Combinatorics, Probability and Computing (2012) 21, 803–815. © Cambridge University Press 2012
doi:10.1017/S0963548312000272

Connectivity for Bridge-Addable Monotone Graph Classes

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Received 12 May 2008; revised 4 June 2012; first published online 30 July 2012

A class $\mathcal{A}$ of labelled graphs is *bridge-addable* if, for all graphs G in $\mathcal{A}$ and all vertices u and v in distinct connected components of G, the graph obtained by adding an edge between u and v is also in $\mathcal{A}$; the class $\mathcal{A}$ is *monotone* if, for all $G \in \mathcal{A}$ and all subgraphs H of G, we have $H \in \mathcal{A}$. We show that for any bridge-addable, monotone class $\mathcal{A}$ whose elements have vertex set $\{1, \ldots, n\}$, the probability that a graph chosen uniformly at random from $\mathcal{A}$ is connected is at least $(1 - o_n(1))\, e^{-\frac{1}{2}}$, where $o_n(1) \to 0$ as $n \to \infty$. This establishes the special case of the conjecture of McDiarmid, Steger and Welsh when the condition of monotonicity is added. This result has also been obtained independently by Kang and Panagiotou.

AMS 2010 *Mathematics subject classification*: Primary 05C80
Secondary 60C05

1. Introduction

Given a class $\mathcal{A}$ of graphs, we say that $\mathcal{A}$ is *bridge-addable* (or *weakly addable*) if, for all graphs G in $\mathcal{A}$ and all vertices u and v in distinct connected components of G, the graph obtained by adding an edge between u and v is also in $\mathcal{A}$. The concept of bridge-addability was introduced in McDiarmid, Steger and Welsh [16] in the course of studying random planar graphs. It was shown there that, for a graph chosen uniformly at random from a finite non-empty bridge-addable class $\mathcal{A}$ of labelled graphs, the probability that it is connected is at least e^{-1}.

As well as the class of planar graphs, other examples of bridge-addable graph classes include forests, graphs with tree-width at most k, graphs embeddable on any fixed surface,

† This research was supported by the Banff International Research Station and the McGill Bellairs Research Institute.

https://doi.org/10.1017/S0963548312000272 Published online by Cambridge University Press

and generally any minor-closed graph class with cut-point-free excluded minors; triangle-free graphs, and more generally H-free graphs for any two-edge-connected graph H; and k-colourable graphs.

It is well known that there are n^{n-2} trees on n labelled vertices: see Cayley [8]. Together with a result of Rényi [19] (see also Moon [18]) that the corresponding number of forests is asymptotic to $e^{\frac{1}{2}}n^{n-2}$, we see that, for a uniformly random forest on n labelled vertices, the probability that it is connected is asymptotically $e^{-\frac{1}{2}}$.

McDiarmid, Steger and Welsh [17] suggested that if connectivity is desired then the worst possible example of a bridge-addable graph class is the class of forests. More precisely, they conjectured that the lower bound of e^{-1} on the probability of connectedness for a bridge-addable graph class can be improved asymptotically to $(1+o(1))e^{-\frac{1}{2}}$. (They assumed also that the graph class was closed under isomorphism.) Recently, Balister, Bollobás and Gerke [1, 2] took a first step towards proving this conjecture, proving an asymptotic lower bound of $e^{-0.7983}$.

Observe that the examples above of bridge-addable graph classes, and many other interesting examples of such graph classes, also satisfy the property of being *monotone* (decreasing); that is, given a graph G in $\mathcal{A}$, each graph obtained by deleting edges from G is also in $\mathcal{A}$. In this paper we investigate the probability of connectedness of a graph chosen uniformly at random from a monotone bridge-addable graph class: for such graph classes, we prove the conjectured lower bound. Our method relies upon a reduction to weighted random forests, and in particular to the properties of weighted random trees.

For several bridge-addable graph classes, including some mentioned above, the asymptotic probability of connectedness has recently been determined: see the last section below.

Let us now state our theorem. A *bridge* in a graph G is an edge e such that $G-e$ has strictly more connected components than G. We say that a class $\mathcal{A}$ of graphs is *bridge-alterable* if, for any graph G and bridge e in G, G is in $\mathcal{A}$ if and only if $G-e$ is. We remark that if $\mathcal{A}$ is bridge-alterable then it is bridge-addable, and if it is both bridge-addable and monotone then it is bridge-alterable.

Theorem 1.1. *For any $\varepsilon > 0$ there exists W_0 such that, if $W \geqslant W_0$ and $\mathcal{A}$ is a non-empty bridge-alterable class of graphs on $\{1,\dots,W\}$, and if $\mathbf{G}$ is a graph chosen uniformly at random from $\mathcal{A}$, then*

$$\mathbb{P}(\mathbf{G} \text{ is connected}) \geqslant (1-\varepsilon)e^{-\frac{1}{2}}. \tag{1.1}$$

This result was announced independently by Kang and Panagiotou while the present paper was under (slow) revision following referee reports, with at that time a weaker form of the above theorem. Their proof in [15] starts like our proof here but then proceeds very differently, with a clever induction involving merging vertices when counting forests with two components (where the vertices correspond to bridgeless graphs).

In the next section we describe a reduction which allows us to establish Theorem 1.1 by proving that, for a random forest with an appropriately chosen distribution, full connectivity (the event that the forest is a tree) is almost twice as likely as the event

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that the forest has precisely two connected components (the precise statement appears as Lemma 2.3, below). In the following two sections we explain how Lemma 2.3 will yield Theorem 1.1, and then prove Lemma 2.3. Finally we make some concluding remarks.

