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ON RELATIVE AGING COMPARISONS OF COHERENT SYSTEMS WITH IDENTICALLY DISTRIBUTED COMPONENTS

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The relative aging is an important notion which is useful to measure how a system ages relative to another one. Among the existing stochastic orders, there are two important orders describing the relative aging of two systems, namely, aging faster orders in the cumulative hazard and the cumulative reversed hazard rate functions. In this paper, we give some sufficient conditions under which one coherent system ages faster than another one with respect to the aforementioned stochastic orders. Further, we show that the proposed sufficient conditions are satisfied for k-out-of-n systems. Moreover, some numerical examples are given to illustrate the applications of proposed results.

Keywords: coherent system, dual distortion/domination function, k-out-of-n system, stochastic orders

1. INTRODUCTION

Stochastic aging is one of the important concepts in reliability theory and survival analysis. It broadly describes the pattern in which a system ages/degrades with time. In the literature, three different types of aging notions are introduced, namely, positive aging, negative aging, and no aging. Positive aging means that the residual lifetime of system decreases with time, stochastically, whereas the negative aging describes the completely opposite phenomenon. On the other hand, no aging means that the system does not age with time. Based on these three aging notions, different types of aging classes (namely, IFR, DFR, IFRA, DFRA, ILR, DLR, to name a few) have been proposed in the literature in order to study different aging characteristics of system. For more discussion on this topic, the reader may go through Barlow and Proschan [3] and Lai and Xie [23]. Apart from these aging concepts, relative aging is another useful notion which has applications in the study of system reliability. Relative aging measures how a system improves/deteriorates over time with respect to another system.

There are many real-life scenarios where we deal with more than one system of the same type (e.g., television sets of different companies, CPUs produced by different brands, etc.). In such cases, we often come across the following question: How to measure if one system

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ages faster compared to others as time progresses? An effective solution to this question is provided through the notion of relative aging. Another aspect of the relative aging is found to be useful when we deal with the phenomenon of crossing hazards/mean residual lifetimes. These kinds of scenarios occur in many real-life situations. For example, Pocock et al. [38] noticed the crossing hazards phenomenon when they were dealing with the survival data on the effects of two different treatments to breast cancer patients. Further, Champlin et al. [7] also reported some instances where the supremacy of a treatment over another treatment stays only for a while. The above discussion suggests that the increasing/decreasing hazard ratio models could be considered as a reasonable choice in many real-life situations. Indeed, Kalashnikov and Rachev [18] introduced a relative aging notion (called aging faster order in the hazard rate) based on monotonicity of the ratio of two hazard rate functions. In the same spirit, Sengupta and Deshpande [42] proposed another notion based on monotonicity of the ratio of two cumulative hazard rate functions. Later, Di Crescenzo [9] developed two relative aging notions, one based on monotonicity of the ratio of two reversed hazard functions and another one based on monotonicity of the ratio of two cumulative reversed hazard functions. Finkelstein [13] proposed a notion based on the ratio of mean residual lifetime functions. Furthermore, Hazra and Nanda [17] introduced some relative aging notions based on some generalized reliability functions.

It is meaningful to study coherent systems (the definition is given in the next section) because these are the skeletons of most of the real-life systems. Thus, the study of different reliability properties of coherent systems is considered as one of the important problems in reliability theory. The problem, which we focus here, is the study of relative aging of coherent systems. There exists a vast literature on various stochastic comparisons of k-out-of-n systems (in particular, series and parallel systems) with independent components (see, for instance, [2,37,39] and the references therein). Further, the same problem, for general coherent systems, is considered in Esary and Proschan [12], Nanda et al. [29], Kochar et al. [21], Belzunce et al. [5], Navarro and Rubio [35], Navarro et al. [31–34], Navarro [30], Samaniego and Navarro [41], Amini-Seresht et al. [1], to name a few. However, all of these studies were done in terms of traditional stochastic orders, namely, usual stochastic order, hazard rate order, reversed hazard rate order, likelihood ratio order, etc. There are a few articles where aging faster orders were considered to compare two coherent systems. The study of k-out-of-n systems using aging faster orders (in the hazard and the reversed hazard rates) was done by Misra and Francis [25], Li and Li [24], and Ding and Zhang [11], whereas a similar study for coherent systems was done by Ding et al. [10] and Hazra and Misra [15]. Recently, Misra and Francis [26] studied k-out-of-n systems with independent components in terms of aging faster orders (in the cumulative hazard and the cumulative reversed hazard rates). It will be interesting to extend these studies to coherent systems with dependent component lifetimes. The goal of this paper is to study the comparison of coherent systems, with dependent component lifetimes, in terms of aging faster orders based on cumulative hazard and cumulative reversed hazard rates.

The rest of the paper is organized as follows. In Section 2, we introduce some notations and definitions that are used in the paper. In this section, we also present some important lemmas that are used in proving the main results of the paper. In Section 3, we discuss the main results. Here, we provide some sufficient conditions under which one coherent system performs better than the other with respect to aging faster orders based on the cumulative hazard and the cumulative reversed hazard rates. Further, we show that the proposed results hold for the well known k-out-of-n systems thereby generalizing the results of Misra and Francis [26] to situations where we have general coherent systems comprising of components with d.i.d. (dependent and identically distributed) lifetimes. We also provide some examples to illustrate the applications of the proposed results.

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All proofs of lemmas and theorems, wherever given, are deferred to the Appendix.

