu_\lambda)_\Omega\big(\varphi_\lambda - (\varphi_\lambda)_\Omega\big) = (\mu_\lambda)_\Omega \int_\Omega \big(\varphi_\lambda - (\varphi_\lambda)_\Omega\big) = 0$ as well as the fact that $\varphi_{0,\lambda}$ is constant and, therefore, coincides with its mean value $(\varphi_{0,\lambda})_\Omega$. Hence, thanks to the estimate (3.11) we deduce that

$$\int_\Omega \beta(\varphi_\lambda(s))(\varphi_\lambda(s) - (\varphi_\lambda(s))_\Omega) \le C(1 + \|\nabla\mu_\lambda(s)\|\,)\,\big\|\varphi_\lambda(s) - \varphi_{0,\lambda}\big\| \quad \forall\, s \in [0, T]. \tag{3.23}$$

Next, we note that (3.17) and (3.18) imply that there exists $T_0 \in (0, T)$ and $r_0 \in (0, 1)$, both independent of λ, such that, for every $\lambda \in (0, \lambda_0)$, it holds that $-1 < (\varphi_\lambda(s))_\Omega < -1 + r_0$ for all $s \in (0, T_0)$ and $-1 + r_0 \le (\varphi_\lambda(s))_\Omega \le 1 - r_0$ for all $s \in [T_0, T]$. Consequently, for $s \in (0, T_0)$ we can make use of estimate (2.20) with $\phi = \varphi_\lambda(s)$ and infer that

$$\begin{aligned}
&c_\beta((\varphi_\lambda(s))_\Omega + 1) \int_\Omega |\beta(\varphi_\lambda(s))| \\
&\quad \le \int_\Omega \beta(\varphi_\lambda(s))(\varphi_\lambda(s) - (\varphi_\lambda(s))_\Omega) \\
&\qquad + C_\beta((\varphi_\lambda(s))_\Omega + 1)|\beta(-1 + ((\varphi_\lambda(s))_\Omega + 1)/2)| \qquad \forall\, s \in (0, T_0).
\end{aligned} \tag{3.24}$$

Analogously, for $s \in [T_0, T]$ we can make use of estimate (2.18) and infer that

$$c_* \int_\Omega |\beta(\varphi_\lambda(s))| \le \int_\Omega \beta(\varphi_\lambda(s))(\varphi_\lambda(s) - (\varphi_\lambda(s))_\Omega) + C_*|\Omega| \qquad \forall\, s \in [T_0, T]. \tag{3.25}$$

Consequently, we can combine (3.23) evaluated at $s \in (0, T_0)$ with inequality (3.24), divide by $(\varphi_\lambda(s))_\Omega + 1$, and exploit the estimates (3.12) and (3.18) to get, for all $s \in (0, T_0)$ and $\alpha \in [0, \frac{3}{4}]$ (resp. $\alpha \in (0, 1)$ if $g_0 \in W$),

$$\begin{aligned}
c_\beta\|\beta(\varphi_\lambda(s))\|_1 &\le C\frac{\big\|\varphi_\lambda(s) - \varphi_{0,\lambda}\big\|}{(\varphi_\lambda(s))_\Omega + 1}(1 + \|\nabla\mu_\lambda(s)\|\,) + C_\beta|\beta(-1 + ((\varphi_\lambda(s))_\Omega + 1)/2)| \\
&\le \frac{CC_\alpha}{2c_1} s^{\alpha-1}(1 + \|\nabla\mu_\lambda(s)\|\,) + C_\beta|\beta(-1 + c_1 s)|.
\end{aligned} \tag{3.26}$$

Analogously, if we combine (3.23) evaluated at $s \in [T_0, T]$ with inequality (3.24) and estimate (3.11), we get, for all $s \in [T_0, T]$,

$$c_*\|\beta(\varphi_\lambda(s))\|_1 \le C(1 + \|\nabla\mu_\lambda(s)\|\,) + C_*|\Omega|. \tag{3.27}$$

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Collecting the above estimates, integrating with respect to $s \in (0,t)$, and using the Young inequality, for all $\delta > 0$, we find a constant $C_\delta > 0$, independent of λ, such that, for every $t \in [0,T]$,

$$\int_0^t \|\beta(\varphi_\lambda(s))\|_1 \mathrm{d}s \le \delta \int_0^t \|\nabla \mu_\lambda(s)\|^2 \,\mathrm{d}s + C_\delta + C_\delta \int_0^{T_0} s^{2(\alpha-1)} \mathrm{d}s + C \int_0^{T_0} |\beta(-1+c_1 s)| \mathrm{d}s.$$

As for the right-hand side, the last term makes sense by **A1** as

$$\int_0^{T_0} |\beta(-1+c_1 s)| \,\mathrm{d}s = \frac{1}{c_1} \int_{-1}^{-1+c_1 T_0} |\beta(s)| \,\mathrm{d}s \le C \,\|\beta\|_{L^1(-1,0)} \le C.$$

By choosing then $\alpha \in (\frac{1}{2}, \frac{3}{4}]$ (resp. $\alpha \in (\frac{1}{2}, 1)$ if $g_0 \in W$), we infer then that

$$\int_0^t \|\beta(\varphi_\lambda(s))\|_1 \mathrm{d}s \le \delta \int_0^t \|\nabla \mu_\lambda(s)\|^2 \,\mathrm{d}s + C_\delta \quad \forall\, t \in [0,T]. \tag{3.28}$$