2. A reduction to weighted forests

In this section we assume that the conditions of Theorem 1.1 are satisfied. For any class $\mathcal{A}$ of graphs and any positive integer W, let $\mathcal{A}_W$ denote the set of graphs in $\mathcal{A}$ on the vertex set $\{1,\dots,W\}$. We shall prove Theorem 1.1 by partitioning $\mathcal{A}_W$ and showing that an inequality such as (1.1) holds for a uniformly random element of each block of the partition. (This step of our proof is essentially Lemma 2.1 of [1].)

Definition. Given a graph G, let $b(G)$ be the graph obtained by removing all bridges from G. We say G and G' are *equivalent* if $b(G) = b(G')$, and in this case write $G \sim G'$. For a graph G, let $[G]$ be the set of graphs G' for which $b(G') = b(G)$.

It is easily seen that $\sim$ is an equivalence relation on graphs, and thus we always have $[G] = [b(G)]$. Furthermore, if $G \in \mathcal{A}_W$ then as $\mathcal{A}$ is closed under deleting bridges, $b(G) \in \mathcal{A}_W$, and as $\mathcal{A}$ is bridge-addable, we have $[G] \subseteq \mathcal{A}_W$. It follows that $\mathcal{A}_W$ can be written as a union of some set of disjoint equivalence classes $[G_1], [G_2], \dots$. To prove Theorem 1.1, we will in fact prove the following.

Claim 2.1. *For any graph $G \in \mathcal{A}_W$, if $\mathbf{H}$ is a uniformly random element of $[G]$ then*

$$\mathbb{P}(\mathbf{H} \text{ is connected}) \geqslant e^{-\frac{1}{2}+o(1)},$$

where $o(1) \to 0$ as $W \to \infty$.

Clearly, Theorem 1.1 immediately follows from Claim 2.1, and it thus remains to prove Claim 2.1. Fix a bridgeless graph G on vertex set $\{1,\dots,W\}$ and let $\mathcal{B} = [G]$. We write $C_1(G),\dots,C_n(G)$ for the connected components of G, and let $w_i = |V(C_i)|$ for $i = 1,\dots,n$, so $W = \sum_{i=1}^{n} w_i$. We remark that since the components $C_1,\dots,C_n$ are bridgeless, either $w_i = 1$ or $w_i \geqslant 3$ for all $i \in \{1,\dots,n\}$ (though we shall not use this fact). We denote by $\vec{w}$ the vector $(w_1,\dots,w_n)$. We use $\vec{w}$ to define a probability measure on the set $\mathcal{F}_n$ of forests with vertex set $\{1,\dots,n\}$. Given $F \in \mathcal{F}_n$, let

$$\text{mass}(F) = \text{mass}_{\vec{w}}(F) = \prod_{i=1}^{n} w_i^{d_F(i)},$$

where $d_F(i)$ denotes the degree of vertex i in the forest F. Also, let $K = \sum_{F\in\mathcal{F}_n} \text{mass}(F)$, and let $\mathbf{F}$ be a random element of $\mathcal{F}_n$ with $\mathbb{P}(\mathbf{F} = F) = \text{mass}(F)/K$ for all $F \in \mathcal{F}_n$. We say $\mathbf{F}$ is *distributed according to* $\vec{w}$; when we wish to highlight the distribution of $\mathbf{F}$, we will sometimes write $\mathbf{F}_{\vec{w}}$ in place of $\mathbf{F}$. For our purposes, the key fact about such a random forest is the following.

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Lemma 2.2. *For a uniformly random element* $\mathbf{H}$ *of* $\mathcal{B}$,

$$\mathbb{P}(\mathbf{H} \text{ is connected}) = \mathbb{P}(\mathbf{F} \text{ is connected}).$$

Proof. Given $H \in \mathcal{B}$, let $f(H)$ be the graph obtained from H by contracting $C_i(G)$ to a single point for each $i = 1, \dots, n$; then $f(H) \in \mathcal{F}_n$, and for each $F \in \mathcal{F}_n$, the set $f^{-1}(F)$ has cardinality precisely $\prod_{i=1}^n w_i^{d_F(i)}$. Since $H \in \mathcal{B}$ in connected if and only if $f(H)$ is connected, it follows that

$$\begin{aligned}\mathbb{P}(\mathbf{H} \text{ is connected}) &= \frac{|\{H \in \mathcal{B} : H \text{ is connected}\}|}{|\mathcal{B}|} \\ &= \frac{\sum_{\{F \in \mathcal{F}_n : F \text{ is connected}\}} |f^{-1}(F)|}{\sum_{F \in \mathcal{F}_n} |f^{-1}(F)|} \\ &= \frac{\sum_{\{F \in \mathcal{F}_n : F \text{ is connected}\}} \text{mass}(F)}{K} \\ &= \mathbb{P}(\mathbf{F} \text{ is connected}),\end{aligned}$$

as required. □

To prove Claim 2.1, it therefore suffices to show that, for such a random forest $\mathbf{F}$,

$$\mathbb{P}(\mathbf{F} \text{ is connected}) \geqslant e^{-\frac{1}{2}+o(1)},$$

where $o(1)$ tends to zero as $W \to \infty$.

