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A PARTICLE SYSTEM APPROACH TO AGGREGATION PHENOMENA

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Abstract

Inspired by a PDE–ODE system of aggregation developed in the biomathematical literature, we investigate an interacting particle system representing aggregation at the level of individuals. We prove that the empirical density of the individual converges to the solution of the PDE–ODE system.

Keywords: Aggregation phenomena; population dynamic; macroscopic limit; particle system; semigroup approach

2010 Mathematics Subject Classification: Primary 35K57; 35Q92; 60K35
Secondary 92B05; 92C17; 92D2

1. Introduction

The mathematical literature applied to biology and social science is rich of models devoted to the description of aggregation. Motivations come from several problems like embryo development, tissue homeostasis, tumor growth, animal swarming, and flocking. The literature presents heterogeneous mathematical tools: discrete and continuous individual-based models, ordinary and partial differential equations (ODEs and PDEs) and a mixture of the two. Also, because of this heterogeneity, an interesting issue is to justify the PDE models through the investigation of scaling limits of models based on interactions between individuals. Following this general program, we propose an individual-based model and prove convergence, when the number of individuals goes to infinity, to a class of PDE–ODE systems that include the so-called Armstrong–Painter–Sherratt model proposed in [1] and [10], which evolved from a previous model given in [11], including, in particular, a form of delay by coupling the system with an ODE.

We assume that individuals interact with one another by looking at the density field produced by the others. Consider, for instance, the motion of animals in a swarm or a flock; presumably, the movement of each animal is driven by a general overview of the others, not computing several pairwise interactions. Let N be the number of individuals and $X_t^{i,N}$, $i = 1, \dots, N$, be their positions. We model this particle-density interaction by the equations:

$$\mathrm{d}X_t^{i,N} = \int_{\mathbb{R}^d} \frac{y - X_t^{i,N}}{|y - X_t^{i,N}|} g(|y - X_t^{i,N}|, u_t^N(y), m_t^N(y))\,\mathrm{d}y\,\mathrm{d}t + \sqrt{2}dB_t^i, \tag{1a}$$

$$\frac{\partial m_t^N(x)}{\partial t} = -\lambda u_t^N(x)(m_t^N(x))^\zeta, \qquad x \in \mathbb{R}^d, \tag{1b}$$

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for $i = 1, \dots, N$ and $t \in [0, T]$, where $u_t^N(y)$ is a density associated with the population of particles, defined below, $m_t^N(x)$ is the field which allows a dependence on the past or may be used to model external effects like those of the extracellular matrix, ζ is typically equal to 1 or 2, and the B_t^i are independent Brownian motions B_t^i accounting for a random component of the motion. Each particle $X_t^{i,N}$ interacts with each location y; the direction of the force is given by the unitary vector $(y - X_t^{i,N})/|y - X_t^{i,N}|$, which spans the line between the particles; the strength of the interaction is $g(|y - X_t^{i,N}|, u_t^N(y))$, namely, it is modulated by the distance $|y - X_t^{i,N}|$, by the density $u_t^N(y)$, and by the external field $m_t^N(y)$. At positions y where $g > 0$, particle $X_t^{i,N}$ moves towards y, namely, it has a tendency to aggregate. Using different functions g, we may describe different kinds of attraction;a wide discussion is presented in Section 5. A technical issue concerns the definition of the density $u_t^N(x)$; see the discussion below. Under suitable assumptions, our main theorem is the convergence of the previous particle model to the PDE–ODE system

$$\frac{\partial u_t}{\partial t} = \Delta u_t - \operatorname{div}(u_t b(u_t, m_t)), \qquad \frac{\partial m_t}{\partial t} = -\lambda u_t m_t^{\zeta} \tag{2}$$

on $[0, T] \times \mathbb{R}^d$, where

$$b(u, m)(x) := \int_{\mathbb{R}^d} \frac{y - x}{|y - x|} g(|y - x|, u(y), m(y))\, \mathrm{d}y. \tag{3}$$

Let us finally discuss the concept of density $u_t^N(x)$. Given the particles $X_t^{i,N}$, we first associate to them the classical concept of empirical measure:

$$S_t^N(\mathrm{d}x) := \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^{i,N}}(\mathrm{d}x).$$

Its direct use, however, in the previous modelling would oblige us to choose functions g depending on measures, instead of functions, which are less easy to formulate in examples. More importantly, we could not speak of $u_t^N(y)$, the density at position y. In numerics it is common to overcome this difficulty by the so-called *kernel smoothing*, which consists in mollifying the measure by convolution with a smooth kernel. We adopt this procedure. We choose a smooth, compactly supported, probability density W (the kernel) and rescale it with N in a suitable way. A general form of rescaling is

$$W_N(x) := N^{\beta} W(N^{\beta/d} x) \quad \text{for some } \beta \in (0, 1),$$

as suggested by Oelschläger [9]. The density $u_t^N(x)$ is thus given by

$$u_t^N(x) := (W_N * S_t^N)(x) = \sum_{i=1}^{N} W_N(x - X^{i,N}).$$

Thanks to the semigroup approach that we implement in the estimates on the particle system, we are able to consider any choice of $\beta \in (0, 1)$. This is not a trivial task, since other approaches require more restrictions on β; see [8] and [9].

The paper is structured as follows. In Section 2 we give some notation, formulate the main result, and prove some preliminary facts. In Section 3 we prove tightness of the density $u_t^N(x)$ in suitable spaces. In Section 4 we show the passage to the limit and complete the proof of the main result. Finally, in Section 5 we discuss several examples of the interaction function g and show by numerical simulations that the previous model may reveal different types of aggregation patterns.

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2. Notation and basic results

2.1. The particle system

For every positive integer N, we consider a particle system described by (1a) coupled with the random field $m_t^N(x)$ satisfying (1b) for some integer $\zeta \geq 1$, with initial conditions $X_0^{i,N} = X_0^i$, $i = 1, \ldots, N$, where B_t^i, $i \in \mathbb{N}$, is a sequence of independent Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$; X_0^i, $i \in \mathbb{N}$, is a sequence of $\mathcal{F}_0$-measurable independent random variables with values in $\mathbb{R}^d$, identically distributed with density u_0; the random function u_t^N is given by $u_t^N(x) := (W_N * S_t^N)(x)$, where $S_t^N = (1/N)\sum_{i=1}^N \delta_{X_t^{i,N}}$ and $W_N(x) := N^\beta W(N^{\beta/d}x)$ for some $\beta \in (0, 1)$; the random fields m_t^N have initial conditions $m_0^N(x) = m_0(x)$, where $m_0 \colon \mathbb{R}^d \to \mathbb{R}$ is a measurable function with $0 \leq m_0 \leq M$, and the functional

$$b \colon L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$$

is given by (3), where $g \colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, $g = g(r, u, m)$, is differentiable, bounded with bounded derivatives, and satisfies

$$|g(r, u, m)| + |\nabla g(r, u, m)| \leq C \exp(-r) \tag{4}$$

for some constant $C > 0$ (where ∇g denotes the gradient in all variables). It follows that, for every pair of measurable functions $u(x)$ and $m(x)$, the aggregation force is bounded:

$$|b(u, m)(x)| \leq \int_{\mathbb{R}^d} |g(|y - x|, u(y), m(y))| \, \mathrm{d}y \leq C \int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|} \, \mathrm{d}y := C' < \infty. \tag{5}$$

We also have

$$|b(u, m)(x) - b(u', m')(x)| \leq C \int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|}(|u(y) - u'(y)| + |m(y) - m'(y)|) \, \mathrm{d}y, \tag{6}$$

and, regarding the derivative, due to the condition on the gradient of g,

$$\begin{aligned} |\nabla_x b(u, m)(x)| &\leq \left| \int_{\mathbb{R}^d} \nabla_x \left(\frac{y - x}{|y - x|} g(|y - x|, u(y), m(y)) \right) \mathrm{d}y \right| \\ &\leq |g(0, u(x), m(x))| + \left| \int_{\mathbb{R}^d} \frac{y - x}{|y - x|} \nabla_x \left(g(|y - x|, u(y), m(y)) \right) \mathrm{d}y \right| \\ &\leq C_1 + \int_{\mathbb{R}^d} |\partial_r g(|y - x|, u(y), m(y))| \, \mathrm{d}y \\ &\leq C_1 + C_2. \end{aligned} \tag{7}$$

Under these assumptions, existence and uniqueness of a solution, for finite N, of system (1) can be proved by classical methods. Let us explain some details. Denote by $C(L^2)$, $C_+(L^2)$, and $C_{0,M}(L^2)$ the spaces

$$\begin{aligned} C(L^2) &:= C([0, T], L^2(\mathbb{R}^d)), \\ C_+(L^2) &:= \{u \in C(L^2) \colon u_t \geq 0 \text{ for all } t \in [0, T]\}, \\ C_{0,M}(L^2) &= \{m \in C(L^2) \colon 0 \leq m_t \leq M \text{ for all } t \in [0, T]\}. \end{aligned}$$

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We say that a random field $m_t^N(x)$, $t \in [0, T]$, $x \in \mathbb{R}^d$, defined on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, is adapted of class $C_{0,M}(L^2)$ if, P-a.s., the functions $(t, x) \mapsto m_t^N(x)$ belong to $C_{0,M}(L^2)$ and, for every $t \in [0, T]$, the function $(x, \omega) \mapsto m_t^N(x, \omega)$ is $(\mathcal{B}(\mathbb{R}^d) \times \mathcal{F}_t)$-measurable. We say that $(X^{1,N}, \dots, X^{N,N}, m^N)$ is a strong solution of system (1) if $X_t^{1,N}, \dots, X_t^{N,N}$ are continuous, adapted processes on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, $m_t^N(x)$ is adapted of class $C_{0,M}(L^2)$, all defined on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, and identities (1a)–(1b) hold, with the equations understood integrated in time. We say that pathwise uniqueness holds if two such solutions are indistinguishable processes.