2. PRELIMINARIES

We will first introduce some notation and acronyms that will be used throughout the paper. For a random variable W (with absolutely continuous distribution function), we denote its probability density function (pdf) by $f_W(\cdot)$, the cumulative distribution function (cdf) by $F_W(\cdot)$, the survival/reliability function by $\bar{F}_W(\cdot) \equiv 1 - F_W(\cdot)$, the hazard rate function by $r_W(\cdot) \equiv f_W(\cdot)/\bar{F}_W(\cdot)$, the reversed hazard rate function by $\tilde{r}_W(\cdot) \equiv f_W(\cdot)/F_W(\cdot)$, the cumulative hazard rate function by $\Delta_W(\cdot) \equiv -\ln \bar{F}_W(\cdot)$, and the cumulative reversed hazard rate function by $\widetilde{\Delta}_W(\cdot) \equiv -\ln F_W(\cdot)$. Further, for any two random variables U and V, we write $U =_{\mathrm{st}} V$ to mean that U and V are identically distributed. Furthermore, we denote the set of real numbers by $\mathbb{R}$. By $a \overset{\mathrm{sgn}}{=} b$, we mean that the real numbers a and b have the same sign, whereas $c \overset{\mathrm{def.}}{=} d$ means that c is defined as d. We use two acronyms i.i.d and d.i.d. for 'independent and identically distributed' and 'dependent and identically distributed', respectively.

Throughout the paper, the words increasing, decreasing, positive, and negative are used in the non-strict sense. All random variables considered in this paper are assumed to have absolutely continuous distribution functions supported on $[0, \infty)$. We use bold symbols to represent vectors of scalars and random vectors (e.g., $\boldsymbol{b} = (b_1, b_2, \ldots, b_p)$, $b_i \in \mathbb{R}$, $i = 1, 2, \ldots, p$, and $\boldsymbol{X} = (X_1, X_2, \ldots, X_m)$).

In what follows, we provide a brief description of coherent systems. A system is said to be coherent if each of its components is relevant, and its structure function is monotonically increasing in each argument (see [3]). The so-called k-out-of-n systems are special cases of the coherent system. A k-out-of-n system is a system, formed by n components, that functions as long as at least k of its n components function. Further, the series (n-out-of-n) and the parallel (1-out-of-n) systems are special cases of k-out-of-n systems. Moreover, the lifetime of a k-out-of-n system is the same as $(n-k+1)$th order statistic of the lifetimes of n components. Thus, the study of a k-out-of-n system and the study of an $(n-k+1)$th order statistic (of nonnegative random variables) are basically the same.

Let $\tau(\boldsymbol{X})$ be a random variable representing the lifetime of a coherent system made up of n components with the lifetime vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$, where X_i's are d.i.d. For the sake of notational simplicity, let us assume that $X_i =_{\mathrm{st}} X$, $i = 1, 2, \ldots, n$, for some nonnegative random variable X. Then the joint reliability function of $\boldsymbol{X}$ can be written as

$$\begin{aligned}\bar{F}_{\mathbf{X}}(x_1, x_2, \ldots, x_n) &= P\left(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n\right) \\ &= K\left(\bar{F}_X(x_1), \bar{F}_X(x_2), \ldots, \bar{F}_X(x_n)\right), \quad x_i \in \mathbb{R},\ i = 1, 2, \ldots, n,\end{aligned}$$

where $K(\cdot, \cdot, \ldots, \cdot)$ is a survival copula. It is worthwhile to mention here that this representation indeed follows from Sklar's Theorem (see [36]). The survival copula describes the dependency structure among components of a system. In the literature, different types of survival copulas have been proposed, for example, Farlie–Gumbel–Morgenstern (FGM) copula, Archimedean copulas with different generators, Clayton-Oakes (CO) copula, etc. (see [36] for more discussion on this topic). Below, we give a useful lemma that represents a relation between the reliability function of a system and the reliability functions of its corresponding components.

LEMMA 2.1 (Navarro et al. [31]): *Let $\tau(\boldsymbol{X})$ be the lifetime of a coherent system formed by n d.i.d. components with the lifetime vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$. Then the reliability*

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function of $\tau(\boldsymbol{X})$ *can be written as*

$$\bar{F}_{\tau(\boldsymbol{X})}(x) = P(\tau(\boldsymbol{X}) > x) = h\left(\bar{F}_X(x)\right), \quad x \in \mathbb{R},$$

where the function $h : [0,1] \to [0,1]$ *(called the domination (or dual distortion) function) depends on the structure function* $\phi(\cdot)$ *(see [3] for definition) and on the survival copula* K *of* $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$*. Furthermore,* $h(\cdot)$ *is an increasing continuous function in* $[0,1]$ *such that* $h(0) = 0$ *and* $h(1) = 1$*.*

Since the domination function $h(\cdot)$ is a linear combination of the copula and its subcopulas, and all of them are Lipschitz functions, it is differentiable almost everywhere on $(0,1)$ and its derivative (wherever it exists) is positive.

In what follows, we discuss some preliminary concepts that will be helpful in better understanding of the paper. We begin with the notion of stochastic orders which are effectively used to compare lifetimes of two systems. In the literature, different types of stochastic orders have been developed to compare two random variables describing two different random phenomena. The detailed discussion on this topic could be found in the monographs written by Shaked and Shanthikumar [43] and Belzunce et al. [6]. Below, we give the definitions of stochastic orders that are used in this paper.

Definition 2.1: *Let* X *and* Y *be two random variables with absolutely continuous distribution functions supported on* $[0,\infty)$*. Then,* X *is said to be smaller than* Y *in the*

a. usual stochastic (st) order, denoted as $X \leq_{\mathrm{st}} Y$*, if*

$$\bar{F}_X(x) \leq \bar{F}_Y(x) \text{ for all, } x \in [0,\infty);$$

b. hazard rate (hr) order, denoted as $X \leq_{\mathrm{hr}} Y$*, if*

$$\bar{F}_Y(x)/\bar{F}_X(x) \text{ is increasing in } x \in [0,\infty);$$

c. reversed hazard rate (rh) order, denoted as $X \leq_{\mathrm{rh}} Y$*, if*

$$F_Y(x)/F_X(x) \text{ is increasing in } x \in [0,\infty).$$

It is worthwhile to mention here that

$$X \leq_{\mathrm{hr[rh]}} Y \implies X \leq_{\mathrm{st}} Y.$$

Apart from above-discussed stochastic orders, there are two more sets of stochastic orders which are frequently used to study the relative agings of two systems. One of them is the set of transform orders (e.g., convex transform order, super-additive order, star-shaped order, DMRL order, s-IFR order, s-IFRA orders, etc.) which are defined based on different aging concepts, namely, increasing failure rate, increasing failure rate in average, new better than used, etc. (see, for instance, [3,4,8,22,28]). The other one is the set of aging faster orders that describe whether a system is aging faster than another one in terms of the hazard rate, the cumulative hazard rate, the reversed hazard rate, the cumulative reversed hazard rate, etc. (see [9,13,14,17,18,20,26,27,40,42] and the references therein). For the sake of completeness in our presentation, we give the definitions of following stochastic orders.