Eventually, recalling that C_* is the constant appearing in (3.22), we multiply (3.28) by $2C*$ and sum it with inequality (3.22). By choosing δ small enough, for instance $\delta := \frac{1}{4C_*}$, and rearranging the terms, we deduce the bound

$$\|\varphi_\lambda\|_{L^\infty(0,T;V)} + \|\widehat{\beta}(\varphi_\lambda)\|_{L^\infty(0,T;L^1(\Omega))} + \|\beta(\varphi_\lambda)\|_{L^1(0,T;L^1(\Omega))} + \|\nabla \mu_\lambda\|_{L^2(0,T;H)} \le C. \tag{3.29}$$

From this, it is a standard matter to derive by comparison in equation (3.1) that

$$\|\partial_t \varphi_\lambda\|_{L^2(0,T;V^*)} \le C. \tag{3.30}$$

Once the above is established, we go back to (3.26)–(3.27), take square powers and exploit the fact that now $t \mapsto \|\nabla \mu_\lambda(t)\|$ is bounded in $L^2(0,T)$. To this end, we note that by **A1** one has

$$\int_0^{T_0} |\beta(-1+c_1 s)|^2 \,\mathrm{d}s = \frac{1}{c_1} \int_{-1}^{-1+c_1 T_0} |\beta(s)|^2 \,\mathrm{d}s \le C \,\|\beta\|^2_{L^2(-1,0)} \le C,$$

while for every $\ell \in (1,4)$ (resp. for every $\ell \in (1,+\infty)$ if $g_0 \in W$) there exists $\alpha \in (0, \frac{3}{4}]$ (resp. $\alpha \in (0,1)$ if $g_0 \in W$) such that $\ell(\alpha-1) > -1$. Hence, for every such ℓ, there exists also a constant $C_\ell > 0$, independent of λ, such that

$$\left\| t \mapsto t^{\alpha-1} \right\|_{L^\ell(0,T_0)} \le C_\ell.$$

Thus, going back to (3.26)–(3.27), we can now infer that

$$\forall\, p \in (1,\overline{p}), \quad \exists\, C_p > 0: \quad \|\beta(\varphi_\lambda)\|_{L^p(0,T;L^1(\Omega))} \le C_p,$$

with $\overline{p}$ being defined in (2.21). Next, comparison in equation (3.2) readily produces

$$\forall\, p \in (1,\overline{p}), \quad \exists\, C_p > 0: \quad \|(\mu_\lambda)_\Omega\|_{L^p(0,T)} \le C_p,$$

whence, using the Poincaré inequality, that

$$\forall\, p \in (1,\overline{p}), \quad \exists\, C_p > 0: \quad \|\mu_\lambda\|_{L^p(0,T;V)} \le C_p. \tag{3.31}$$

By monotonicity and comparison in (3.2), this leads by classical techniques also that

$$\forall\, p \in (1,\overline{p}), \quad \exists\, C_p > 0: \quad \|\beta(\varphi_\lambda)\|_{L^p(0,T;H)} \le C_p. \tag{3.32}$$

Lastly, arguing as in ([39], Lemmas 7.3–7.4) and using interpolation, we read (3.2) as a one-parameter family of time-dependent system of elliptic equation with maximal monotone nonlinearity β and infer that

$$\forall\, p \in (1,\overline{p}), \quad \exists\, C_p > 0: \quad \|\varphi_\lambda\|_{L^{2p}(0,T;H^2(\Omega)) \cap L^p(0,T;W^{2,\gamma}(\Omega))} \le C_p, \tag{3.33}$$

with γ being defined in (2.21).

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3.4.2 Nonlocal case

We can argue as above. Again, we test (3.5) by μ_λ, (3.6) by $-\partial_t\varphi_\lambda$ and add the resulting equalities. This produces

$$\frac{1}{4}\int_{\Omega\times\Omega} K(x-y)|\varphi_\lambda(t,x)-\varphi_\lambda(t,y)|^2\mathrm{dx}\,\mathrm{dy}+\int_\Omega F(\varphi_\lambda(t))+\int_0^t \|\nabla\mu_\lambda(s)\|^2\,\mathrm{ds}$$
$$=|\Omega|F(-1+\lambda)+\int_0^t\int_\Omega S(\cdot,\varphi_\lambda(s),g(s))\mu_\lambda(s)\,\mathrm{ds}-\int_0^t\int_\Omega g(s)\partial_t\varphi_\lambda(s)\,\mathrm{ds},$$

where the last term on the right-hand side has to be interpreted rigorously as before. On account of A4, we have

$$\int_{\Omega\times\Omega} K(x-y)|\varphi_\lambda(t,x)-\varphi_\lambda(t,y)|^2\mathrm{dx}\,\mathrm{dy}\le 2|\Omega|a^*\left\|\varphi_\lambda-\varphi_{\lambda,0}\right\|^2_{L^\infty(0,T;H)}.$$