Proposition 1. *Given any positive integer N and any function $m_0 \in L^2(\mathbb{R}^d)$ such that $0 \le m_0 \le M$, there exists a strong solution of system (1) and pathwise uniqueness holds.*

Proof. The proof is classical, we explain only the idea. Given an integer $\zeta \ge 1$, u^N and a.e. $x \in \mathbb{R}^d$, the solution of (1b) is global, unique, and explicit:

$$m_t^N(x) = F_\zeta\left(m_0(x), \int_0^t u_s^N(x)\,\mathrm{d}s\right), \qquad F_\zeta(a, b) = a\widetilde{F}_\zeta(a, b),$$

$$\widetilde{F}_\zeta(a, b) = \begin{cases} \exp(-\lambda b) & \text{if } \zeta = 1, \\ \dfrac{1}{[a^{\zeta-1}(\zeta-1)\lambda b + 1]^{1/(\zeta-1)}} & \text{if } \zeta \ge 2. \end{cases}$$

The function $\widetilde{F}_\zeta\colon [0, M] \times [0, \infty) \to \mathbb{R}$ is bounded and the function $F_\zeta\colon [0, M] \times [0, \infty) \to \mathbb{R}$ is Lipschitz continuous, with at most linear growth in a, uniformly in b. Then we may consider the system of integral equations

$$X_t^{i,N} = X_0^i + \int_0^t b\left(u_s^N, F_\zeta\left(m_0(\cdot), \int_0^s u_r^N(\cdot)\,\mathrm{d}r\right)\right)(X_s^{i,N})\,\mathrm{d}s + \sqrt{2}B_t^i, \qquad i = 1, \dots, N, \quad (8)$$

as a closed system, with only the variables $X_t^{1,N}, \dots, X_t^{N,N}$. It is a path-dependent equation: the past appears in the drift; but this does not change the way the contraction principle applies. We can check that strong existence and pathwise uniqueness for the original system in the variables $(X^{1,N}, \dots, X^{N,N}, m^N)$ is equivalent to strong existence and pathwise uniqueness for this reduced path-dependent system in the variables $(X^{1,N}, \dots, X^{N,N})$ only; property $m^N \in C_{0,M}(L^2)$ is deduced from the explicit formula. Let us now prove existence and uniqueness for (8). Thanks to property (7), the drift of (8) is globally Lipschitz continuous. We define the family of maps J^i as

$$J^i\colon E \to \mathbb{R},$$

$$J^i(Y) := X_0^i + \int_0^t b\left(u_s^N, F_\zeta\left(m_0(\cdot), \int_0^s u_r^N(\cdot)\,\mathrm{d}r\right)\right)(Y)\,\mathrm{d}s + \sqrt{2}B_t^i, \qquad i = 1, \dots, N,$$

where $E = L^2_{\mathcal{F}}(\Omega, C([0, T'], \mathbb{R}^d))$, with $T' < T$. Then with classical computation, we find that J^i is a contraction on the space E,

$$||J^i(Y) - J^i(Y')||_E \le CT'||Y - Y'||_E,$$

choosing $CT' < 1$. Hence, local existence and uniqueness of strong solutions is proved. Iterating this argument we can obtain the global existence result, because the amplitude of the interval of iteration depends only on CT', namely, it is fixed for each iteration. □

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Remark 1. Existence and uniqueness of solution of system (1) could be obtained following another approach. With less effort, it could be possible to obtain just weak existence and uniqueness in law for system (1): the method of creating a weak solution to SDEs is transformation of drift via Girsanov's theorem; see [6]. With the drift b bounded, see condition (5), hypotheses of Proposition 3.6 and Proposition 3.10 of [6] are verified, and existence and uniqueness of the system is obtained. Then $X_t^{i,N}$ is a solution of (1a). Thus, m_t^N also exists, is unique, and explicit. This kind of existence would be enough for the purpose of the paper, but we still decide to emphasize in Proposition 1 that a stronger result is attainable.

2.2. Main results

Once the indentity in Lemma 3 below for the empirical measure is proved, it is natural to conjecture that the limit of the pair $(u_t^N(x), m_t^N(x))$ solves system (2) with initial condition (u_0, m_0), where u_0 is the density of the random variables (RVs) X_0^i and m_0 is the limit of m_0^N. We interpret the first equation of this system in the so called mild form and the second equation in integral form. Concerning the initial conditions, we make a choice of simplicity. We assume that $u_0 \colon \mathbb{R}^d \to \mathbb{R}$ (the initial distribution of individuals) is a probability density of class C^1 with compact support, see Lemma 6. We assume that $m_0 \colon \mathbb{R}^d \to \mathbb{R}$ is of class $L^2(\mathbb{R}^d)$ and that $0 \le m_0 \le M$.

Definition 1. By a mild solution of system (2) we mean a pair (u, m) belonging to $C_+(L^2) \times C_{0,M}(L^2)$ such that

$$u_t(x) = \mathrm{e}^{t\Delta} u_0 + \int_0^t \nabla \cdot \mathrm{e}^{(t-s)\Delta}(u_s b(u_s, m_s))\,\mathrm{d}s,$$

$$m_t(x) = m_0(x) - \int_0^t \lambda u_s(x) m_s^\zeta(x)\,\mathrm{d}s.$$

Here e^{tA} denotes the heat semigroup, more precisely defined in Section 2.3. Note that the $L^2(\mathbb{R}^d)$-norm of $u_s b(u_s, m_s)$ is bounded, since b is bounded and $u \in C(L^2)$. Hence, $\nabla \cdot \mathrm{e}^{(t-s)\Delta}(u_s b(u_s, m_s))$ is integrable by property (3). Convergence of the particles system is proved only locally in space; hence, we introduce the space

$$C(L^2_{\mathrm{loc}}) := C([0, T], L^2_{\mathrm{loc}}(\mathbb{R}^d)),$$

where the topology on $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ is given by the metric

$$d_{L^2_{\mathrm{loc}}}(f, g) = \sum_{n=1}^{\infty} 2^{-n}(\|f - g\|_{L^2(B(0,n))} \wedge 1).$$

Theorem 1. *System (2) has one and only one mild solution (u, m) in $C_+(L^2) \times C_{0,M}(L^2)$, and the pair (u^N, m^N) converges to (u, m) in $C(L^2_{\mathrm{loc}}) \times C(L^2_{\mathrm{loc}})$, in probability.*

2.3. Some useful properties of the analytic semigroup

We denote by $W^{\alpha,2}(\mathbb{R}^d)$ the fractional Sobolev space, which is a Banach space with the norm

$$\|f\|_{W^{\alpha,2}(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)} + [f]_{\alpha,2,\mathbb{R}^d},$$

where

$$[f]_{\alpha,2,\mathbb{R}^d} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^2}{|x - y|^{2\alpha+d}}\,\mathrm{d}x\,\mathrm{d}y.$$

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Or, equivalently,

$$||u_0^N||_{W^{\alpha,2}} = \|u_0^N\|_{L^2(\mathbb{R}^d)} + \|(-\Delta)^\alpha u_0^N\|_{L^2(\mathbb{R}^d)},$$

where the fractional Laplacian can be characterized by the following lemma.

Lemma 1. *Let $s \in (0, 1)$. Then there exists a constant $C(s, d)$ such that*

$$(-\Delta)^s f(x) = C(s, d) \int_{\mathbb{R}^d} \frac{f(x+y) + f(x-y) - 2f(x)}{|y|^{d+2s}}\, \mathrm{d}y, \qquad x \in \mathbb{R}^d,$$

for every compact support twice differentiable function f.

Note that boundedness of f guarantees integrability at infinity, while twice differentiability implies that the numerator is, for small $|y|$, infinitesimal of order two, which compensates the singularity of the denominator. Another very useful property on the fractional Laplacian is that it is a local operator, namely, it preserves the compact support of functions.

Let us recall some well-known properties of analytical semigroups. The family of operators

$$(\mathrm{e}^{tA} f)(x) := \int_{\mathbb{R}^d} \frac{1}{(2\pi\sigma^2 t)^{d/2}} \mathrm{e}^{-|x-y|^2/2\sigma^2 t} f(y)\, \mathrm{d}y \quad \text{for } t \geq 0$$

defines an analytic semigroup (the heat semigroup) on the space $W^{\alpha,2}(\mathbb{R}^d)$ for every $\alpha \geq 0$. The infinitesimal generator in $L^2(\mathbb{R}^d)$ is the operator $A\colon D(A) \subset L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $D(A) = W^{2,2}(\mathbb{R}^d)$, given by $Af = \frac{1}{2}\sigma^2 \Delta f$. It is possible to define the fractional power of the operator $(I - A)^\delta$ for $\delta \in \mathbb{R}$ and a well-known fact is the equivalence of norms:

$$\|(I - A)^{\delta/2} f\|_{L^2} \sim \|f\|_{W^{\delta,2}}. \tag{9}$$

Another property, often used in the sequel, is that, for every $\delta, T > 0$, there is a constant $C_{\delta,T}$ such that, for $t \in (0, T]$,

$$\|(I - A)^\delta \mathrm{e}^{tA}\|_{L^2 \to L^2} \leq \frac{C_{\delta,T}}{t^\delta}. \tag{10}$$

Finally, we remark that the operator $\nabla(I - A)^{-1/2}$ is bounded in L^2,

$$\|\nabla(I - A)^{-1/2}\|_{L^2 \to L^2} \leq C, \tag{11}$$

where, here and below, we continue to simply write L^2 when the functions are also vector valued, as in the case of $\nabla(I - A)^{-1/2} f$.

It will be useful to know the following result on the improvement of regularity.

Lemma 2. *If $u \in L^p(0, T; L^2(\mathbb{R}^d))$ for some $p > 2$ and satisfies*

$$u_t(x) = \mathrm{e}^{tA} u_0 + \int_0^t \nabla \cdot \mathrm{e}^{(t-s)A}(u_s b_s)\, \mathrm{d}s$$

for some bounded measurable function b, then $u \in C([0, T], L^2(\mathbb{R}^d))$.

Proof. The product ub is in $L^p(0, T; L^2(\mathbb{R}^d))$. Using the bound

$$\|\nabla \cdot \mathrm{e}^{(t-s)A}\|_{L^2 \to L^2} = \|\nabla \cdot (I - A)^{-1/2}(I - A)^{1/2} \mathrm{e}^{(t-s)A}\|_{L^2 \to L^2} \leq \frac{C}{(t-s)^{1/2}},$$

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we deduce that $t \mapsto \int_0^t \nabla \cdot \mathrm{e}^{(t-s)A}(u_s b_s)\,\mathrm{d}s$ is of class $C([0,T], L^2(\mathbb{R}^d))$; the same is true for $t \mapsto \mathrm{e}^{tA}u_0$ because $u \in L^2(\mathbb{R}^d)$ as a byproduct of our assumptions. Hence, $u \in C([0,T], L^2(\mathbb{R}^d))$. □

2.4. Preliminary results

Lemma 3. *For every $\phi \in C^2([0,T]\times\mathbb{R}^d)$, S_t^N satisfies the identity*

$$\langle S_t^N, \phi_t\rangle - \langle S_0^N, \phi_0\rangle = \int_0^t \left\langle S_s^N, \frac{\partial \phi_s}{\partial s}\right\rangle \mathrm{d}s + \frac{\sigma^2}{2}\int_0^t \langle S_s^N, \Delta\phi_s\rangle\,\mathrm{d}s + \int_0^t \langle S_s^N, \nabla\phi_s b(u_s^N, m_s^N)\rangle\,\mathrm{d}s + M_t^{N,\phi},$$

where

$$M_t^{N,\phi} = \frac{\sigma}{N}\sum_{i=1}^N \int_0^t \nabla\phi(X_s^{i,N})\cdot\,\mathrm{d}B_s^i.$$

In particular, choosing $\phi(\cdot) = \phi_x(\cdot) = W_N(\cdot - x)$ for $x \in \mathbb{R}^d$, we obtain

$$u_t^N(x) - u_0^N(x) = \frac{\sigma^2}{2}\int_0^t \Delta u_s^N(x)\,\mathrm{d}s + \int_0^t \mathrm{div}(W_N * (b(u_s^N, m_s^N)S_s^N))(x)\,\mathrm{d}s + M_t^N(x),$$

where

$$M_t^N(x) = \frac{\sigma}{N}\sum_{i=1}^N \int_0^t \nabla W_N(x - X_s^{i,N})\cdot\,\mathrm{d}B_s^i.$$

Proof. The proof follows by Itô's formula and the Gauss Green formula. □

Concerning the family of mollifiers, we have the following useful result, whose proof is an elementary computation; see, for instance, [5].