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DEFINITION 2.2: *Let X and Y be two random variables with absolutely continuous distribution functions supported on $[0,\infty)$. Then X is said to be aging faster than Y in the*

a. hazard rate, denoted as $X \le_c Y$, if

$$r_X(x)/r_Y(x) \text{ is increasing in } x \in [0,\infty);$$

b. reversed hazard rate, denoted as $X \le_b Y$, if

$$\tilde{r}_X(x)/\tilde{r}_Y(x) \text{ is decreasing in } x \in [0,\infty);$$

c. cumulative hazard rate, denoted as $X \le_{c^*} Y$, if

$$\Delta_X(x)/\Delta_Y(x) \text{ is increasing in } x \in [0,\infty);$$

d. cumulative reversed hazard rate, denoted as $X \le_{b^*} Y$, if

$$\widetilde{\Delta}_X(x)/\widetilde{\Delta}_Y(x) \text{ is decreasing in } x \in [0,\infty).$$

It is worthwhile to mention here that $X =_c Y$ (i.e., $r_X(x)/r_Y(x)$ is constant) does not imply $X =_{\text{st}} Y$. In fact, $X =_c Y$ implies $\bar{F}_Y \equiv \bar{F}_X^{\alpha}$, for some $\alpha > 0$. Similar interpretation can be given for other aging faster stochastic orders. It is known that

$$X \le_c Y \implies X \le_{c^*} Y, \quad \text{and} \quad X \le_b Y \implies X \le_{b^*} Y.$$

Relative aging of k-out-of-n systems comprising of components having i.i.d. lifetimes has been studied by many researchers (for references, see the paragraph preceding Definition 2.2). Under the i.i.d. set-up, Misra and Francis [26] established the following results:

a. $X \le_{c^*} Y$ and $Y \le_{\text{st}} X \implies X_{k:n} \le_{c^*} Y_{k:n}$, $k = 1, 2, \ldots, n$,
b. $X \le_{b^*} Y$ and $X \le_{\text{st}} Y \implies X_{k:n} \le_{b^*} Y_{k:n}$, $k = 1, 2, \ldots, n$,

where $X_{k:n}$ (resp. $Y_{k:n}$) denotes the kth order statistic based on i.i.d. observations $X_1, X_2, \ldots, X_n$ (resp. $Y_1, Y_2, \ldots, Y_n$). The purpose of this paper is to derive more general results that accommodate general coherent systems comprising of components having d.i.d. lifetimes.

In what follows, we give the definitions of TP_2 and RR_2 functions which are used in the proofs of the main results (cf. [19]).

DEFINITION 2.3: *Let $\mathcal{X}$ and $\mathcal{Y}$ be two linearly ordered sets. Then, a real-valued and nonnegative function $\kappa(\cdot,\cdot)$, defined on $\mathcal{X} \times \mathcal{Y}$, is said to be TP_2 (resp. RR_2) if*

$$\kappa(x_1, y_1)\kappa(x_2, y_2) \ge (\text{resp. } \le)\kappa(x_1, y_2)\kappa(x_2, y_1),$$

for all $x_1 \le x_2$ and $y_1 \le y_2$.

Now, we will present some lemmas which will be used in proving the main results. The first lemma is borrowed from Karlin [19, Thm. 11.2, pp. 324–325] and Hazra and Nanda [16, Lemma 3.5], whereas next two lemmas are taken from Hazra and Misra [15, Lemma 2.3 (ii) and (iii), Lemma 2.4 (ii) and (iii)] (also see [32, pp. 447–448]).

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LEMMA 2.2: *Let $\kappa(x,y) > 0$, defined on $\mathcal{X} \times \mathcal{Y}$, be RR_2 (resp. TP_2), where $\mathcal{X}$ and $\mathcal{Y}$ are subsets of the real line. Assume that a function $f(\cdot,\cdot)$ defined on $\mathcal{X} \times \mathcal{Y}$ is such that*

i. for each $x \in \mathcal{X}$, $f(x,y)$ changes sign at most once, and if the change of sign does occur, it is from positive to negative, as y traverses $\mathcal{Y}$;

ii. for each $y \in \mathcal{Y}$, $f(x,y)$ is increasing (resp. decreasing) in $x \in \mathcal{X}$;

iii. for any nonnegative sigma-finite measure μ, $\int_{\mathcal{Y}} \kappa(x,y)|f(x,y)|\,d\mu(y) < \infty$ for every $x \in \mathcal{X}$, and $\omega(x) = \int_{\mathcal{Y}} \kappa(x,y) f(x,y)\,d\mu(y)$ defines a continuous function on $\mathcal{X}$.

Then $\omega(x)$ changes sign at most once, and if the change of sign does occur, it is from negative (resp. positive) to positive (resp. negative), as x traverses $\mathcal{X}$.

LEMMA 2.3: *Let $h_{k|n}(\cdot)$ and $h_{l|m}(\cdot)$ be the reliability functions of the k-out-of-n and the l-out-of-m systems with i.i.d. component lifetimes, respectively, where $1 \le k \le n$ and $1 \le l \le m$. Further, let $H_{k|n}(p) = ph'_{k|n}(p)/h_{k|n}(p)$ and $H_{l|m}(p) = ph'_{l|m}(p)/h_{l|m}(p)$, $p \in (0,1)$, $1 \le k \le n$ and $1 \le l \le m$. Then*

i. $H_{k|n}(p)/H_{l|m}(p)$ is decreasing in $p \in (0,1)$, for all $l - k \ge \max\{0, m-n\}$;

ii. $(1-p)H'_{k|n}(p)/H_{k|n}(p)$ is negative and decreasing in $p \in (0,1)$, for all $1 \le k \le n$.