Thus, reasoning as above and exploiting (3.15), we find that, for any $p\in(1,2)$, there exists $C_p>0$ such that

$$\|\partial_t\varphi_\lambda\|_{L^2(0,T;V^*)}+\|\varphi_\lambda\|_{L^\infty(0,T;H)}+\|\widehat{\beta}(\varphi_\lambda)\|_{L^\infty(0,T;L^1(\Omega))}+\|\beta(\varphi_\lambda)\|_{L^p(0,T;H)}$$
$$+\|\mu_\lambda\|_{L^p(0,T;V)}+\|\nabla\mu_\lambda\|_{L^2(0,T;H)}\le C_p. \tag{3.34}$$

3.5 Step (V)

The final step consists in passing to the limit as $\lambda\to 0$ along a suitable subsequence. Thus, everything that we are going to present from now on has to be in principle considered for a suitable subsequence. However, in order to keep notation as light as possible, we stipulate that every passage to the limit has to be intended this way.

3.5.1 Local case

Accounting for (3.11) and (3.30), we infer the existence of a function φ such that

$$\varphi\in H^1(0,T;V^*)\cap L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega)),$$

and that, as $\lambda\to 0$,

$$\varphi_\lambda\to\varphi\quad\text{weakly in}\quad H^1(0,T;V^*)\cap L^2(0,T;H^2(\Omega)),$$
$$\varphi_\lambda\to\varphi\quad\text{weakly-star in}\quad L^\infty(0,T;V).$$

The classical Aubin–Lions theorem then yields that, as $\lambda\to 0$,

$$\varphi_\lambda\to\varphi\quad\text{strongly in}\quad C^0([0,T];H)\cap L^2(0,T;V)$$

so that the initial condition $\varphi(0)=-1$ is fulfilled. Our final goal is to pass to the limit, as $\lambda\to 0$, in the weak formulation associated with (3.1)–(3.4) to show that φ_λ and μ_λ converge to some limit φ and μ that yield a solution to the weak formulation of problem (1.6)–(1.9). As far as β is concerned, estimate (3.32), the strong convergence of φ_λ, and the strong-weak closure of β as a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ yield that

$$\beta(\varphi_\lambda)\to\beta(\varphi)\quad\text{weakly in}\quad L^p(0,T;H)\quad\forall\,p\in(1,\overline{p}).$$

Let us stress that this latter holds along a suitable subsequence for every $p\in(1,\overline{p})$, in the sense that the choice of the subsequence can be done uniformly with respect to $p\in(1,\overline{p})$. This fact can be shown rigorously in the following way. First, by exploiting the estimate (3.32) with $p\in(1,\overline{p})$ (e.g., $p=\frac{\overline{p}+1}{2}$), one can select a suitable subsequence along which it holds that $\beta(\varphi_\lambda)\to\beta(\varphi)$ weakly in $L^{\frac{\overline{p}}{2}}(0,T;H)$. Secondly, let us fix such subsequence and prove that along such fixed subsequence it also holds that $\beta(\varphi_\lambda)\to\beta(\varphi)$ weakly in $L^p(0,T;H)$ for all $p\in(1,\overline{p})$. To this end, let $p\in(1,\overline{p})$ be arbitrary and let $(\beta(\varphi_{\lambda_k}))_k$ be an

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arbitrary subsequence of $(\beta(\varphi_\lambda))_\lambda$. Thanks to (3.32) it follows immediately that $(\beta(\varphi_{\lambda_k}))_k$ has a further sub-subsequence $(\beta(\varphi_{\lambda_{k_j}}))_j$ weakly converging to some limit in $L^p(0,T;H)$ as $j \to 0$. As we know already that $\beta(\varphi_\lambda) \to \beta(\varphi)$ weakly in $L^{\frac{\overline{p}}{2}}(0,T;H)$, it necessarily holds that $\beta(\varphi_{\lambda_{k_j}}) \to \beta(\varphi)$ weakly in $L^p(0,T;H)$ as $j \to 0$. This shows that every subsequence of $(\beta(\varphi_\lambda))_\lambda$ admits a further sub-subsequence weakly converging in $L^p(0,T;H)$ to $\beta(\varphi)$: this implies that the whole sequence $(\beta(\varphi_\lambda))_\lambda$ weakly converges to $\beta(\varphi)$ in $L^p(0,T;H)$ as $\lambda \to 0$, as desired. As for $\pi(\varphi_\lambda)$, we use the above strong convergence of the order parameter, along with its Lipschitz continuity, to infer that

$$\pi(\varphi_\lambda) \to \pi(\varphi) \quad \text{strongly in} \quad C^0([0,T];H),$$

and (see (2.13))

$$m(\cdot)\varphi_\lambda \to m(\cdot)\varphi \quad \text{strongly in} \quad C^0([0,T];H),$$

Besides, using the continuity of h with respect to its first argument, the boundedness of h, and the dominated convergence theorem, we get that, as $\lambda \to 0$,

$$S(\cdot,\varphi_\lambda,g) \to S(\cdot,\varphi,g) \quad \text{strongly in} \quad L^2H.$$

For the chemical potential we owe to (3.31) to infer that, as $\lambda \to 0$,

$$\begin{aligned} &\mu_\lambda \to \mu && \text{weakly in } L^p(0,T;V) \cap L^2(\sigma,T;V) && \forall\, p \in (1,\overline{p}),\ \forall\, \sigma \in (0,T), \\ &\nabla\mu_\lambda \to \nabla\mu && \text{weakly in } L^2(0,T;H). \end{aligned}$$