For $i = 1, \dots, n$, let $\mathcal{F}_{n,i}$ be the set of elements of $\mathcal{F}_n$ with i components (so $F \in \mathcal{F}_{n,1}$ precisely if F is connected). It turns out that bounds on $\mathbb{P}(\mathbf{F}$ is connected) follow from bounds on the ratio between $\mathbb{P}(F \in \mathcal{F}_{n,2})$ and $\mathbb{P}(F \in \mathcal{F}_{n,1})$. More precisely, Claim 2.1 follows from Lemma 2.2 and the following lemma.

Lemma 2.3. *For all* $\varepsilon > 0$, *for* W *sufficiently large, and for all* $\overrightarrow{w} = (w_1, \dots, w_n)$ *with* $\sum_{j=1}^n w_j = W$,

$$\mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,2}) \leqslant (1+\varepsilon)\frac{1}{2}\,\mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,1}). \tag{2.1}$$

In Section 3 we explain how to use Lemma 2.3 to prove Claim 2.1; in Section 4 we prove Lemma 2.3.

3. Proof of Claim 2.1 assuming Lemma 2.3

The proof of Claim 2.1 proceeds somewhat differently depending on the value of the ratio of n and W. When W is much larger than n, the proof is rather straightforward, and in fact does not require Lemma 2.3 at all, but rather Lemma 3.2 below. In both cases, however, we compare the probability masses of $\mathcal{F}_{n,i}$ and of $\mathcal{F}_{n,i+1}$ by double-counting edge-weights in a bipartite graph.

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Given a graph G, let $c(G)$ be the set of connected components of G. Given a forest $F \in \mathcal{F}_n$ and $T \in c(F)$, let $w(T) = \sum_{i \in V(T)} w_i$. Consider forests $F, F' \in \mathcal{F}_n$ such that F' can be obtained from F by the addition of an edge e. Writing $T \neq T' \in c(F)$ as shorthand for $\{\{T, T'\} \subseteq c(F) : T \neq T'\}$, we let

$$\varphi(F', F) = \frac{\text{mass}(F')}{\sum_{T \neq T' \in c(F)} w(T)w(T')}. \tag{3.1}$$

For all other pairs F, F', we let $\varphi(F', F) = 0$. We use φ in a standard double-counting argument. We will use the next preliminary lemma also in Section 4.

Lemma 3.1. *For all positive integers W and positive integer weight vectors $\overrightarrow{w} = (w_1, \ldots, w_n)$ with $\sum_{j=1}^n w_j = W$, and for each $i = 1, \ldots, n-1$*

$$\sum_{F' \in \mathcal{F}_{n,i}} \sum_{F \in \mathcal{F}_{n,i+1}} \varphi(F', F) = \sum_{F \in \mathcal{F}_{n,i+1}} \text{mass}(F) = K \cdot \mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,i+1}). \tag{3.2}$$

Proof. Given $i \in \{1, \ldots, n-1\}$ and $F \in \mathcal{F}_{n,i+1}$, if $F' \in \mathcal{F}_{n,i}$ is obtained from F by the addition of edge uv, then $\text{mass}(F') = \text{mass}(F) \cdot w_u \cdot w_v$. We thus have

$$\begin{aligned}
&\sum_{F' \in \mathcal{F}_{n,i}} \varphi(F', F) \\
&\quad = \left(\sum_{T \neq T' \in c(F)} \sum_{u \in V(T), v \in V(T')} \text{mass}(F) \cdot w_u \cdot w_v \right) \cdot \left(\frac{1}{\sum_{T \neq T' \in c(F)} w(T)w(T')} \right) \\
&\quad = \frac{\text{mass}(F)}{\sum_{T \neq T' \in c(F)} w(T)w(T')} \cdot \left(\sum_{T \neq T' \in c(F)} \sum_{u \in V(T), v \in V(T')} w_u \cdot w_v \right) = \text{mass}(F).
\end{aligned}$$

Equation (3.2) now follows on summing over F. □

Lemma 3.2. *For all positive integers W and positive integer weight vectors $\overrightarrow{w} = (w_1, \ldots, w_n)$ with $\sum_{j=1}^n w_j = W$, and for each $i = 1, \ldots, n-1$,*

$$\mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,i+1}) \leqslant \frac{\mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,i})(n/W)}{i}, \tag{3.3}$$

Proof. Fix i with $1 \leqslant i \leqslant n-1$. By the definition of φ, for all $F' \in \mathcal{F}_{n,i}$ we have

$$\sum_{F \in \mathcal{F}_{n,i+1}} \varphi(F', F) = \text{mass}(F') \cdot \sum_{e \in E(F')} \frac{1}{\sum_{T \neq T' \in c(F'-e)} w(T)w(T')}. \tag{3.4}$$

It is well known that for any set of positive integers $a_1, \ldots, a_{i+1}$ with $\sum_{j=1}^{i+1} a_j = W$,

$$\sum_{1 \leqslant j < k \leqslant i+1} a_j a_k \geqslant i(W - i) + \binom{i}{2}.$$

(To see this, if $a_1 \geqslant a_2 \geqslant 2$ then let $a'_1 = a_1 + 1$, $a'_2 = a_2 - 1$ and $a'_j = a_j$ for each $j \geqslant 3$. Then with sums as above, $\sum_{j<k} a'_j a'_k - \sum_{j<k} a_j a_k = a'_1 a'_2 - a_1 a_2 = -a_1 + a_2 - 1 < 0$.