LEMMA 2.4: *Let $h_{k|n}(\cdot)$ and $h_{l|m}(\cdot)$ be the reliability functions of the k-out-of-n and the l-out-of-m systems with i.i.d. component lifetimes, respectively, where $1 \le k \le n$ and $1 \le l \le m$. Further, let $R_{k|n}(p) = (1-p)h'_{k|n}(p)/(1-h_{k|n}(p))$ and $R_{l|m}(p) = (1-p)h'_{l|m}(p)/(1-h_{l|m}(p))$, $p \in (0,1)$, $1 \le k \le n$ and $1 \le l \le m$. Then*

i. $R_{k|n}(p)/R_{l|m}(p)$ is increasing in $p \in (0,1)$, for all $k - l \ge \max\{0, n-m\}$;

ii. $pR'_{k|n}(p)/R_{k|n}(p)$ is positive and decreasing in $p \in (0,1)$, for all $1 \le k \le n$.

3. MAIN RESULTS

Let $\tau_1(\boldsymbol{X})$ and $\tau_2(\boldsymbol{Y})$ be lifetimes of two coherent systems formed by two different sets of d.i.d. components with lifetime vectors $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ and $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_m)$, respectively. For the sake of simplicity of notation, let us assume that $X_i =_{st} X$, $i = 1,2,\ldots,n$, and $Y_j =_{st} Y$, $j = 1,2,\ldots,m$, for some nonnegative random variables X and Y. Further, let $h_1(\cdot)$ and $h_2(\cdot)$ be the domination functions of $\tau_1(\boldsymbol{X})$ and $\tau_2(\boldsymbol{Y})$, respectively. In what follows, we use the following notation:

$$H_i(p) = \frac{ph'_i(p)}{h_i(p)} \quad \text{and} \quad R_i(p) = \frac{(1-p)h'_i(p)}{1-h_i(p)}, \quad p \in (0,1),\ i = 1,2.$$

Since $h_i(\cdot)$, i=1,2, are increasing and positive functions, it follows that $H_i(\cdot)$ and $R_i(\cdot)$ are positive functions. Further, we write $H_1(\cdot) = H_2(\cdot) = H(\cdot)$ (say) and $R_1(\cdot) = R_2(\cdot) = R(\cdot)$ (say) whenever $\tau_1(\boldsymbol{X})$ and $\tau_2(\boldsymbol{Y})$ have the same domination function, that is, $h_1(\cdot) = h_2(\cdot) = h(\cdot)$ (say).

3.1. Relative Aging Based on Cumulative Hazard Functions

In this subsection, we compare two coherent systems in terms of aging faster order in the cumulative hazard rate.

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In the following proposition, we consider a coherent system and provide sufficient conditions under which the use of components with lifetime vector $\boldsymbol{X}$ in the coherent system makes it age faster than if components with lifetime vector $\boldsymbol{Y}$ are used in the coherent system.

PROPOSITION 3.1: *Assume that two coherent systems have the same domination function. Further, suppose that the following conditions hold:*

i. $(1-p)H'(p)/H(p)$ *is negative and decreasing in* $p \in (0,1)$*;*
ii. $X \leq_{c^*} Y$ *and* $Y \leq_{\mathrm{st}} X$*.*

Then $\tau_1(\boldsymbol{X}) \leq_{c^*} \tau_2(\boldsymbol{Y})$*.*

Before stating the next result, we give the following lemma obtained in Hazra and Misra [15, Proposition 3.1].

LEMMA 3.1: *Let* X_i*'s be identically distributed. Then* $\tau_1(\boldsymbol{X}) \leq_c \tau_2(\boldsymbol{X})$ *if and only if* $H_1(p)/H_2(p)$ *is decreasing in* $p \in (0,1)$*.*

In the following theorem, we compare two coherent systems with different domination functions based on the same survival copula. The proof follows from Proposition 3.1, Lemma 3.1, and by using the facts that $\leq_{c^*}$ is transitive and $\leq_c \Longrightarrow \leq_{c^*}$.

THEOREM 3.1: *Assume that two coherent systems have the same survival copula. Further, suppose that* $\{(i),(ii),(iv)\}$ *or* $\{(i),(iii),(iv)\}$ *below holds:*

i. $H_1(p)/H_2(p)$ *is decreasing in* $p \in (0,1)$*;*
ii. $(1-p)H_1'(p)/H_1(p)$ *is negative and decreasing in* $p \in (0,1)$*;*
iii. $(1-p)H_2'(p)/H_2(p)$ *is negative and decreasing in* $p \in (0,1)$*;*
iv. $X \leq_{c^*} Y$ *and* $Y \leq_{\mathrm{st}} X$*.*

Then $\tau_1(\boldsymbol{X}) \leq_{c^*} \tau_2(\boldsymbol{Y})$*.*

Remark 3.1: It is worth mentioning here that the results stated in Proposition 3.1 and Theorem 3.1 also follow from Theorem 3.1 of Hazra and Misra [15] under a similar set of conditions, as in Theorem 3.1, except that there condition (iv) of Theorem 3.1 is replaced by the condition $X \leq_c Y$ and $Y \leq_{rh} X$. Note that this assumption is much stronger as compared to (iv) because $\leq_c \Longrightarrow \leq_{c^*}$ and $\leq_{rh} \Longrightarrow \leq_{st}$, and the reverse implications are not necessarily true. The methodology used in the proofs of this paper differs from that in Hazra and Misra [15]. Thus, it is meaningful to study the results given in Proposition 3.1 and Theorem 3.1.