Exploiting again the strong convergence of φ_λ, we find from (3.33) that

$$\varphi \in \bigcap_{p\in(1,\overline{p})} L^{2p}(0,T;H^2(\Omega)) \cap L^p(0,T;W^{2,\gamma}(\Omega)), \tag{3.35}$$

with γ as in the statement. Observe that, arguing as above, one can actually select the converging subsequences uniformly with respect to p and σ. Next, we use all the above convergences to pass to the limit $\lambda \to 0$ in the variational formulation associated with (3.1) and find that

$$\langle \partial_t\varphi, v\rangle_V + \int_\Omega \nabla\mu \cdot \nabla v = \int_\Omega S(\cdot,\varphi,g)v,$$

for every test function $v \in V$, and almost everywhere in $(0,T)$. Testing (3.2) for an arbitrary test function $v \in V$ and integrating by parts leads us to

$$\int_\Omega \mu_\lambda v = \int_\Omega \nabla\varphi_\lambda \cdot \nabla v + \int_\Omega \beta(\varphi_\lambda)v + \int_\Omega \pi(\varphi_\lambda)v + \int_\Omega gv, \quad \text{a.e. in } (0,T).$$

Exploiting the above convergences, we pass to the limit as $\lambda \to 0$ and get the boundary condition

$$\partial_{\mathbf{n}}\varphi = 0, \quad \text{a.e. on } \Sigma,$$

along with the pointwise formulation

$$\mu = -\Delta\varphi + \beta(\varphi) + \pi(\varphi) + g, \quad \text{a.e. in } Q.$$

Thus (φ,μ) is a variational solution to the original problem (1.6)–(1.9) and the proof of Theorem 2.5 is finished.

3.5.2 Nonlocal case

The situation is similar to the previous case, so we just give a sketch of the proof. From (3.15) and (3.34), we obtain the existence of a function φ such that

$$\varphi \in H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V),$$

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and, as $\lambda \to 0$,

$$\begin{aligned}
&\varphi_\lambda \to \varphi \quad \text{weakly in} \quad H^1(0,T;V^*) \cap L^2(0,T;V),\\
&\varphi_\lambda \to \varphi \quad \text{weakly-star in} \quad L^\infty(0,T;H),\\
&\varphi_\lambda \to \varphi \quad \text{strongly in} \quad C^0([0,T];H),
\end{aligned}$$

which yields that the initial condition $\varphi(0) = -1$ is fulfilled in H. Here we have lower regularity with respect to the previous case. Nonetheless, the above strong convergence is enough to handle the nonlinear terms in the same way. The only difference concerns the convolution kernel for which we have, on account of A4, that, as $\lambda \to 0$,

$$K * \varphi_\lambda \to K * \varphi \quad \text{strongly in} \quad L^2(0,T;V).$$

The rest of the proof goes as above and no repetition is needed here. The only difference is that, here, we can no longer infer the regularity properties (3.35). This concludes the proof of Theorem 2.6.

4. Uniqueness of weak solutions

We will now proceed to demonstrate Theorems 2.8 and 2.9. We will give the full details of the former proof only, since the latter proof goes along the same lines.

Proof of Theorem 2.8. First, we set $\varphi = \varphi_2 - \varphi_1$ and consider the weak formulation for the difference that reads, recalling that h is constant, as

$$\langle \partial_t \varphi, v \rangle_V + \int_\Omega \nabla \mu \cdot \nabla v = -\int_\Omega m(\cdot)\varphi v, \tag{4.1}$$

for every test function $v \in V$, and almost everywhere in $(0,T)$, where

$$\mu = -\Delta \varphi + F'(\varphi_2) - F'(\varphi_1), \quad \text{a.e. in } Q.$$

Taking $v = |\Omega|^{-1}$ in (4.1), we get

$$\frac{d}{dt}\varphi_\Omega = -(m(\cdot)\varphi)_\Omega. \tag{4.2}$$

Multiplying now the above equation by $v = \varphi_\Omega + \operatorname{sign} \varphi_\Omega$, we obtain

$$\frac{d}{dt}\Big(\frac{1}{2}|\varphi_\Omega|^2 + |\varphi_\Omega|\Big) \le C(|\varphi_\Omega|^2 + |\varphi_\Omega|). \tag{4.3}$$

We then subtract equation (4.2) from (4.1), test the resulting identity with $v = \mathcal{N}(\varphi - \varphi_\Omega)$ and use the equation for the chemical potential to express the term $\int_\Omega \nabla \mu \cdot \nabla \mathcal{N}(\varphi - \varphi_\Omega)$. Upon rearranging some terms, this gives, for almost any $t \in (0,T)$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\varphi - \varphi_\Omega\|_*^2 + \|\nabla \varphi\|^2 + \int_\Omega (\beta(\varphi_2) - \beta(\varphi_1))\varphi\\
&\quad = -\int_\Omega m(\cdot)\varphi \mathcal{N}(\varphi - \varphi_\Omega) - \int_\Omega (\pi(\varphi_2) - \pi(\varphi_1))(\varphi - \varphi_\Omega)\\
&\qquad - \int_\Omega (\beta(\varphi_2) - \beta(\varphi_1))\varphi_\Omega =: \sum_{i=1}^3 I_i.
\end{aligned} \tag{4.4}$$