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Hence the sum is minimized when there are i entries 1 and one entry $W-i$.) It follows that, for any $F' \in \mathcal{F}_{n,i}$ and any $e \in E(F')$, we have

$$\sum_{T \neq T' \in c(F'-e)} w(T)w(T') \geqslant i(W-i) + \binom{i}{2} \geqslant i(W-i). \tag{3.5}$$

Since, if $F' \in \mathcal{F}_{n,i}$ then F' has exactly $n-i$ edges, it follows from (3.4) and (3.5) that for all $F' \in \mathcal{F}_{n,i}$,

$$\sum_{F \in \mathcal{F}_{n,i+1}} \varphi(F', F) \leqslant \mathrm{mass}(F') \cdot (n-i) \cdot \frac{1}{i(W-i)} \leqslant \mathrm{mass}(F') \cdot \frac{1}{i} \cdot \frac{n}{W},$$

so

$$\begin{aligned} \sum_{F' \in \mathcal{F}_{n,i}} \sum_{F \in \mathcal{F}_{n,i+1}} \varphi(F', F) &\leqslant \sum_{F' \in \mathcal{F}_{n,i}} \mathrm{mass}(F') \cdot \frac{1}{i} \cdot \frac{n}{W} \\ &= K \cdot \mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,i}) \cdot \frac{1}{i} \cdot \frac{n}{W}, \end{aligned}$$

and (3.3) follows by combining this last result and Lemma 3.1. □

The case $n/W \leqslant \frac{1}{2}$ of Lemma 2.3 follows immediately from the above lemma, and indeed in this case we may complete the proof directly without using the next two lemmas: see the last sentence of this section. To explain why Lemma 2.3 implies Claim 2.1 when n is not much smaller than W, it turns out to be useful to prove a slightly more general implication.

For each finite non-empty set V of positive integers, let $\mathcal{G}(V)$ be the set of all graphs on the vertex set V, and let $\mathcal{G}^k(V)$ be the set of all graphs in $\mathcal{G}(V)$ with exactly k components. Also, write $\mathcal{G}_n$ for $\mathcal{G}(\{1,\ldots,n\})$, and $\mathcal{G}_n^k$ for $\mathcal{G}^k(\{1,\ldots,n\})$. For each positive integer n, let μ_n be a measure on the set of all graphs with vertex set a subset of $\{1,\ldots,n\}$, which is multiplicative on components (that is, if G has components $H_1,\ldots,H_k$, then $\mu_n(G) = \prod_{i=1}^k \mu_n(H_i)$).

Lemma 3.3. *Suppose there exist $x > 0$ and integers $n \geqslant m_0 \geqslant 1$ such that*

$$\mu_n(\mathcal{G}^2(V)) \leqslant x\mu_n(\mathcal{G}^1(V)) \quad \textit{for all} \quad V \subseteq \{1,\ldots,n\} \textit{ with } |V| \geqslant m_0. \tag{3.6}$$

Let k be a positive integer and suppose that $n \geqslant km_0$. Then

$$\mu_n(\mathcal{G}_n^{k+1}) \leqslant \frac{x}{k}\mu_n(\mathcal{G}_n^k). \tag{3.7}$$

Proof. Let $\mathcal{A}$ be the collection of all sets $\{H_1,\ldots,H_{k-1}\}$ of $k-1$ connected graphs such that the vertex sets $V(H_i)$ are pairwise disjoint subsets of $\{1,\ldots,n\}$ and

$$\max_{1 \leqslant i \leqslant k-1} |V(H_i)| < n - \sum_{i=1}^{k-1} |V(H_i)|.$$

Let $\mathcal{H} = \{H_1,\ldots,H_{k-1}\} \in \mathcal{A}$, let $V_{\mathcal{H}} = \{1,\ldots,n\} \setminus \left(\bigcup_{i=1}^{k-1} V(H_i)\right)$, and note that $|V_{\mathcal{H}}| > \max_{1 \leqslant i \leqslant k-1} |V(H_i)|$ and $|V_{\mathcal{H}}| \geqslant m_0$. For $j = k$ and $k+1$, let $\mathcal{G}_n^j(\mathcal{H})$ denote the set of

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all graphs G in $\mathcal{G}_n^j$ such that $H_1, \ldots, H_{k-1}$ are each components of G. Then, letting $\alpha = \prod_{i=1}^{k-1} \mu_n(H_i)$, by the multiplicativity of μ_n and by (3.6) we have

$$\mu_n(\mathcal{G}_n^{k+1}(\mathcal{H})) = \alpha \cdot \mu_n(\mathcal{G}^2(V_{\mathcal{H}})) \leqslant x\alpha \cdot \mu_n(\mathcal{G}^1(V_{\mathcal{H}})) = x \cdot \mu_n(\mathcal{G}_n^k(\mathcal{H})). \tag{3.8}$$

Next, consider any graph $G \in \mathcal{G}_n^{k+1}$, and suppose that G has components $G_1, \ldots, G_{k+1}$, where $|V(G_1)| \leqslant \ldots \leqslant |V(G_{k+1})|$. For each set $\mathcal{H}$ formed by picking any $k-1$ of the graphs $G_1, \ldots, G_k$, we have $\mathcal{H} \in \mathcal{A}$ and $G \in \mathcal{G}_n^{k+1}(\mathcal{H})$. It follows that

$$k \cdot \mu_n(\mathcal{G}_n^{k+1}) \leqslant \sum_{\mathcal{H} \in \mathcal{A}} \mu_n(\mathcal{G}_n^{k+1}(\mathcal{H})). \tag{3.9}$$