Lemma 4. *Recall that $W_N(x) = N^\beta W(N^{\beta/d}x)$. Then*

$$\|W_N\|_{L^2}^2 \le CN^\beta, \qquad \|W_N\|_{W^{\gamma,2}}^2 \le CN^{\gamma^*},$$

with $\gamma^* = \beta(2\gamma + d)/d$.

We shall also use the following tightness result.

Lemma 5. *Let X_1 and X_2 be two metric spaces with their Borel σ-fields $\mathcal{B}_1$ and $\mathcal{B}_1$, and let $\phi: X_1 \to X_2$ be a continuous function. Let $\mathcal{G}_1$ be a family of probability measures on $(X_1, \mathcal{B}_1)$. Denote by $\mathcal{G}_2$ the family of probability measures on $(X_2, \mathcal{B}_2)$ obtained as image laws of the measures in $\mathcal{G}_1$ under the map ϕ. If $\mathcal{G}_1$ is tight then $\mathcal{G}_2$ is tight.*

Proof. Given $\varepsilon > 0$, let $K_1^\varepsilon \subset X_1$ be a compact set such that $\mu(K_1^\varepsilon) > 1 - \varepsilon$ for every $\mu \in \mathcal{G}_1$. Set $K_2^\varepsilon = \phi(K_1^\varepsilon)$; it is a compact set of X_2 and, for every $\nu \in \mathcal{G}_2$, with μ a measure in $\mathcal{G}_1$ such that ν is the image of μ under ϕ, we have

$$\nu(K_2^\varepsilon) = \mu(K_1^\varepsilon) > 1 - \varepsilon.$$

This proves tightness of $\mathcal{G}_2$. □

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Regarding the initial condition, we state the following result, that will be useful in the proof of tightness.

Lemma 6. *Assume that X_0^i, $i = 1, \dots, N$, are independent, identically distributed RVs with common probability density $u_0 \in C^2(\mathbb{R}^d)$. Then, on u_0^N, defined as $u_0^N(x) = (W^N * u_0)(x)$, we obtain the uniform bounds*

$$\mathbb{E}[\|u_0^N\|_{W^{\alpha,2}}^p] \le C_{u_0,\alpha,p} \quad \text{for } p > 1,$$

where C is a constant depending on p and α.

Proof. By the definition of the norm in the fractional Sobolev space, we need to estimate uniformly in N:

$$\mathbb{E}[\|u_0^N\|_{W^{\alpha,2}}^p] = \mathbb{E}[\|u_0^N\|_{L^2(\mathbb{R}^d)}^p] + \mathbb{E}[\|(-\Delta)^\alpha u_0^N\|_{L^2(\mathbb{R}^d)}^p]. \tag{12}$$

We recall that u_0 is compactly supported and, moreover, that the fractional Laplacian is a local operator, namely, it preserves the compactness properties of functions. Then, for $p \ge 2$,

$$\|u_0^N\|_{L^2(\mathbb{R}^d)}^p \le \int_{\mathcal{B}_1} |u_0^N(x)|^p \, \mathrm{d}x,$$
$$\|(-\Delta)^\alpha u_0^N\|_{L^2(\mathbb{R}^d)}^p \le \int_{\mathcal{B}_1} |\,(-\Delta)^\alpha \, u_0^N(x)|^p \, \mathrm{d}x,$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ are respectively compact supports of u_0^N and $(-\Delta)^\alpha \, u_0^N$. Assuming that

$$Y^i = Y^i(x) = W_N(x - X_0^i), \qquad \widetilde{Y}^i = \widetilde{Y}^i(x) = (-\Delta)^\alpha W_N(x - X_0^i),$$

we can write the right-hand side (RHS) estimates of (12) as follows:

$$\text{RHS of (12)} \le \mathbb{E}\left[\int_{\mathcal{B}_1} \left| \frac{1}{N} \sum_{i=1}^N Y^i(x) \right|^p \mathrm{d}x \right] + \mathbb{E}\left[\int_{\mathcal{B}_2} \left| \frac{1}{N} \sum_{i=1}^N \widetilde{Y}^i(x) \right|^p \mathrm{d}x \right].$$

Then we need to estimate

$$\mathbb{E}\left[\left| \frac{1}{N} \sum_{i=1}^N Y^i(x) \right|^p \right] + \mathbb{E}\left[\left| \frac{1}{N} \sum_{i=1}^N \widetilde{Y}^i(x) \right|^p \right].$$

With $Y^i \ge 0$ on the first summand, we have

$$\begin{aligned}
\mathbb{E}\left[\left| \frac{1}{N} \sum_{i=1}^N Y^i \right|^p \right] &= \int_0^\infty \mathbb{P}\left(\left(\frac{1}{N} \sum_{i=1}^N Y^i \right)^p > t \right) \mathrm{d}t \\
&= \int_0^\infty \mathbb{P}\left(\frac{1}{N} \sum_{i=1}^N Y^i > t^{1/p} \right) \mathrm{d}t \\
&= \int_0^\infty \mathbb{P}\left(\exp\left(\frac{1}{N} \sum_{i=1}^N Y^i \right) > \exp\left(t^{1/p}\right) \right) \mathrm{d}t
\end{aligned}$$

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$$\leq \int_0^\infty \exp(-t^{1/p}) \mathbb{E}\left[\exp\left(\left(\frac{1}{N}\right)\sum_{i=1}^N Y^i\right)\right] \mathrm{d}t$$
$$= \mathrm{e}^{N\log \mathbb{E}[\mathrm{e}^{Y/N}]} \int_0^\infty \exp(-t^{1/p})\, \mathrm{d}t,$$

where Y has the same law as Y^i. Note that the equality

$$\mathbb{E}\left[\exp\left(\left(\frac{1}{N}\right)\sum_{i=1}^N Y^i\right)\right] = \mathrm{e}^{N\log \mathbb{E}[\mathrm{e}^{Y/N}]}$$

follows easily from the fact that the Y^i are independent and identically distributed (i.i.d.). Because $\tilde{Y}^i \geq 0$ also, the same result holds for the second term. Then

$$\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N Y^i(x)\right|^p\right] + \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N \widetilde{Y}^i(x)\right|^p\right]$$
$$\leq (\mathrm{e}^{N\log \mathbb{E}[\mathrm{e}^{Y/N}]} + \mathrm{e}^{N\log \mathbb{E}[\mathrm{e}^{\tilde{Y}/N}]}) \int_0^\infty \exp(-t^{1/p})\, \mathrm{d}t.$$

We know estimate the first term (the same result will hold for the second term). We recall the basic inequalities $\log(1+x) \leq x$ and $\mathrm{e}^x - 1 \leq x\mathrm{e}^x$ for $x \geq 0$. Then

$$\log \mathbb{E}[\mathrm{e}^{Y(x)/N}] = \log(1 + \mathbb{E}[\mathrm{e}^{Y(x)/N} - 1])$$
$$\leq \mathbb{E}[\mathrm{e}^{Y(x)/N} - 1]$$
$$\leq \mathbb{E}\left[\frac{Y(x)}{N}\mathrm{e}^{Y(x)/N}\right].$$

We have to estimate

$$\mathbb{E}[Y(x)\mathrm{e}^{Y(x)/N}] \quad \text{and} \quad \mathbb{E}[\tilde{Y}(x)\mathrm{e}^{\tilde{Y}(x)/N}].$$

Recalling the definition of Y^i,

$$\frac{Y(x)}{N} = N^{-1} W_N(x - X_0) = N^{\beta-1} W(N^{\beta/d}(x - X_0)) \leq C.$$

With W bounded, $Y(x)/N$ is bounded. Now we just need to estimate

$$\mathbb{E}[Y(x)] = (W^N * u_0)(x).$$

The last term is bounded because u_0 is. Let us analyze the second term, which is a bit more delicate. By the definition of $\tilde{Y}^i$,

$$\frac{\widetilde{Y}(x)}{N} = N^{-1}(-\Delta)^\alpha W_N(x - X_0) \leq CN^{-1+\beta}N^{2\alpha\beta/d}.$$

Choose an α small enough that the term $\widetilde{Y}(x)/N$ is bounded. At the end we need to prove a uniform estimate on $\mathbb{E}[\tilde{Y}(x)]$. We have

$$\mathbb{E}[\widetilde{Y}(x)] = \mathbb{E}[(-\Delta)^\alpha W_N(x - X_0)]$$
$$= \int [(-\Delta)^\alpha W_N](x - x_0) u_0(x_0)\, \mathrm{d}x_0$$

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$$
\begin{aligned}
&= -\int (-\Delta)^{\alpha} W_N(x_0') u_0(x - x_0')\ \mathrm{d}x_0' \quad (x_0' = x - x_0)\\
&= -\langle (-\Delta)^{\alpha} W_N,\, u_0(x - \cdot)\rangle_{L^2}\\
&= -\langle W_N,\, (-\Delta)^{\alpha} u_0(x - \cdot)\rangle_{L^2}\\
&= -\int W_N(x_0')[(-\Delta)^{\alpha} u_0](x - x_0')\ \mathrm{d}x_0'\\
&= [W_N * ((-\Delta)^{\alpha} u_0)]\,(x).
\end{aligned}
$$

With $(-\Delta)^{\alpha} u_0$ compactly supported and continuous, $W_N * ((-\Delta)^{\alpha} u_0)$ is also uniformly bounded. In summary,

$$\mathbb{E}[\|u_0^N\|_{W^{\alpha,2}}^p] \le C_{\mathcal{B}_1,\mathcal{B}_2,u_0,\alpha,p}.$$ □

3. Tightness

3.1. Compactness of function spaces

We use Corollary 9 of [12], using as far as possible the notation of that paper, for ease of reference. Given a ball $B_R := B(0, R)$ in $\mathbb{R}^d$, letting $\alpha > \varepsilon > 0$, consider the spaces

$$X = W^{\alpha,2}(B_R), \qquad B = W^{\alpha-\varepsilon,2}(B_R), \qquad Y = W^{-2,2}(B_R).$$

We have

$$X \subset B \subset Y$$

with compact dense embeddings. Moreover, we have the interpolation inequality (see Theorem 6.4.5 of [2])

$$\|f\|_B \le C_R \|f\|_X^{1-\theta} \|f\|_Y^{\theta}$$

for all $f \in X$, with

$$\theta = \frac{\varepsilon}{2+\alpha}.$$

These are the preliminary assumptions of Corollary 9 of [12]. The corollary states that the embedding of

$$\mathcal{W}_R := L^{r_0}(0, T; X) \cap W^{s_1,r_1}(0, T; Y)$$

is relatively compact in $C([0, T]; B)$ if $s_1 r_1 > 1$ and r_0 is large enough that $s_\theta > 1/r_\theta$, where (always following the notation of [12]) $s_\theta = \theta s_1$ and $1/r_\theta = (1-\theta)/r_0 + \theta/r_1$. Below we shall choose, for instance, $s_1 = \frac{1}{3}$ (any number smaller than $\frac{1}{2}$) and $r_1 = 4$, so $s_1 r_1 > 1$ is fulfilled. Then we need

$$\frac{\theta}{3} > \frac{1-\theta}{r_0} + \frac{\theta}{4}.$$

The logical sequence of our choices is as follows. Given $\beta \in (0, 1)$ (consider β close to 1, which is the most difficult choice), we shall choose $\alpha > 0$ small enough to satisfy a condition related to β which appears in the proof of Lemma 7 below (when β is close to 1, we have to choose small α). Given this small α, we choose $\varepsilon \in (0, \alpha)$ and then $\theta = \varepsilon/(2+\alpha)$ is

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determined, typically very small. Now, we choose r_0 large enough that $\theta/3 > (1-\theta)/r_0 + \theta/4$. Summarising, we choose $(\alpha, s_1, r_1, r_0, \varepsilon)$, in the following way:

- α is determined by β;
- (s_1, r_1) is determined (almost) *a priori* (in the proof of Proposition 2 below $s_1 r_1 - r_1/2 < 0$);
- r_0 is chosen large enough that $\theta s_1 > (1-\theta)/r_0 + \theta/r_1$;
- $\varepsilon < \alpha$ arbitrarily small.