The following corollary follows from Theorem 3.1 and Lemma 2.3. It shows that the result stated in Theorem 3.1 indeed holds for k-out-of-n systems with i.i.d. components. Further, it is noted that Theorem 3.1 of Misra and Francis [26] is a particular case of this corollary ($k=l$ and $m=n$). In the sequel, we use $\tau_{k|n}$ and $\tau_{l|m}$ to represent the lifetimes of k-out-of-n and l-out-of-m systems, respectively, where k, l, m, and n are positive integers such that $1 \leq k \leq n$ and $1 \leq l \leq m$.

COROLLARY 3.1: *Suppose that* X_i*'s are i.i.d., and that* Y_j*'s are i.i.d. If* $X \leq_{c^*} Y$ *and* $Y \leq_{\mathrm{st}} X$*, then* $\tau_{k|n}(\boldsymbol{X}) \leq_{c^*} \tau_{l|m}(\boldsymbol{Y})$*, whenever* $l-k \geq \max\{0, m-n\}$*.*

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Remark 3.2: Let the assumptions of Corollary 3.1 hold. Then

i. $\tau_{k|n}(\boldsymbol{X}) \leq_{c^*} \tau_{l|n}(\boldsymbol{Y})$ for $1 \leq k \leq l \leq n$;
ii. $\tau_{k|n}(\boldsymbol{X}) \leq_{c^*} \tau_{k|m}(\boldsymbol{Y})$ for $1 \leq k \leq m \leq n$;
iii. $\tau_{l-r|m-r}(\boldsymbol{X}) \leq_{c^*} \tau_{l|m}(\boldsymbol{Y})$ for $1 \leq r \leq l \leq m$.

Below, we give a couple of examples to demonstrate applications of Proposition 3.1 and Theorem 3.1.

EXAMPLE 3.1: *Let us consider two coherent systems* $\tau_1(\boldsymbol{X}) = \min\{X_1, \max\{X_2, X_3\}\}$ *and* $\tau_2(\boldsymbol{Y}) = \min\{Y_1, Y_2, Y_3\}$ *formed by two different sets of components with lifetime vectors* $\boldsymbol{X} = (X_1, X_2, X_3)$ *and* $\boldsymbol{Y} = (Y_1, Y_2, Y_3)$, *respectively. Assume that* $X_i =_{st} X$ *and* $Y_i =_{\mathrm{st}} Y$, $i = 1, 2, 3$, *for some nonnegative random variables* X *and* Y. *Further, assume that* X_1, X_2, *and* X_3 *are independent, and* Y_1, Y_2, *and* Y_3 *are also independent. Then the domination functions of* $\tau_1(\boldsymbol{X})$ *and* $\tau_2(\boldsymbol{Y})$ *are given by*

$$h_1(p) = 2p^2 - p^3 \quad \textit{and} \quad h_2(p) = p^3, \quad 0 < p < 1,$$

respectively. From the above expressions, we get

$$\frac{(1-p)H_2'(p)}{H_2(p)} = 0 \quad \textit{and} \quad \frac{H_1(p)}{H_2(p)} = \frac{4-3p}{6-3p}, \quad 0 < p < 1.$$

Now, it can easily be verified that conditions (i) and (iii) of Theorem 3.1 hold (see also Example 3.1 of Navarro [31]). Using Theorem 3.1, we conclude that $\tau_1(\boldsymbol{X}) \leq_{c^*} \tau_2(\boldsymbol{Y})$ *holds for all* X *and* Y *such that* $X \leq_{c^*} Y$ *and* $Y \leq_{\mathrm{st}} X$. *An example of distribution functions of random variables* X *and* Y *satisfying* $X \leq_c Y$ *(and* $X \leq_{c^*} Y$ *in consequence) and* $Y \leq_{\mathrm{st}} X$ *is* $\bar{F}_X(x) = \exp\{-\alpha_1(x + \beta x^2)\}$, $x \geq 0$, *and* $\bar{F}_Y(x) = \exp\{-\alpha_2(x + \beta x^2)\}$, $x \geq 0$, *where* $0 < \alpha_1 \leq \alpha_2$ *and* $\beta > 0$.

EXAMPLE 3.2: *Let us consider two coherent systems* $\tau_1(\boldsymbol{X}) = \max\{X_1, X_2\}$ *and* $\tau_2(\boldsymbol{Y}) = \min\{Y_1, Y_2\}$ *formed by two different sets of components with lifetime vectors* $\boldsymbol{X} = (X_1, X_2)$ *and* $\boldsymbol{Y} = (Y_1, Y_2)$, *respectively. Assume that* $X_i =_{\mathrm{st}} X$ *and* $Y_i =_{\mathrm{st}} Y$, $i = 1,2$, *for some nonnegative random variables* X *and* Y *such that* $X \leq_{c^*} Y$ *and* $Y \leq_{\mathrm{st}} X$. *Further, assume that the distributions of* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are described by the same Clayton survival copula (with parameter 1)*

$$K(p_1, p_2) = \frac{p_1 p_2}{p_1 + p_2 - p_1 p_2}, \quad 0 < p_i < 1, \ i = 1, 2.$$

Then the domination functions of $\tau_1(\boldsymbol{X})$ *and* $\tau_2(\boldsymbol{Y})$ *are given by*

$$h_1(p) = K(p, 1) + K(1, p) - K(p, p) = \frac{3p - 2p^2}{2 - p}, \quad 0 < p < 1,$$

and

$$h_2(p) = K(p, p) = \frac{p}{2 - p}, \quad 0 < p < 1,$$

respectively. From the above expressions, we get

$$H_1(p) = \frac{2(1-p)(3-p)}{(3-2p)(2-p)}, \quad 0 < p < 1, \quad \textit{and} \quad H_2(p) = \frac{2}{2-p}, \quad 0 < p < 1.$$

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Consequently,

$$\frac{(1-p)H_1'(p)}{H_1(p)} = \frac{p^2-3}{(3-p)(2-p)(3-2p)}, \quad 0 < p < 1,$$

and

$$\frac{H_1(p)}{H_2(p)} = \frac{(1-p)(3-p)}{3-2p}, \quad 0 < p < 1.$$

By plotting $(1-p)H_1'(p)/H_1(p)$ *and* $H_1(p)/H_2(p)$ *over* $p \in (0,1)$*, it can be shown that conditions (i) and (ii) of Theorem 3.1 hold. Hence* $\tau_1(\boldsymbol{X}) \leq_{c^*} \tau_2(\boldsymbol{Y})$ *follows from Theorem 3.1.*

Remark 3.3: It is noted that the assumption $Y \leq_{\text{st}} X$, used in Proposition 3.1, Theorem 3.1, Corollary 3.1, and Remark 3.2, cannot be dropped as illustrated in Example 3.2 of Misra and Francis [26].