The third term on the left-hand side is nonnegative due to the monotonicity of β, whereas the terms on the right-hand side can be bounded as follows. Recalling the interpolation inequality:

$$\forall \delta > 0, \quad \exists\, C_\delta > 0 : \quad \|\varphi\|^2 \le \delta \|\nabla \varphi\|^2 + C_\delta \|\varphi - \varphi_\Omega\|_*^2 + C|\varphi_\Omega|^2,$$

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and using the Young inequality and the above estimate, we infer that

$$I_1 + I_2 \leq C \|\varphi\|^2 + C \|\varphi - \varphi_\Omega\|_*^2 \leq \delta \|\nabla \varphi\|^2 + C_\delta \|\varphi - \varphi_\Omega\|_*^2 + C|\varphi_\Omega|^2,$$
$$I_3 \leq C(\|\beta(\varphi_2)\|_1 + \|\beta(\varphi_1)\|_1)|\varphi_\Omega|.$$

Next, we add (4.4) with (4.3) and observe that $t \mapsto C(\|\beta(\varphi_2(t))\|_1 + \|\beta(\varphi_1(t))\|_1)$ belongs to $L^1(0,T)$. Thus, integrating over time, choosing δ small enough, and applying Gronwall's lemma we deduce (2.22). This concludes the proof. □

Proof of Theorem 2.9. Here, we just emphasize the key differences with respect to the previous proof. Also in this case, setting $\varphi = \varphi_2 - \varphi_1$, we have (cf. (4.1))

$$\langle \partial_t \varphi, v \rangle_V + \int_\Omega \nabla \mu \cdot \nabla v = -\int_\Omega m(\cdot)\varphi v,$$

for every test function $v \in V$, and almost everywhere in $(0,T)$, where now

$$\mu = a\varphi - K * \varphi + F'(\varphi_2) - F'(\varphi_1), \quad \text{a.e. in } Q. \tag{4.5}$$

Arguing as in the proof above, we derive (4.2) and (4.3). We then repeat the previous computation with the test function $v = \mathcal{N}(\varphi - \varphi_\Omega)$, using (4.5). Thus, for almost any $t \in (0,T)$, we get

$$\frac{1}{2}\frac{d}{dt} \|\varphi - \varphi_\Omega\|_*^2 + \|\varphi\|^2 \leq \int_\Omega (K * \varphi)(\varphi - \varphi_\Omega) + \int_\Omega a\varphi(\varphi - \varphi_\Omega)$$
$$- \int_\Omega m(\cdot)\varphi \mathcal{N}(\varphi - \varphi_\Omega) - \int_\Omega (\pi(\varphi_2) - \pi(\varphi_1))(\varphi - \varphi_\Omega)$$
$$- \int_\Omega (\beta(\varphi_2) - \beta(\varphi_1))\varphi_\Omega,$$

where the second term on the left-hand side arises from $\int_\Omega (a\varphi + \beta(\varphi_2) - \beta(\varphi_1))\varphi$ using A1, and A4. Observe now that, using the Young inequality, we have

$$\int_\Omega (K * \varphi)(\varphi - \varphi_\Omega) \leq \|K * \varphi\|_V \|\varphi - \varphi_\Omega\|_* \leq \|K\|_{W^{1,1}(\Omega)} \|\varphi\| \|\varphi - \varphi_\Omega\|_*$$
$$\leq \delta \|\varphi\|^2 + C_\delta \|\varphi - \varphi_\Omega\|_*^2,$$

for a positive δ yet to be selected. The remaining terms on the right-hand side can be readily controlled again by the Young inequality. Thus, we can proceed arguing as above to get (2.23).

□

5. Application to tumour growth models

This section is devoted to a nontrivial application of the existence results analysed above in the context of tumour growth models. The diffuse interface approach is nowadays broadly used in biological modelling, in particular, to describe (solid) tumour growth dynamics (see, e.g., [13, 31]). The prototypical scenario that they try to capture may be schematized as follows: a tumour mass is embedded in a region rich in nutrients and it expands through a proliferation mechanism which is possible due to the consumption of the surrounding nutrient. In the literature, there are several models, but their common denominator is a coupling between a Cahn–Hilliard equation with a source term, accounting for the evolution of the tumour mass, with a reaction-diffusion equation for some nutrients species n. Therefore, problems (1.6)–(1.9) and (1.10)–(1.13) are related to those models. The role of the source term is to encapsulate nontrivial interactions between the tumour cells and the nutrient such as the proliferation mechanism. More precisely, in this context, the order parameter φ describes the presence of tumour cells with the convention that the level set $\{\varphi = -1\} := \{x \in \Omega : \varphi(x) = -1\}$ describes the healthy region, whereas $\{\varphi = 1\}$ the tumour region. The free energies (1.1) and (1.3) account for cell-to-cell adhesion effects, which may be of local or nonlocal nature, respectively.