The final step consists in taking $\mathbb{R}^d$ instead of B_R. We denote by $W^{\alpha,2}_{\mathrm{loc}}(\mathbb{R}^d)$ the space of functions $f \in \bigcap_{R>0} W^{\alpha,2}(B_R)$ and we endow this space with the metric

$$d_{W^{\alpha,2}_{\mathrm{loc}}}(f,g) = \sum_{n=1}^{\infty} 2^{-n}(\|f-g\|_{W^{\alpha,2}(B_n)} \wedge 1).$$

Under the same conditions on the indexes,

$$\mathcal{W} := L^{r_0}(0,T; W^{\alpha,2}(\mathbb{R}^d)) \cap W^{s_1,r_1}(0,T; W^{-2,2}(\mathbb{R}^d))$$

is compactly embedded into $C([0,T]; W^{\alpha-\varepsilon,2}_{\mathrm{loc}}(\mathbb{R}^d))$.

3.2. Main estimate on the empirical density u^N

Before examining in detail the derivation of the main estimates for the empirical density, we state the mild formulation for u^N_t (see Lemma 3 for the identity for u^N_t):

$$u^N_t = \mathrm{e}^{tA}u^N_0 + \int_0^t \mathrm{e}^{(t-s)A}\mathrm{div}(W_N * (b(u^N_s, m^N_s)S^N_s))(x)\,\mathrm{d}s + \int_0^t \mathrm{e}^{(t-s)A}\,\mathrm{d}M^N_s.$$

Lemma 7. *Given $\beta \in (0,1)$, there exists $\alpha > 0$ small enough such that the following assertion holds: for every $p > 1$, there is a constant $C_p > 0$ such that*

$$\sup_{t\in[0,T]} \mathbb{E}[\|u^N_t\|^p_{W^{\alpha,2}(\mathbb{R}^d)}] \le C_p$$

independently of N.

Proof. Step 1: preliminary estimates. We shall use the equivalence between norms (9):

$$\|(I-A)^{\alpha/2} f\|_{L^2(\mathbb{R}^d)} \sim \|f\|_{W^{\alpha,2}(\mathbb{R}^d)}.$$

Then, up to a constant, letting $f^N_s(x) = \mathrm{div}(W_N * (b(u^N_s, m^N_s)S^N_s))(x)$,

$$\|u^N_t\|_{W^{\alpha,2}(\mathbb{R}^d)} \le \|(I-A)^{\alpha/2}\mathrm{e}^{tA}u^N_0\|_{L^2(\mathbb{R}^d)} + \|(I-A)^{\alpha/2}\int_0^t \mathrm{e}^{(t-s)A}f^N_s\,\mathrm{d}s\|_{L^2(\mathbb{R}^d)}$$
$$+ \|(I-A)^{\alpha/2}\int_0^t \mathrm{e}^{(t-s)A}\,\mathrm{d}M^N_s\|_{L^2(\mathbb{R}^d)}.$$

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For the first term, using (10), we prove the estimate

$$\begin{aligned}\mathbb{E}[\|(I-A)^{\alpha/2}\mathrm{e}^{tA}u_0^N\|_{L^2(\mathbb{R}^d)}^p] &\le \|\mathrm{e}^{tA}\|_{L^2\to L^2}^p\mathbb{E}[\|(I-A)^{\alpha/2}u_0^N\|_{L^2(\mathbb{R}^d)}^p]\\ &\le C\mathbb{E}[\|u_0^N\|_{W^{\alpha,2}(\mathbb{R}^d)}^p].\end{aligned}$$

The last expected value is bounded by the assumption that u_0 is a C^1 compact support: in this case we can show convergence of the empirical means of the i.i.d. RVs X_0^i, which implies a uniform in N bound on $\mathbb{E}[\|u_0^N\|_{W^{\alpha,2}(\mathbb{R}^d)}^p]$ for every p (see [5] for similar results).

For the third term, we use the following fact. For every $p>1$, there is a constant $C_p>0$ such that, if $\Phi_t^1,\dots,\Phi_t^N$ are adapted square-integrable processes with values in a Hilbert space H,

$$\mathbb{E}\left[\left\|\sum_{i=1}^N\int_0^T\Phi_t^i\,\mathrm{d}B_t^i\right\|_H^p\right]\le C_p\mathbb{E}\left[\left(\sum_{i=1}^N\int_0^T\left\|\Phi_t^i\right\|_H^2\,\mathrm{d}t\right)^{p/2}\right].$$

Therefore,

$$\begin{aligned}&\mathbb{E}\left[\left\|\int_0^t(I-A)^{\alpha/2}\mathrm{e}^{(t-s)A}\,\mathrm{d}M_s^N\right\|_{L^2(\mathbb{R}^d)}^p\right]\\ &\quad=\mathbb{E}\left[\left\|\frac{\sigma}{N}\sum_{i=1}^N\int_0^t(I-A)^{\alpha/2}\mathrm{e}^{(t-s)A}\nabla W_N(\cdot-X_s^{i,N})\,\mathrm{d}B_s^i\right\|_{L^2(\mathbb{R}^d)}^p\right]\\ &\quad\le C_p\mathbb{E}\left[\left(\frac{\sigma^2}{N^2}\sum_{i=1}^N\int_0^T\|(I-A)^{\alpha/2}\mathrm{e}^{(t-s)A}\nabla W_N(\cdot-X_s^{i,N})\|_{L^2(\mathbb{R}^d)}^2\,\mathrm{d}s\right)^{p/2}\right]\\ &\quad=C_p\mathbb{E}\left[\left(\frac{\sigma^2}{N^2}\sum_{i=1}^N\int_0^T\|(I-A)^{\alpha/2}\mathrm{e}^{(t-s)A}\nabla W_N\|_{L^2(\mathbb{R}^d)}^2\,\mathrm{d}s\right)^{p/2}\right]\\ &\quad=C_p\left(\frac{\sigma^2}{N}\int_0^T\|(I-A)^{\alpha/2}\mathrm{e}^{(t-s)A}\nabla W_N\|_{L^2(\mathbb{R}^d)}^2\,\mathrm{d}s\right)^{p/2}.\end{aligned}$$

Moreover, the gradient commutes with the heat semigroup and the fractional powers of the Laplacian. Hence, using (10) and (11), the integrand can be estimated as

$$\begin{aligned}&\|(I-A)^{\alpha/2}\mathrm{e}^{(t-s)A}\nabla W_N\|_{L^2(\mathbb{R}^d)}^2\\ &\quad\le[\|\nabla(I-A)^{-1/2}\|_{L^2\to L^2}\|(I-A)^{(1-\varepsilon)/2}\mathrm{e}^{(t-s)A}\|_{L^2\to L^2}\|(I-A)^{(\alpha+\varepsilon)/2}W_N\|_{L^2(\mathbb{R}^d)}]^2\\ &\quad\le\frac{c}{(t-s)^{1-\varepsilon}}\|W_N\|_{W^{\alpha+\varepsilon,2}(\mathbb{R}^d)}^2.\end{aligned}$$

From Lemma 4 we obtain

$$\|(I-A)^{\alpha/2}\mathrm{e}^{(t-s)A}\nabla W_N\|_{L^2(\mathbb{R}^d)}^2\le\frac{c}{(t-s)^{1-\varepsilon}}N^{(\alpha+\varepsilon)^*},$$

and thus we can estimate the martingale term in the following way:

$$\begin{aligned}\mathbb{E}\left[\left\|\int_0^t(I-A)^{\alpha/2}\mathrm{e}^{(t-s)A}\,\mathrm{d}M_s^N\right\|_{L^2(\mathbb{R}^d)}^p\right]&\le C_p\left(\frac{\sigma^2}{N}\int_0^T\frac{c}{(t-s)^{1-\varepsilon}}N^{(\alpha+\varepsilon)^*}\,\mathrm{d}s\right)^{p/2}\\ &=C_{p,T}\left(\frac{N^{(\alpha+\varepsilon)^*}}{N}\right)^{p/2}.\end{aligned}$$

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Choosing α small enough that $(\alpha+\varepsilon)^* \le 1$, i.e. $\beta \le d/(2(\alpha+\varepsilon)+d) < 1$, we obtain a uniform bound on the martingale term.

Finally, thanks to the boundness on b we obtain the estimate

$$\begin{aligned}|W_N * (b(u_t^N, m_t^N)S_t^N)(x)| &= \left| \int_{\mathbb{R}^d} W_N\,(x-y)\, b(u_t^N, m_t^N)(y) S_t^N\,(\mathrm{d}y) \right| \\ &\le \int_{\mathbb{R}^d} W_N\,(x-y)\, |b(u_t^N, m_t^N)(y)| S_t^N\,(\mathrm{d}y) \\ &\le C \int_{\mathbb{R}^d} W_N\,(x-y)\, S_t^N\,(\mathrm{d}y) \\ &= C u_t^N\,(x);\end{aligned}$$

hence,

$$\|W_N * (b(u_t^N, m_t^N)S_t^N)\|^2_{L^2(\mathbb{R}^d)} \le C \|u_t^N\|_{L^2(\mathbb{R}^d)}.$$

Step 2: estimate in $L^2(\mathbb{R}^d)$. Consider the $\alpha = 0$ case in the previous computations. We have proved, with the notation $H = L^2(\mathbb{R}^d)$, that

$$\|u_t^N\|_{L^p(\Omega;H)} \le C + \left\| \int_0^t \mathrm{e}^{(t-s)A} f_s^N \,\mathrm{d}s \right\|_{L^p(\Omega;H)}.$$

Thus,

$$\|u_t^N\|_{L^p(\Omega;H)} \le C + \int_0^t \|\mathrm{e}^{(t-s)A} f_s^N\|_{L^p(\Omega;H)}\,\mathrm{d}s.$$

We have

$$\begin{aligned}\|\mathrm{e}^{(t-s)A} f_s^N\|_{L^p(\Omega;H)} &= \mathbb{E}[\|\nabla \cdot \mathrm{e}^{(t-s)A}(W_N * (b(u_t^N, m_t^N)S_t^N)\|^p_{L^2(\mathbb{R}^d)}]^{1/p} \\ &\le \|\nabla \cdot \mathrm{e}^{(t-s)A}\|_{L^2 \to L^2} \mathbb{E}[\|W_N * b(u_t^N, m_t^N)S_t^N\|^p_{L^2(\mathbb{R}^d)}]^{1/p} \\ &\le \frac{C}{(t-s)^{1/2}} \|u_s^N\|_{L^p(\Omega;H)},\end{aligned}$$

using properties on the analytical semigroup, (10) and (11), and the last bound of step 1. Therefore,

$$\|u_t^N\|_{L^p(\Omega;H)} \le C + \int_0^t \frac{C}{(t-s)^{1/2}} \|u_s^N\|_{L^p(\Omega;H)}\,\mathrm{d}s.$$

A generalised form of the Gronwall lemma implies that

$$\sup_{t\in[0,T]} \|u_t^N\|_{L^p(\Omega;H)} \le C,$$

where the constant C depends on p but not on N.