3.2. Relative Aging Based on Cumulative Reversed Hazard Functions

In this subsection, we compare two coherent systems with respect to the aging faster order in the cumulative reversed hazard rate.

In the following proposition, we consider that two coherent systems have the same domination function. We derive sufficient conditions under which the use of components having lifetime vector $\boldsymbol{X}$ makes the system age faster than if the components with lifetime vector $\boldsymbol{Y}$ are used.

PROPOSITION 3.2: *Assume that two coherent systems have the same domination function. Further suppose that the following conditions hold:*

i. $pR'(p)/R(p)$ *is positive and decreasing in* $p \in (0,1)$*;*

ii. $X \leq_{b^*} Y$ *and* $X \leq_{\text{st}} Y$*.*

Then $\tau(\boldsymbol{X}) \leq_{b^*} \tau(\boldsymbol{Y})$*.*

The following lemma, obtained in Hazra and Misra [15, Proposition 3.2], will be used in proving the next result.

LEMMA 3.2: *Let* X_i*'s be identically distributed. Then* $\tau_1(\boldsymbol{X}) \leq_b \tau_2(\boldsymbol{X})$ *if and only if* $R_1(p)/R_2(p)$ *is increasing in* $p \in (0,1)$*.*

The following theorem is an extension of the above result. Here, we compare two coherent systems with different domination functions based on the same survival copula. The proof follows from Proposition 3.2, Lemma 3.2 , and by using the facts that $\leq_{b^*}$ is transitive and $\leq_b \implies \leq_{b^*}$.

THEOREM 3.2: *Assume that two coherent systems have the same survival copula. Further, suppose that* $\{(i),(ii),(iv)\}$ *or* $\{(i),(iii),(iv)\}$ *below holds:*

i. $R_1(p)/R_2(p)$ *is increasing in* $p \in (0,1)$*;*

ii. $pR_1'(p)/R_1(p)$ *is positive and decreasing in* $p \in (0,1)$*;*

iii. $pR_2'(p)/R_2(p)$ *is positive and decreasing in* $p \in (0,1)$*;*

iv. $X \leq_{b^*} Y$ *and* $X \leq_{\text{st}} Y$*.*

Then $\tau_1(\boldsymbol{X}) \leq_{b^*} \tau_2(\boldsymbol{Y})$*.*

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Remark 3.4: It is mentioned here that the results given in Proposition 3.2 and Theorem 3.2 also follow from Theorem 3.2 of Hazra and Misra [15] under a similar set of conditions, as in Theorem 3.2, except that there condition (iv) above is replaced by $X \leq_b Y$ and $X \leq_{\mathrm{hr}} Y$. Note that this assumption is much stronger as compared to (iv) because $\leq_b \Longrightarrow \leq_{b^*}$ and $\leq_{\mathrm{hr}} \Longrightarrow \leq_{\mathrm{st}}$, and the reverse implications are not necessarily true. Thus, the study of the results given in Proposition 3.2 and Theorem 3.2 is worthy of investigation.

The following corollary immediately follows from Theorem 3.2 and Lemma 2.4. Note that Theorem 4.1 of Misra and Francis [26] is a particular case of this corollary ($k = l$ and $m = n$).

COROLLARY 3.2: *Suppose that X_i's are i.i.d., and that Y_j's are i.i.d. If $X \leq_{b^*} Y$ and $X \leq_{\mathrm{st}} Y$, then $\tau_{k|n}(\boldsymbol{X}) \leq_{b^*} \tau_{l|m}(\boldsymbol{Y})$, whenever $k - l \geq \max\{0, n - m\}$.*

Remark 3.5: Let the assumptions of Corollary 3.2 hold. Then

i. $\tau_{k|n}(\boldsymbol{X}) \leq_{b^*} \tau_{l|n}(\boldsymbol{Y})$ for $1 \leq l \leq k \leq n$;
ii. $\tau_{k|n}(\boldsymbol{X}) \leq_{b^*} \tau_{k|m}(\boldsymbol{Y})$ for $1 \leq k \leq n \leq m$;
iii. $\tau_{k|n}(\boldsymbol{X}) \leq_{b^*} \tau_{k-r|n-r}(\boldsymbol{Y})$ for $1 \leq r \leq k \leq n$.

The following example illustrates applications of results given in Theorem 3.2.