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Let us now introduce some meaningful models which fit the framework of the Cahn–Hilliard equations analysed so far. Usually, the evolution of the nutrient n is ruled by a reaction-diffusion equation of the form

$$\partial_t n - \operatorname{div}\big(\eta(\varphi)\nabla n\big) = S_n(\varphi, n) \quad \text{in } Q, \tag{5.1}$$

endowed with no-flux or Robin boundary conditions as well as a suitable initial condition. Here, $S_n(\varphi, n)$ stands for a nutrient source term, and $\eta(\varphi)$ for the nutrient mobility. For more details on the modelling we refer, for instance, to [31] and references therein.

Thus, we can introduce the following initial and boundary value problems for tumour growth systems whose well-posedness readily frame in the setting of the above results

$$\partial_t \varphi - \Delta(A_j\varphi + F'(\varphi) - \chi n) = S(\varphi, n) \qquad \text{in } Q, \quad j = 1, 2, \tag{5.2}$$

$$\partial_t n - \Delta n = S_n(\varphi, n) \qquad \text{in } Q, \tag{5.3}$$

$$\partial_{\mathbf{n}}(A_j\varphi + F'(\varphi) - \chi n) = \partial_{\mathbf{n}} n = 0 \qquad \text{on } \Sigma, \tag{5.4}$$

$$\varphi(0) = -1, \quad n(0) = n_0 \qquad \text{in } \Omega, \tag{5.5}$$

where A_1 is the usual (minus) Laplace operator with homogeneous Neumann boundary condition, while A_2 is the linear nonlocal operator from H to H defined by $A_2\varphi = a\varphi - K * \varphi$. Besides, $\chi \geq 0$ denotes a nonnegative constant related to the chemotaxis sensitivity. Typical choices for the source terms S and S_n are the following (see, e.g., [2, 3]):

$$S(\varphi, n) = -m\varphi + (n - \delta_n)_+ \mathbb{h}(\varphi), \quad S_n(\varphi, n) = \mathcal{B}(n_B - n) - \mathcal{C}\mathbb{h}(\varphi)n, \tag{5.6}$$

for prescribed constants $m > 0$, and $\delta_n \in [0, 1]$. From now on, we postulate (5.6) for the source terms. The capital letters $\mathcal{B}$ and $\mathcal{C}$ indicate nonnegative constants taking into account consumption rate of the nutrient with respect to a pre-existing concentration n_B, and the nutrient consumption rate, respectively. Here, $\mathbb{h}$ plays the role of a truncation so to have the action of $\mathcal{C}\mathbb{h}(\varphi)n$ only in the spatial region where the tumour cells are located. Namely, we postulate that $\mathbb{h}(-1) = 0$, $\mathbb{h}(1) = 1$, and $\mathbb{h}$ is suitably interpolated in between if $-1 < \varphi < 1$. Finally, the source term $\mathcal{B}(n_B - n)$ in (5.3) models the nutrient supply from the blood vessels if $n_B > n$ and the nutrient transport away from the domain for $n_B < n$.

The novelty of our approach consists in enabling the initial condition φ_0 to be chosen as the pure healthy phase. We somehow show that even if the tissue of the patient is completely healthy at the beginning, corresponding to $\varphi_0 = -1$, the presence of the disease (captured by the structure of the source term) might trigger the evolution and the growth of the tumour. Below, we provide existing results for the above local and nonlocal problems where the initial datum φ_0 can correspond to the healthy state, i.e., $\varphi_0 = -1$ in Ω.

With no claim to be exhaustive, let us briefly review some related works dealing with singular potentials like (1.2). For those models, a crucial point is to handle the mass evolution; in particular, one would avoid inconsistency of the model and have non-physical solutions φ that satisfy $|(\varphi(t))_\Omega| > 1$ for some time $t \in [0, T]$. To overcome this possible situation, one usually proceeds as we did and requires that the source term S, which rules the mass evolution of the order parameter, is given by (5.6). In this direction, we refer to [25, 40, 56]. Alternatively, some authors suggest to introduce some regularization terms in the system to allow for more general potentials and source terms (see, for instance, [12, 57] and the references therein). Finally, let us point out that nonlocal models in the context of phase field tumour growth models are also investigated as well (see, e.g., [26, 57]).

As a direct consequence of the previous sections, we can deduce the validity of the following weak existence results.

Theorem 5.1. *Suppose that **A1–A3** hold. Let the initial data fulfil*

$$\varphi_0 \in V, \quad F(\varphi_0) \in L^1(\Omega), \quad (\varphi_0)_\Omega \in [-1, 1], \quad n_0 \in V \cap L^\infty(\Omega),$$
$$0 \leq n_0(x) \leq 1 \quad \textit{for a.e.} x \in \Omega,$$