Step 3: estimate in $W^{\alpha,2}(\mathbb{R}^d)$. Similarly to the beginning of step 2, we have

$$\|u_t^N\|_{L^p(\Omega;\widetilde{H})} \le C + \int_0^t \|\mathrm{e}^{(t-s)A} f_s^N\|_{L^p(\Omega;\widetilde{H})}\,\mathrm{d}s,$$

where now $\widetilde{H} = W^{\alpha,2}(\mathbb{R}^d)$; recalling some properties of the analytical semigroup (see (9),(10), and (11)), we obtain

$$\|\mathrm{e}^{(t-s)A} f_s^N\|_{L^p(\Omega;\widetilde{H})} \le \frac{C}{(t-s)^{(\alpha+1)/2}} \|u_s^N\|_{L^p(\Omega;H)}.$$

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However, from step 2 we know that $\|u_s^N\|_{L^p(\Omega;H)}$ is uniformly bounded; hence, for $\alpha < 1$, we deduce the claim of the lemma. □

3.3. Tightness of (u_t^N, m_t^N)

Recall from Section 3.1 that the space there denoted by $\mathcal{W}$ is compactly embedded in $C([0,T];W^{\alpha-\varepsilon,2}_{\mathrm{loc}}(\mathbb{R}^d))$ when $s_1 = \frac{1}{3}$ and $r_1 = 4$, and when, having chosen α small enough related to the original choice of β such that the result of Lemma 7 holds, we take r_0 large enough.

In order to prove tightness of the family of laws of u_t^N in $\mathcal{W}$, we have to prove that u_t^N is bounded in probability in $L^{r_0}(0,T;W^{\alpha,2}(\mathbb{R}^d))$ and in $W^{s_1,r_1}(0,T;W^{-2,2}(\mathbb{R}^d))$. For the first claim, it is sufficient to prove that

$$\mathbb{E}\int_0^T \|u_t^N\|^{r_0}_{W^{\alpha,2}(\mathbb{R}^d)}\,\mathrm{d}t \le C,$$

which is true by Lemma 7 because

$$\mathbb{E}\int_0^T \|u_t^N\|^{r_0}_{W^{\alpha,2}(\mathbb{R}^d)}\,\mathrm{d}t = \int_0^T \mathbb{E}[\|u_t^N\|^{r_0}_{W^{\alpha,2}(\mathbb{R}^d)}]\,\mathrm{d}t \le \sup_{t\in[0,T]} \mathbb{E}[\|u_t^N\|^{r_0}_{W^{\alpha,2}(\mathbb{R}^d)}].$$

The second claim is proved in the next proposition.

Proposition 2. *The family $\{u_t^N\}_N$ is bounded in probability in $W^{s_1,r_1}(0,T;W^{-2,2}(\mathbb{R}^d))$.*

Proof. Let us recall that a norm on $W^{s_1,r_1}(0,T;W^{-2,2}(\mathbb{R}^d))$ is given by the sum

$$\left(\int_0^T \|f_t\|^{r_1}_{W^{-2,2}(\mathbb{R}^d)}\,\mathrm{d}t\right)^{1/r_1} + \left(\int_0^T\int_0^T \frac{\|f_t - f_s\|^{r_1}_{W^{-2,2}(\mathbb{R}^d)}}{|t-s|^{1+s_1 r_1}}\,\mathrm{d}t\,\mathrm{d}s\right)^{1/r_1}.$$

The property

$$\mathbb{E}\int_0^T \|u_t^N\|^{r_1}_{W^{-2,2}(\mathbb{R}^d)}\,\mathrm{d}t \le C$$

is a consequence of Lemma 7 because $\|u_t^N\|_{W^{-2,2}}$ is a weaker norm than $\|u_t^N\|_{W^{\alpha,2}}$. We have to prove that

$$\mathbb{E}\int_0^T\int_0^T \frac{\|u_t^N - u_s^N\|^{r_1}_{W^{-2,2}(\mathbb{R}^d)}}{|t-s|^{1+s_1 r_1}}\,\mathrm{d}t\,\mathrm{d}s \le C.$$

Thus, for $t > s$, we have to estimate

$$\mathbb{E}[\|u_t^N - u_s^N\|^{r_1}_{W^{-2,2}(\mathbb{R}^d)}].$$

From the equation satisfied by u_t^N, proved in Lemma 3, and Hölder's inequality, we have

$$\begin{aligned}
&\mathbb{E}[\|u_t^N - u_s^N\|^{r_1}_{W^{-2,2}(\mathbb{R}^d)}] \\
&\quad\le C(t-s)^{r_1-1}\mathbb{E}\int_s^t \|Au_r^N\|^{r_1}_{W^{-2,2}(\mathbb{R}^d)}\,\mathrm{d}r + C\mathbb{E}[\|M_t^N - M_s^N\|^{r_1}_{W^{-2,2}(\mathbb{R}^d)}] \\
&\qquad + C(t-s)^{r_1-1}\mathbb{E}\int_s^t \|\mathrm{div}(W_N * b(u_r^N, m_r^N)S_r^N)\|^{r_1}_{W^{-2,2}(\mathbb{R}^d)}\,\mathrm{d}r.
\end{aligned}$$

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With A a bounded operator from L^2 to $W^{-2,2}$, we have

$$C(t-s)^{r_1-1}\mathbb{E}\int_s^t \|Au_r^N\|^{r_1}_{W^{-2,2}(\mathbb{R}^d)}\,\mathrm{d}r \le C(t-s)^{r_1-1}\int_s^t \mathbb{E}[\|u_r^N\|^{r_1}_{L^2(\mathbb{R}^d)}]\,\mathrm{d}r \le C(t-s)^{r_1}$$

thanks to the estimate in Lemma 7. Note that the spaces L^2 and $W^{1,2}$ are continuously embedded in $W^{-2,2}$, namely, there exists a constant $C>0$ such that $\|f\|_{W^{-2,2}} \le C\|f\|_{L^2}$ and $\|f\|_{W^{-2,2}} \le C\|f\|_{W^{-1,2}}$. We shall use this in the following computations. We have

$$\begin{aligned}\|\mathrm{div}(W_N * b(u_r^N, m_r^N)S_r^N)\|_{W^{-2,2}(\mathbb{R}^d)} &\le C\|(W_N * b(u_r^N, m_r^N)S_r^N\|_{W^{-1,2}(\mathbb{R}^d)}\\ &\le C\|(W_N * b(u_r^N, m_r^N)S_r^N)\|_{L^2(\mathbb{R}^d)}\\ &\le C\|u_r^N\|_{L^2(\mathbb{R}^d)},\end{aligned}$$

where the last inequality is similar to that proved in Lemma 7. Hence,

$$\begin{aligned}&C(t-s)^{r_1-1}\mathbb{E}\int_s^t \|\mathrm{div}(W_N * b(u_r^N, m_r^N)S_r^N)\|^{r_1}_{W^{-2,2}(\mathbb{R}^d)}\,\mathrm{d}r\\ &\le C(t-s)^{r_1-1}\int_s^t \mathbb{E}[\|u_r^N\|^{r_1}_{L^2(\mathbb{R}^d)}]\,\mathrm{d}r\\ &\le C(t-s)^{r_1},\end{aligned}$$

as above. Therefore, we have proved that

$$\mathbb{E}[\|u_t^N - u_s^N\|^{r_1}_{W^{-2,2}(\mathbb{R}^d)}] \le C(t-s)^{r_1} + C\mathbb{E}[\|M_t^N - M_s^N\|^{r_1}_{W^{-2,2}(\mathbb{R}^d)}].$$

Estimating the martingale as in Lemma 7, we have

$$\begin{aligned}\mathbb{E}\left[\|M_t^N - M_s^N\|^{r_1}_{W^{-2,2}(\mathbb{R}^d)}\right] &= \mathbb{E}\left[\left\|\frac{\sigma}{N}\sum_{i=1}^N\int_s^t \nabla W_N(\cdot - X_u^{i,N})\,\mathrm{d}B_u^i\right\|^{r_1}_{W^{-2,2}(\mathbb{R}^d)}\right]\\ &\le C_{r_1}\mathbb{E}\left[\left(\frac{\sigma^2}{N^2}\sum_{i=1}^N\int_s^t \|\nabla W_N(\cdot - X_u^{i,N})\|^2_{W^{-2,2}(\mathbb{R}^d)}\,\mathrm{d}u\right)^{r_1/2}\right]\\ &\le C'_{r_1}\mathbb{E}\left[\left(\frac{\sigma^2}{N^2}\sum_{i=1}^N\int_s^t \|W_N(\cdot - X_u^{i,N})\|^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}u\right)^{r_1/2}\right]\\ &= C'_{r_1}\mathbb{E}\left[\left(\frac{\sigma^2}{N}\|W_N\|^2_{L^2(\mathbb{R}^d)}\right)^{r_1/2}\right]\\ &\le C'_{r_1}\mathbb{E}\left[\left(\frac{\sigma^2}{N}N^\beta\right)^{r_1/2}\right](t-s)^{r_1/2},\end{aligned}$$

by Lemma 4; hence (being $\beta<1$),

$$\mathbb{E}[\|M_t^N - M_s^N\|^{r_1}_{W^{-2,2}(\mathbb{R}^d)}] \le C\,(t-s)^{r_1/2}.$$

Summarising,

$$\mathbb{E}[\|u_t^N - u_s^N\|^{r_1}_{W^{-2,2}(\mathbb{R}^d)}] \le C\,(t-s)^{r_1/2}.$$

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It follows that

$$\mathbb{E}\int_0^T\int_0^T \frac{\|u_t^N - u_s^N\|^{r_1}_{W^{-2,2}(\mathbb{R}^d)}}{|t-s|^{1+s_1 r_1}}\,\mathrm{d}t\,\mathrm{d}s \le \mathbb{E}\int_0^T\int_0^T \frac{C}{|t-s|^{1+s_1 r_1 - r_1/2}}\,\mathrm{d}t\,\mathrm{d}s,$$

which is finite if $s_1 r_1 - r_1/2 < 0$; with our choices $s_1 = \frac{1}{3}$ and $r_1 = 4$, such an assertion holds. □

Corollary 1. *The family $\{u_t^N\}_N$ is bounded in probability in $\mathcal{W}$, and, therefore, the family of laws of $\{u_t^N\}_N$ is tight in $C([0,T];W^{\alpha-\varepsilon,2}_{\mathrm{loc}}(\mathbb{R}^d))$. In particular, it is tight in $C(L^2_{\mathrm{loc}})$. If Q_u is any limit measure of this family and u is a RV with law Q_u, we also have the property that*

$$\mathbb{E}\int_0^T \|u_t\|^{r_0}_{L^2(\mathbb{R}^d)}\,\mathrm{d}t < \infty,$$

namely, Q_u is supported on $L^p(0,T;L^2(\mathbb{R}^d))$ for some $p > 2$. Therefore, if we prove that Q_u is supported on mild solutions, by Lemma 2 we deduce that Q_u is supported on $C(L^2)$.