Applying (3.8) to bound the right-hand side of (3.9), we obtain

$$k \cdot \mu_n(\mathcal{G}_n^{k+1}) \leqslant x \cdot \sum_{\mathcal{H} \in \mathcal{A}} \mu_n(\mathcal{G}_n^k(\mathcal{H})). \tag{3.10}$$

Furthermore, the sets $\{\mathcal{G}_n^k(\mathcal{H}) : \mathcal{H} \in \mathcal{A}\}$ are pairwise disjoint subsets of $\mathcal{G}_n^k$, so

$$\sum_{\mathcal{H} \in \mathcal{A}} \mu_n(\mathcal{G}_n^k(\mathcal{H})) \leqslant \mu_n(\mathcal{G}_n^k),$$

which combined with (3.10) yields that

$$k \cdot \mu_n(\mathcal{G}_n^{k+1}) \leqslant x \cdot \mu_n(\mathcal{G}_n^k),$$

which completes the proof. □

Lemma 3.3 allows us to derive bounds on the ratio between $\mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,i+1})$ and $\mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,i})$ for $i > 1$. In the following lemma, for convenience write $\mathcal{F}_{n,i} = \emptyset$ for $i > n$.

Lemma 3.4. *Suppose that there exist $0 < \gamma < 1$ and $m_0 > 0$, such that, for any positive integer weights $\vec{w} = (w_1, \ldots, w_n)$ with $\sum_{k=1}^n w_k \geqslant m_0$, $\mathbb{P}(\mathbf{F}_{\vec{w}} \in \mathcal{F}_{n,2}) \leqslant \gamma \mathbb{P}(\mathbf{F}_{\vec{w}} \in \mathcal{F}_{n,1})$. Fix any positive integer j. Then, for W sufficiently large, for all integers i with $1 \leqslant i \leqslant j$ and any positive integer weights $\vec{w} = \{w_1, \ldots, w_n\}$ with $\sum_{k=1}^n w_k = W$,*

$$\mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,i+1}) \leqslant \frac{\gamma \mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,i})}{i}. \tag{3.11}$$

Proof. Suppose γ and m_0 satisfy the hypotheses of the lemma, and fix some positive integer j. Observe first that, by Lemma 3.2, the inequality (3.11) holds if $n \leqslant \gamma W$. We may thus assume that $n > \gamma W$.

Let $n \geqslant m_0$ and consider any weights $w_1, \ldots, w_n$. Define $\mu_n(G)$ for each graph G with vertex set $V \subseteq \{1, \ldots, n\}$ by setting $\mu_n(G) = \prod_{i \in V} w_i^{d_G(i)}$ if G is a forest and $\mu_n(G) = 0$ otherwise. Then μ_n is multiplicative on components, and by the hypotheses of the lemma, for each $V \subseteq \{1, \ldots, n\}$ with $\sum_{i \in V} w_i \geqslant m_0$ we have

$$\mu_n(\mathcal{G}^2(V)) \leqslant \gamma \mu_n(\mathcal{G}^1(V)).$$

Now we may use Lemma 3.3 to obtain

$$\mu_n(\mathcal{G}_n^{i+1}) \leqslant \frac{\gamma}{i} \mu_n(\mathcal{G}_n^i)$$

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whenever $n \geqslant im_0$. For $1 \leqslant i \leqslant j$, we have $n \geqslant im_0$ whenever $W \geqslant jm_0/\gamma$, and Lemma 3.4 follows. □

Proof of Claim 2.1 assuming Lemma 2.3. Fix α with $0 < \alpha < 1$, and choose j large enough that $2/j! \leqslant \alpha/2$. Let $\varepsilon > 0$ be small enough that $(1-\alpha/2)/(1+\varepsilon)^j \geqslant 1-\alpha$. We apply Lemma 3.4 with $\gamma = (1+\varepsilon)^{\frac{1}{2}}$ (Lemma 2.3 guarantees that there exists $m_0 > 0$ such that the hypotheses of Lemma 3.4 hold with this choice of m_0 and γ). It follows that for W large enough, for all i with $1 \leqslant i \leqslant j$ we have

$$\mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,i+1}) \leqslant (1+\varepsilon)^i \frac{\mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,1})}{2^i i!}.$$

Furthermore, writing $\kappa(F)$ for the number of connected components of F,

$$\begin{aligned} 1 &= \sum_{i=0}^{n-1} \mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,i+1}) \\ &\leqslant (1+\varepsilon)^j \sum_{i=0}^{j-1} \frac{\mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,1})}{2^i i!} + \mathbb{P}(\kappa(\mathbf{F}) \geqslant j+1). \end{aligned} \tag{3.12}$$

By Lemma 3.2, for all $i \geqslant 1$,

$$\mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,i+1}) \leqslant \frac{(n/W)^i}{i!} \leqslant \frac{1}{i!},$$

from which it follows that for all $k \geqslant 1$,

$$\mathbb{P}(\kappa(\mathbf{F}) \geqslant k+1) \leqslant \sum_{i \geqslant k} \frac{1}{i!} \leqslant \frac{2}{k!}.$$

Combining the latter equation with (3.12) yields that

$$1 \leqslant (1+\varepsilon)^j e^{\frac{1}{2}} \mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,1}) + \alpha/2,$$

so

$$\mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,1}) \geqslant \frac{1-\alpha/2}{(1+\varepsilon)^j e^{\frac{1}{2}}} \geqslant \frac{1-\alpha}{e^{\frac{1}{2}}}. \tag{3.13}$$

As $\alpha > 0$ was arbitrary, (3.13) implies that $\mathbb{P}(\mathbf{F} \text{ is connected}) \geqslant e^{-\frac{1}{2}+o(1)}$, which combined with Lemma 2.2 proves Claim 2.1. □

A simpler version of the above argument lets us deduce directly from inequality (3.3) in Lemma 3.2 the non-asymptotic result, that $\mathbb{P}(\mathbf{F} \text{ is connected}) > e^{-n/W}$ for all positive integers W and all weight vectors $\vec{w} = (w_1, \ldots, w_n)$ with $\sum_{j=1}^n w_j = W$.