EXAMPLE 3.3: *Let us consider two coherent systems $\tau_1(\boldsymbol{X}) = \min\{X_1, X_2, \ldots, X_m\}$ and $\tau_2(\boldsymbol{Y}) = \min\{Y_1, Y_2, \ldots, Y_n\}$ formed by two different sets of components with lifetime vectors $\boldsymbol{X} = (X_1, X_2, \ldots, X_m)$ and $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)$, respectively, where $n \leq m$. Assume that $X_i =_{\mathrm{st}} X$, $i = 1, 2, \ldots, m$, and $Y_j =_{\mathrm{st}} Y$, $j = 1, 2, \ldots, n$, for some nonnegative random variables X and Y. Further, assume that the distribution of $\boldsymbol{X} = (X_1, X_2, \ldots, X_m)$ is described by the Gumbel–Hougaard survival copula*

$$K(p_1, p_2, \ldots, p_m) = \exp\left\{-\left(\sum_{i=1}^{m}(-\ln p_i)^{\theta}\right)^{1/\theta}\right\}, \quad 0 < p_i < 1,\ i = 1, 2, \ldots, m,\ \theta \geq 1,$$

and that of $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)$ is described by the Gumbel–Hougaard survival copula

$$K(p_1, p_2, \ldots, p_n) = \exp\left\{-\left(\sum_{i=1}^{n}(-\ln p_i)^{\theta}\right)^{1/\theta}\right\}, \quad 0 < p_i < 1,\ i = 1, 2, \ldots, n,\ \theta \geq 1.$$

Then the domination functions of $\tau_1(\boldsymbol{X})$ and $\tau_2(\boldsymbol{Y})$ are given by $h_1(p) = p^a$ and $h_2(p) = p^b$, respectively, where $a = m^{1/\theta}$, $b = n^{1/\theta}$, and $1 \leq b \leq a < \infty$. Thus, we have

$$R_1(p) = \frac{a(1-p)p^{a-1}}{1-p^a} \quad \text{and} \quad R_2(p) = \frac{b(1-p)p^{b-1}}{1-p^b}, \quad 0 < p < 1.$$

Let $d = a - b$ $(0 \leq d < a)$. Then, from the above expressions, we get

$$l(p) \overset{\text{def.}}{=} \frac{R_1(p)}{R_2(p)} = \frac{a}{b}\left(1 - \frac{1-p^d}{1-p^a}\right), \quad 0 < p < 1,$$

and

$$\frac{pR_1'(p)}{R_1(p)} = \frac{a - 1 - ap + p^a}{1 - p - p^a + p^{a+1}}, \quad 0 < p < 1.$$

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Clearly,

$$l'(p) = \frac{a}{b}\left[\frac{d(1-p^a)p^{d-1} - a(1-p^d)p^{a-1}}{(1-p^a)^2}\right], \quad 0 < p < 1.$$

Since, for $0 \le d < a$,

$$\frac{1-p^a}{1-p^d} \ge \frac{a}{d}p^{a-d}, \quad 0 < p < 1,$$

it follows that $l'(p) \ge 0$, *for all* $0 < p < 1$, *and hence* $R_1(p)/R_2(p)$ *is increasing in* $p \in (0,1)$. *Further, Example 4.1 of Hazra and Misra [15] shows that* $pR_1'(p)/R_1(p)$ *is positive and decreasing in* $p \in (0,1)$. *Now, using Theorem 3.2, we conclude that* $\tau_1(\boldsymbol{X}) \le_{b^*} \tau_2(\boldsymbol{Y})$ *holds for all random variables* X *and* Y *such that* $X \le_{b^*} Y$ *and* $X \le_{\mathrm{st}} Y$. *An example of distribution functions of random variables* X *and* Y *satisfying* $X \le_b Y$ *(and* $X \le_{b^*} Y$ *in consequence) and* $X \le_{\mathrm{st}} Y$ *is* $\bar{F}_X(x) = \exp\{-3x\}$, $x \ge 0$, *and* $\bar{F}_Y(x) = \exp\{-2x\}$, $x \ge 0$.

Remark 3.6: It is noted that the assumption $X \le_{\mathrm{st}} Y$, used in Proposition 3.2, Theorem 3.2, Corollary 3.2, and Remark 3.5, cannot be relaxed as shown in Example 4.2 of Misra and Francis [26].

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APPENDIX

PROOF: For any fixed $x \in (0,\infty)$, we have

$$\Delta_{\tau(\boldsymbol{X})}(x) = \int_0^x r_{\tau(\boldsymbol{X})}(u)\,du = \int_0^x r_X(u) H(\bar{F}_X(u))\,du.$$

Using the change of variable $z = -\ln \bar{F}_X(u)/(-\ln \bar{F}_Y(x))$, and noting that $-\ln \bar{F}_X(x)/(-\ln \bar{F}_Y(x)) \leq 1$ (since $Y \leq_{st} X$), we get

$$\Delta_{\tau(\boldsymbol{X})}(x) = \left(-\ln \bar{F}_Y(x)\right) \int_0^1 H\left(\bar{F}_Y^z(x)\right) I_{\left[0, \frac{-\ln \bar{F}_X(x)}{-\ln \bar{F}_Y(x)}\right]}(z)\,dz, \quad x > 0,$$

where $I_{[0,-\ln \bar{F}_X(x)/(-\ln \bar{F}_Y(x))]}(\cdot)$ be an indicator function of the set $[0, -\ln \bar{F}_X(x)/(-\ln \bar{F}_Y(x))]$, $x > 0$. Replacing $\boldsymbol{X}$ by $\boldsymbol{Y}$ in the above yields

$$\Delta_{\tau(\boldsymbol{Y})}(x) = \left(-\ln \bar{F}_Y(x)\right) \int_0^1 H\left(\bar{F}_Y^z(x)\right) dz, \quad x > 0.$$

For every fixed real number $\alpha \in (-\infty, \infty)$, we will analyze the sign changes of the difference

$$\frac{\Delta_{\tau(\boldsymbol{X})}(x)}{\Delta_{\tau(\boldsymbol{Y})}(x)} - \alpha,$$

or equivalently,

$$\Delta_{\tau(\boldsymbol{X})}(x) - \alpha\Delta_{\tau(\boldsymbol{Y})}(x) = \left(-\ln \bar{F}_Y(x)\right) \int_0^1 \xi_1(x,z)\eta_1(x,z)\,dz,$$

where, for $(x,z) \in (0,\infty) \times (0,1)$,

$$\xi_1(x,z) = H\left(\bar{F}_Y^z(x)\right)$$

and

$$\eta_1(x,z) = I_{\left[0, \frac{-\ln \bar{F}_X(x)}{-\ln \bar{F}_Y(x)}\right]}(z) - \alpha.$$