Proposition 3. *The family of laws of $\{m_t^N\}_N$ is tight in $C(L^2_{\mathrm{loc}})$. If Q_m is any limit measure of this family and m is a RV with law Q_m, we also have the property that*

$$\mathbb{E}\int_0^T \|m_t\|^{r_0}_{L^2(\mathbb{R}^d)}\,\mathrm{d}t < \infty,$$

namely, Q_m is supported on $L^p(0,T;L^2(\mathbb{R}^d))$ for some $p > 2$. Therefore, if we prove that Q_m is supported on mild solutions, by Lemma 2 we deduce that Q_m is supported on $C(L^2)$.

Proof. Let $C_+(L^2_{\mathrm{loc}})$ be the space of nonnegative functions of class $C(L^2_{\mathrm{loc}})$. Recall the explicit form of the solution of (1b) given in Proposition 1. We want to apply Lemma 5 with $X_1 = C_+(L^2_{\mathrm{loc}})$, $X_2 = C_+(L^2_{\mathrm{loc}})$, $\mathcal{G}_1$ given by the family of laws of $\{u_t^N\}_N$, $\mathcal{G}_2$ given by the family of laws of $\{m_t^N\}_N$, and ϕ given by (for $f \in C_+(L^2_{\mathrm{loc}})$, it is here that we use nonnegativity)

$$(\phi f)_t(x) := F_\zeta\left(m_0(x), \int_0^t f_s(x)\,\mathrm{d}s\right),$$

where $F_\zeta(a,b)$ was introduced in Proposition 1. Tightness of the family $\mathcal{G}_1$ is given by Proposition 2. To prove continuity of ϕ, we first note that the map

$$f \mapsto \int_0^t f(s,x)\,\mathrm{d}s$$

is continuous from $C(L^2_{\mathrm{loc}})$ to $C(L^2_{\mathrm{loc}})$, and then we compose it with a bounded continuous map. □

4. Passage to the limit

Denote by Q^N the law of (u^N, m^N) on the space $C(L^2_{\mathrm{loc}}) \times C(L^2_{\mathrm{loc}})$. We have proved that the family $\{Q^N\}$ is tight. Hence, by Prohorov's theorem, there is a subsequence Q^{N_k} which converges weakly to some probability measure Q on $C(L^2_{\mathrm{loc}}) \times C(L^2_{\mathrm{loc}})$. Moreover, from Corollary 1, the marginal Q_u on the first component is supported on the space $L^p(0,T;L^2(\mathbb{R}^d))$

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for some $p > 2$. We first prove that Q is supported on the class of mild solutions of system (2). We then prove that this class has a unique element (u, m); it will follow that the full sequence $\{Q^N\}$ converges to $\delta_{(u,m)}$ in the weak sense of measures, and that (u^N, m^N) converges in probability to (u, m) because the limit is deterministic. This will complete the proof of Theorem 1; verification of the properties $u_t \geq 0$ and $0 \leq m_t \leq M$ for every $t \in [0, T]$ are done with the same technique as used in the proof of the next proposition, through suitable continuous functionals; we omit the details. The regularity $C(L^2) \times C(L^2)$ of (u, m) comes from Corollary 1 and Proposition 3. In the following proof, we prove that Q is supported on the class of mild solutions of system (2). The proof of the following result is quite classical. It has been widely used in the mean-field theory; see [13]. Our case is very close to the mean-field framework , but it cannot be considered a particular case of the known mean-field theories, in particular because of the presence of u^N and the dependence of the function g on a density of particles. To prove this step, we adopt the approach of [7, Chapter 4], although presumably it can be proved along several classical lines; see [13]. Before going in to the details of the proof, we introduce a family of functionals which characterizes the solution of the system

$$
\begin{aligned}
&(u, m) \to \Psi_\phi(u, m) \\
&\quad := \sup_{t\in[0,T]} \left| \langle u_t - u_0, \phi\rangle - \frac{\sigma^2}{2} \int_0^t \langle u_s, \Delta\phi\rangle \, \mathrm{d}s - \int_0^t \langle u_s, b(u_s, m_s) \cdot \nabla\phi\rangle \, \mathrm{d}s \right| \\
&\qquad + \sup_{t\in[0,T]} \left| \langle m_t(\cdot) - F_\zeta\left(m_0(\cdot), \int_0^t u_s(\cdot)\, \mathrm{d}s\right), \phi\rangle \right|,
\end{aligned}
$$

where $\phi \in C_c^\infty(\mathbb{R}^d)$. On these family we prove a preliminary result to Proposition 4 below.

Lemma 8. *Let Q^{N_k} be the subsequence of measure of Q^N that converges to Q on $C(L^2_{\mathrm{loc}}) \times C(L^2_{\mathrm{loc}})$. Then*

$$
\lim_{k\to\infty} Q^{N_k}((u, m)\colon \Psi_\phi(u, m) > \delta) = 0.
$$

Proof. We have

$$
\begin{aligned}
&Q^{N_k}((u, m)\colon \Psi_\phi(u, m) > \delta) \\
&\quad = \mathbb{P}\left(\sup_{t\in[0,T]} \left| \langle u_t^{N_k} - u_0^{N_k}, \phi\rangle - \frac{\sigma^2}{2} \int_0^t \langle u_s^{N_k}, \Delta\phi\rangle \, \mathrm{d}s - \int_0^t \langle u_s^{N_k}, b(u_s^{N_k}, m_s^{N_k}) \cdot \nabla\phi\rangle \, \mathrm{d}s \right| \right. \\
&\qquad \left. + \sup_{t\in[0,T]} \left| \langle m_t^{N_k}(\,\cdot\,) - F_\zeta\left(m_0(\,\cdot\,), \int_0^t u_s^{N_k}(\,\cdot\,)\, \mathrm{d}s\right), \phi\rangle \right| > \delta \right).
\end{aligned}
$$

The second term of the functional is clearly 0, because of the equation satisfied by $m_t^{N_k}$. Using the identity satisfied by $u_t^{N_k}$, we obtain

$$
\begin{aligned}
&Q^{N_k}((u, m)\colon \Psi_\phi(u, m) > \delta) \\
&\quad \leq \mathbb{P}\left(\sup_{t\in[0,T]} \left| \left\langle \int_0^t [W_{N_k} * (b(u_s^{N_k}, m_s^{N_k}) S_s^{N_k}) - u_s^{N_k} b(u_s^{N_k}, m_s^{N_k})] \, \mathrm{d}s, \nabla\phi \right\rangle \right. \right. \\
&\qquad \left. \left. + \langle M_t^{N_k}, \phi\rangle \right| > \delta \right).
\end{aligned}
$$

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Hence, it is sufficient to prove that, for given $\delta > 0$, both the probabilities

$$\mathbb{P}\Big(\int_0^T |\langle W_{N_k} * (b(u_s^{N_k}, m_s^{N_k}) S_s^{N_k}) - u_s^{N_k} b(u_s^{N_k}, m_s^{N_k}), \nabla\phi \rangle| \, \mathrm{d}s > \delta \Big) \tag{13}$$

and

$$\mathbb{P}\Big(\sup_{t\in[0,T]} |\langle M_t^{N_k}, \phi\rangle| > \delta \Big)$$

converge to 0 as $k \to \infty$. The first probability is bounded above:

$$(13) \le \mathbb{P}\Big(C \int_0^T \| W_{N_k} * (b(u_s^{N_k}, m_s^{N_k}) S_s^{N_k}) - u_s^{N_k} b(u_s^{N_k}, m_s^{N_k}) \|_{L^2} \, \mathrm{d}s > \delta \Big).$$

We have

$$\begin{aligned}
&|(W_{N_k} * (b(u_s^{N_k}, m_s^{N_k}) S_s^{N_k}))(x) - u_s^{N_k}(x) b(u_s^{N_k}, m_s^{N_k})(x)| \\
&\quad \le \int W_{N_k}(x-y) |b(u_s^{N_k}, m_s^{N_k})(x) - b(u_s^{N_k}, m_s^{N_k})(y)| S_s^{N_k}(\mathrm{d}y) \\
&\quad \le C'' \int W_{N_k}(x-y) \, |x-y| \, S_s^{N_k}(\mathrm{d}y) \quad \text{(using property (7))} \\
&\quad \le C N_k^{-\beta/d} \int W_{N_k}(x-y) \, S_s^{N_k}(\mathrm{d}y) \, .
\end{aligned}$$

Defining $W_N(x) \coloneqq N^\beta W(N^{\beta/d} x)$ and using the property of compact support of W,

$$C N_k^{-\beta/d} \int W_{N_k}(x-y) \, S_s^{N_k}(\mathrm{d}y) = C N_k^{-\beta/d} u_s^{N_k}(x).$$

Hence,

$$(13) \le \mathbb{P}\Big(C N_k^{-\beta/d} \int_0^T \| u_s^{N_k} \|_{L^2} \, \mathrm{d}s > \delta \Big) \le \frac{C N_k^{-\beta/d}}{\delta} \mathbb{E} \int_0^T \| u_s^{N_k} \|_{L^2} \, \mathrm{d}s,$$

which goes to 0 (recall the bound of Lemma 7).