4. Proof of Lemma 2.3

As already noted, Lemma 2.3 follows immediately from Lemma 3.2 in the special case when $n \leqslant \frac{1}{2}W$: here we will prove the full result. Let W be a positive integer and consider any positive integer weight vector $\vec{w} = (w_1, \ldots, w_n)$ with $\sum_i w_i = W$. We may assume that

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$n \geqslant 2$. Given a tree T with vertex set $[n] := \{1, \ldots, n\}$ and an edge $e \in T$, we denote by $s(T, e)$ the smaller weight component of $T - e$, or the component of T containing vertex 1 if the components have equal weights. We call the components of $T - e$ *pendant subtrees of* T. For $i = 1, \ldots, \lfloor W/2 \rfloor$, denote by $r(T, i)$ the quantity $|\{e \in T : w(s(T, e)) = i\}|$.

Recall that $K = \sum_{F \in \mathcal{F}_n} \text{mass}(F)$, and let $K' = \sum_{T \in \mathcal{F}_{n,1}} \text{mass}(T) = K \cdot \mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,1})$. Let $\mathbf{T}$ be a random tree with vertex set $\{1, \ldots, n\}$ and such that

$$\mathbb{P}(\mathbf{T} = T) = \frac{\text{mass}(T)}{K'}. \tag{4.1}$$

Our proof of Lemma 2.3 starts with the identity in Lemma 4.1 below, which expresses the ratio of $\mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,2})$ to $\mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,1})$ as a weighted sum of the values $\mathbb{E}(r(\mathbf{T}, i))$. We then see that we can generate the random tree $\mathbf{T}$ in a natural way using Prüfer codes: see Lemma 4.2. This lets us obtain a good upper bound on the probability that $\mathbf{T}$ has a pendant subtree with a given set of vertices (inequality (4.5) in Lemma 4.3); and using (4.5) we can upper-bound the weighted sum mentioned above, and thus complete the proof.

Lemma 4.1.

$$\mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,2}) = \mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,1}) \cdot \sum_{i=1}^{\lfloor W/2 \rfloor} \frac{\mathbb{E}(r(\mathbf{T}, i))}{i(W - i)}. \tag{4.2}$$

Proof. By Lemma 3.1 and the definition of the function φ given in (3.1),

$$\begin{aligned} K \cdot \mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,2}) &= \sum_{F' \in \mathcal{F}_{n,1}} \sum_{F \in \mathcal{F}_{n,2}} \varphi(F', F) \\ &= \sum_{T \in \mathcal{F}_{n,1}} \text{mass}(T) \cdot \sum_{e \in T} \frac{1}{s(T, e)(W - s(T, e))} \\ &= \sum_{i=1}^{\lfloor W/2 \rfloor} \frac{1}{i(W - i)} \sum_{T \in \mathcal{F}_{n,1}} \text{mass}(T) \cdot r(T, i). \end{aligned}$$

Also, for each $i = 1, \ldots, \lfloor W/2 \rfloor$,

$$\begin{aligned} \sum_{T \in \mathcal{F}_{n,1}} \text{mass}(T) \cdot r(T, i) &= K \cdot \sum_{T \in \mathcal{F}_{n,1}} \mathbb{P}(\mathbf{T} = T) \cdot \mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,1}) \cdot r(T, i) \\ &= K \cdot \mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,1}) \cdot \mathbb{E}(r(\mathbf{T}, i)). \end{aligned}$$

Combining these results proves the lemma. □

Lemma 4.1 allows us to understand the ratio between $\mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,2})$ and $\mathbb{P}(\mathbf{F} \in \mathcal{F}_{n,1})$ by studying the values $\mathbb{E}(r(\mathbf{T}, i))$ for $1 \leqslant i \leqslant \lfloor W/2 \rfloor$. Observe that, for any $k_0 > 0$, for $W \geqslant 2k_0$,

$$\sum_{k=k_0}^{\lfloor W/2 \rfloor} \frac{\mathbb{E}(r(\mathbf{T}, k))}{k(W - k)} \leqslant \frac{n - 1}{k_0(W - k_0)} \leqslant \frac{2}{k_0}.$$

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Thus it suffices to show that for any $\varepsilon > 0$, for W sufficiently large we have

$$\sum_{k=1}^{k_0} \frac{\mathbb{E}(r(\mathbf{T},k))}{k(W-k)} \leqslant (1+\varepsilon)\cdot\frac{1}{2}. \tag{4.3}$$

Consider a sequence of i.i.d. random variables $Z_1, Z_2, \ldots, Z_{n-2}$ with $\mathbb{P}(Z_1 = i) = \frac{w_i}{W}$ for $i = 1, \ldots, n$, and let $\mathbf{Z} = (Z_1, \ldots, Z_{n-2})$. Given $\mathbf{z} = (z_1, \ldots, z_{n-2}) \in [n]^{n-2}$, let $n_i(\mathbf{z}) = |\{j : z_j = i\}|$ for each $i = 1, \ldots, n$. We remind the reader that the *Prüfer code* construction (see for example the book by West [20]) gives a bijection $\phi : [n]^{n-2} \to \mathcal{F}_{n,1}$ such that if $\phi(\mathbf{z}) = T$ then $d_T(i) = n_i(\mathbf{z}) + 1$ for each i. (Recall that $\sum_j d_T(j) - 1 = n - 2$ for any tree T on $[n]$.)