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and assume that $\mathbb{h}$ *is continuous and uniformly bounded,* χ *is a nonnegative constant, and* $n_B \in L^\infty(Q)$ *with* $0 \le n_B \le 1$ *a.e. in* Q*. Then, problem (5.2)–(5.4) with* $j = 1$ *admits a weak solution* (φ, μ, n) *such that*

$$\begin{aligned}
&\varphi \in H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;W),\\
&\varphi \in L^4(\sigma,T;W) \cap L^2(\sigma,T;W^{2,\gamma}(\Omega)) \quad \forall \sigma \in (0,T),\\
&\varphi \in L^{2p}(0,T;W) \cap L^p(0,T;W^{2,\gamma}(\Omega)) \quad \forall p \in (1,\overline{p}),\\
&\varphi \in L^\infty(Q), \quad |\varphi(x,t)| < 1 \quad \textit{for a.a. } (x,t) \in Q,\\
&\mu \in L^p(0,T;V) \cap L^2(\sigma,T;V) \quad \forall p \in (1,\overline{p}) \quad \forall \sigma \in (0,T),\\
&\nabla \mu \in L^2(0,T;H),\\
&n \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W),\\
&n \in L^\infty(Q), \quad 0 \le n(x,t) \le 1 \quad \textit{for a.a. } (x,t) \in Q,
\end{aligned}$$

where

$$\gamma \in \begin{cases} [1,6] & \textit{if } d = 3,\\ [1,+\infty) & \textit{if } d = 2, \end{cases} \qquad \overline{p} := \begin{cases} \frac{4}{3} & \textit{if } n_0 \in V \setminus W,\\ 2 & \textit{if } n_0 \in W, \end{cases}$$

where

$$\mu = -\Delta\varphi + F'(\varphi) - \chi n, \quad \textit{a.e. in} Q,$$

and the following identities

$$\begin{aligned}
&\langle \partial_t \varphi, v \rangle_V + \int_\Omega \nabla\mu \cdot \nabla v = \int_\Omega S(\varphi, n) v,\\
&\int_\Omega \partial_t n\, v + \int_\Omega \nabla n \cdot \nabla v + \int_\Omega \mathbb{h}(\varphi) n v = \mathcal{B} \int_\Omega (n_B - n) v,
\end{aligned}$$

are satisfied for every test function $v \in V$*, and almost everywhere in* $(0,T)$*. Moreover, it holds that*

$$\varphi(0) = \varphi_0, \quad n(0) = n_0, \quad \textit{a.e. in } \Omega.$$

Theorem 5.2. *Suppose that* ***A1–A4*** *hold. Let the initial data fulfil*

$$\begin{aligned}
&\varphi_0 \in H, \quad F(\varphi_0) \in L^1(\Omega), \quad (\varphi_0)_\Omega \in [-1,1), \quad n_0 \in W,\\
&0 \le n_0(x) \le 1 \quad \textit{for a.e. } x \in \Omega,
\end{aligned}$$

and assume that $\mathbb{h}$ *is continuous and uniformly bounded,* χ *is a nonnegative constant, and* $n_B \in L^\infty(Q)$ *with* $0 \le n_B \le 1$ *a.e. in* Q*. Then, problem (5.2)–(5.5) with* $j = 2$ *admits a weak solution* (φ, μ, n) *such that*

$$\begin{aligned}
&\varphi \in H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V),\\
&\varphi \in L^\infty(Q), \quad |\varphi(x,t)| < 1 \quad \textit{for a.a. } (x,t) \in Q,\\
&\mu \in L^p(0,T;V) \cap L^2(\sigma,T;V) \quad \forall p \in (1,2) \quad \forall \sigma \in (0,T),\\
&\nabla \mu \in L^2(0,T;H),\\
&n \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W),\\
&n \in L^\infty(Q), \quad 0 \le n(x,t) \le 1 \quad \textit{for a.a. } (x,t) \in Q,
\end{aligned}$$

where

$$\mu = a\varphi - K * \varphi + F'(\varphi) - \chi n, \quad \textit{a.e. in} Q,$$

and the identities

$$\begin{aligned}
&\langle \partial_t \varphi, v \rangle_V + \int_\Omega \nabla\mu \cdot \nabla v = \int_\Omega S(\varphi, n) v,\\
&\int_\Omega \partial_t n\, v + \int_\Omega \nabla n \cdot \nabla v + \int_\Omega \mathbb{h}(\varphi) n v = \mathcal{B} \int_\Omega (n_B - n) v,
\end{aligned}$$

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are satisfied for every test function $v \in V$, and almost everywhere in $(0, T)$. Moreover, it holds that

$$\varphi(0) = \varphi_0, \quad n(0) = n_0, \quad \text{a.e. in } \Omega.$$

Remark 5.3. *If the function $\mathbb{h}$ does not depend on the phase variable φ, then continuous dependence estimates can be proven by arguing along the lines of Theorem 2.8 and Theorem 2.9. In this case, both the local and the nonlocal problem are well posed.*

Proof. We just give a sketch of both the proofs since they are natural extensions of the previous ones with the choice $g = -\chi n$.

The difference is that here we do not know, a priori, whether $g = -\chi n$ fulfils A2 or not. However, a similar parabolic-type estimate as (3.12) ((3.16) resp.) can be derived for n and the strategy of the previous proof can be carried out with minor modifications. Rigorously speaking, one should introduce a Faedo–Galerkin scheme, let k denote the discretization parameter, for the system (5.2)–(5.5) along with the same λ-approximation introduced before. Then, the discretized version of (5.3) can be tested by n_k. This gives

$$\frac{1}{2}\frac{d}{dt}\|n_k\|^2 + \|\nabla n_k\|^2 \leq C(\|n_k\|^2 + 1),$$

so that

$$\|n_k\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq C,$$

whence, as before, we deduce

$$\forall\, \alpha \in [0,1), \quad \exists\, C_\alpha > 0: \quad \|n_k\|_{L^\infty(0,t;H)} \leq C_\alpha t^\alpha \quad \forall\, t \in [0,T],$$

with $C_\alpha > 0$ that are independent of k. Testing then by $\partial_t n_k$ and recalling that $n_0 \in V$, we find

$$\|n_k\|_{H^1(0,t;H)\cap L^\infty(0,t;V)\cap L^2(0,t;W)} \leq C,$$

which, by the same argument, yields

$$\forall\, \alpha \in [0,1), \quad \exists\, C_\alpha > 0: \quad \|n_k\|_{L^\infty(0,t;V)} \leq C_\alpha t^\alpha, \quad \forall\, t \in [0,T],$$

matching finally A2. The rest of the proof goes as in Section 3.