Finally,

$$\begin{aligned}
\mathbb{P}\Big(\sup_{t\in[0,T]} |\langle M_t^{N_k}, \phi\rangle| > \delta \Big) &\le \mathbb{P}\Big(\sup_{t\in[0,T]} \| M_t^{N_k} \|_{L^2} > \delta \Big) \\
&\le \frac{1}{\delta^2} \mathbb{E}\Big[\sup_{t\in[0,T]} \| M_t^{N_k} \|_{L^2}^2 \Big] \\
&\le \frac{C_T}{\delta^2} \frac{N_k^{\varepsilon *}}{N_k},
\end{aligned}$$

as in the proof of Lemma 7 (with $\alpha = 0$, $p = 2$, without semigroup, and using Doob's inequality), which goes to 0. □

Proposition 4. *Let Q be the limit probability measure of some subsequence Q^{N_k}. Then Q is supported on the set of mild solutions of system (2).*

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Proof. We first observe that the functional is continuous with respect to the topology of $C(L^2_{\mathrm{loc}}) \times C(L^2_{\mathrm{loc}})$. This holds because ϕ is a compact support with its derivatives (this is sufficient to treat the terms $\langle u_t, \phi\rangle$ and $\int_0^t \langle u_s, \Delta\phi\rangle \, \mathrm{d}s$), by property (6) (this fact together with the previous facts is used to treat the term $\int_0^t \langle u_s, b(u_s, m_s) \cdot \nabla\phi\rangle \, \mathrm{d}s$), and the facts that F_ζ is continuous and $\widetilde{F}_\zeta$ is bounded (these facts are used to deal with the m term). Moreover, $b(u, m)$ converges locally uniformly in space when (u, m) converges in $C(L^2_{\mathrm{loc}}) \times C(L^2_{\mathrm{loc}})$. This last point is a delicate one, so in the next lines we prove it. Let us consider a sequence (u^N, m^N) converging in L^2_{loc}; for fixed $\varepsilon > 0$ and $x \in B(0, K)$, with $K > 0$

$$
\begin{aligned}
&|b(u^N, m^N)(x) - b(u, m)(x)| \\
&\quad \le \int_{\mathbb{R}^d} |g(|y - x|, u^N(y), m^N(y)) - g(|y - x|, u(y), m(y))| \, \mathrm{d}y \\
&\quad = \int_{B(0,R)} |g(|y - x|, u^N(y), m^N(y)) - g(|y - x|, u(y), m^N(y))| \, \mathrm{d}y \\
&\qquad + \int_{B(0,R)^c} |g(|y - x|, u^N(y), m^N(y)) - g(|y - x|, u(y), m(y))| \, \mathrm{d}y \\
&\quad = I_1 + I_2(R, N),
\end{aligned}
$$

where, thanks to hypothesis (4), R can be chosen such that $I_2(R, N) \le \varepsilon/2$ uniformly in x. Regarding I_1, there exists N_0 such that

$$
I_1 \le \|Dg\| \int_{B(0,R)} |u^N(y) - u(y)| \, \mathrm{d}y \le \frac{\varepsilon}{2} \quad \text{for all } N > N_0.
$$

Computations including the time component are straightforward. So, $b(u^N, m^N)$ converges locally uniformly in space to $b(u, m)$.

Thanks to continuity of the functional Ψ_ϕ, by the Portmanteau theorem,

$$
Q((u, m)\colon \Psi_\phi(u, m) > \delta) \le \liminf_k Q^{N_k}((u, m)\colon \Psi_\phi(u, m) > \delta).
$$

Then, by Lemma 8,

$$
Q((u, m)\colon \Psi_\phi(u, m) > \delta) = 0
$$

for every $\delta > 0$. By a classical argument, see [7],

$$
Q((u, m)\colon \Psi_\phi(u, m) = 0 \text{ for all } \phi \in \mathcal{D}) = 1.
$$

Thus, Q is supported on weak solutions. In addition, by Corollary 1, u is also of class $L^p(0, T; L^2(\mathbb{R}^d))$ for some $p > 2$. With a proper choice of ϕ related to the heat kernel $\mathrm{e}^{-|x-y|^2/(4t)}/(4\pi t)^{d/2}$, we prove that u satisfies the mild formulation; it is straightforward to see that m satisfies the differential equation. Hence, we have proved that Q is supported by the set of mild solutions. □

Proposition 5. *Assume that u^1, m^1, u^2, and m^2 are functions of class $C(L^2)$ such that (u^1, m^1) and (u^2, m^2) are mild solutions of system (2) corresponding to the same initial condition (u_0, m_0), with $u_t, m_t \ge 0$ for every $t \in [0, T]$. Then $(u^1, m^1) = (u^2, m^2)$.*

Proof. Each u^i, $i = 1, 2$, satisfies the identity

$$
u^i_t(x) = \mathrm{e}^{tA} u_0 + \int_0^t \nabla \cdot \mathrm{e}^{(t-s)A} \left(u^i_s b\left(u^i_s, F_\zeta\left(m_0(x), \int_0^s u^i_r(x) \, \mathrm{d}r \right) \right) \right) \mathrm{d}s,
$$

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where we have used the explicit formula for (1b). This is a closed equation and we are going to prove from it that $u^1 = u^2$. A fortiori we also obtain $m^1 = m^2$, again from the explicit formula for (1b).

Assume by contradiction that $u^1 \neq u^2$. Let $t_0 \in [0, T)$ be the infimum of all $t \in [0, T]$ such that $u_t^1 \neq u_t^2$. On $[0, t_0]$ we have $(u^1, m^1) = (u^2, m^2)$. On $[t_0, T]$ we use the mild formula and property (3) to obtain

$$\begin{aligned}
&\|u_t^1 - u_t^2\|_{L^2} \\
&\quad \leq \int_{t_0}^t \|\nabla \cdot \mathrm{e}^{(t-s)A}(u_s^1 b(u_s^1, m_s^1) - u_s^2 b(u_s^2, m_s^2))\|_{L^2}\,\mathrm{d}s \\
&\quad \leq \int_{t_0}^t \frac{C}{|t-s|^{1/2}} \|u_s^1 b(u_s^1, m_s^1) - u_s^2 b(u_s^2, m_s^2)\|_{L^2}\,\mathrm{d}s \\
&\quad \leq \int_{t_0}^t \frac{C}{|t-s|^{1/2}} (\|b(u_s^1, m_s^1)\|_\infty \|u_s^1 - u_s^2\|_{L^2} + \|u_s^2\|_{L^2} \|b(u_s^1, m_s^1) - b(u_s^2, m_s^2)\|_{L^\infty})\,\mathrm{d}s.
\end{aligned}$$

Recall that b is bounded, see (5), and that $\|u_s^2\|_{L^2}$ is bounded by assumption. Hence,

$$\left\|u_t^1 - u_t^2\right\|_{L^2} \leq \int_{t_0}^t \frac{C}{|t-s|^{1/2}} (\|u_s^1 - u_s^2\|_{L^2} + \|b(u_s^1, m_s^1) - b(u_s^2, m_s^2)\|_{L^\infty})\,\mathrm{d}s.$$

From property (6) and Hölder's inequality we have

$$\begin{aligned}
|b(u_s^1, m_s^1)(x) - b(u_s^2, m_s^2)(x)| &\leq C \int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|} (|u_s^1(y) - u_s^2(y)| + |m_s^1(y) - m_s^2(y)|)\,\mathrm{d}y \\
&\leq C\|u_s^1 - u_s^2\|_{L^2} + C\|m_s^1 - m_s^2\|_{L^2};
\end{aligned}$$

hence,

$$\left\|u_t^1 - u_t^2\right\|_{L^2} \leq \int_{t_0}^t \frac{C}{|t-s|^{1/2}} (\|u_s^1 - u_s^2\|_{L^2} + \|m_s^1 - m_s^2\|_{L^2})\,\mathrm{d}s.$$

Recalling the explicit formula for (1b), we have

$$\begin{aligned}
|m_s^1(x) - m_s^2(x)| &= \left| F_\zeta\left(m_0(x), \int_0^s u_r^1(x)\,\mathrm{d}r\right) - F_\zeta\left(m_0(x), \int_0^s u_r^2(x)\,\mathrm{d}r\right)\right| \\
&\leq \|\partial_b F_\zeta\|_\infty \left| \int_0^s u_r^1(x)\,\mathrm{d}r - \int_0^s u_r^2(x)\,\mathrm{d}r\right| \\
&\leq C \int_0^s |u_r^1(x) - u_r^2(x)|\,\mathrm{d}r \\
&= C \int_{t_0}^s |u_r^1(x) - u_r^2(x)|\,\mathrm{d}r,
\end{aligned}$$

whence

$$\|m_s^1 - m_s^2\|_{L^2} \leq C_T \int_{t_0}^s \|u_r^1 - u_r^2\|_{L^2}\,\mathrm{d}r.$$

Summarising, the function $v_t := \|u_t^1 - u_t^2\|_{L^2}$ satisfies

$$v_t \leq \int_{t_0}^t \frac{C}{|t-s|^{1/2}} \left(v_s + C_T \int_{t_0}^s v_r\,\mathrm{d}r\right)\mathrm{d}s.$$

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Given $t_1 \in [t_0, T]$, we set $A(t_1) := \sup_{t\in[t_0,t_1]} v_t$. Then, on the interval $t \in [t_0, t_1]$, we have

$$v_t \le \int_{t_0}^{t} \frac{C}{|t-s|^{1/2}} (A(t_1) + C'_T A(t_1))\, \mathrm{d}s = C'' A(t_1)\, (t-t_0)^{1/2}.$$

It follows that

$$A(t_1) \le C'' A(t_1)\, (t_1 - t_0)^{1/2}.$$

If $t_1 - t_0 > 0$ is small enough, we deduce that $A(t_1) = 0$; hence, $u^1 = u^2$ on $[t_0, t_1]$, in contradiction with the definition of t_0. □

5. Simulations

The aim of this section is to show the flexibility of the model, namely how it may reveal different types of aggregations. For instance, we avoid an arbitrary concentration (even with infinitesimal noise), opposite to most of the models in the literature, but cover the case of concentration for the cases of single and multiple concentration points. Each numerical simulation shown below is given for the following choice of parameters: number of particles $N = 100$, parameter of diffusion $\sigma^2 = 0.1$, discretization of time $\mathrm{d}t = 10^{-4}$, kernel smoothing parameter $\beta = 0.9$, and, for the initial condition, we made a simple choice, choosing just a realization of the uniform distribution on the square $[0, 2] \times [0, 2]$.

5.1. Degenerate aggregation

Let us start from the most basic example, the case when each particle is attracted by the others. Recall that we model the interaction between individuals and the density of the population; hence, each individual is pushed to high population density regions. A standard choice for g could be

$$g(r, u) = \mathrm{e}^{-r} \frac{u}{1+u} \quad \text{or} \quad g(r, u) = \mathrm{e}^{-r} \tanh(u).$$

Note that, with this choice, cells continue to aggregate even at high density. The population mass tends to concentrate into a single point (see Figure 1).

5.2. Moderate aggregation

Let us now include a repulsive component in the force, to avoid collapse of the total mass (see Figure 2). The function $g(r, u)$ we look for should have the following features (see Figure 3).

- Given the distance r, $g(r, u)$ is such that the force is aggregative for small density and repulsive for large density. This behavior is natural for certain animals: each individual is attracted by its similar, but this changes when there are many individuals.
- There is an issue when we quantify small and large densities: this quantification should depend on distance. At large distances, we expect aggregation to be more relevant, with the individual tending to avoid only very large densities. On the other hand, at short distances, each individual is attracted only by very small densities.