Lemma 4.2. *The random variables $\phi(\mathbf{Z})$ and $\mathbf{T}$ are identically distributed.*

Proof. Fix any tree T_0 on $[n]$. There is exactly one sequence $\mathbf{z} \in [n]^{n-2}$ with $\phi(\mathbf{z}) = T$, and this sequence must contain exactly $d_{T_0}(j) - 1$ co-ordinates j for each $j \in [n]$. We thus have

$$\mathbb{P}(\phi(\mathbf{Z}) = T_0) = \mathbb{P}(\mathbf{Z} = \mathbf{z}) = \prod_{j=1}^{n} \left(\frac{w_j}{W}\right)^{d_{T_0}(j)-1} = \frac{\mathrm{mass}(T_0)}{(\prod_j w_j)\cdot W^{n-2}}.$$

We must then have $K' = (\prod_j w_j)\cdot W^{n-2}$ and, comparing with (4.1), the result follows. □

Definition. Given $I \subseteq \{1, \ldots, n\}$, let P_I be the event that $\mathbf{T}$ contains a pendant subtree T with $V(T) = I$, and write $w(I) = \sum_{i \in I} w_i$.

Lemma 4.3. *For each $I \subseteq [n]$,*

$$\mathbb{P}(P_I) = \left(\frac{w(I)}{W}\right)^{|I|-1} \left(1 - \frac{w(I)}{W}\right)^{n-|I|-1}. \tag{4.4}$$

Thus, for any $\delta > 0$ and $w_0 > 0$, there is a W_0 such that, for each $W \geqslant W_0$ and each $I \subseteq [n]$ with $w(I) \leqslant w_0$,

$$\mathbb{P}(P_I) \leqslant (1+\delta)\left(\frac{w(I)}{W}\right)^{|I|-1} e^{-\frac{nw(I)}{W}}. \tag{4.5}$$

Proof. Let $I = \{1, \ldots, i\} \subseteq [n]$. According to the Prüfer bijection, I is the set of vertices in a pendant subtree of $\phi(\mathbf{z})$, where $\mathbf{z} = (z_1, \ldots, z_{n-2})$, if and only if each of $z_1, \ldots, z_{i-1}$ is in I and none of $z_i, \ldots, z_{n-2}$ is in I. Hence the probability that I is the set of vertices in a pendant subtree of $\phi(\mathbf{Z})$ equals the right-hand side of (4.4). This probability depends on the weights of the vertices in I but does not depend on their labels, so (4.4) must hold for each set $I \subseteq [n]$. Further, the inequality $1 - x \leqslant e^{-x}$ gives

$$\left(1 - \frac{w(I)}{W}\right)^{n-|I|-1} \leqslant e^{-\frac{nw(I)}{W}} \cdot e^{\frac{w(I)(|I|+1)}{W}},$$

and so (4.4) yields (4.5). □

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Now a little manipulation of generating functions will allow us to use (4.5) to upper-bound the sum on the right-hand side of (4.2), and thus complete the proof.

Let $F(y) = \sum_{i=1}^{n} y^{w_i}$ and let $G(y) = \frac{1}{n}F(y)$. Next, let X be uniformly distributed on $[n]$, and let $Y = w_X$, so that Y has probability generating function $G(y)$. Let $Y_1, Y_2, \dots$ be independent, each distributed like Y, and for $i \geqslant 1$ let $Z_i = Y_1 + \cdots + Y_i$. Then, for each positive integer k,

$$[y^k]G(y)^i = \mathbb{P}(Z_i = k),$$

and we write $p_i(k) = \mathbb{P}(Z_i = k)$. Thus

$$\begin{aligned}
\sum_{I\subseteq[n],w(I)=k} x^{|I|} &= [y^k]\sum_{I\subseteq[n]} x^{|I|}y^{w(I)} = [y^k]\sum_{i\geqslant 1} x^i \sum_{I\subseteq[n],|I|=i} y^{w(I)} \\
&= [y^k]\sum_{i\geqslant 1}\frac{x^i}{i!}\sum_{x_1,\dots,x_i\in[n]\text{ distinct}} y^{\sum_j w_{x_j}} \\
&\leqslant [y^k]\sum_{i\geqslant 1}\frac{x^i}{i!}\sum_{x_1,\dots,x_i\in[n]} y^{\sum_j w_{x_j}} = [y^k]\sum_{i\geqslant 1}\frac{x^i}{i!}\,F(y)^i \\
&= \sum_{i\geqslant 1}\frac{(nx)^i}{i!}[y^k]G(y)^i = \sum_{i\geqslant 1}\frac{(nx)^i}{i!}\,p_i(k).
\end{aligned}$$

(Since $p_i(k) = 0$ for $i > k$ we could replace $i \geqslant 1$ in the sum above by $1 \leqslant i \leqslant k$.)