Concerning the minimum and maximum principles for n, one can argue as done in [57]. The final step consists in pass to the limit first as $k \to \infty$ and then as $\lambda \to 0$, along suitable subsequences. □

6. Final comments and future work

The present results are a first step towards a more general characterization of source terms which allow a pure initial datum. Another interesting, and perhaps nontrivial, issue is the extension of our results to other models. For instance, Cahn–Hilliard–Navier–Stokes system with sources (see [43] and references therein), fractional Cahn–Hilliard equations with sources (see, for instance, [28, 30] for the conserved case), or multi-component Cahn–Hilliard equations with sources (see [17], see also [29] for the general conserved case).

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A. Appendix

Proof of Proposition 2.4. Property (P1) follows from ([36], Section 5) and [54]. Let us focus only on the proof of property (P2). Since $\widehat{\beta}$ is strictly convex with $\widehat{\beta}(0) = 0$, one has that β is increasing, $\beta(0) = 0$, with $\beta < 0$ in $(-1, 0)$, and $\beta > 0$ in $(0, 1)$.

Let $\delta \in (0, 1)$ be fixed, and let $r \in (-1, 1)$ and $r_0 \in (-1, -1 + \delta)$ as in property (P2). We define now

$$\underline{r} := \frac{r_0 - 1}{2} = -1 + \frac{1}{2}(r_0 + 1), \qquad \overline{r} := -1 + \frac{1}{\delta}(r_0 + 1),$$

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so that it holds

$$-1 < \underline{r} < r_0 < \overline{r} < 0, \qquad \beta(\underline{r}), \beta(r_0), \beta(\overline{r}) < 0.$$

Let us consider the quantity

$$\beta(r)(r - r_0)$$

and estimate it from below in the four cases $r \in (-1, \underline{r})$, $r \in [\underline{r}, \overline{r}]$, $r \in (\overline{r}, 0]$, and $r \in (0, 1)$.

- $r \in (-1, \underline{r})$: one has that

$$\beta(r)(r - r_0) = |\beta(r)||r - r_0| = (r_0 - r)|\beta(r)| \geq (r_0 - \underline{r})|\beta(r)| = \frac{r_0 + 1}{2}|\beta(r)|.$$

- $r \in [\underline{r}, \overline{r}]$: one has that

$$\begin{aligned}\beta(r)(r - r_0) &\geq -|r - r_0| \max_{x \in [\underline{r}, \overline{r}]} |\beta(x)| \geq -(\overline{r} - \underline{r}) \max_{x \in [\underline{r}, \overline{r}]} |\beta(x)| \\ &\geq \frac{r_0 + 1}{2}|\beta(r)| - \left[\frac{r_0 + 1}{2} + (\overline{r} - \underline{r})\right] \max_{x \in [\underline{r}, \overline{r}]} |\beta(x)| \\ &= \frac{r_0 + 1}{2}|\beta(r)| - \frac{1}{\delta}(r_0 + 1)|\beta(\underline{r})|.\end{aligned}$$

- $r \in (\overline{r}, 0]$: one has that

$$\begin{aligned}\beta(r)(r - r_0) &\geq -|r - r_0||\beta(r)| = -(r - r_0)|\beta(r)| = r_0|\beta(r)| - r|\beta(r)| \\ &= (r_0 + 1)|\beta(r)| - (r + 1)|\beta(r)| \\ &\geq \frac{r_0 + 1}{2}|\beta(r)| - (r + 1) \max_{x \in [\overline{r}, 0]} |\beta(x)| \\ &= \frac{r_0 + 1}{2}|\beta(r)| - (r + 1)|\beta(\overline{r})| \geq \frac{r_0 + 1}{2}|\beta(r)| - (r + 1)|\beta(\underline{r})|.\end{aligned}$$

- $r \in (0, 1)$: one has that

$$\beta(r)(r - r_0) = |\beta(r)|(r - r_0) \geq -r_0|\beta(r)| \geq \frac{1 - \delta}{\delta}(r_0 + 1)|\beta(r)|,$$

  where we have used the fact that $r_0 \in (-1, -1 + \delta)$ implies $-r_0 > \frac{1-\delta}{\delta}(r_0 + 1)$.

Putting everything together, we obtain

$$\min\left\{\frac{1}{2}, \frac{1 - \delta}{\delta}\right\}(r_0 + 1)|\beta(r)| \leq \beta(r)(r - r_0) + \left[(r + 1) + \frac{1}{\delta}(r_0 + 1)\right]|\beta(\underline{r})|$$

and the thesis follows. □

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