The function we propose is

$$g(r, u) := \frac{u \log(r/u)}{1 - u \log(r/u)}.$$

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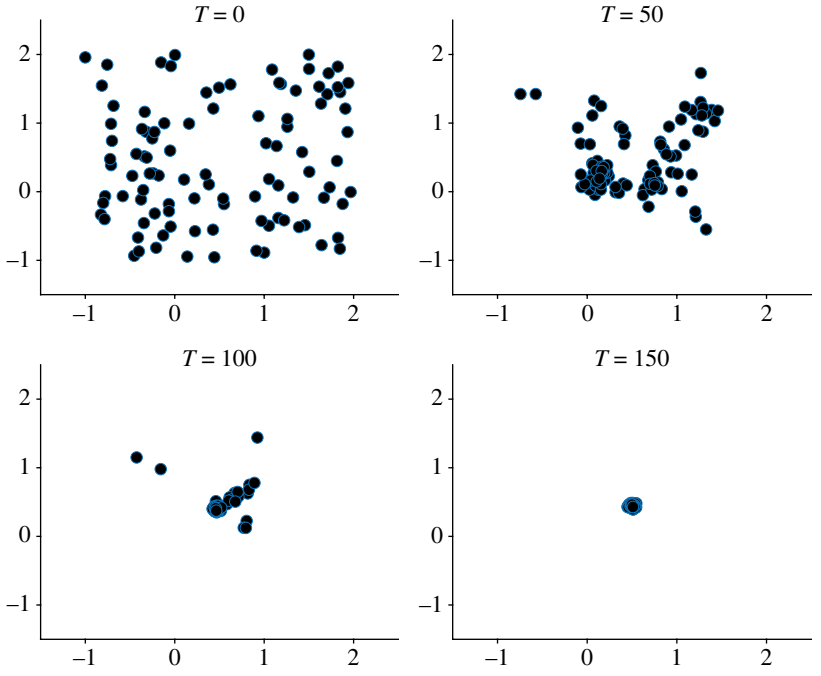


FIGURE 1: Configuration of 100 particles at times $T = 0$, 50, 100, and 150 with $g(r, u) = \mathrm{e}^{-r}u/(1+u)$.

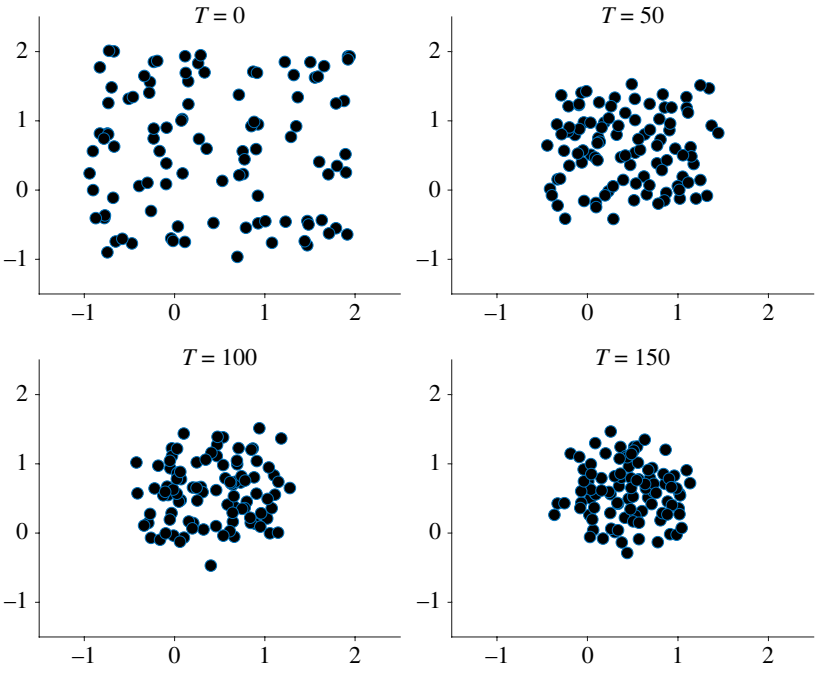


FIGURE 2: Configuration of 100 particles at times $T = 0$, 50, 100, and 150 with $g(r, u) \coloneqq u \log(r/u)/(1 - u \log(r/u))$.

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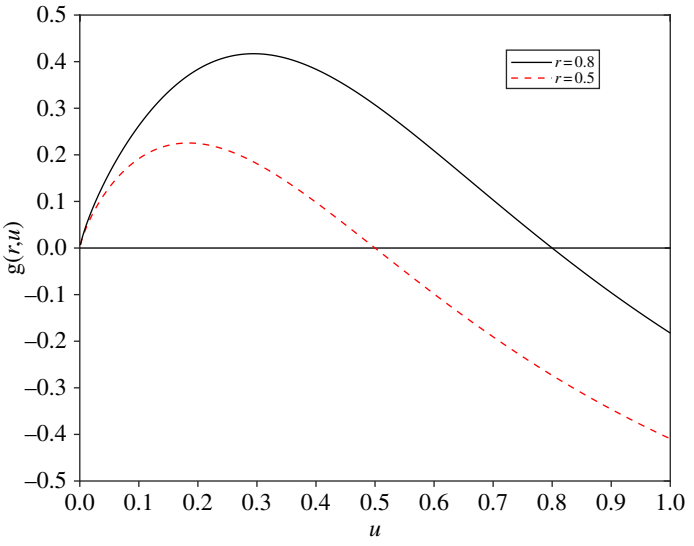


FIGURE 3: Plot of the the function $r \mapsto g(r, u)$ for $r = 0.8$ (*solid line*) and $r = 0.5$ (*dotted line*).

Another example could be

$$g(r, u) := \mathrm{e}^{-r} \frac{u(\alpha - u)}{1 + u},$$

where the parameter α can be interpreted as an index of overcrowding; choosing α properly, particles aggregate, without collapsing. The main drawback we have observed in simulations for this alternative is its strong sensitivity to the choice of parameter α with respect to the initial configuration. The first option we propose is more stable.

Note that the functions $g(r, u)$ of this subsection are not products of functions of two single variables, namely, $g(r, u) = g_1(r)\, g_2(u)$.

5.3. Aggregation in clusters

Returning to the first model, $g(r, u) = \mathrm{e}^{-r} u/(1 + u)$, an interesting variant is when attraction happened only up to a certain distance (see Figure 4):

$$g_R(r, u) := \frac{u}{1 + u} \exp\left(-\frac{r^2}{R}\right).$$

With this choice we observe the formation of clusters of individuals. Clearly, the parameter R influences the number of clusters that are generated: for large R, the population aggregates into a reduced number of clusters.

When t goes to infinity, if the noise is infinitesimal, each cluster reduces to a point, and maybe due to the noise, different clusters may meet and collapse.

5.4. Moderate aggregation in clusters

We may mix-up the previous two features. The following example has a tendency to construct clusters (see Figure 5), but they remain of finite size (independently of the noise):

$$g_R(r, u) := \frac{u(\alpha - u)}{1 + u} \exp\left(-\frac{r^2}{R}\right).$$

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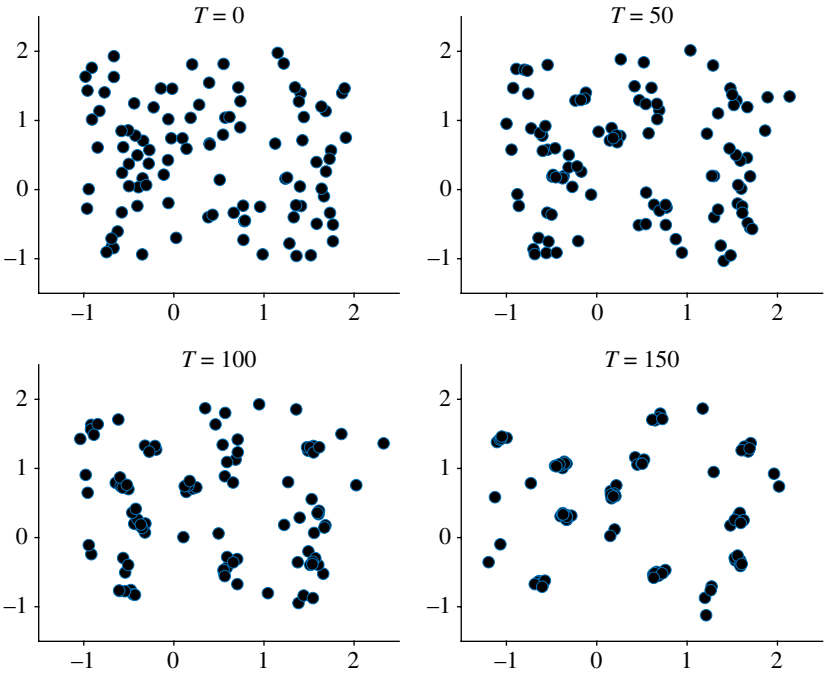


FIGURE 4: Configuration of 100 particles at times $T = 0, 50, 100,$ and 150 with $g_R(r, u) := (u/(1+u)) \exp(-r^2/R)$ and $R = 0.3$.

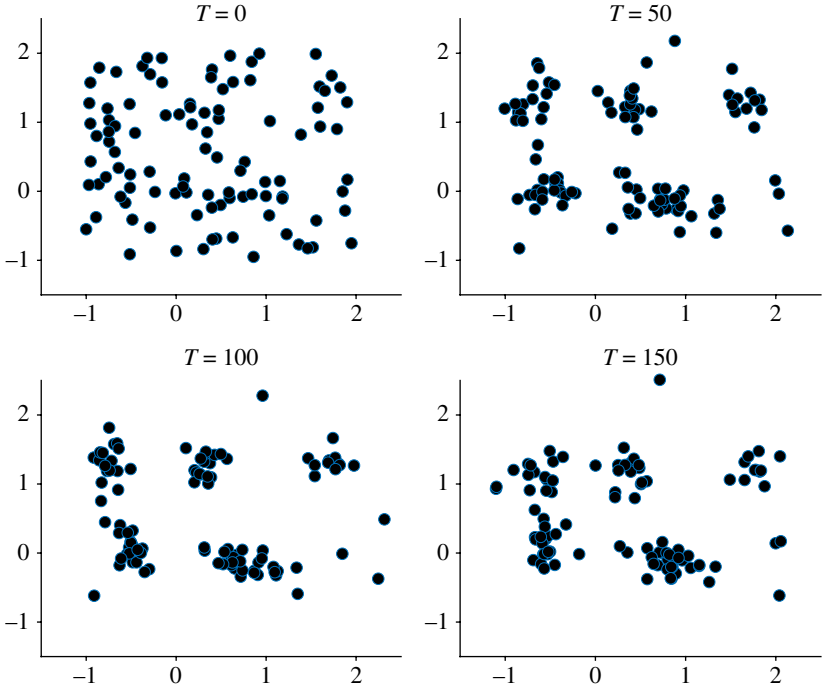


FIGURE 5: Configuration of 100 particles at times $T = 0, 50, 100,$ and 150 with $g_R(r, u) := (u(\alpha - u)/(1+u)) \exp(-r^2/R)$, $R = 0.3$, and $\alpha = 1.3$.

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