Since $-\ln \bar{F}_Y(x) > 0$, for all $x \in (0,\infty)$, it suffices to focus on the sign changes of the integral

$$\mathbb{I}_1(x) \stackrel{\text{def.}}{=} \int_0^1 \xi_1(x,z)\eta_1(x,z)\,dz, \quad \text{as } x \text{ traverses from 0 to } \infty.$$

To prove the result, it suffices to show that the integral $\mathbb{I}_1(x)$ changes sign at most once, and if the change of sign does occur, it is from negative to positive, as x traverses from 0 to ∞. We will make use of Lemma 2.2 to prove the aforesaid. We will first show that $\xi_1(x,z)$ is RR_2 in $(x,z) \in (0,\infty) \times (0,1)$, that is, for all $0 < z_1 < z_2 < 1$,

$$\frac{H\left(\bar{F}_Y^{z_2}(x)\right)}{H\left(\bar{F}_Y^{z_1}(x)\right)} \text{ is decreasing in } x \in (0,\infty).$$

This is equivalent to proving that, for all $x \in (0,\infty)$,

$$\tilde{r}_{W_2}(x)\left[(1-\bar{F}_{W_2}(x))\frac{H'(\bar{F}_{W_2}(x))}{H(\bar{F}_{W_2}(x))}\right] \geq \tilde{r}_{W_1}(x)\left[(1-\bar{F}_{W_1}(x))\frac{H'(\bar{F}_{W_1}(x))}{H(\bar{F}_{W_1}(x))}\right], \tag{A.1}$$

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where W_i is a random variable with the survival function $\bar{F}_{W_i}(\cdot) \equiv \bar{F}_Y^{z_i}(\cdot)$, for i=1,2. It can easily be verified that $W_2 \leq_{lr} W_1$. Consequently, $\tilde{r}_{W_2}(x) \leq \tilde{r}_{W_1}(x)$ and $\bar{F}_{W_2}(x) \leq \bar{F}_{W_1}(x)$, for all $x > 0$. On using this and condition (i), we get (A.1) and hence $\xi_1(x,z)$ is RR_2 in $(x,z) \in (0,\infty) \times (0,1)$. Also using condition (ii), we have $X \leq_{c^*} Y$, that is,

$$\frac{-\ln \bar{F}_X(x)}{-\ln \bar{F}_Y(x)} \text{ is increasing in } x \in (0,\infty).$$

This implies that

$$\eta_1(x,z) \text{ is increasing in } x \in (0,\infty), \quad \text{for all } z \in (0,1). \tag{A.2}$$

Further, note that

$$I_{\left[0, \frac{-\ln \bar{F}_X(x)}{-\ln \bar{F}_Y(x)}\right]}(z) \text{ is decreasing in } z \in (0,1), \quad \text{for all } x \in (0,\infty),$$

which is equivalent to the fact that, for every fixed $x \in (0,\infty)$, $\eta_1(x,z)$ changes sign at most once, and if the change of sign does occur, it is from positive to negative, as z traverses from 0 to 1. Now using Lemma 2.2, we conclude that $\mathbb{I}_1(x)$ changes sign at most once, and if the change of sign does occur, it is from negative to positive, as x traverses from 0 to ∞. Hence the result follows. ■

Proof of Proposition 3.2.: As the proof of this proposition is similar to that of Proposition 3.1, we will only provide an outline of the proof. For any fixed $x \in (0,\infty)$, we have

$$\widetilde{\Delta}_{\tau(\boldsymbol{X})}(x) = [-\ln F_Y(x)] \int_0^1 R\left(1 - F_Y^z(x)\right) I_{\left[0, \frac{-\ln F_X(x)}{-\ln F_Y(x)}\right]}(z)\, dz$$

and

$$\widetilde{\Delta}_{\tau(\boldsymbol{Y})}(x) = [-\ln F_Y(x)] \int_0^1 R\left(1 - F_Y^z(x)\right) dz.$$

For every fixed real number $\beta \in (-\infty,\infty)$, we analyze the sign changes of the difference

$$\widetilde{\Delta}_{\tau(\boldsymbol{X})}(x) - \beta\widetilde{\Delta}_{\tau(\boldsymbol{Y})}(x) = [-\ln F_Y(x)] \int_0^1 \xi_2(x,z)\eta_2(x,z)\, dz,$$

where, for $(x,z) \in (0,\infty) \times (0,1)$,

$$\xi_2(x,z) = R\left(1 - F_Y^z(x)\right)$$

and

$$\eta_2(x,z) = I_{\left[0, \frac{-\ln F_X(x)}{-\ln F_Y(x)}\right]}(z) - \beta.$$

Since $-\ln F_Y(x) > 0$, for all $x \in (0,\infty)$, it suffices to focus on the sign changes of the integral

$$\mathbb{I}_2(x) \overset{\text{def.}}{=} \int_0^1 \xi_2(x,z)\eta_2(x,z)\, dz, \quad \text{as } x \text{ traverses from 0 to } \infty.$$

On using condition (i), it can be shown that

$$\xi_2(x,z) \text{ is } \mathrm{TP}_2 \text{ in } (x,z) \in (0,\infty) \times (0,1). \tag{A.3}$$

Further, condition (ii) implies that

$$\eta_2(x,z) \text{ is decreasing in } x \in (0,\infty), \quad \text{for all } z \in (0,1). \tag{A.4}$$

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Again, note that

$$I_{\left[0,\frac{-\ln F_X(x)}{-\ln F_Y(x)}\right]} \text{ is decreasing in } z \in (0,1), \quad \text{for all } x \in (0,\infty).$$

This implies that, for all $x \in (0,\infty)$, $\eta_2(x,z)$ changes sign at most once, and if the change of sign does occur, it is from positive to negative, as z traverses from 0 to 1. On using this together with (A.3) and (A.4) in Lemma 2.2, we conclude that $\mathbb{I}_2(x)$ changes sign at most once, and if the change of sign does occur, it is from positive to negative, as x traverses from 0 to ∞. Hence the result follows. ■

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