We will use this result with $x = \frac{k}{W}$. Let $\alpha = \frac{n}{W}$, so that $nx = \alpha k$. By the last inequality and (4.5) in Lemma 4.3, for each $\delta > 0$ and each k, for W sufficiently large (both for (4.5) and so that $\left(1 - \frac{k}{W}\right)^{-1} \leqslant (1+\delta)$), we have

$$\begin{aligned}
\frac{\mathbb{E}(r(\mathbf{T},k))}{k(W-k)} &= \frac{1}{k(W-k)}\sum_{I\subseteq[n],w(I)=k}\mathbb{P}(P_I) \\
&\leqslant (1+\delta)\left(1-\frac{k}{W}\right)^{-1} k^{-2}e^{-\frac{nk}{W}}\sum_{I\subseteq[n],w(I)=k}\left(\frac{k}{W}\right)^{|I|} \\
&\leqslant (1+\delta)^2 k^{-2}e^{-\alpha k}\sum_{i\geqslant 1}\frac{(\alpha k)^i}{i!}p_i(k).
\end{aligned}$$

Now fix $\varepsilon > 0$ and $k_0 > 0$. By the preceding inequality, for W sufficiently large, for all $k \leqslant k_0$ we have

$$(1+\varepsilon)^{-1}\cdot\frac{\mathbb{E}(r(\mathbf{T},k))}{k(W-k)} \leqslant k^{-2}e^{-\alpha k}\sum_{i\geqslant 1}\frac{(\alpha k)^i}{i!}p_i(k),$$

and thus

$$\begin{aligned}
(1+\varepsilon)^{-1}\cdot\sum_{k=1}^{k_0}\frac{\mathbb{E}(r(\mathbf{T},k))}{k(W-k)} &\leqslant \sum_{k=1}^{k_0}k^{-2}e^{-\alpha k}\sum_{i\geqslant 1}\frac{(\alpha k)^i}{i!}p_i(k) \\
&= \sum_{1\leqslant i\leqslant k_0}\frac{\alpha^i}{i!}\sum_{k=i}^{k_0}k^{i-2}e^{-\alpha k}p_i(k)
\end{aligned}$$

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$$\leqslant \sum_{i\geqslant 1} \frac{\alpha^i}{i!} i^{-2} \sum_{k\geqslant i} k^i e^{-\alpha k} p_i(k)$$
$$= \sum_{i\geqslant 1} \frac{\alpha^i}{i!} i^{-2} \mathbb{E}(Z_i^i e^{-\alpha Z_i}).$$

For $i = 1, 2, \ldots$, let $f_i(x) = x^i e^{-\alpha x}$ for $x > 0$. Then $f_i'(x) = x^{i-1} e^{-\alpha x}(i - \alpha x)$. Thus $f_i(x)$ takes its maximum value at $x = i/\alpha$, and its maximum value is $\left(\frac{i}{\alpha e}\right)^i$. Hence, for W sufficiently large,

$$(1+\varepsilon)^{-1} \cdot \sum_{k=1}^{k_0} \frac{\mathbb{E}(r(\mathbf{T}, k))}{k(W-k)} \leqslant \sum_{i\geqslant 1} \frac{\alpha^i}{i!} i^{-2} \mathbb{E}(f_i(Z_i))$$
$$\leqslant \sum_{i\geqslant 1} \frac{\alpha^i}{i!} i^{-2} \left(\frac{i}{\alpha e}\right)^i = \sum_{i\geqslant 1} \frac{i^{i-2}}{i! e^i} = \frac{1}{2}.$$

The final equality above can be found, for example, in Bollobás [7, p. 109]. We have now established (4.3); this completes the proof of Lemma 2.3, and thus of Theorem 1.1.

5. Concluding remarks

We have proved the conjecture from McDiarmid, Steger and Welsh [17] that the class of forests is asymptotically the worst possible example of a bridge-addable graph class, from the point of view of connectivity, but only in the special case when the graph class is bridge-alterable, and so also when the class is monotone as well as bridge-addable. Recently, a substantial amount of work has gone into counting the number of random graphs in a variety of graph classes that are bridge-addable; in some cases, this has also led to precise estimates on the probability of connectedness. Giménez and Noy [13] have shown that for a uniformly random planar graph, the probability of connectedness is approximately 0.963253 (correct to 6 decimal places, as are all the figures in this paragraph). The same results hold for the class of graphs embeddable on any fixed surface (Bender and Gao [3], Chapuy, Fusy, Giménez, Mohar and Noy [9]). Similarly, Bodirsky, Giménez, Kang and Noy [5, 6] have shown that for series-parallel graphs and outerplanar graphs, the probabilities of connectedness are approximately 0.889038 and 0.862082, respectively, and Gerke, Giménez, Noy and Weißl [11] have shown that for random $K_{3,3}$-minor-free graphs, this probability is approximately 0.963262. For further related results see [4, 10, 12, 14]. All these results are for monotone graph classes.

The original conjecture from [17] still is open. We venture the following, stronger conjecture.

Conjecture 5.1. *For any n and any non-empty bridge-addable set $\mathcal{A}$ of graphs on $\{1, \ldots, n\}$, if $\mathbf{G}$ is a uniformly random element of $\mathcal{A}$ and $\mathbf{F}$ is a uniformly random element of $\mathcal{F}_n$, then*

$$\mathbb{P}(\mathbf{G} \text{ is connected}) \geqslant \mathbb{P}(\mathbf{F} \text{ is connected}). \tag{5.1}$$

This conjecture would of course yield the original conjecture.

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Acknowledgements

The authors would like to thank two anonymous referees for comments that substantially improved the presentation of the paper.

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